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Two-Dimensional Transport of Tokamak Plasmas

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ABSTRACT

A reduced set of two-fluid transport equations is obtained from the conservation equations describing the time evolution of the differential particle number, entropy, and magnetic fluxes in an axisymmetric toroidal plasma with nested magnetic surfaces. Expanding in the small ratio of perpendicular to parallel mobilities and thermal conductivities yields as solubility constraints one-dimensional equations for the surface-averaged thermodynamic variables and magnetic fluxes. Since Ohm's law $\mathbf{E} + \mathbf{u} \times \mathbf{B} = \mathbf{R}'$, where R' accounts for any nonideal effects, only determines the particle flow relative to the diffusing magnetic surfaces, it is necessary to solve a single two-dimensional generalized differential equation, $(\partial/\partial t)\nabla\psi\cdot(\nabla p - \mathbf{J} \times \mathbf{B}) = 0$, to find the absolute velocity of a magnetic surface enclosing a fixed toroidal flux. This equation is linear but nonstandard in that it involves flux surface averages of the unknown velocity. Specification of $\mathbf{R}'$ and the cross-field ion and electron heat fluxes provides a closed system of equations. A time-dependent coordinate transformation is used to describe the diffusion of plasma quantities through magnetic surfaces of changing shape.

I. INTRODUCTION

The importance of developing a self-consistent two-dimensional description of the temporal evolution of axisymmetric tokamak equilibria on a resistive time scale which is long compared to the Alfvén transit time has been previously recognized.[1] Recent theoretical investigations of stable "high β" configurations have intensified interest in this subject. Indeed, since the Shafranov shift $\Delta_s \sim (\beta_p \epsilon) a$, where a is the plasma radius, β_p is the poloidal β, and $\epsilon = a/x_0$ is the inverse aspect ratio, we see that for $\beta_p \gtrsim \epsilon^{-1}$, the standard low β one-dimensional model of concentric circular flux surfaces ceases to be an adequate approximation.

Here, a two-dimensional (2-D) configuration is defined to be one in which the flux function $\psi(x, z)$ (magnetic surfaces are defined by ψ = constant) depends in an irreducible way on both the radial and vertical cylindrical coordinates (x, z). The concept of 2-D becomes particularly important when the surfaces change shape in time. There are at least four distinct reasons for requiring a two-dimensional description of plasma transport.

First, the 2-D (noncircular) geometry of the flux surfaces modifies the value of the plasma transport coefficients. The resistive transport coefficients for diffusion normal to magnetic surfaces involve surface averages of various quantities which are sensitive to the variation of the magnitude of $\underset{\sim}{B}$ along $\underset{\sim}{B}$. Thus, for example, in high β equilibria, where mod $\underset{\sim}{B}$ contours tend to align with the magnetic field, the neoclassical transport coefficients, as well as anomalous coefficients driven by magnetically trapped particles, could be substantially reduced from their usual 1-D values. Vertical elongation of the plasma can also reduce the transport coefficients from their circular value.

Secondly, knowledge of the two-dimensional configuration of the plasma is necessary to apply boundary conditions and to compute the effects on the plasma of externally maintained heat and particle sources. Magnetic boundary conditions must be ultimately related to the currents in the poloidal field coils, whose spatial distribution is inherently two dimensional. The location of pressure surfaces, which approximately correspond with density contours, is important to determine heating and particle deposition profiles due to neutral beam injection.

Thirdly, a self-consistent two-dimensional transport calculation is necessary to obtain realistic pressure and magnetic field profiles to use in magnetohydrodynamic (MHD) stability calculations. Once a stable initial equilibrium configuration is specified, the time dependent transport equations determine how that equilibrium evolves in time. In particular, the steady state solutions of the transport equations correspond to a unique prescription for the equilibrium profiles. The importance of these resistive steady states in determining stable stationary profiles of the otherwise arbitrary pressure and toroidal magnetic field functions $p(\psi)$ and $g(\psi) \equiv xB_T$ has been previously emphasized.[1,2]

Finally, a self-consistent 2-D calculation includes potentially important inductive contributions to the flux surface-averaged normal plasma flow arising from $\partial B/\partial t$. The maintenance of the lowest order (in resistivity η) pressure balance $\nabla p = J \times B$ on the resistive time scale can lead to deformations of the plasma magnetic surfaces, with associated self-consistent inductive $E^{(A)} \times B$ flows $(E^{(A)} = -\partial A/\partial t)$ which may be comparable to the resistive diffusion during the fast transient phase of the discharge.[1] This inductive convection of the magnetic surfaces must be determined before the absolute particle flux can be evaluated.

The inductive convection is generally separable from the resistive plasma motion (although it is itself affected self-consistently by diffusion) and is shown in this paper to satisfy a linear 2-D integro-differential equation. This decomposition is physically fundamental, for it allows the treatment of very general forms of Ohm's law for which previous methods are not readily adaptable. Thus, our procedure embraces neoclassical transport,[3] for which the stress anisotropy is an important part of Ohm's law, with no additional difficulty than that required for the simple collisional transport models used previously.[1,2,4,5,6] In addition, it is shown that only the surface averages of the components of Ohm's law within the magnetic surfaces are needed to close the moment equations. This is significant since spatially local expressions for the fluxes in the long mean free path regimes are generally difficult to obtain.

A time dependent magnetic flux coordinate transformation is used to obtain coupled 1-D evolution equations for the plasma and the fields, and a single linear 2-D equation to determine the flux surface velocity. Our particular choice of coordinates is motivated by the tokamak ordering $\epsilon \ll 1$ and enables isolation of physically distinct processes leading to plasma transport. For example, the velocity of a fluid element is shown explicitly to consist of three parts: (i) resistive diffusion relative to the flux surfaces; (ii) distortions in which flux surfaces change their shape while conserving toroidal flux; (iii) a slower motion in which the amount of toroidal flux within these surfaces changes.

The coordinate transformation determining the magnetic surface geometry evolves simultaneously in time with the plasma and the fields. The conversion from thermodynamic quantities (which are required to evaluate the particle and heat fluxes) to adiabatic variables (which are advanced in time by 1-D resistive diffusion equations) is determined by the coordinate mapping x = x(, ,) where labels a magnetic surface and θ is a poloidal angle coordinate (cf., Section III). The real space distribution of the plasma, which is needed to apply the boundary conditions induced by the external circuit, is also determined once the coordinate transformation is known.

In Section II, we give the primitive form of the field equations and Maxwell's equations. A time-dependent magnetic flux coordinate transformation is introduced in Section III. This is used in Sections IV and V to reduce the time-dependent transport problem to the solution of five coupled one-dimensional time evolution equations together with a single linear two-dimensional equation. We summarize these equations in Section V and discuss their closure in Section VI. Closure requires additional results from a kinetic treatment. Explicit closure models for collisional (Pfirsch-Schlüter) and collisionless (banana-regime) plasmas are presented in Section VIII. Both conducting wall and "free boundary" boundary conditions are discussed in Section VII. Finally, in Section IX, we summarize and compare the present approach with earlier work.

II. FUNDAMENTAL EQUATIONS

The basic equations describing the time evolution of a magnetically confined plasma are comprised of field equations (Maxwell's equations):

$$(\partial B/\partial t) = -\nabla \times E \quad , \tag{1a}$$

$$4\pi J = \nabla \times B \quad , \tag{1b}$$

$$\nabla \cdot B = 0 \quad , \tag{1c}$$

and fluid equations, which are the appropriate velocity moments of the Vlasov-Fokker Planck equations for electrons and ions. The fluid equations are distinguished by their parity in v (the microscopic particle velocity). The even parity moment equations represent the conservation of particles and energy:

$$(\partial n_j/\partial t) + \nabla \cdot (n_j u_j) = S_{nj} \quad , \tag{2a}$$

$$(3/2)(\partial p/\partial t) + \nabla \cdot \sum_j [q_j + (5/2) p_j u_j] = J \cdot E + S_p + \Pi \quad , \tag{2b}$$

$$(3/2)(\partial p_e/\partial t) + \nabla \cdot [q_e + (5/2) p_e u_e] = J \cdot E + Q_{\Delta e} + S_e + \Pi - u_i \cdot \nabla p_i + \pi_i : \nabla u_i \quad . \tag{2c}$$

Here, n_j is the particle density, u_j is the macroscopic fluid velocity, q_j is the conductive heat flux, p_j is the pressure, $p = p_e + p_i$, S_{nj}, S_p, S_e are particle and pressure sources,

$$\Pi = -\sum_{e,i} \nabla\cdot(\underset{\sim}{u}_j\cdot\underset{\approx}{\pi}_j) \quad , \tag{2d}$$

$$Q_{\Delta e} = 3(m_e/m_i)(n_e/\tau_{ei})(T_i - T_e) \quad , \tag{2e}$$

and $\underset{\approx}{\pi}_j$ is the viscous stress tensor. The relation $p_j >> (1/2)m_j n_j u_j^2$ has been assumed for the subsonic flows of interest.

The odd parity moments yield the force balance equation for each species:

$$\underset{\sim}{E} + \underset{\sim}{u}_j \times \underset{\sim}{B} = -\underset{\sim}{R}_j/(n_j e_j) + m_j \underset{\sim}{T}(\underset{\sim}{u}_j)/e_j \tag{3a}$$

where

$$\underset{\sim}{T}(\underset{\sim}{u}) = (\partial \underset{\sim}{u}/\partial t) + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} \quad , \tag{3b}$$

$$\underset{\sim}{R}_j = -\nabla p_j - \nabla\cdot\underset{\approx}{\pi}_j + \underset{\sim}{F}_j \quad , \tag{3c}$$

$$\underset{\sim}{F}_j = \int m_j \underset{\sim}{v} C_j \, d\underset{\sim}{v} \quad , \tag{3d}$$

and C_j is the collision operator of species (j). $\underset{\sim}{R}_j$ represents the departure from ideal ($\underset{\sim}{E} \times \underset{\sim}{B}$) drift motion, $\underset{\sim}{T}$ is the inertial term, and $\underset{\sim}{F}_j$ is the collisional friction between electrons and ions. Equation (3a) for electrons is the general form of Ohm's law. Summing $e_j n_j \times$ Eq. (3a) and using quasineutrality, $\sum_{e,i} n_j e_j = 0$, yields the net force balance

$$\rho \underset{\sim}{T}(\underset{\sim}{u}) + \nabla\cdot(\underset{\approx}{\pi}_i + \underset{\approx}{\pi}_e) = -\nabla p + \underset{\sim}{J} \times \underset{\sim}{B} \quad , \tag{4}$$

where $\rho = \sum_{e,i} m_j n_j$, $\underset{\sim}{u} = \sum_{e,i} m_j n_j \underset{\sim}{u}_j/\rho$, and $\underset{\sim}{J} = \sum_{e,i} n_j e_j \underset{\sim}{u}_j$ is the current density.

