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MASTER

Vacuum Periodicity in the Instanton Angle, Chiral Perturbation Theory, and Fractional Topological Charge

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Abstract

Following Crewther, the methods of chiral perturbation theory and Dashen's theorem are applied to the QCD quark mass matrix. Stable domains of θ-vacua are found. The period of the system with respect to the instanton angle is 2π. It is concluded that non-integer topological charge is not required by chiral symmetry breaking. An argument to the contrary by the present authors and by Crewther is refuted.

DISCLAIMER

I. Introduction

In this note, following Crewther,[1] we vary the vacuum energy of QCD with respect to arbitrary chiral SU(N) rotations at fixed instanton angle θ. According to Dashen's[2] theorem this identifies the correct perturbative Hamiltonian and the perturbative vacuum when an absolute minimum is achieved. For $\theta = 0$ and $\theta = \pm N\pi$ this repeats Dashen's calculation and for θ infinitesimal this repeats the calculation of Baluni[3] and Crewther et al.[4] for the CP violating Hamiltonian. In all cases Nuyts'[5] theorem is verified. The vacuum energy in the stable θ-vacua is calculated for various mass cases. It is found that the vacuum periodicity is 2π in θ. We comment on confusions in the literature concerning chirality selection rules and fractional topological charge. We show that all physical matrix elements have period 2π in θ and that fractional topological charge is not required.

II. Notation

The chiral symmetry breaking Hamiltonian is

$$\epsilon H' = \bar{q}_L M q_R + \text{h.c.} = \bar{q}\frac{M+M^+}{2}q + \bar{q}\frac{M-M^+}{2}\gamma_5 q \quad . \tag{1}$$

It is assumed that chiral symmetry but not CP is spontaneously broken in $O(\epsilon^0)$ with condensate condition

$$\langle \bar{q}_{Li}\, q_{Rj}\rangle = -\frac{\Delta}{2}\,\delta_{ij} \,, \qquad \Delta > 0 \tag{2}$$

in the perturbative vacuum. There are chiral SU(N) and a U(1) instanton phase possible which may be assigned either to the mass matrix or the condensate condition: We prefer the former procedure. (In the notation of Baluni[3] our θ is his observable $\bar{\theta}$. In section IV we discuss this distinction more precisely).

The chiral U(1) current,

$$j_\mu^5 = \sum_i \bar{q}_i \gamma_\mu \gamma_5 q_i = \sum \bar{q}_i \gamma_\mu \gamma_5 (q_{iL} + q_{iR}) = j_\mu^R - j_\mu^L \qquad (3)$$

generates the phase transformation

$$e^{-iQ^5\alpha_o} \bar{q}_{Li} M_{ij} q_{Rj} e^{+iQ^5\alpha_o} = e^{2i\alpha_o} \bar{q}_{Li} M_{ij} q_{Rj} \qquad (4)$$

where the instanton angle is θ, with $e^{i\alpha_o Q^5} |\theta\rangle = |\theta + 2N\alpha_o\rangle$ and $e^{i(\theta/2N)Q^5} | 0\rangle = |\theta\rangle$.

With respect to the condensate condition (2) it is sufficient to consider a positive mass matrix rotated by the (fixed) instanton angle θ and variable diagonal phases, $M_{ij} = \delta_{ij} \exp\left[i\frac{\theta}{N} + i\alpha_i\right] m_i$ subject to the unimodular condition $\sum \alpha_i = 0 \bmod (2\pi)$.

Dashen's theorem[1] requires that

$$\langle 0|\epsilon H'|0\rangle = \langle 0|\bar{q}_{Li} M_{ij} q_{Rj}|0\rangle + \text{h.c.} = -\frac{\Delta}{2} \operatorname{tr}(M + M^+) \equiv E(\theta;\alpha_i) \qquad (5)$$

be an absolute minimum at some $\{\bar{\alpha}_i\}$ where the correct perturbative Hamiltonian is $\bar{q}_L M(\theta;\bar{\alpha}_i) q_R + \text{h.c.}$ and the vacuum energy is

$$E(\theta) \equiv E(\theta;\bar{\alpha}_i)$$

Maintaining Dashen's constraint for all θ, assures that all θ-vacua are stable against the addition of quark mass term to the theory.

