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Nonlinear Rayleigh-Taylor Stability with Mass and
Heat Transfer (II): Finite Fluid Layers

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Abstract

The nonlinear Rayleigh-Taylor stability of a liquid layer over a finite vapor layer is studied using a simplified formulation of the problem. The method of multiple scale expansion is employed for the investigation. It is found when the heat transfer rate is strong enough, the classically unstable system is stabilized by the nonlinear effect and the effect of mass and heat transfer across the interface. A simple non-dimensional parameter is also found to characterize the stability of the system.

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I. Introduction.

In a previous paper^[1], referred later as P1, we have presented a study of nonlinear Rayleigh-Taylor stability with mass and heat transfer for two semi-infinite fluids. In this paper we shall investigate the corresponding case of finite fluid layers. Again let both fluids be taken as incompressible and inviscid, let the density of the fluid on top be $\rho^{(2)}$, the density of the fluid underneath $\rho^{(1)}$, and let the two fluids be confined between two parallel planes $y = -h_1$ and $y = h_2$. The equilibrium interface is taken to be located at $y = 0$ (See Figure 1). Then the critical wave number is again given by

$$k_c = [g(\rho^{(2)} - \rho^{(1)})/\sigma]^{1/2}, \quad (1)$$

where g is the gravitation constant and σ , the surface tension coefficient.

Now let the interface be given by

$$y = \eta(x, t), \quad (2)$$

and let $\phi^{(1)}(x, y, t)$ and $\phi^{(2)}(x, y, t)$ be the velocity potential of the two fluids, then we have

$$\nabla^2 \phi^{(1)} = 0, \quad -h_1 < y < \eta, \quad (3)$$

$$\nabla^2 \phi^{(2)} = 0, \quad \eta < y < h_2. \quad (4)$$

The Bernoulli equations are

$$\frac{p^{(j)}}{\rho^{(j)}} + \frac{1}{2}(\nabla \phi^{(j)})^2 + gy + \frac{\partial \phi^{(j)}}{\partial t} = 0, \quad j = 1, 2 \quad (5)$$

where $p^{(j)}$ is the pressure in the fluid.

The interfacial conditions on $y = \eta(x, t)$, as those given in Pl, are

$$\rho^{(1)} [\eta_t + \phi_x^{(1)} \eta_x - \phi_y^{(1)}] = \{2\}. \quad (6)$$

$$\begin{aligned} \rho^{(1)} [\phi_t^{(1)} + \frac{1}{2}(\phi_x^{(1)})^2 + \phi_y^{(1)2} + g\eta \\ - \frac{1}{1+\eta_x} (\phi_x^{(1)} \eta_x - \phi_y^{(1)}) (\eta_t + \phi_x^{(1)} \eta_x - \phi_y^{(1)})] \\ = \{2\} + \sigma \eta_{xx} (1 + \eta_x^2)^{-3/2}, \end{aligned} \quad (7)$$

and

$$\rho^{(1)} [\eta_t + \phi_x^{(1)} \eta_x - \phi_y^{(1)}] = -\alpha(\eta + \alpha_2 \eta^2 + \alpha_3 \eta^3), \quad (8)$$

where the notation $\{2\}$ on the right hand side of the equations is used to denote the same expression as that of the left, except changing the superscript (1) to (2), and moreover we have included higher order terms in the right hand side of (8). For a finite layer, we also have the boundary condition that normal fluid velocities vanish in both the upper and lower walls.

$$\frac{\partial \phi^{(1)}}{\partial y} = 0, \quad \text{on } y = -h_1, \quad (9)$$

$$\frac{\partial \phi^{(2)}}{\partial y} = 0, \quad \text{on } y = h_2. \quad (10)$$

It is to be noted that according to the quasi-equilibrium approximation, the coefficients α and α_2 are given by

$$\alpha = \frac{G}{L} \left(\frac{1}{h_1} + \frac{1}{h_2} \right),$$

and

$$\alpha_2 = -\left(\frac{1}{h_1} - \frac{1}{h_2} \right),$$

where G is the equilibrium heat flux and L is the latent heat of phase transformation.

II. Multi-scale Expansion Near the Critical Wave Number.

Following the development presented in P1, we employ now the multi-scale expansion method to investigate the nonlinear stability of the system. Introducing ϵ as a small parameter, we assume the following expansion of the variables:

$$\eta = \sum_{n=1}^3 \epsilon^n \eta_n(x, T_0, T_1, T_2) + O(\epsilon^4), \quad (11)$$

$$\phi^{(j)} = \sum_{n=1}^3 \epsilon^n \phi_n^{(j)}(x, y, T_0, T_1, T_2) + O(\epsilon^4), \quad (12)$$

where $T_n = \epsilon^n t$.

We assume also that α , α_2 and α_3 are all of the order $O(1)$.

Substituting (11) and (12) into (3), (4) and (6)-(10), and equating like powers of ϵ , we obtain for $n = 1, 2, 3$, three sets of equations for $\phi_n^{(j)}$ and η_n . These equations are collected in the Appendix.