We seek to solve Eqs. (1) through (4) on the time scale $\tau_R \sim (\partial/\partial t)^{-1} \sim \eta^{-1}$ characterizing resistive diffusion. Here, η is the resistivity. This time is much longer than the Alfvén transit time $\tau_A \sim a/(B^2/4\pi\rho)^{1/2}$. Indeed, using the classical resistive time to estimate τ_R yields

$$\tau_A/\tau_R \sim \delta \sim (\rho_e/a)(\nu_{ii}/\Omega_i)\beta^{1/2} << 1 \quad , \tag{5}$$

where $\rho_e = (2T_e/m_e)^{1/2}\ \Omega_e^{-1}$ is the electron thermal gyroradius, $\Omega_j = e_jB/m_j$, ν_{ii} is the ion-ion collision frequency, and $\beta = nT_e/(B^2/8\pi) \sim 1$ (in the δ expansion). We introduce the small parameter $\Delta \sim \rho_e/a$, and note that $\delta \sim \Delta^2$ for a magnetized plasma. Here $\nu_{ii}/\Omega_i \sim \Delta$ if the square root of the ion-electron mass ratio is neglected in the gyroradius and collision frequency expansion. Adopting the drift-ordering $|u_j^S|/v_A \sim \Delta$, $|u_j^N|/v_A \sim \delta$, $|p_{\parallel j} - p_{\perp j}|/p_j \sim \Delta$, $\tau_A(\partial/\partial t) \sim \delta$, where u_j^S, u_j^N are the components of velocity within and normal to a magnetic surface, respectively, we find to lowest order in Δ :

$$\nabla\cdot(n_j u_{j0}^S) = 0 \quad , \tag{6a}$$

$$\nabla\cdot[q_{j0} + (5/2)p_j u_{j0}^S] = 0 \quad , \tag{6b}$$

$$\nabla p = J \times B \quad , \tag{6c}$$

where $u_j^S = u_{j0}^S + u_{j1}^S$ with $u_{j1}^S/u_{j0}^S \sim \Delta$. Equations (6a) and (6b) describe the inability of density and temperature perturbations within a magnetic surface to be maintained for times longer than τ_A (cf., Section IV). On the resistive time scale, inertial terms in Eq. (4) are negligible as $\delta \to 0$, and the plasma evolves in time through a sequence of equilibria satisfying the static force balance Eq. (6c). Note that by eliminating the fast Alfvénic motion from Eq. (4), the mathematical structure of the problem has been altered from a "marching" hyperbolic equation for u (on the fast time scale) to the constraint Eq. (6c) whose satisfaction on the longer resistive time scale determines, in the sense to be described, the normal component of u as the solution to a global boundary value problem.[1]

To order , Eqs. (1) through (3) remain unaltered, with $u_j^S = u_{j1}^S$ in the left hand side of Eq. (2) and $T(u_j) = 0$ in Eq. (3a). The inertial and viscous terms in Eq. (4) determine the poloidal and toroidal plasma angular velocities, which are indeterminate surface constants[2,5] to order Δ. For classical collisional transport, these surface functions cancel from the expression for the normal mass flux.[1,2,5] However, in neoclassical transport for a long mean free path plasma, the determination of the poloidal rotation is necessary and has been calculated elsewhere.[7]

III. MAGNETIC FLUX COORDINATES AND EQUILIBRIUM RELATIONS

The equilibrium Eq. (6b) implies $B\cdot\nabla p = 0$. If the constant pressure contours form nested toroidal surfaces, a time dependent coordinate transformation may be constructed utilizing $\psi(x, t)$ as a coordinate to label magnetic flux and pressure surfaces, $B\cdot\nabla\psi = 0$; θ as an angle variable which increases by 2π the short way around the torus (poloidal direction); and ϕ, the toroidal (symmetry) angle which increases by 2π the long way around the torus. The vectors $\nabla\psi$ and $\nabla\theta$ are both orthogonal to $\nabla\phi$, but $\nabla\psi\cdot\nabla\theta \neq 0$ in general. The poloidal angle is chosen so that the Jacobian of the transformation from x to (ψ, θ, ϕ),

$$J \equiv (\nabla\psi\cdot\nabla\theta\times\nabla\phi)^{-1} = (\partial x/\partial\psi)\cdot(\partial x/\partial\theta \times \partial x/\partial\phi) \quad , \tag{7}$$

is everywhere finite. Then, the most general form of the magnetic field, consistent with Eq. (6c), Maxwell's equations, and toroidal symmetry, can be written[8]

$$B = (2\pi)^{-1}\,\psi'(\psi, t)\nabla\phi\times\nabla\psi + q(\psi, t)\nabla\phi \quad . \tag{8}$$

Here, $\psi' \equiv 2\pi J B\cdot\nabla\theta$ is the poloidal flux density and $q = x^2 B\cdot\nabla\phi$, where $x = |\nabla\phi|^{-1}$ is the distance from the toroidal symmetry axis. The current can be written in contravariant form, noting $J\cdot\nabla\psi = 0$,

$$J = J[(J\cdot\nabla\Theta)\nabla\phi \times \nabla\psi + (J\cdot\nabla\phi)\nabla\psi \times \nabla\Theta] \quad , \tag{9}$$

where $J\nabla\psi \times \nabla\Theta = x^2\nabla\phi$. The components of J are obtained from Ampere's law, Eq. (1b):

$$J\cdot\nabla\phi = -J^{-1}g_{\theta}/4\pi \ , \tag{10a}$$

$$J\cdot\nabla\theta = \nabla\cdot(\chi' x^{-2}\nabla\psi)/8\pi^{2} \ . \tag{10b}$$

Combining this with the $\nabla\phi\cdot\nabla\psi$ component of the pressure balance, Eq. (6c), yields the Grad-Shafranov equation

$$\Delta^{*}\chi + (2\pi)^{2}(gg_{\psi} + 4\pi x^{2}p_{\psi})(\chi')^{-1} = 0 \ , \tag{11a}$$

where

$$\chi = \int_{0}^{\psi}\chi'(\psi, t)\,d\psi + \chi(0, t) \tag{11b}$$

and

$$\Delta^{*}\chi = x^{2}\nabla\cdot(x^{-2}\nabla\chi)$$
$$= x^{2}J^{-1}\left[(h^{\psi\psi}\chi_{\psi} + h^{\psi\theta}\chi_{\theta})_{\psi} + (h^{\theta\psi}\chi_{\psi} + h^{\theta\theta}\chi_{\theta})_{\theta}\right] \ . \tag{11c}$$

Here, the metric elements $h^{ij} = x^{-2}J\nabla i\cdot\nabla j$ can be expressed as derivatives of the cylindrical spatial coordinates (x, z):

$$h^{\psi\psi} = J^{-1}|\partial \underset{\sim}{x}/\partial\theta|^{2} = J^{-1}(x_{\theta}^{2} + z_{\theta}^{2}) \ , \tag{12a}$$

$$h^{\theta\theta} = J^{-1}|\partial \underset{\sim}{x}/\partial\psi|^{2} = J^{-1}(x_{\psi}^{2} + z_{\psi}^{2}) \ , \tag{12b}$$

$$h^{\psi\theta} = -J^{-1}(\partial \underset{\sim}{x}/\partial\psi)\cdot(\partial \underset{\sim}{x}/\partial\theta) = -J^{-1}(x_{\theta}x_{\psi} + z_{\theta}z_{\psi}) \ . \tag{12c}$$

Subscripts have been used to denote differentiation. Comparing Eqs. (11a) and (10b), we note that

$$\underset{\sim}{J}\cdot\nabla\phi = -2\pi[(4\pi x^{2})^{-1}gg_{\psi} + p_{\psi}](\chi')^{-1} \ . \tag{13a}$$

The parallel current, $\underset{\sim}{J}\cdot\underset{\sim}{B} = J[2\pi\underset{\sim}{J}\cdot\nabla\theta B_{p}^{2}/\chi' + \underset{\sim}{J}\cdot\nabla\phi g]$, where $B_{p} = \chi'|\nabla\psi|/(2\pi x)$, is

$$\underset{\sim}{J}\cdot\underset{\sim}{B} = -(2\pi)[B^{2}g_{\psi}/(4\pi) + gp_{\psi}](\chi')^{-1} \ . \tag{13b}$$

Equation (11) is a two-dimensional, nonlinear differential constraint relating the surface quantities $p(\psi, t)$, $q(\psi, t)$, $\ldots$, to the magnetic surface geometry at all times. The incorporation of this constraint into the time evolution equations is the central problem in the theory of self-consistent two-dimensional transport. Numerical procedures for solving Eq. (11a) have been previously discussed.[9,10] Here we note that it is sufficient to time differentiate the constraint, Eq. (11a), and use the resulting linear equation to determine the velocity of the constant ψ surfaces, and the velocities of the magnetic fluxes (poloidal flux) and $(2\pi)^{-1}\int_0^{\psi} B\cdot\nabla\zeta\, dx$ (toroidal flux) relative to the constant ψ surfaces.[2,5]

The coordinate velocity can be defined using the transformation of time derivatives between the fixed (x, z) and moving (ψ, Θ) coordinate frames:

$$(\partial/\partial t)\big|_{\underset{\sim}{x}} = (\partial/\partial t)\big|_{\psi,\Theta} - \underset{\sim}{u}_g\cdot\nabla \quad . \tag{14a}$$

Here the coordinate (or grid) velocity is

$$\underset{\sim}{u}_g = J\,[(-\partial\psi/\partial t)\big|_{\underset{\sim}{x}}\,\nabla\Theta \times \nabla\phi + (-\partial\Theta/\partial t)\big|_{\underset{\sim}{x}}\,\nabla\phi \times \nabla\psi\,] \,. \tag{14b}$$

Operating on $\underset{\sim}{x}$ with Eq. (14a) yields

$$(\partial\underset{\sim}{x}/\partial t)\big|_{\psi,\Theta} = \underset{\sim}{u}_g \tag{14c}$$

which can be used to advance $\underset{\sim}{x}(\psi, \Theta, t)$ in time once $\underset{\sim}{u}_g$ is known.

