Effects of an RF Phase Feedback System on the Coherent Tunes and Damping Rates

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One possible feedback system, designed mainly for damping the longitudinal dipole oscillations,^{1,2} utilizes a beam position monitor and an rf cavity. The horizontal displacement of the beam is measured at the monitor and the measurement is sent to the rf cavity. The phase of the cavity voltage is then adjusted so that an electron changes its energy by the additional amount of $\Delta\epsilon = \zeta E_0 x_{\text{monitor}}$. This FB system introduces damping or anti-damping to the horizontal betatron oscillation and the longitudinal synchrotron oscillation. Although approximate expressions for the associated damping constants $\alpha_{x,s}$ can be obtained by elementary considerations,¹ it is perhaps constructive to have an exact calculation available as well. In the following, we will (i) describe the exact calculation; (ii) obtain approximate expressions of $\alpha_{x,s}$ from the exact calculation; (iii) obtain approximate expressions of $\Delta\nu_{x,s}$, the coherent tune shifts caused by the FB system; and (iv) numerically compare the exact and approximate results under various conditions. We assume that there is only one active rf cavity in the storage ring and that the monitor signal reaches the rf cavity before the beam completes one turn. Let X be a vector defined by³

$$X = \begin{bmatrix} x \\ x' \\ z \\ \epsilon/E_0 \end{bmatrix} \quad (1)$$

The transformation of X for one complete revolution starting from the rf cavity is given by

$$T_{\text{tot}} = T_{\text{cav}} T_0 \quad (2)$$

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where T_{cav} is the transformation across the cavity and T_0 is the transformation for the rest of the storage ring.

The energy kick at the rf cavity is given by

$$\Delta\epsilon/E_0 = \underbrace{\frac{2\pi v_s^2}{\alpha R}}_{\text{rf focusing term}} z + \underbrace{\zeta x_{\text{monitor}}}_{\text{FB term}}, \quad (3)$$

where v_s is the unperturbed synchrotron tune,⁴ α is the momentum compaction factor and $2\pi R$ is the machine circumference. The transformation T_{cav} can, therefore, be written as

$$T_{cav} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2\pi v_s^2}{\alpha R} & 1 \end{bmatrix} + \zeta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} T_{CM}^{-1} \quad (4)$$

with T_{CM} the transformation from the monitor to the cavity. We notice that⁵

$$T_0 = \begin{bmatrix} r_{11} & r_{12} & 0 & \eta_c - r_{11} & \eta_c - r_{12}\eta_c' \\ r_{21} & r_{22} & 0 & \eta_c' - r_{21}\eta_c - r_{22}\eta_c' \\ a & b & 1 & -2\pi\alpha R - a\eta_c - b\eta_c' \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (5)$$

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and

$$T_{CM}^{-1} T_O = T_{MC} = \begin{bmatrix} R_{11} & R_{12} & 0 & \eta_m - R_{11} \eta_c - R_{12} \eta_c' \\ R_{21} & R_{22} & 0 & \eta_m' - R_{21} \eta_c - R_{22} \eta_c' \\ A & B & 1 & C - A \eta_c - B \eta_c' \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

where we have defined

$$r_{11} = \cos 2\pi v_x + \alpha_c \sin 2\pi v_x$$

$$r_{12} = \beta_c \sin 2\pi v_x$$

$$r_{21} = -\frac{1}{\beta_c} (1 + \alpha_c^2) \sin 2\pi v_x$$

$$r_{22} = \cos 2\pi v_x - \alpha_c \sin 2\pi v_x$$

$$a = r_{21} \eta_c - r_{11} \eta_c' + \eta_c'$$

$$b = -r_{12} \eta_c' + r_{22} \eta_c - \eta_c$$

$$R_{11} = \sqrt{\frac{\beta_m}{\beta_c}} \left[\cos (\psi_m - \psi_c) + \alpha_c \sin (\psi_m - \psi_c) \right]$$

$$R_{12} = \sqrt{\beta_c \beta_m} \sin (\psi_m - \psi_c)$$

$$R_{21} = -\frac{1}{\sqrt{\beta_c \beta_m}} \left[(1 + \alpha_c \alpha_m) \sin (\psi_m - \psi_c) - (\alpha_c - \alpha_m) \cos (\psi_m - \psi_c) \right]$$