III. Some Specific Examples.

Consider the illuminating special case

$$M = e^{i(\theta/3)} \begin{pmatrix} e^{i\alpha} m_\ell & 0 & 0 \\ 0 & e^{i\beta} m_\ell & 0 \\ 0 & 0 & e^{-i(\alpha+\beta)} m_s \end{pmatrix} \qquad (6)$$

with

$$E(\theta;\alpha,\beta) = -\Delta\left[m_\ell \cos\left(\frac{\theta}{3}+\alpha\right) + m_\ell \cos\left(\frac{\theta}{3}+\beta\right) + m_s \cos\left(\frac{\theta}{3}-\alpha-\beta\right)\right]. \quad (7)$$

The extremal conditions, $\frac{\partial E}{\partial\alpha} = \frac{\partial E}{\partial\beta} = 0$, are solved by $\alpha = \beta$ and an implicit relation between θ and α,

$$m_\ell \sin\left(\frac{\theta}{3}+\alpha\right) = m_s \sin\left(\frac{\theta}{3}-2\alpha\right) \quad . \quad (8)$$

The condition for a minimum is that the eigenvalues of the matrix $\partial^2 E/\partial\alpha_i\partial\alpha_j$ be positive:

$$\lambda^- = \Delta m_\ell \cos\left(\frac{\theta}{3}+\alpha\right) > 0 \quad (9)$$

$$\lambda^+ = \Delta\left[m_\ell \cos\left(\frac{\theta}{3}+\alpha\right) + 2m_s \cos\left(2\alpha-\frac{\theta}{3}\right)\right] > 0 \quad . \quad (10)$$

When $m_\ell = m_s = m$ the solutions to Eq.(8) are $\bar{\alpha}_k = \frac{2\pi k}{3}$, $k = -1,0,+1$, resulting in the stability conditions

$$\frac{\lambda^+}{3} = \lambda^- = \Delta m \cos\left(\frac{\theta}{3}+\frac{2\pi k}{3}\right) > 0 \quad (11)$$

and vacuum energy

$$E(\theta,\bar{\alpha}_k) = -3\Delta m \cos\left(\frac{\theta}{3}+\frac{2\pi k}{3}\right) \quad . \quad (12)$$

Referring to Fig. 1 it is seen that the absolute minimum of E corresponds to $k=-1$, $-3\pi<\theta<-\pi$; $k=0$, $-\pi<\theta<+\pi$; and $k=1$, $\pi<\theta<3\pi$. This behavior then repeats for all θ. The period of the stable vacuum is thus 2π, and not the naive 6π periodicity of the unconstrained mass matrix. The natural domain of θ is the unit circle with a discontinuity in $\frac{dE}{d\theta}$ at π (mod 2π).

The above behavior is easily generalized to N degenerate quarks with $\bar{\alpha} = \frac{2\pi k}{N}$, $k = 0,1,2,\ldots,N-1$. The stable vacuum energy is the minimum with respect to k of

$$E(\theta;\bar{\alpha}_k) = -\Delta N \cos\left(\frac{\theta}{N} + \frac{2\pi k}{N}\right) \quad , \tag{13}$$

which has period 2π. Note in the limit of large N (number of flavors) the stable vacuum energy becomes independent of θ.

In the interesting case $m_\ell/m_s << 1$ Eq.(8) may be solved perturbatively; when the resulting relations between $\bar{\alpha}$ and θ are imposed, we find that the vacuum energy is the minimum with respect to $k = 0,1$ of

$$E(\theta) = -\Delta m_s \left[1 + \frac{2m_\ell}{m_s} (-1)^k \cos\frac{\theta}{2} + \frac{1}{2}\left(\frac{m_\ell}{m_s}\right)^2 \sin^2\frac{\theta}{2} + \ldots\right] \tag{14}$$

where the CP violating part of the Hamiltonian is, in the interval $-\pi < \theta < \pi$,

$$\epsilon H'_{CP} = \bar{q}\frac{M - M^+}{2}\gamma_5 q = im_\ell \sin\frac{\theta}{2}\left(1 - \frac{1}{2}\cos\frac{\theta}{2}\frac{m_\ell}{m_s} + \ldots\right)\bar{q}\gamma_5 q \tag{15}$$

verifying Nuyts'[5] theorem. For $\theta << 1$,

$$\epsilon H'_{CP} \approx im_\ell \frac{\theta}{2}\left(1 - \frac{1}{2}\frac{m_\ell}{m_s} + \ldots\right)\bar{q}\gamma_5 q \tag{16}$$

which agrees with Baluni[3] and Crewther et al.,[4] in this special case.