Physically, it is clear that the equilibrium equation, (11a), can be used to advance the coordinates only in the direction normal to the flux surfaces (cf., Section V). advancement of the poloidal angle contours depends on the ch of the Jacobian J. Note from Eq. (7) that J evolves in flux surface frame as

$$J_t = J\nabla\cdot u_g$$

where J_t ($\partial J/\partial t$)... To guarantee the rigidity, i. "incompressibility" of the moving ψ, θ mesh for all times (is clearly desirable for computational reasons), the Jacobian chosen to have the form[11]

$$J = x^m \psi^n$$

where m, n are integers which will be specified in Section V. then follows from Eqs. (14a) and (15) that $\nabla\cdot(x^{-m}\, u_g) = 0$, a stream function ξ for the two-dimensional coordinate velo exists,

$$u_g = x^m \nabla \times (\xi \nabla\phi) \quad . \qquad ()$$

Thus,

$$u_g\cdot\nabla\psi \equiv -(\partial\psi/\partial t)|_x = x^m J^{-1}\xi_\Theta$$

$$u_g\cdot\nabla\Theta \equiv -(\partial\Theta/\partial t)|_x = -x^m J^{-1}\xi_\psi \quad .$$

Taking components of Eq. (14c) and using these results yield evolution equations for the coordinate transformation:

$$x_t \quad \cdot^{-m} \quad y \quad - \quad x \quad) \ , \tag{18a}$$

$$\quad_t \quad \cdot^{-m} \quad \quad - \quad \quad , \tag{18b}$$

where subscript t denotes time differentiation at fixed , . The determination of the coordinate velocity stream function from the time derivative of the equilibrium constraint will be considered in Section V.

Finally, note that Eq. (17b) implies a "solubility" constraint for the normal component of the coordinate velocity; i.e.,

$$x^{-m} \quad \cdot \quad = 0 \ . \tag{19}$$

Here, brackets denote the flux surface average operator,

$$\langle a \rangle \quad \int_0^{2\pi} d \quad Ja \Big/ \int_0^{2\pi} d \quad J \ . \tag{20}$$

This constraint allows the determination of the motion of an arbitrary surface function $a(\psi, t)$ relative to the constant ψ surfaces once its absolute velocity (at fixed $\underset{\sim}{x}$) is known. For, from Eqs. (14a) and (19), we find

$$(\partial a/\partial t)\big|_{\underset{\sim}{x}} = a_t - a' \underset{\sim}{u}_g \cdot \nabla\psi \ , \tag{21}$$

where $a_t = (\partial a/\partial t)\big|_{\psi,\Theta} = \langle x^{-m}\ \partial a/\partial t|_{\underset{\sim}{x}}\rangle / \langle x^{-m}\rangle$ and a prime denotes $\partial/\partial\psi$.

Physically, the coordinate velocity is nonvanishing only if there is a change in the flux surface geometry. The relative motion of ψ and $a(\psi, t)$ contours (measured by a_t) corresponds to a relabeling of ψ surfaces with different values of a without any geometry change. For, if $\partial a/\partial t|_{\underset{\sim}{x}} = a_t(\psi, t)$, Eq. (21)

indicates that $u_g \cdot \nabla\psi = 0$. Thus, the ψ-coordinate system remains unchanged whenever the magnetic flux simply moves through space without alteration of the surface geometry. Such a coordinate system has some features of a fixed Eulerian system. However, the mesh is allowed to deform consistent with $B \cdot \nabla\psi = 0$ and the maintenance of static pressure balance, Eq. (11a).

IV. DERIVATION OF 1-D TRANSPORT EQUATIONS

The conservation equations given in Section II are spatially local relations for advancing the thermodynamic variables in time. In a confined tokamak plasma, the motion of particles and heat along B is rapid compared to the resistive motion across magnetic surfaces. This leads to density and temperature profiles which are nearly uniform on a flux surface. To show this, we write for the leading order surface flows appearing in Eqs. (6a) and (6b)

$$n_j u_j^S = -(D_{\parallel j}/T_j) B B\cdot(\nabla p_j + e n_j \nabla\Phi)/B^2 + (D_{\perp j}/T_j) B \times (\nabla p_j + e_j n_j \nabla\Phi)/B \ , \tag{22a}$$

$$q_j^S = -\kappa_{\parallel j} B(B\cdot\nabla T_j)/B^2 + \kappa_{\perp j} B \times \nabla T_j/B \ . \tag{22b}$$

Here, $D_{\parallel j} = T_j/(m_j \nu_{j1})$ is the parallel particle diffusion coefficient, where ν_{j1} is the momentum exchange frequency of species (j): $D_{\perp j} = D_{\parallel j}(\nu_{j1}/\Omega_j) \sim D_{\parallel j}\Delta$ is the orthogonal diffusion coefficient, $\kappa_{\parallel j} = p_j/(m_j \nu_{j2})$ is the parallel heat conductivity, and $\kappa_{\perp j} = (5/2)\kappa_{\parallel j}(\nu_{j2}/\Omega_j)$ is the orthogonal heat conductivity,[12] where ν_{j2} is the heat "friction" frequency.[7] Equation (22a) is obtained as the lowest order (in Δ) expansion of the momentum balance equation, Eq. (3a). Using Eqs. (22a) and (22b) in Eqs. (6a) and (6b) and invoking quasineutrality to eliminate the electrostatic potential yields

$$p_j(\underset{\sim}{x}, t) = p_j(\psi, t) + \tilde{p}_j(\psi, \Theta, t) \ , \tag{23a}$$

$$T_j(\underset{\sim}{x}, t) = T_j(\psi, t) + \tilde{T}_j(\psi, \Theta, t) \ , \tag{23b}$$

where $\tilde{p}_j/p_j \sim D_{\perp j}/D_{\parallel j} \sim \Delta \ll 1$ and $\tilde{T}_j/T_j \sim \chi_{\perp j}/\chi_{\parallel j} \sim \Delta$.

Here,

$$p_j(\psi, t) \equiv \langle p_j(\underset{\sim}{x}, t)\rangle \quad , \tag{23c}$$

$$T_j(\psi, t) \equiv \langle T_j(\underset{\sim}{x}, t)\rangle \quad . \tag{23d}$$

Thus, in view of Eq. (23), the local conservation laws contain more information than is necessary to advance the dominant surface averaged part of the thermodynamic variables. This extraneous information can be annihilated by flux surface averaging the conservation equations. Consider, for example, the particle continuity equation, Eq. (2a), which can be written as

$$(\partial n_j/\partial t)|_{\underset{\sim}{x}} + J^{-1}(\partial/\partial\psi)(n_j\underset{\sim}{u}_j\cdot\nabla\psi) - S_{nj} \equiv K_j(\underset{\sim}{x}, t)$$
$$= -\underset{\sim}{B}\cdot\nabla(n_j\underset{\sim}{u}_j\cdot\nabla\Theta/\underset{\sim}{B}\cdot\nabla\Theta) \quad . \tag{24}$$

Here, $\underset{\sim}{u}_j\cdot\nabla\Theta = \underset{\sim}{u}^S_{j1}\cdot\nabla\Theta$ is the first order surface component of the velocity. This is a magnetic differential equation for $n_j\underset{\sim}{u}_j\cdot\nabla\Theta/\underset{\sim}{B}\cdot\nabla\Theta$, whose solution requires the solubility constraint

$$\int_0^{2\pi}[K_j(\underset{\sim}{x}, t)/\underset{\sim}{B}\cdot\nabla\Theta]\, d\Theta = 0 \quad . \tag{25}$$

The chain rule, Eq. (14a), together with Eq. (15), is used to commute the Θ integration and time differentiation in performing the Θ average of K_j in Eq. (24). This yields the desired 1-D continuity equation for the differential particle number $N'_j \equiv n_j(\psi, t)V'$, where $V'(\psi, t) = \partial(\int_0^{\psi} dx)/\partial\psi = 2\pi \int_0^{2\pi} J\, d\theta$:

$$(N'_j)_t + [N'_j \langle(u_j - u_g)\cdot\nabla\psi\rangle]_\psi = V'\langle S_{nj}\rangle \quad . \tag{26}$$

Applying this procedure to the pressure equations, Eqs. (2b) and (2c), we obtain conservation equations for the differential plasma and electron entropy:

$$(5/2)(p/\sigma')\sigma'_t = -S^* \quad , \tag{27a}$$

$$(5/2)(p_e/\sigma'_e)(\sigma'_e)_t = -S_e^* \quad . \tag{27b}$$

Here, the entropy variables are

$$\sigma' = p^{3/5}\, V' \quad , \tag{28a}$$

$$\sigma'_e = p_e^{3/5}\, V' \quad , \tag{28b}$$

where $p = p(\psi, t)$ and $p_e = p_e(\psi, t)$. The entropy sources are

$$S^*(u_g\cdot\nabla\psi) = 1/V'\{V'\sum_j [\langle \underset{\sim}{q}_j\cdot\nabla\psi\rangle + (5/2)p_j\langle(\underset{\sim}{u}_j - \underset{\sim}{u}_g)\cdot\nabla\psi\rangle]\}_\psi$$
$$-\langle \underset{\sim}{J}\cdot\underset{\sim}{E}\rangle + p_\psi\langle \underset{\sim}{u}_g\cdot\nabla\psi\rangle - \langle S_p\rangle \quad , \tag{29a}$$

$$S_e^*(\underset{\sim}{u}_g\cdot\nabla\psi) = 1/V'\{V'[\langle \underset{\sim}{q}_e\cdot\nabla\psi\rangle + (5/2)p_e\langle(\underset{\sim}{u}_e - \underset{\sim}{u}_g)\cdot\nabla\psi\rangle]\}_\psi$$

$$-\langle \underset{\sim}{J}\cdot\underset{\sim}{E}\rangle + p_\psi\langle \underset{\sim}{u}_g\cdot\nabla\psi\rangle + (p_i)_\psi\langle(\underset{\sim}{u}_i - \underset{\sim}{u}_g)\cdot\nabla\psi\rangle$$

$$-\langle Q_{\Delta e}\rangle - \langle S_e\rangle + \langle \underset{\sim}{u}_i\cdot\nabla\cdot\underset{\approx}{\pi}_i\rangle \quad . \tag{29b}$$

Finally, consider the three components of Faraday's law (magnetic flux conservation), Eq. (1a)

$$\partial/\partial t(\underset{\sim}{B}\cdot\nabla\phi) = \partial/\partial t(gx^{-2}) = -\nabla\cdot(\underset{\sim}{E}\times\nabla\phi) \quad , \tag{30a}$$

$$-\underset{\sim}{B}\cdot\nabla\partial\psi/\partial t = -\nabla\cdot(\underset{\sim}{E}\times\nabla\psi) \quad , \tag{30b}$$

$$\nabla\Theta\cdot\partial\underset{\sim}{B}/\partial t = -\nabla\cdot(\underset{\sim}{E}\times\nabla\Theta) \quad . \tag{30c}$$

The last two components imply that

$$(\partial\chi/\partial t)\big|_{\underset{\sim}{x}} - \chi'\,\underset{\sim}{u}_{\chi}\cdot\nabla\psi = 2\pi x^2\underset{\sim}{E}\cdot\nabla\phi + c(t) \quad . \tag{31a}$$

The constant of integration c(t) is related to the voltage applied at the plasma surface. Differentiating Eq. (31a) with respect to ψ removes this integration constant and yields a conservation equation for the poloidal flux density χ' ,

$$\chi'_t + [\chi'<(\underset{\sim}{u}_{\chi} - \underset{\sim}{u}_g)\cdot\nabla\psi>]_{\psi} = 0 \quad . \tag{31b}$$