$$R_{22} = \sqrt{\frac{\beta_c}{\beta_m}} \left[\cos (\psi_m - \psi_c) - \alpha_m \sin (\psi_m - \psi_c) \right]$$

$$A = R_{21} \eta_m - R_{11} \eta_m' + \eta_c'$$

$$B = -R_{12} \eta_m' + R_{22} \eta_m - \eta_c$$

$$C = - \int_{s_c}^{s_m} ds \frac{\eta(s)}{\rho(s)}$$

$\eta_{c,m}$ = dispersion functions

$\alpha_{c,m}$ = alpha functions

$\beta_{c,m}$ = beta functions

$\psi_{c,m}$ = horizontal betatron phases, $2\pi\nu_x > \psi_m > \psi_c \geq 0$.

We have shown the explicit expression of T_{MC} for completeness. To obtain T_{tot} , we need only to know the first row of T_{MC} . Substituting Eqs. (4), (5), (6) into (2) yields

$$T_{tot} = \begin{bmatrix} r_{11} & r_{12} & 0 & \eta_c - r_{11}\eta_c - r_{12}\eta_c' \\ r_{21} & r_{22} & 0 & \eta_c' - r_{21}\eta_c - r_{22}\eta_c' \\ a & b & 1 & -2\pi\alpha R - a\eta_c - b\eta_c' \\ \frac{2\pi\nu_s^2}{\alpha R} a + \zeta R_{11} & \frac{2\pi\nu_s^2}{\alpha R} b + \zeta R_{12} & \frac{2\pi\nu_s^2}{\alpha R} & T_{tot,44} \end{bmatrix}, \quad (7)$$

with

$$T_{tot,44} = 1 - \frac{2\pi\nu_s^2}{\alpha R} (2\pi\alpha R + a\eta_c + b\eta_c') + \zeta(\eta_m - R_{11}\eta_c - R_{12}\eta_c').$$

The first three rows of T_{tot} are the same as those of T_0 . The eigenvalues λ of T_{tot} are determined from the secular equation

$$\begin{aligned}
 & (1 - 2\lambda \cos 2\pi\nu_x + \lambda^2) \left[(1 - \lambda)^2 + 4\pi^2 \nu_s^2 \lambda \right] \\
 & - \frac{2\pi\nu_s^2 \sin 2\pi\nu_x}{\alpha_R \beta_C} \left[\eta_C^2 + (\alpha_C \eta_C + \beta_C \eta_C')^2 \right] \lambda (1 - \lambda)^2 \\
 & + \zeta \left\{ \eta_m (1 - \lambda) (1 - 2\lambda \cos 2\pi\nu_x + \lambda^2) + (1 - \lambda)^2 \left[Q + (R_{11} \eta_C + R_{12} \eta_C') \lambda \right] \right\} = 0, \quad (8)
 \end{aligned}$$

with

$$Q = \sqrt{\beta_C \beta_m} \eta_C' \sin \Delta\psi - \sqrt{\frac{\beta_m}{\beta_C}} \eta_C (\cos \Delta\psi - \alpha_C \sin \Delta\psi)$$

and $\Delta\psi = 2\pi\nu_x - \psi_m + \psi_C$ the betatron phase advance from the monitor to the cavity. The first term on the LHS of Eq.(8) is the unperturbed contribution. The second term is the synchrobetatron coupling term found in Ref. 5. The last term, being proportional to ζ , is the contribution from the FB system. Numerically solving Eq.(8) for λ , expressed as $\exp[-\alpha_k \pm i2\pi(\nu_k + \Delta\nu_k)]$ with $k = x, s$, yields an exact solution for the damping constants $\alpha_{x,s}$ and the tune shifts $\Delta\nu_{x,s}$. Furthermore, the following identity results from the property of eigenvalues:

$$\begin{aligned}
 e^{-2\alpha_x - 2\alpha_s} &= \det(T_{\text{tot}}) \\
 &= 1 + \zeta [\eta_m + Q].
 \end{aligned}$$

For weak damping, $|\alpha_{x,s}| \ll 1$, we have the following sum rule

$$\alpha_x + \alpha_s = -\frac{\zeta}{2} (\eta_m + Q). \quad (9)$$

If the cavity and the monitor are located at the same position ($\Delta\psi = 0$), we find $\alpha_x + \alpha_s = 0$ and thus damping for one mode necessarily means anti-damping for the other mode.