Referring to the sketch of Fig. 2 it is again verified that the periodicity of the energy density in the stable vacua is 2π.

IV. General Proof of 2π Periodicity

The most general hermitean chiral symmetry breaking hamiltonian can be written

$$\epsilon H'(W,\bar{\theta}) = e^{i\bar{\theta}/N}\bar{q}_L W_L^\dagger V_L m_d V_R^\dagger W_R q_R + \text{h.c.} \tag{17}$$

where

$$M = e^{i(\bar{\theta}-\theta)/N}\, V_L\, m_d\, V_R^\dagger ; \qquad V_{L(R)} \in SU(N) \tag{18}$$

and $\bar{\theta} = \theta + \arg\det M$. m_d is a real non-negative diagonal matrix. Using Eq.(2), we find

$$E(W,\bar{\theta}) = -\frac{\Delta}{2} e^{i\bar{\theta}/N}\mathrm{Tr}\,(m_d V_R^\dagger W V_L) + \text{h.c.} \tag{19}$$

where

$W = W_R W_L^\dagger$. The W which minimizes Eq.(19) is $W = U = U_R U_L^\dagger$ and must have the form

$$V_R^\dagger U V_L = \delta_{ij} e^{i\bar{\alpha}_i} ; \qquad \sum \bar{\alpha}_i = 0 \bmod (2\pi) \tag{20}$$

Thus

$$E(U,\bar{\theta}) = -\Delta \sum_{i=1} m_d \quad \cos\left(\frac{\bar{\theta}}{N} + \bar{\alpha}_i\right). \tag{21}$$

Eq.(20) follows from Nuyts' Theorem, which fixes the imaginary part of $e^{i\bar{\theta}/N}\, U_L^\dagger V_L\, m_d\, V_R^\dagger U_R \equiv e^{i\bar{\theta}/N}\mathcal{M}$, and the fact that $V_R^\dagger W V_L$ is unitary.

It is important to note that while minimizing E fixes U it does not fix U_R and U_L separately; there remains an SU(N) invariance X which is seen by replacing $U_{R,L}$ by $\overline{U}_{R(L)} = U_{R(L)} X^{\dagger}$. (Alternatively, this also follows from Nuyts' theorem

$$\mathcal{M}e^{i\overline{\theta}/N} - \mathcal{M}_e^{\dagger} e^{-i\overline{\theta}/N} = i\mu I$$

Where I is the identify.)

We now have the stable perturbation $\epsilon H'(U,\overline{\theta}) = e^{i\overline{\theta}/N}\overline{q}_L \mathcal{M} q_R + e^{-i\overline{\theta}/N}\overline{q}_R \mathcal{M}^{\dagger} q_L$. Again, by Nuyts' theorem $[\mathcal{M}^{\dagger},\mathcal{M}] = 0$ and we can now choose the free SU(N) matrix X to diagonalize $\mathcal{M}$, resulting in the form

$$\epsilon H'\left(U(\overline{\alpha}_i),\overline{\theta}\right) = \sum_{i=1}^{N} \overline{q}_{L,i}\,(m_d)_i\, e^{i(\overline{\theta}/N - \overline{\alpha}_i)} q_{R,i} + \text{h.c.}$$