The toroidal component of Faraday's law, Eq. (30a), yields a conservation relation for the toroidal flux density

$$\Psi' \equiv g<x^{-2}>V'/(2\pi) = \chi' q(\psi) \quad , \tag{32}$$

where

$$q(\psi) = d\Psi/d\chi = \int (d\Theta/2\pi)(\underset{\sim}{B}\cdot\nabla\phi/\underset{\sim}{B}\cdot\nabla\Theta) \tag{33}$$

is the safety factor. The result is:

$$\Psi'_t + [\Psi'<(\underset{\sim}{u}_{\Psi} - \underset{\sim}{u}_g)\cdot\nabla\psi>]_{\psi} = 0 \quad , \tag{34a}$$

where the toroidal flux velocity is

$$\underset{\sim}{u}_{\Psi}\cdot\nabla\psi = 2\pi(-x^2\underset{\sim}{E}\cdot\nabla\phi + <\underset{\sim}{E}\cdot\underset{\sim}{B}>/<\underset{\sim}{B}\cdot\nabla\phi>)(\chi')^{-1}$$

$$\equiv -(\Psi')^{-1}(\partial\Psi/\partial t)\big|_{\underset{\sim}{x}} \quad . \tag{34b}$$

Note that at the magnetic axis where $B_p = 0$, $u_\psi \cdot \nabla\psi = 0$, so that the toroidal flux remains zero there for all time. Finally, the Joule heating term in Eq. (29) can be evaluated using $J = B \times \nabla p/B^2 + J\cdot B\, B/B^2$, where $J\cdot B$ is given in Eq. (13b). Then

$$\langle J\cdot E\rangle - p_\psi \langle u_g \cdot \nabla\psi\rangle = \langle J\cdot \nabla\phi\rangle(\langle E\cdot B\rangle/\langle B\cdot \nabla\phi\rangle) + p_\psi \langle (u_\psi - u_g)\cdot \nabla\psi\rangle \ . \tag{35}$$

Equations (26), (27a), (27b), (31b), and (34a) comprise the one-dimensional conservation equations for the "adiabatic" variables N', σ', σ_e', Ψ', and χ' . The adiabatic variables are constant in time on each flux surface during motions of the plasma which are rapid compared to the resistive time τ_R . The closure of these equations requires transport relations for the relative fluxes and viscous heating and also equations for the coordinate velocity and the motion of toroidal or poloidal flux surfaces relative to the coordinates.

V. DETERMINATION OF MAGNETIC FLUX AND COORDINATE VELOCITIES

The transport equations derived in Section IV can now be used to obtain an equation for the magnetic flux velocities $u_\chi \cdot \nabla\psi$ and $u_\Psi \cdot \nabla\psi$, defined in Eqs. (31a) and (34b). Once these are known, Eq. (21), with $a = \chi$ or Ψ , can be used to determine the coordinate velocity.

Taking the time derivative of the Grad-Shafranov Eq. (11a) and using the convective derivative Eq. (14a) yields

$$(16\pi^3 x^2)^{-1} \chi' \Delta^*(\chi' u_\chi \cdot \nabla\psi) + \chi' u_g \cdot \nabla\psi \, (p_\psi/\chi')' + (4\pi x^2)^{-1}(gg_\psi/\chi')'$$

$$(5/3)\, p(\chi'_t/\chi' - V'_t/V')_\psi$$

$$+ (4\pi x^2)^{-1}[g^2(\Phi'_t/\Phi' - (V'\langle x^{-2}\rangle)_t/(V'\langle x^{-2}\rangle))]_\psi + (2\pi)^{-1} \chi'_t J \cdot \nabla\phi \,. \tag{36}$$

The transport equations (27a), (31b), and (34b), as well as the $\oint d\Theta\, (1, x^{-2})$ moments of Eq. (15),

$$(V')_t = (V' \langle u_g \cdot \nabla\psi \rangle)_\psi \,, \tag{37a}$$

$$(V' \langle x^{-2} \rangle)_t = (V' \langle x^{-2} u_g \cdot \nabla\psi \rangle)_\psi \,, \tag{37b}$$

are used to evaluate the time derivatives in Eq. (36). This yields a linear integro-differential equation relating the two magnetic flux velocities and the grid velocity

$$(16\pi^3 x^2)^{-1} \chi' \Delta^*(\chi' u_\chi \cdot \nabla\psi) + L_0(\chi' u_\Psi \cdot \nabla\psi) + L_1(\chi' u_g \cdot \nabla\psi)$$

$$= -(4\pi x^2)^{-1}[gg_\psi \langle (u_\Psi - u_g) \cdot \nabla\psi \rangle]_\psi - (2/3) S^*_\psi$$

$$+ (2\pi)^{-1} J \cdot \nabla\phi \langle \chi' (u_g - u_\chi) \cdot \nabla\psi \rangle_\psi \,. \tag{38}$$

Here, $S^*(u_g \cdot \nabla\psi)$ is defined in Eq. (29a) and

$$L_0(a) = [illegible] (4\pi x^2)^{-1} g^2 (V' x^{-2} [illegible]) (V' [illegible]) x^{-2} a [illegible] , \qquad (39a)$$

$$L_1(a) = (p [illegible] + (4\pi x^2)^{-1} (gg [illegible]) a + \frac{5}{3}\left[\frac{p}{V'}\left(V' [illegible] a \right) [illegible] \right] . \qquad (39b)$$

Note that $L_0 / L_1 \sim B_T^2 / p$ and $(16 [illegible] x^2 L_0 [illegible]) [illegible]^{*} \sim B_T^2/B_p^2$ where B_T [illegible] $\times B \cdot$ [illegible] is the magnitude of the toroidal field. Thus, for a low [illegible] tokamak plasma, L_0 is the dominant operator in Eq. (38). Using Eqs. (31a) and (34b) to express the poloidal flux velocity in terms of the toroidal flux velocity, Eq. (38) becomes a linear equation for the normal component of the toroidal flux velocity, $u_{\chi} \cdot$ [illegible] :

$$[illegible] (4\pi x^2)^{-1} [illegible]^{*} (\chi' u_{\chi} \cdot \nabla\psi) + (L_0 + L_1)(\chi' u_{\chi} \cdot \nabla\psi)$$

$$= 2J \cdot [illegible] (E^{*}) [illegible] + (B_p^2/2) (E^{*}_{\parallel}) [illegible] / [illegible] - (2/3) S [illegible] . \qquad (40)$$

Here,

$$E^{*}_{\parallel} = \langle E \cdot B \rangle / \langle B \cdot \nabla [illegible] \rangle , \qquad (41a)$$

and

$$S = S^{*}(u_{\chi} \cdot \nabla\psi)$$

$$= -\langle J \cdot \nabla [illegible] \rangle E^{*}_{\parallel} - \langle S_p \rangle$$

$$+ (1/V') [V' \sum_j \langle q_j \cdot \nabla\psi + (5/2) p_j (u_j - u_{\psi}) \cdot \nabla\psi \rangle]_{\psi} . \qquad (41b)$$

In deriving Eq. (40), it was noted from Eqs. (21) and (34b) that

$$u_g \cdot \nabla\psi = u_{\psi} \cdot \nabla\psi + \Psi_t / \Psi' \quad , \qquad (42a)$$

and the identity [valid for an arbitrary surface function $u(\psi, t)$]

$$L_1(\chi' u) - (4\pi x^2)^{-1} (g g_{\psi} u)_{\psi} - (2\pi)^{-1} J \cdot \nabla\phi (\chi' u)_{\psi}$$

$$= \frac{2}{3} \left(\frac{5}{2} (pV'u)_{\psi} / V' - p_{\psi} u \right)_{\psi} \qquad (42b)$$

was used to eliminate u_t . Note that the relative particle flow $(u_j - u_\psi)\cdot\nabla\psi$ appearing in ω is purely resistive and not explicitly dependent on the motion of the flux surfaces $u_\psi\cdot\nabla\psi$ (cf., Section VI).

Equation (40) is a linear two-dimensional equation determining the motion of toroidal flux surfaces as the plasma evolves in time. Mathematically, Eq. (40) is a linear generalized differential equation (GDE).[9] Since the L^* operator is elliptic, the value of $u_\psi\cdot\nabla\psi$ is required on the plasma surface. The operators L_0 and L_1, which involve integrals over the coordinate θ and derivatives with respect to the coordinate ψ, are essentially ordinary integro-differential operators and thus require a boundary condition at the origin. Fourier analysis in θ of Eq. (40) leads to a standard linear system of coupled second order ordinary differential equations[11] in ψ for the Fourier coefficients of $u_\psi\cdot\nabla\psi$. This system, of course, still requires a single boundary condition at the origin for the lowest order harmonic.

The numerical advantage of using the toroidal flux velocity rather than the poloidal flux velocity as the dependent variable in Eq. (38) arises from the dominance of the L_0 operator. This is a feature of conventional tokamak plasmas where the large externally maintained toroidal field provides "rigidity" to the plasma. The toroidal flux is comparatively less mobile than the poloidal flux, which is tied to self-consistent plasma currents. The [illegible] manifests itself in Eq. (40) where the source terms (right side) are all of comparable magnitude (for $\beta_p \sim 1$). The analogous equation for the poloidal flux velocity $u_\psi \cdot \nabla\psi$ includes an additional large source term, arising from [illegible], which exceeds the remaining sources by factors ϵ^{-1} and B^2/B_p^2, thus obscuring the slow diffusion of the toroidal flux. More explicitly, comparing the toroidal and poloidal "slippage" velocities relative to the ideal surfaces moving with $u_E = E \times B/B^2$, we find

$$\frac{\langle (u_E - u_\Psi)\cdot\nabla\Psi\rangle}{\langle (u_E - u_\psi)\cdot\nabla\psi\rangle} = \frac{\langle (E\cdot B/B^2)(1 - B^2/\langle B_T^2\rangle)\rangle}{\langle E\cdot B/B^2\rangle} \sim \frac{\langle B_p^2\rangle}{\langle B_T^2\rangle} \ll 1 . \tag{43}$$

Finally, it remains to determine the relation between the coordinate velocity stream function ξ, introduced in Eq. (17), and the toroidal flux velocity which is computed from Eq. (40). From Eq. (42a) and the constraint Eq. (19), we obtain

$$u_q\cdot\nabla\psi = u_\psi\cdot\nabla\psi - \langle x^{-m} u_\psi\cdot\nabla\psi\rangle/\langle x^{-m}\rangle = x^m J^{-1}\xi_\Theta \quad . \tag{44a}$$

Note that the relative velocity of ψ and Ψ surfaces is

$$u(\psi,\ t) = \langle x^{-m}\ u_\psi\cdot\nabla\psi\rangle/\langle x^{-m}\rangle \ , \tag{44b}$$

where m is still to be determined. Eq. (44a) can be integrated to obtain the coordinate stream function up to a constant of integration on each flux surface. Physically, this constant corresponds to an arbitrary rigid rotation of the Θ = constant coordinate lines on each ψ constant surface. This constant is chosen to make $\oint$ d 0 and thereby remove any net poloidal rotation of the constant surfaces. This uniquely defines , which can then be used in Eq. (18) to advance the coordinate transformation in time.