It turns out that approximate closed-form expressions for $\alpha_{x,s}$ and $\Delta v_{x,s}$ can be obtained from Eq.(8) provided that

- (i) $2\pi v_s \ll 1$,
- (ii) the synchrotron coupling effect, i.e., the second term on the LHS of Eq.(8), can be ignored.
- (iii) the FB strength is reasonably weak.

α_x and Δv_x . Let $\lambda = (1-\Delta)e^{i2\pi v_x}$, $|\Delta| \ll 1$. Substituting into Eq.(8) and keeping terms to first order in Δ and ζ , we get

$$\Delta \approx - \frac{i\zeta}{2 \sin 2\pi v_x} (Qe^{-i2\pi v_x} + R_{11}\eta_c + R_{12}\eta_c')$$

and therefore,

$$\alpha_x \approx \operatorname{Re}(\Delta) \approx - \frac{\zeta Q}{2} \quad (10)$$

$$\Delta v_x \approx - \operatorname{Im}(\Delta) \approx \frac{\zeta}{2 \sin 2\pi v_x} \left[Q \cos 2\pi v_x + R_{11}\eta_c + R_{12}\eta_c' \right].$$

α_s and Δv_s . Letting $\lambda = (1-\Delta)e^{i2\pi v_s}$, $|\Delta| \ll 1$, we obtain

$$\alpha_s \approx - \frac{\zeta \eta_m}{2} \quad (11)$$

$$\Delta v_s \approx - \frac{\pi \zeta v_s}{2} \left[\eta_m + \frac{1}{\cos 2\pi v_s - \cos 2\pi v_x} (Q \cos 2\pi v_s + R_{11}\eta_c + R_{12}\eta_c') \right].$$

The results of $\alpha_{x,s}$ differ from Ref. 1 by factors of 2. The values of $\Delta v_{x,s}$ give the coherent tune shifts caused by the FB system. Eqs. (10) and (11) agree with the sum rule, Eq.(9), as they should.

In Figs. 1 to 6, calculations using Eqs.(10) and (11) are compared with the exact solutions of Eq.(8). For all cases considered, the configuration has $\nu_x = 21.15$, $2\pi nR = 6.38m$, and the cavity is located near the 12 o'clock position with $\beta_c = 230m$, $\eta_c = -0.92m$, $\eta_c' = -0.039$, $\alpha_c = -5.16$ and $\psi_c = 1.363$.

Case 1. Figs. 1 and 2. The monitor is located at the symmetry point at the 11 o'clock position, with $\eta_m = 1.8m$, $\theta_m = 26m$ and $\psi_m = 121.815$. The synchrotron tune ν_s is 0.07. The range of ζ covers $0 > \zeta > -0.1m^{-1}$. The sign of ζ is chosen so that, according to Eq.(11), the synchrotron mode will be damped. Solid and dotted curves are the exact and approximate values, respectively. In this case, the synchrotron mode is damped, while the betatron mode is anti-damped. The nonzero values of $\Delta\nu_{x,s}$ at $\zeta = 0$ are due to the synchrobetatron coupling.

Case 2. Figs. 3 and 4. All input parameters are the same as Case 1 except that the monitor is moved to the symmetry point at the 9 o'clock position with $\psi_m = 99.667$. In this case, both modes are damped.

Case 3. Figs. 5 and 6. The approximate expressions break down near a synchrobetatron resonance even outside of the stopband. Case 3 is the same as Case 2 except that $\nu_s = 0.13$, which is 0.02 away from the resonance line.

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References

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2. J.-L. Pellegrin, private communication (1978).
3. We have defined x to be the total horizontal displacement which includes both the synchrotron and the betatron contributions.
4. M. Sands, SLAC-121 (1970).
5. A. Piwinski and A. Wrulich, DESY 76/07 (1976).

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β_1 [m⁻¹]

Fig. 3