In this form it is clear that if $\overline{\alpha}_i$ minimizes $E(W,\overline{\theta})$ then because $\sum_i \overline{\alpha}_i = 0 \bmod (2\pi)$ we have an equivalent set of $\overline{\alpha}_i'$, $\overline{\alpha}_i' = \overline{\alpha}_i - \frac{2\pi}{N}$, that minimizes $E(W;\overline{\theta}+2\pi)$ so that the stable pertubations at $\overline{\theta}$ and $\overline{\theta}+2\pi$ are equal:

$$\epsilon H'(U;\overline{\theta}+2\pi) = \epsilon H'(U;\theta)$$

V. Chirality Selection Rules and Fractional Topological Charge

The unconstrained vacua depend on θ and α_i, $|\theta;\alpha_i\rangle$, whereas the stable physical vacua depend only on θ, $|\theta;\overline{\alpha}_i(\theta)\rangle$. Taking matrix elements of

$$e^{-i(\theta/2N)Q_o^5}\,\overline{q}\,\frac{1\pm\gamma_5}{2}\,q\,e^{i(\theta/2N)Q_o^5} = e^{\pm i(\theta/N)}\,\overline{q}\,\frac{1\pm\gamma_5}{2}\,q \tag{23}$$

in the unconstrained states we have

$$\langle\theta;\alpha_i|\overline{q}\,\frac{1\pm\gamma_5}{2}\,q\,|\theta;\alpha_i\rangle = e^{\pm i(\theta/N)}\,\langle 0;\alpha_i|\overline{q}\,\frac{1\pm\gamma_5}{2}\,q\,|0;\alpha_i\rangle \tag{24}$$

from which one correctly concludes that the unphysical vacua have fundamental period $2\pi N$. In the basis of topological charge

$$|\theta;\alpha_i\rangle = \sum_\nu e^{i\nu\theta}|\nu;\alpha_i\rangle$$

we concluded in Ref. 8, as Crewther did in Ref. 9 that $\nu = N^{-1}$ is required to sustain this period. These statements are formally correct but they apply to the wrong vacuum. In a physical vacuum the matrix element of Eq.(23) yields

$$\langle\theta;\bar{\alpha}_i(0)|\bar{q}\,\frac{1\pm\gamma_5}{2}\,q|\theta;\bar{\alpha}_i(0)\rangle = e^{\pm i(\theta/N)}\langle 0;\bar{\alpha}_i(0)|\bar{q}\,\frac{1\pm\gamma_5}{2}\,q|0;\bar{\alpha}_i(0)\rangle \qquad (25)$$

from which one can no longer make the $2\pi N$ periodicity claim because the vacuum at the left is not physical; the same statement is true for the corresponding chirality selection rule (also appearing in Refs. 8, 9, and 10)

$$(2N\nu \mp 2)\langle\bar{q}\,\frac{1\pm\gamma_5}{2}\,q\rangle_\nu = 0 \qquad (26)$$

which is the ν-space statement of Eq.(24).

All other conclusions in Ref. 8 concerning how the anomalous Ward identities are satisfied without the need of a ninth Goldstone particle are correct; a careful review of those arguments show that they continue to carry through in the neighborhood of the physical vacuum.

A seemingly more subtle argument for fractional topological charge is advanced in Refs. 1 and 9, where the two vacua at π approached from below and above are studied. Allowing for the possible need for fraction charge, Crewther writes

$$|vac\rangle_1 = \oint_m e^{im\pi}|m\rangle$$

$$|vac\rangle_2 = \oint_m e^{-im\pi}|m\rangle \quad . \qquad (27)$$

He then properly requires $|vac\rangle_1 \neq e^{i\xi}|vac\rangle_2$ and concludes from

$$\sum_m \{e^{im\pi} - e^{i(\xi - m\pi)}\}|m\rangle \neq 0 \tag{28}$$

that $\nu = m' - m \neq$ integer when m and m′ contribute to (27). This argument is false because the physical vacua need to be parameterized not only by θ but also by $\bar{\alpha}_1$, or equivalently, the integer k discussed above which numbers the elements of Figs. 1 and 2. Then

$$|vac\rangle_1 = \sum_m e^{im\pi}|m;k=0\rangle$$

$$|vac\rangle_2 = \sum_m e^{-im\pi}|m;k=1\rangle$$

and no longer does a form like Eq.(28) hold.