It is apparent from Eq. (40) that, in general, uncoupled equations cannot be obtained for the relative flow u(ψ, t) and the - varying deformation flow $u_q \cdot \nabla$. This is due to the poloidal harmonic coupling introduced by the Δ^* operator. However, since the L_0 operator is the largest in Eq. (40), it is possible to obtain a separation through O(β) by choosing m = 2, n = 0 in Eqs. (16) and (44). With this choice for the Jacobian, $J \sim x^2$, the dominant homogeneous term in Eq. (40) involves only u(ψ, t):

$$L_0(\chi' u_\psi \cdot \nabla\psi) = (4\pi x^2)^{-1}(q^2/(V'\langle x^{-2}\rangle)[V'\langle x^{-2}\rangle u(\psi, t)]_\psi]_\psi \quad . \tag{45}$$

This separation leads to an efficient numerical scheme[13] for solving Eq. (40). It is also clear from Eqs. (40) and (45) that the relative motion of ψ and Ψ surfaces is small compared to their deformation rate; i.e., $u/\xi_0 \sim O(\beta)$. For $J \sim x^2$, the coordinate $\psi \sim \int x^{-2}\, d\underset{\sim}{x} \sim \Psi_V$ is proportional to the external (i.e., vacuum, $\underset{\sim}{B}_V \cdot \nabla\phi \sim x^{-2}$) toroidal flux enclosed by a magnetic surface. For low β tokamak plasmas, Ψ_V is nearly an adiabatic invariant, which accounts for its desirability as an independent radial coordinate. The choice $J \sim x^2$ has also been useful for stability calculations.[14] [Other choices for J which have been used are (i) J = 1 , the Hamada system for which [illegible], and [illegible] = x. , where [illegible] r is an effective minor radius. These prescriptions for [illegible] produce coupling of u(., t) and [illegible] directly through the I_0 term in Eq. (40).]

Finally, we conclude this section by evaluating the one-dimensional transport equations derived in Section IV for [illegible] in Eq. (44a):

$$(N'_j)_t + (N'_j u)_{.} = -(V' [illegible]_j)_{.} + V' S_{nj} \quad , \tag{46a}$$

$$(5/2)(p [illegible])_t + ([illegible] u)_{.} = -S \quad , \tag{46b}$$

$$(5/2)(p_e [illegible]) ([illegible]_e)_t + ([illegible]_e u)_{.} = -S_e \quad , \tag{46c}$$

$$[illegible]_t + ([illegible] u)_{.} = 2 [illegible] (E^*_{\parallel})_{.} \quad , \tag{46d}$$

$$[illegible]_t + ([illegible] u)_{.} = 0 \quad , \tag{46e}$$

where S is given in Eq. (41b),

$$S_e \quad S^*_e(\underset{\sim}{u}_\psi \cdot \nabla\psi)$$

$$= -\langle J \cdot [illegible] E^*_{\parallel} + (p_i)_{.} \langle (\underset{\sim}{u}_i - \underset{\sim}{u}_\psi) \cdot \nabla\psi \rangle$$

$$+ \langle \underset{\sim}{u}_i \cdot \nabla \cdot \underset{\approx}{\pi}_i \rangle - \langle Q_{\Delta e} \rangle - \langle S_{pe} \rangle$$

$$+ (1/V')[V'(\langle \underset{\sim}{q}_e \cdot \nabla\psi \rangle + (5/2)\Gamma_e T_e)]_\psi \quad , \tag{46f}$$

and the normal component of the particle flux is

$$\Gamma_j \equiv \langle n_j (\underset{\sim}{u}_j - \underset{\sim}{u}_\psi) \cdot \nabla\psi \rangle \quad . \tag{46g}$$

Here, u [illegible] $-\psi_t/\psi'$ [illegible] $\chi^{-2} u_\psi \cdot$ [illegible] $/\chi^{-2}$, and $u_\psi \cdot \nabla\psi$ is determined from the solution of the two-dimensional Eq. (40). Equations (18a) and (18b) with [illegible]$^{-n}$ [illegible] 1 are used to evolve the coordinate grid.

Note that there are three distinct contributions to the changes in the actual particle density $n(\psi, t)$, or pressure $p(\psi, t)$, apparent in Eqs. (46) in addition to the sources:

(1) Expansion or compression associated with the deformation at constant "area," $\int dx\, J^{-1}$, of flux surfaces; i.e., $(N_j')_t = 0$ or

$$V'(n_j)_t/n_j = -(V' u_g \cdot \nabla\psi)_\psi \quad . \tag{47}$$

Thus, the change in magnetic surface geometry is implicit in Eq. (46) through the use of adiabatic dependent variables and flux surface coordinates.

(2) Changes due to the slow motion of the toroidal flux surface relative to the coordinate grid (based on the vacuum toroidal flux). That is, the plasma and the magnetic fluxes are "frozen" together in the absence of resistive processes.

(3) Resistive diffusion of the plasma relative to the toroidal flux surfaces, driven by the normal component of $(\tilde{u}_j - \tilde{u}_\psi) \cdot n$ (cf., Section VI). It is somewhat remarkable that in the absence of resistivity or anomalous diffusion, the adiabatic variables are preserved in spite of complex guiding center particle motion.[15]

The motions (1) and (2) result from the presence of a non-trivial $\tilde{u}_\psi \cdot \nabla\psi$ in Eq. (40), which can arise either from nonzero [illegible] values (adiabatic motion) or from a nonzero right-hand side (diffusive motion of magnetic surfaces or pressure [illegible].

VI. CLOSURE OF MOMENT EQUATIONS

The 1-D conservation Eqs. (46) and the 2-D Eq. (40) for the toroidal flux velocity can be integrated only when the relative fluxes of particles (Γ_j) heat ($\langle q_j\cdot\nabla\psi\rangle$) poloidal flux ($E_\parallel^*$) and the viscous heating term $\langle u_i\cdot\nabla\cdot\pi_i\rangle$ have been expressed in terms of the thermodynamic forces. The thermodynamic forces are, apart from geometric factors, simply the ψ derivatives of the adiabatic variables which satisfy the conservation equations. This is the goal of transport theory[3] and will now be briefly reviewed.

A. Relative Particle Flux

Taking the toroidal component of Eq. (3a), averaging over a flux surface, and using Eqs. (34b) and (46g) yields

$$(2\pi)^{-1}\chi' e_j\Gamma_j = -(\langle x^2\nabla\phi\cdot R_j\rangle + e_j n_j E_\parallel^*) + \langle n_j e_j x^2\nabla\phi\cdot\nabla\Phi\rangle \quad , \qquad (47)$$

where $E_\parallel^*$ is defined in Eq. (41a). Note that only the induced electric field entered the flux velocity defined in Eqs. (31a) and (34b). In an axisymmetric plasma (i.e., neglecting symmetry-breaking, anomalous perturbations), $\nabla\phi\cdot\nabla\Phi = 0$ and the quantity $\langle x^2\nabla\phi\cdot\nabla\cdot\pi_j\rangle = 0$, since π_j has the Chew, Goldberger, Low form for neoclassical and low frequency, long wavelength fluctuation phenomena. Thus

$$\langle x^2\nabla\phi\cdot R_j\rangle = \langle x^2\nabla\phi\cdot F_j\rangle \qquad (48)$$

where F_j is the friction force defined in Eq. (3d). This result shows that the classical relative particle flux results entirely from the collisional friction. Now, writing [illegible] $B^2 + \pi B^2$, the particle flux can be conveniently decomposed as

$$\Gamma_j = \Gamma_j^{class} + \Gamma_j^{nc} + \Gamma_j^{anom} - (2\pi n_j E_{\|}^{*}/\chi')\langle B_p^2\rangle/\langle B^2\rangle \quad . \tag{49}$$

Here, the classical diffusion flux is

$$\Gamma_j^{class} = -\langle F_j \cdot B \times \nabla\chi/e_j B^2\rangle \; ; \tag{50a}$$

the neoclassical diffusion flux is[3]

$$\Gamma_j^{nc} = -(2\pi q/\chi')(\langle F_j \cdot B/e_j B^2\rangle + n_j\langle E \cdot B\rangle/\langle B^2\rangle) \; ; \tag{50b}$$

and the anomalous flux due to symmetry breaking perturbations is

$$\Gamma_j^{anom} = 2\pi\langle x^2 \nabla\chi \cdot (\nabla p_j + \nabla \cdot \pi_j + n_j e_j \nabla\Phi)\rangle/(e_j \chi') \, . \tag{50c}$$

[An average over wavenumbers is also implied in Eq. (50c).] Thus, except for the small term $-(2\pi n_j E_{\|}^{*}/\chi')\langle B_p^2\rangle/\langle B^2\rangle$ in Eq. (49), which represents the classical $E_T \times B_p$ radial pinch, the relative particle flux in the frame moving with the toroidal flux is just that flux which is usually (though incorrectly) interpreted as the particle flux in the fixed laboratory frame; i.e., neglecting the magnetic surface motion. Grad and Hogan[1] and more recently, Pao,[5,16] have established the importance of inductive motions in a collisional plasma. The classical and neoclassical diffusion formulae, derived for stationary flux surfaces, are nevertheless approximately valid for circular, low β tokamaks in which the strong toroidal field makes the plasma behave like a rigid conductor [i.e., the toroidal flux stationary as $L_0 \to \infty$ in Eq. (40)].

The particle flux relative to the poloidal magnetic surf... contains a contribution proportional to $E_{\parallel}^{*}$ which is larger by a factor $\langle B^2\rangle/\langle B_p^2\rangle$ than the last term in Eq. (49). ... spurious advective flux, corresponding to the rapid motion ... poloidal flux (cf., Eq. (43)], would completely dominate, ... therefore obscure, the resistive contributions to Eq. (49).

In a similar way, the conductive heat flux, which is fr... invariant and thus independent of the flux surface motion, can ... related to resistive forces.[1,7]

B. Ohm's Law

The parallel electric field $\langle \underset{\sim}{E}\cdot\underset{\sim}{B}\rangle$, which plays the role ... a thermodynamic flux for χ' (rotational transform) in E... (46d), can be eliminated from the moment equations using generalized form of Ohm's law:

$$\langle \underset{\sim}{E}\cdot\underset{\sim}{B}\rangle = \eta_{\parallel}\,(1 + L_{33})^{-1}(\langle \underset{\sim}{J}\cdot\underset{\sim}{B}\rangle - \sum_{b;j=1,2} L_{3j}^{b}A_{j}^{b}) \quad . \tag{51}$$

Here, $\eta_{\parallel} = 0.51\, m_e(ne^2\tau_{ei})^{-1}$ is the parallel resistivity for $Z_i = 1$, L_{3j}^{b} are the bootstrap current coefficients, and L_{33} accounts for the neoclassical (trapped particle) and anomalous conductivity reduction. The generalized forces are defined to be

$$A_1^a \equiv \partial \ln p_a/\partial\psi \quad , \tag{52a}$$

$$A_2^a \equiv \partial \ln T_a/\partial\psi \quad . \tag{52b}$$

equation (16b) can be used in Eq. (51) to express $\langle E\cdot B\rangle$ in terms of the thermodynamic forces A_i^a and q'. Alternatively, the equilibrium Eq. (62) can be used to eliminate p' giving

$$\langle J\cdot B\rangle \quad (q^2/8\pi^2 V') \ [illegible] \ {}^2/x^2\rangle \ . \qquad (52c)$$

C. Viscous Heating

The ion viscous heating term which appears in the electron entropy Eq. (46c), can be evaluated through order ϵ as

$$\langle u_i\cdot\nabla\cdot\pi_i\rangle = \langle u_i\cdot\nabla\Theta/B\cdot\nabla\Theta\rangle\langle B\cdot\nabla\cdot\pi_i\rangle + \Pi_i^{anom} \ , \qquad (53)$$

where Π_i^{anom} represents any anomalous viscous heating. Classically, $\langle B\cdot\nabla\cdot\pi_i\rangle = 3\mu_{i1}\langle(n\cdot\nabla B)^2\rangle\langle u_i\cdot\nabla\Theta/B\cdot\nabla\Theta\rangle$ where μ_{i1} is the neoclassical ion viscosity.[7] Thus, the viscous heating in a tokamak represents the randomization of the ordered poloidal flow due to collisions and magnetic pumping. From the parallel component of the force balance, Eq. (3a), it is noted that

$$\langle B\cdot\nabla\cdot\pi_i\rangle = -\langle B^2\rangle g^{-1}[(2\pi)^{-1}\chi' e_i\Gamma_i^{bp}] \ , \qquad (54a)$$

where the long mean free path contribution to the particle flux, driven by the stress anisotropy, can be expressed in terms of the parallel component of the resistive friction force,

$$(2\pi)^{-1}\chi' e_j \Gamma_j^{bp} = -g\langle (F_j + e_j n_j E)\cdot B\rangle/\langle B^2\rangle \quad . \qquad (54b)$$

Thus, the net neoclassical particle flux is $\Gamma_j^{nc} = \Gamma_j^{bp} + \Gamma_j^{PS}$ and is comprised of Γ_j^{bp} and a Pfirsch-Schlüter flux,

$$(2\pi)^{-1}\chi' e_j \Gamma_j^{PS} = -g\langle (F_j\cdot B/B^2)(1 - B^2/\langle B^2\rangle)\rangle \quad . \qquad (54c)$$

Here, a small flux driven by $n(x) - \langle n(x)\rangle$ has been neglected.

Finally, the poloidal rotation can be expressed in terms of the long mean free path contributions to the particle and heat fluxes as follows:

$$\frac{\langle u_j\cdot\nabla\theta\rangle}{\langle B\cdot\nabla\theta\rangle} = \frac{(2\pi)^{-1} e_j \chi' \langle B^2\rangle}{3g\langle (n\cdot\nabla B)^2\rangle}\left(\frac{-\mu_{j3}\Gamma_j^{bp} + \mu_{j2}\, q_j^{bp}/T_j}{\mu_{j1}\mu_{j3} - \mu_{j2}^2}\right) \quad , \qquad (54d)$$

where μ_{jk}, $(k = 1 - 3)$ are the viscosity coefficients[7] of species (j).

Explicit evaluation of the resistive fluxes for a neoclassical transport model will be given in Section VIII.

VII. BOUNDARY CONDITIONS

Each of the five one-dimensional Eqs. (46a-e) requires a boundary value at $\psi = \psi_{max}$. The toroidal field function g is set to its vacuum value $g_v(t)$. To obtain numerical solutions, the electron and ion pressure and density are prescribed to be held constant at a pedestal which is some small fraction of their central value (0.001 to 0.1). The actual boundary conditions which apply at a physical plasma boundary are generally not consistent with the ϵ - expansion introduced in Section II. For example, a perfectly conducting material limiter (Subsection A) is an equipotential, but this is not compatible[17] with the presence of a Pfirsch-Schlüter flux driven by $\underset{\sim}{B}_p \cdot \nabla \Phi$. A boundary layer analysis is required in a small region about the wall where inertial terms become important.[18] In practical applications,[13] only interior solutions are computed and the results of the boundary layer theory are used to justify the imposition of pedestal pressure and density boundary conditions.

The boundary condition for χ' can be determined self-consistently from external currents (Subsection B) or can be prescribed arbitrarily. It is often desirable to specify the total plasma toroidal current,

$$I_T = (2\pi)^{-1} \int \underset{\sim}{J} \cdot \nabla\phi \, d\underset{\sim}{x} \quad ,$$

as a boundary condition. This determines the boundary value for χ'

$$\chi'(\psi_{max}\ ,\ t) = 16\pi^3\ I_T\ (V'\langle|\nabla\psi|^2/x^2\rangle)^{-1}\ ,$$

where the averages are evaluated at the surface ψ_{max} and at time t.

We consider two types of boundary conditions for Eq. (40) corresponding to (a) a conducting wall at the plasma boundary, and (b) a free boundary in which the plasma is surrounded by a vacuum with discrete external poloidal field coils carrying currents which vary in time.

A. Conducting Wall Boundary

The requirement that the boundary does not change its shape is satisfied by setting the coordinate velocity stream function $\xi = 0$ at $\psi = \psi_{max}$. The relative flow $u \equiv \langle x^{-2}\underset{\sim}{u}_\psi\cdot\nabla\psi\rangle/\langle x^{-2}\rangle$ must also be given at the boundary. In principle this may be specified arbitrarily, but three cases of special physical significance occur:

(i) $u(\psi_{max}\ ,\ t) = 0$.

This corresponds to no toroidal magnetic flux crossing the wall [cf., Eq. (46e)].

(ii) $u(\psi_{max}\ ,\ t) = (2\pi E^*_{\parallel}/\chi')|_{\psi_{max}}$.

In this case, no poloidal magnetic flux crosses the wall [cf., Eq. (46d)].

(iii) $u(\psi_{max}\ ,\ t) = -\langle(\underset{\sim}{u}_j - \underset{\sim}{u}_\psi)\cdot\nabla\psi\rangle$.

This corresponds to no net flux of species (j) across the wall [cf., Eq. (46a)].

The specification of u and ζ determines $u_\psi \cdot \nabla\psi$ at the boundary from Eq. (44a) and is thus the appropriate boundary condition for the elliptic operator Δ^* in Eq. (40). The operators L_0 and L_1 in Eqs. (39) and (40) also require a boundary value at the origin, ψ = 0 . The physical requirement that no toroidal flux arise at the magnetic axis [cf., Eq. (34b)] corresponds to u = 0 at ψ = 0 .

B. Free Boundary

In the free boundary problem, the plasma is surrounded by a vacuum region which extends to infinity. There are M axisymmetric line current sources at fixed locations $[(x_m, z_m);\ m = 1, M]$ in the vacuum with strengths $[I_m(t)]$. The time derivative of the poloidal flux at a fixed point in space $\underset{\sim}{x} = (x, z)$, $\dot{\chi}$, satisfies

$$(4\pi x^2)^{-1}\,\Delta^*\dot{\chi} = 2\pi x^{-1} \sum \dot{I}_m\,\delta(\underset{\sim}{x} - \underset{\sim}{x}_m) \quad . \tag{55}$$

in the vacuum region and

$$(16\pi^3 x^2)^{-1}\chi'\,\Delta^*[\dot{\chi} - \dot{\chi}(0)] + (L_0 + L_1)[\dot{\chi} - \dot{\chi}(0)] = Q \tag{56a}$$

in the plasma region where $[\dot{\chi} - \dot{\chi}(0)] = -\underset{\sim}{u}_\psi \cdot \nabla\psi\chi' + 2\pi[E_{||}^* - E_{||}^*(0)]$. Here, $\dot{\chi}(0)$ and $E_{||}^*(0)$ are the values of $\dot{\chi}$ and $\dot{E}_{||}^*$ at the magnetic axis, $\dot{I}_m \equiv \partial I_m/\partial t$, and Q is the source term

$$Q = \frac{5}{3}\left[p'\left(\frac{2 - \gamma' \; E_1^{\star} - E_1^{\star}(0)}{[illegible]}\right)\right] + (4\,x^2)^{-1}\left[\frac{g^2}{[illegible]}\left(\frac{2 - \gamma' \; E_1 - E_1^{\star}(0)}{[illegible]}\right)\right] + [illegible] \qquad (56b)$$

Here, use was made of Eqs. (39) through (42), and of Eqs. (46d) and (46e).

We now introduce the free space toroidal Green's function,

$$G(x, x') = -(xx')^{1/2}\left[(2 - k^2)K(k^2) - 2E(k^2)\right]/k \ , \qquad (57a)$$

where $K(k^2)$ and $E(k^2)$ are complete elliptic integrals and

$$k^2 = 4xx'/\left[(x + x')^2 + (z - z')^2\right] \ . \qquad (57b)$$

This satisfies

$$\Delta^{\star} G(x, x') = 2\pi x\, \delta(x - x') \ . \qquad (57c)$$

By using Green's theorem, it can easily be shown that $\dot{\chi}$ at a boundary point x_b satisfies the integral equation

$$\dot{\chi}(x_b) = 4\pi \sum_{m=1}^{M} I_m G(x_m, x_b) + \int 8\pi^2 \, dA \; xG(x, x_b)\{Q - (L_0 + L_1)[\dot{\chi} - \dot{\chi}(0)]\}/\chi' \qquad (58)$$

where the surface integral is over the plasma cross section. This is converted to a line integral over the plasma boundary by introducing a function $v(\psi, \theta)$ inside the plasma which satisfies

$$(16\pi^3 x^2)^{-1}\chi'\Delta^{\star} v = Q - (L_0 + L_1)[\dot{\chi} - \dot{\chi}(0)] \qquad (59)$$

with $v = 0$ on the plasma-vacuum boundary. Another application of Green's theorem gives the identity

$$\oint \frac{d\ell}{2\pi x} G(x, x_1) \frac{\partial v}{\partial n} = 8\pi^2 \int dA \, x G(x, x_1) [\text{[illegible]} - (L_0 + L_1)(\psi - \psi(0))] \text{[illegible]} \tag{60}$$

which can be used to eliminate the surface integral in Eq. (58).