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References

1. R. J. Crewther, Phys. Lett. 93B, 75 (1980).

2. R. Dashen, Phys. Rev. D 3, 1879 (1971).

3. V. Baluni, Phys. Rev. D 19, 2227 (1979).

4. R. J. Crewther, P. DiVecchia, G. Veneziano, and E. Witten, Phys. Lett. 88B, 123 (1979).

5. J. Nuyts, Phys. Rev. Lett. 26, 1604 (1971).

6. C. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).

7. We thank K. Lane for suggesting the general arguments of this section.

8. W. Palmer and S. Pinsky, Phys. Rev. D 22, 1437 (1980).

9. R. Crewther, International Summer Institute on Theoretical Physics, Kaiserslautern, Germany, August, 1979.

10. S. Coleman, "The Uses of Instantons," 1977 International School of Subnuclear Physics, Ettore Majorana.

Figure Captions

Fig. 1 Vacuum energy for three equal mass quarks. The solid lines are the physical vacuum.

Fig. 2 Vacuum energy for two light, one heavy quark. The solid lines are the physical vacuum.

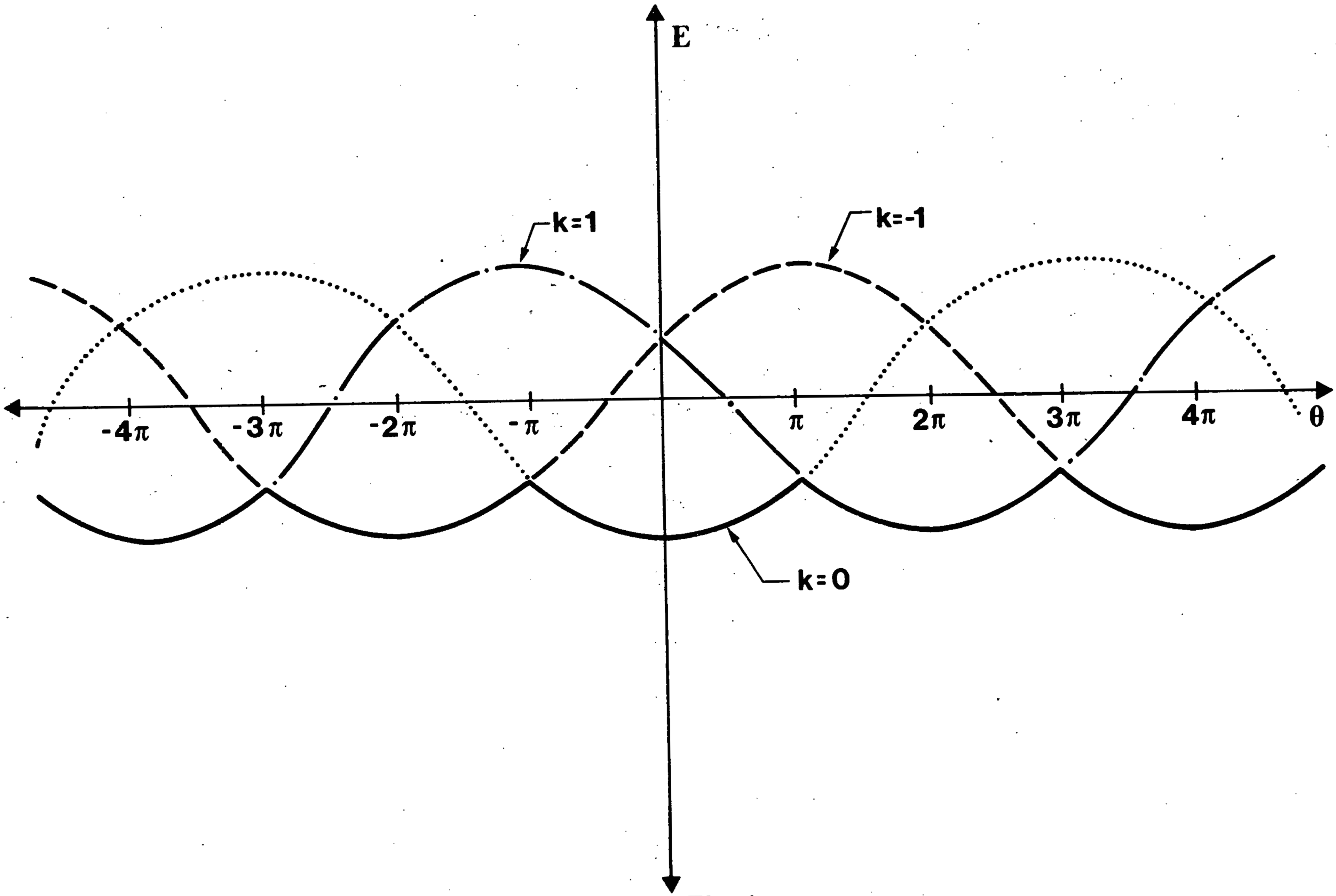

Fig.1

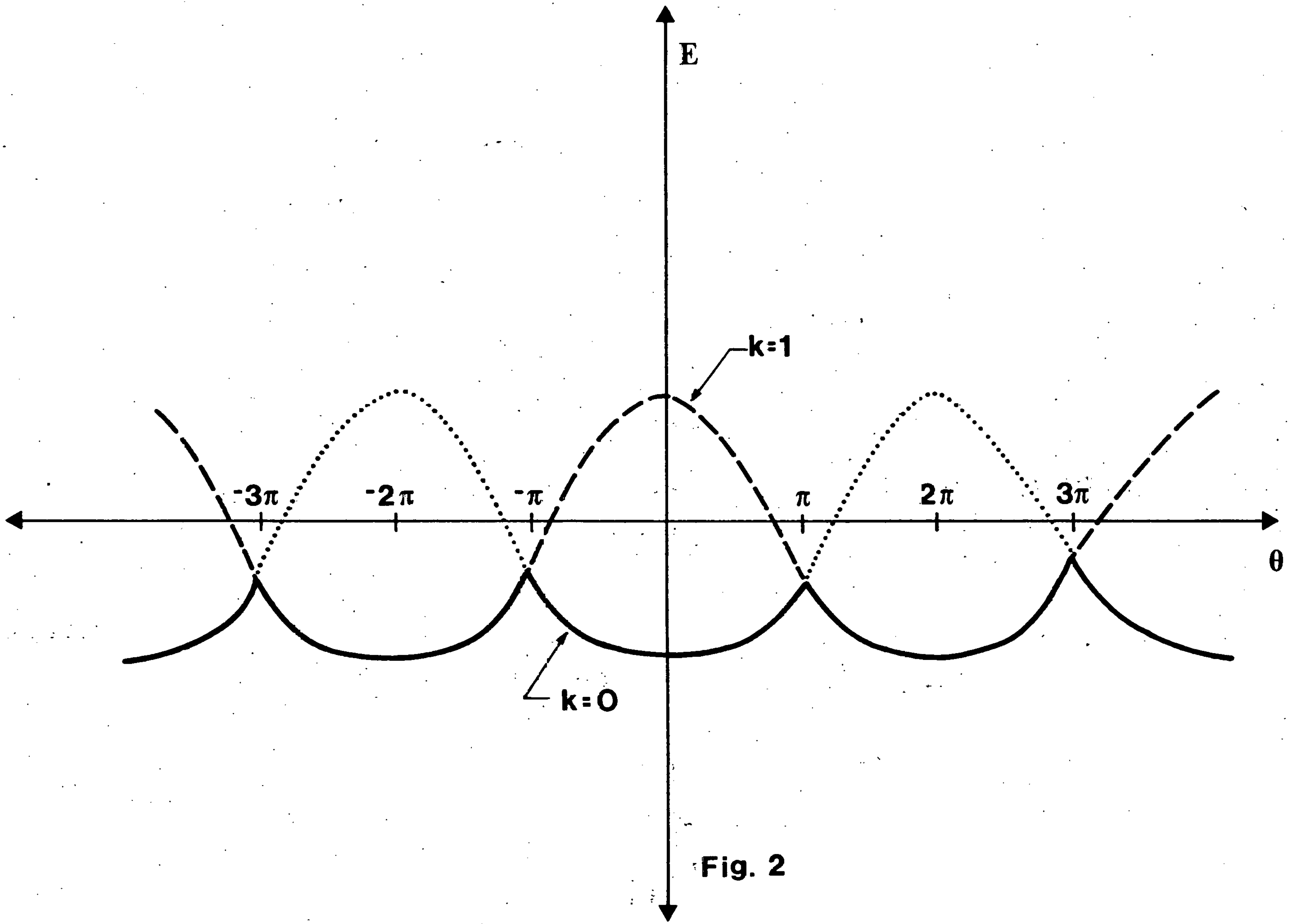

Fig. 2