A more convenient form for numerical iteration is obtained by defining another function H([illegible]),

$$H = \psi - \psi(0) - v$$

$$= \chi' u_\psi \cdot \text{[illegible]} + 2\pi [E_\psi^\star - E_\psi^\star(0)] - v \, . \tag{61}$$

The functions v and H then satisfy the coupled system of equations

$$(16\pi^4 x^2)^{-1} \Delta^\star v + (L_0 + L_1) v = Q - (L_0 + L_1) H \quad , \tag{62a}$$

$$\Delta^\star H = 0 \quad , \tag{62b}$$

with boundary conditions

$$v(x_b) = 0 \tag{63a}$$

$$H(x_b) = 4\pi \sum_{m=1}^{M} I_m [G(x_m, x_b) - G(x_m, x_0)]$$

$$+ \oint (d\ell/2\pi x) [G(x, x_b) - G(x, x_0)] \frac{\partial v}{\partial n} + H(x_0) \, . \tag{63b}$$

Here, x_0 is the position of the magnetic axis, $\psi = 0$. Once v and H are obtained from Eqs. (62) and (63), Eq. (61) can be used to obtain $\chi' u_\psi \cdot \nabla\psi$ at the plasma-vacuum interface. This serves as the boundary condition for Eq. (40).

The point $\psi = 0$ is an interior point for Eq. (62b) and thus does not take a boundary value. As discussed previously, the operators L_0 and L_1 in Eq. (62a) do require a boundary condition at the origin. Setting $v = -H$ at $\psi = 0$ corresponds to no toroidal flux change there. The system of Eqs. (61) through (63) is mathematically equivalent to Eq. (58) but is in a form more suitable for solution by iteration.

Finally, we consider the self-consistent determination of χ' at the plasma boundary in terms of currents in the poloidal field coils. If χ is the poloidal flux function which vanishes at infinity, then application of Green's theorem to the vacuum region yields for every boundary point $\underset{\sim}{x}_b$,[11]

$$\chi(x_b) = 4\pi \sum_{m=1}^{M} I_m(t) G(x_m, x_b) + \oint (d\ell/2\pi x)\, G(x, x_b)(\partial\chi/\partial n) . \qquad (64)$$

Noting that $\chi(\underset{\sim}{x}_b) = \chi(\psi_{max}, t) \equiv \chi_p(t)$, $\partial\chi/\partial n = \chi'(\psi_{max}, t)|\nabla\psi|$, and $d\ell = |\nabla\psi|(J/x)\, d\theta$, we can average Eq. (64) over the angle θ to obtain

$$\chi'(\psi_{max}, t) = \left[\chi_p(t) - 4\pi \sum_{m=1}^{M} I_m \oint \frac{d\theta}{2\pi}\, G(\underset{\sim}{x}_m, \underset{\sim}{x}')\right]$$

$$\times \left[\oint \frac{d\theta'}{2\pi} \oint \frac{d\theta}{2\pi} \left(\frac{|\nabla\psi|^2 J}{x^2}\right) G(\underset{\sim}{x}, \underset{\sim}{x}')\right]^{-1} . \qquad (65)$$

In Eq. (65), $\chi_p(t)$ is obtained by integrating Eq. (46d).

VIII. EVALUATION OF RESISTIVE CONTRIBUTION TO FLUXES

The resistive transport coefficients for an electron-ion plasma will now be explicitly evaluated for the collision dominated regime (Pfirsch-Schlüter regime) and the long mean free path banana regime.

A. Pfirsch-Schlüter Regime

In the collisional regime, parallel viscosity is negligible,[12] $\langle B\cdot\nabla\cdot\pi_i\rangle \approx 0$. Thus, $\Gamma_j^{nc} = \Gamma_j^{PS}$, and the transport fluxes are (for $Z_i = 1$)[19]:

$$\Gamma_e^{class} + \Gamma_e^{PS} = \Gamma_i^{class} + \Gamma_i^{PS} = -L_{11} p_\psi + L_{12} n_e (T_e)_\psi \quad (66a)$$

$$\langle q_e\cdot\nabla\psi\rangle/T_e = L_{12} p_\psi - L_{22} n_e (T_e)_\psi \ , \quad (66b)$$

$$\langle E\cdot B\rangle = -(2\pi\eta_{\|}/\psi')(g p_\psi + \langle B^2\rangle g_\psi/4\pi)$$

$$= (\eta_{\|} g^2/8\pi^2 V')\,[(V'\psi'/g)\langle|\nabla\psi|^2/x^2\rangle]_\psi \ , \quad (66c)$$

$$\langle q_i\cdot\nabla\psi\rangle/T_i = -L_i n_i (T_i)_\psi \ . \quad (66d)$$

Equation (52c) was used in obtaining the second form for Eq. (66c). The transport coefficients are:

$$L_{11} = L_0[1 + 2.65\ (\eta_{\|}/\eta_\perp)\ q_*^2] \ , \quad (67a)$$

$$L_{12} = (3/2)L_0[1 + 1.47\ (\eta_{\|}/\eta_\perp) q_*^2] \ , \quad (67b)$$

$$L_{22} = 4.66\ L_0[1 + 1.67\ (\eta_{\|}/\eta_\perp) q_*^2] \ , \quad (67c)$$

$$L_i = \sqrt{2}\ L_0 (m_i/m_e)^{1/2}\ (T_e/T_i)^{3/2}\ (\ 1 + 1.60\ q_*^2) \ , \quad (67d)$$

$$L_0 = n_e \eta_\perp \langle |\nabla\psi|^2/B^2\rangle \quad , \qquad (67e)$$

$$\eta_\perp = m_e (n_e e^2 \tau_{ei})^{-1} \quad , \qquad (67f)$$

$$q_*^2 = (1/2) q^2 \langle |\nabla\psi|^2/B^2\rangle^{-1} [\langle B^{-2}\rangle - \langle B^2\rangle^{-1}] \quad , \qquad (67g)$$

where $\eta_{\|}/\eta_\perp = 0.51$.

As an application of this Pfirsch-Schlüter model, consider the resistive diffusion of particles, heat, and magnetic fluxes determined by Eq. (46). Equations (46d) and (46e) combine to become a diffusion equation[20] for the rotational transform $\iota \equiv 1/q$ (i.e., the poloidal magnetic field), where q is the safety factor defined in Eq. (33) :

$$\left.\frac{\partial \iota}{\partial t}\right|_\psi = \frac{\partial}{\partial \Psi} (V_\psi \langle \underset{\sim}{E} \cdot \underset{\sim}{B}\rangle) = \frac{1}{2} \frac{\partial}{\partial \Psi} \eta_{\|} g^2 \frac{\partial}{\partial \Psi} [(V_\psi q^2 \langle B_p^2\rangle g^{-1})\iota] \quad , \qquad (68a)$$

where $V_\psi = \partial V/\partial\Psi$. Here, the second form of Ohm's law, Eq. (66c), was used to express $\langle \underset{\sim}{E} \cdot \underset{\sim}{B}\rangle$ in terms of ι . In the Pfirsch-Schlüter regime, the electrons and ions may be assumed to be equilibrated, $T_e \approx T_i \equiv T$. Then, the particle and density Eqs. (46a) and (46b) become, with $n_e = n_i = n$, $S_p = S_n = 0$, $N_\psi \equiv nV_\psi$ and $\sigma_\psi \equiv p^{3/5} V_\psi$:

$$\left.\frac{\partial}{\partial t}\right|_\psi N_\psi = \frac{\partial}{\partial \Psi}\{N_\psi[(\Psi')^2 (L_{11}/n) p_\psi - (\Psi')^2 (L_{12}/n) nT_\psi + V_\psi \langle \underset{\sim}{E} \cdot \underset{\sim}{B}\rangle q (\langle B_p^2\rangle/\langle B^2\rangle)]\} , \qquad (68b)$$

$$\frac{3}{2} \ldots \frac{\partial}{\partial t} \ldots = \frac{1}{4\pi} \langle E\cdot B\rangle \ldots \langle B_p^2\rangle + \frac{1}{2V_\psi}\frac{\partial}{\partial \psi} V_\psi (\chi')^2 L_i \ldots \quad . \qquad \text{[illegible]}$$

(68c)

In deriving Eq. (68c) it was noted that $L_i \ll L_{mn}$.

There are three characteristic time scales in Eqs. (68a-c):

(i) $\tau_0 \approx a^2/\eta_\parallel \approx (a/\delta_e)^2 \nu_{ei}^{-1}$, the classical skin penetration time of the poloidal magnetic field. Here, $\delta_e = c/\omega_{pe}$ is the collisionless skin depth.

(ii) $\tau_1 \approx a^2/L_i(\chi')^2 T \approx \epsilon^{-1}(m_e/m_i)^{1/2} (1 + 1.6\, q_*^2)^{-1}\, \tau_0$, the Pfirscn-Schlüter heat conduction time.

(iii) $\tau_2 \approx (L_i/L_\parallel)\, \tau_1 \approx (m_i/m_e)^{1/2}\, \tau_1$, the particle transport time.

For low $\epsilon\,\beta^2$ tokamak plasmas, the ordering $\tau_1 \ll \tau_0 \ll \tau_2$ seems most appropriate. Thus, ion heat conduction balances Joule heating at the end of the most rapid τ_1 time scale. This determines the temperature profile, but the MHD functions $p(\Psi)$ and $g(\Psi)$, which determine the equilibrium configuration, remain arbitrary. On the τ_0 time scale, which was called the fast time scale in Ref. 1 (where heat conduction was ignored), the poloidal flux χ diffuses through a stationary density profile. Equation (68a) yields an asymptotic constraint between p and g; i.e., $V_\psi\langle E\cdot B\rangle = c_1(t)$, where c_1 varies on the τ_2 time scale. Using Eq. (66c) implies[1,2]

$$q p_\psi + \langle B^2\rangle q_\psi/4\pi + c_1(t)/(2\langle qV\rangle) = 0 \ . \qquad (69a)$$

Furthermore, the steady state energy Eq. (68c) can now be integrated once to yield the temperature profile on the τ_0 time scale for an ohmically heated, collisional plasma,

$$(1/4\pi) c_1 q \langle B_p^2\rangle + (1/2) L_i (\Psi')^2 p T_\psi = 0 \ . \qquad (69b)$$

The boundary condition $B_p = T_\psi = 0$ at the magnetic axis has been used. Finally, on the τ_2 (slow[1]) time scale, the density tends to a stationary value determined by the vanishing of the particle flux,

$$p_\psi + nqc_\perp \langle L_{11}(\Psi')^2\rangle^{-1} [(2L_{12}/L_i)\langle B_p^2\rangle/(4\pi p) + \langle B_p^2\rangle/\langle B^2\rangle] = 0 \ , \qquad (69c)$$

where Eq. (69b) was used to eliminate T_ψ . Equation (69c) is a second relation determining the profile of the MHD function $p(\Psi)$. Combining Eqs. (69a) and (69c) yields a relation for the local (in Ψ) poloidal beta $\beta_p = 8\pi p/\langle B_p^2\rangle$ in a stationary, Pfirsch-Schlüter plasma. With $p_\psi(V_\psi q^{-1} q\langle B_p^2\rangle)_\psi^{-1} = -p(V_\psi g^{-1} q\langle B_p\rangle)^{-1}$ we obtain

$$\beta_p - \beta_*^2/\beta_p - \beta_{GH} = 0 \ , \qquad (70a)$$

where

$$\beta_* = \{16\pi\, n\eta_{\parallel}[(\Psi')^2 L_i]^{-1}(L_{12}/L_{11}) V_\psi g q^2\}^{1/2}$$

$$= [(4/\pi)(m_e/m_i)^{1/2} g V_\psi (\langle x^2 B_p^2/B^2\rangle)^{-1}(\eta_\perp/\eta_{\parallel} + 2.65\, q_*^2)^{-1}]^{1/2} . \qquad (70b)$$

[illegible] 4 [illegible] $^{-1}$ [illegible] B_p^2 [illegible] B^2 [illegible]

$$(1 \ [\text{illegible}]) \ [\text{illegible}] \, (\ [\text{illegible}] \, B_p^2 / B^2 \ [\text{illegible}]) \, (\langle B_p^2 \rangle / \langle B^2 \rangle) \, (\ [\text{illegible}]_{\perp} / \ [\text{illegible}] + 2.65 \, q_*^2)^{-1} \ . \tag{70c}$$

The first term in Eq. (70a) arises from the pressure gradient driven outward particle flux. The middle term arises from finite [illegible] $^{-1}$ [illegible] thermal particle diffusion [illegible]. The $_{CH}$ term is due to the classical [illegible] inward [illegible]. Thus, the [illegible][1] [illegible] - limit, [illegible] $+ 2.65\, q_*^2)^{-1}$ [illegible], arises from the balance of inward pinch and outward diffusion. However, since

$$[\text{illegible}]_* / [\text{illegible}]_{GH} \ [\text{illegible}] (\ [\text{illegible}]_{\perp} / \ [\text{illegible}] + 2.6 \, q_*^2)^{1/2} \ [\text{illegible}] \ 1 \tag{71}$$

for [illegible] $^{-1} (m_e/m_i)^{1/4}$ [illegible] 1, the presence of ion heat conductior leads [illegible] balance between the thermal pinch and normal outward diffusion [illegible] a larger [illegible] limit,[21] $[\text{illegible}]_p = [\text{illegible}]_*$ [illegible], than previously obtained.

Steady state profiles for p and g can be obtained in terms of $\langle B_p^2 \rangle$ and [illegible] by solving the transport constraint Eqs. (69a) and (69c). These profiles must be inserted into the equilibrium Eq. (11), thus yielding a nonlinear GDE[9] for the self-consistent flux surface geometry. The determination and stability of steady state 2-D resistive profiles is considered in Ref. 13, together with a complete analysis of the time evolution of a collisional and collisicnless plasma.

B. Banana Regime

The transport coefficients for a banana-regime simple plasma have been recently computed for arbitrary aspect ratio and cross section shape,[22] with the restriction that B has only a single maximum on each flux surface. In previous calculations,[23] the values of the banana transport coefficients were obtained by a linear superposition of large and small aspect ratio limit results. The present results are based on an aspect ratio fit for the banana regime viscosity coefficients μ_j,

$$\mu_j = f_c^{-2}\left[\left(f_t \mu_j'(0)\right)^{-1} + \left(f_c \mu_j'(1)\right)^{-1} \right]^{-1} , \qquad (72a)$$

where

$$f_t = 1 - (3/4)\langle B^2\rangle \int_0^{B_c^{-1}} \lambda\, d\lambda / \langle (1 - \lambda B)^{1/2}\rangle , \qquad (72b)$$

is the fraction of trapped particles (B_c is the maximum value of B on a flux surface), $f_c = 1 - f_t$ is the fraction of circulating particles, and

$$\mu_j'(0) = \lim_{f_t \to 0} \mu_j / f_t , \qquad (72c)$$

$$\mu_j'(1) = \lim_{f_t \to 1} f_c \mu_j . \qquad (72d)$$

The quantity $\Delta\mu_j \equiv \mu_j'(0) - \mu_j'(1)$ is nonzero due to like-particle collisions (*i.e.*, $\Delta\mu_j \to 0$ in the Lorentz limit). Neglecting $\Delta\mu_j$ leads to the approximate form $\mu_j = f_t \mu_j'(0)/f_c$, which was used to evaluate the transport fluxes:

$$\Gamma_e^{bp} = \Gamma_i^{bp} = -L_{11}^{bp}[p_\psi + y n_i (T_i)_\psi] + L_{12}^{bp} n_e (T_e)_\psi - L_{13}\langle \underset{\sim}{E}\cdot\underset{\sim}{B}\rangle/\langle B^2\rangle , \qquad (73a)$$

$$[illegible]^{class}_e + [illegible]^{PS}_e = [illegible]^{class}_i + [illegible]^{PS}_i = -L_{11} p' + L_{12} n_e (T_e)' \quad , \tag{73b}$$

$$\langle q_e \cdot [illegible] \rangle / T_e = (L^{bp}_{11} + L_{12}) p' + L^{bp}_{12} y n_i (T_i)' - L^{nc}_{22} n_e (T_e)' + L_{23} \langle E \cdot B \rangle / \langle B^2 \rangle \quad , \tag{73c}$$

$$\langle q_i \cdot [illegible] \rangle / T_i = -L^{nc}_i n_i (T_i)' \quad , \tag{73d}$$

$$\langle u_i \cdot [illegible] B \cdot [illegible] \rangle = -y [illegible] (\langle [illegible] e_i \rangle \langle B^2 \rangle)^{-1} (T_i)' \quad , \tag{73e}$$

$$\langle E \cdot B \rangle = [illegible]_{\parallel} (1 + L_{33})^{-1} [illegible] -2 [illegible] q p' + (\langle B^2 \rangle / 4 [illegible]) a' / [illegible]$$

$$+ L_{13} [p' + y n_i (T_i)'] - L_{23} n_e (T_e)' [illegible] \quad . \tag{73f}$$

Here the transport coefficients are

$$L^{bp}_{11} = L_*(1.53 - 0.53\ f_t) \quad , \tag{74a}$$

$$L^{bp}_{12} = L_*(2.13 - 0.63\ f_t) \quad , \tag{74b}$$

$$L_{13} = (2 [illegible] / [illegible]') nqf_t(1.68 - 0.68\ f_t) \quad , \tag{74c}$$

$$L_{23} = 1.25\ (2 [illegible] / [illegible]') nqf_t(1 - f_t) \quad , \tag{74d}$$

$$L_{11} = L_0(1 + 2q_*^2) \quad , \tag{74e}$$

$$L_{12} = (3/2) L_0 (1 + 2q_*^2) \quad , \tag{74f}$$

$$L^{nc}_{22} = 4.66\ [L_0(1 + 2q_*^2) + L_*] \quad , \tag{74g}$$

$$L^{nc}_i = \sqrt{2}\ (m_i/m_e)^{1/2}\ (T_e/T_i)^{3/2}\ [L_0(1 + 2q_*^2) \tag{74h}$$

$$+ 0.46\ L_*(1 - 0.54\ f_t)^{-1}] \quad , \tag{74i}$$

$$y = -1.17\ (1 - f_t)(1 - 0.54\ f_t)^{-1} \quad , \tag{74j}$$

$$L_{33} = -1.26\ f_t(1 - 0.18\ f_t) \quad . \tag{74k}$$

$$L_* = f_t(2\pi g/\chi')^2 nn_\perp/\langle B^2 \rangle \quad . \tag{74l}$$

IX. DISCUSSION AND SUMMARY

The theory of self-consistent two-dimensional [illegible] transport has been presented in a manner which is both [illegible] and amenable to efficient numerical solution. The reduced set of 1-D evolution Eqs. (46a) through (46e), together with the [illegible] Eq. (40), comprise a complete description of resistive [illegible] scale phenomena for an arbitrary transport model. Particular closure models have been explicitly evaluated for the neoclassical collision-dominated and long mean free path regimes. Some steady state solutions have been discussed. Complete analytical and numerical solutions using these transport models will be [illegible] [illegible].[13]

It has long been recognized that the MHD equilibrium equation $J \times B = \nabla p$ constrains the resistive evolution of surface averaged thermodynamic and magnetic variables in a plasma.[8] The work of Grad and Hogan[1] discussed consequences of this constraint and thereby renewed interest in a self-consistent treatment of diffusion in a deformable plasma. As a result, iterative numerical methods for solving a highly nonlinear form of the equilibrium (Grad-Shafranov) Eq. (11a) simultaneously with a simple collisional diffusion model have been developed.[4,24-27] A numerical method for extending this to an arbitrary transport model has recently been described by Hogan.[28] In that work, each numerical time step is divided (split) into two parts: one in which plasma and field profiles are advanced with fixed geometry, and the other in which the nonlinear equilibrium equation is iterated to obtain a new flux surface geometry and hence adiabatically increment each plasma variable.

Instead of using the nonlinear Grad-Shafranov equation, we have taken the time differentiated (linearized) form, Eq. (40), to obtain the velocity of toroidal flux surfaces. Time differentiating the equilibrium equation was suggested by several authors.[1,2,3,4] In these works the resulting equation for the poloidal flux velocity was a generalized differential equation,[4] since flux surface averages of the unknown 2-D velocity contributed ordinary differential terms (in ψ) to the elliptic terms (in x) arising from the local variation of the velocity. However, no tractable means of numerical solution was suggested. In addition, the equation discussed by these authors used the poloidal flux velocity as the dependent variable. As discussed in Section V, it is preferable, for both physical and computational reasons, to compute directly the toroidal flux velocity. The time dependent magnetic flux coordinate transformation introduced in Section III enables efficient numerical solution of the formidable[2,3] constraint, Eq. (40), since the surface integrals reduce to simple averages over a single coordinate (θ).

A reconciliation between two different interpretations of the plasma mass flow in a resistive plasma has been suggested. One viewpoint[1] considers the mass flow as a state variable, for which a boundary value problem must be solved. The other interprets the cross-field plasma flow as a transport quantity,[3] determined by <u>local</u> gradients of thermodynamic variables. The "paradox" is resolved by noting that in the frame moving with the toroidal magnetic flux, the mass flow is indeed a transport quantity, essentially equal to the flux computed with $\partial B/\partial t = 0$ (at least in the collisional regime). However, the motion of flux surfaces in space is determined by Eq. (40) and requires the solution of a global boundary value problem for its determination.

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