To solve these equations in the neighborhood of the linear critical wave number k_c , we assume that the critical wave number, because of the nonlinear effect, will shift to

$$k = k_c + \epsilon^2 \mu. \quad (13)$$

The first order solutions will reproduce the linear result for the critical case, and they are given by:

$$\eta_1 = A(T_1, T_2) e^{ikx} + \bar{A}(T_1, T_2) e^{-ikx}, \quad (14)$$

$$\phi_1^{(1)} = \frac{\alpha}{\rho^{(1)}_k} \frac{\cosh k(y+h_1)}{\sinh kh_1} [Ae^{ikx} + \bar{A}e^{-ikx}], \quad (15)$$

$$\phi_1^{(2)} = - \frac{\alpha}{\rho^{(2)}_k} \frac{\cosh k(y-h_2)}{\sinh kh_2} [Ae^{ikx} + \bar{A}e^{-ikx}], \quad (16)$$

where a bar denotes complex conjugate quantities. We also recover from (A-6) the critical dispersion relation $k = k_c$, to this order.

The second order equations for η_2 , $\phi_2^{(1)}$, and $\phi_2^{(2)}$ have non-homogeneous expressions in terms of the first order solutions. In order to avoid the inconsistency that η_2/η_1 may become unbounded as x goes to infinity, we obtain from (A-7) - (A-9):

$$\frac{\partial A}{\partial T_1} = 0. \quad (17)$$

Then the solutions for η_2 , $\phi_2^{(1)}$ and $\phi_2^{(2)}$ are given by:

$$\eta_2 = -2\alpha_2 |A|^2 + A_2 e^{2ikx} + \bar{A}_2 e^{-2ikx} \quad (18)$$

$$\phi_2^{(1)} = b^{(1)}(T_0, T_1, T_2) + [B_2 e^{2ikx} + \bar{B}_2 e^{-2ikx}] \cosh 2k(y+h_1), \quad (19)$$

$$\phi_2^{(2)} = b^{(2)}(T_0, T_1, T_2) + [C_2 e^{2ikx} + \bar{C}_2 e^{-2ikx}] \cosh 2k(y-h_2), \quad (20)$$

where

$$A_2 = NA^2, \quad (21)$$

$$B_2 = \frac{\alpha A^2}{2k \rho^{(1)}_k \sinh 2kh_1} [kN + \alpha_2 - 2k \coth kh_1], \quad (22)$$

$$C_2 = - \frac{\alpha A^2}{2k\rho^{(2)} \sinh 2kh_2} [kN + \alpha_2 + 2k \coth kh_2], \quad (23)$$

$$N = \frac{\alpha^2}{6kg(\rho^{(2)} - \rho^{(1)})} \left[\frac{1 + \coth^2 kh_1}{\rho^{(1)}} - \frac{1 + \coth^2 kh_2}{\rho^{(2)}} \right], \quad (24)$$

and

$$\begin{aligned} \rho^{(1)} \frac{\partial b^{(1)}}{\partial T_0} - \rho^{(2)} \frac{\partial b^{(2)}}{\partial T_0} &= \left(\frac{1}{\rho^{(2)} \sinh^2 kh_2} - \frac{1}{\rho^{(1)} \sinh^2 kh_1} \right) \alpha^2 |A|^2 \\ &+ 2g(\rho^{(1)} - \rho^{(2)}) \alpha_2 |A|^2. \end{aligned} \quad (25)$$

Substitute the first and second order solutions into the third order equation. In order to avoid the nonuniformity of the expansion, we again impose the condition that secular terms vanish. Then we found from (A-12):

$$\frac{\alpha}{k} [\coth kh_1 + \coth kh_2] \frac{\partial A}{\partial T_2} + 2\sigma k_c \mu A + \left(\nu - \frac{3}{2} \sigma k_c^4 \right) |A|^2 A = 0, \quad (26)$$

where

$$\begin{aligned} \nu &= \alpha^2 k \left\{ \left(N + \frac{\alpha_2}{k} \right) \left[\frac{1}{\rho^{(1)}} (\coth kh_1 \coth 2kh_1 - 1) - \frac{1}{\rho^{(2)}} (\coth kh_2 \coth 2kh_2 - 1) \right] \right. \\ &\quad \left. - 2 \left[\frac{1}{\rho^{(1)}} (\coth kh_1 \coth 2kh_1 - 1) \coth kh_1 + \frac{1}{\rho^{(2)}} (\coth kh_2 \coth 2kh_2 - 1) \right] \right\} \end{aligned} \quad (27)$$

From (26), we can see that a sufficient condition for stability of the system is

$$\nu > \frac{3}{2} \sigma k_c^4. \quad (28)$$

When the system is linearly unstable, i.e. when $\mu < 0$, the asymptotic value of A for large time will be given by:

$$|A|^2 = -2\sigma k_c \mu / (\nu - \frac{3}{2} \sigma k_c^4). \quad (29)$$

Therefore the range of spectrum of the stable wavelength is enlarged by the finite amplitude effect. From practical considerations, we would expect that as the amplitude exceeds half the wavelength, or the thickness of the fluid layers, then there is a tendency for bubbles to form and detach from the interface, or to cause the rupture of the fluid layers. Thus more stringent practical stability criterions than (28) are required. When the fluid layers are sufficiently thick, an estimate of the radius R of the bubbles to detach from the interface is given by:

$$R^2 = 2\sigma k_c (k_c - \frac{\pi}{R}) / (\nu - \frac{3}{2} \sigma k_c^4). \quad (30)$$

Suppose h_1 is the thinner of the fluid layer, then the criterion against the rupture of the fluid layer is given by:

$$h_1^2 > -2\sigma k_c \mu / (\nu - \frac{3}{2} \sigma k_c^4). \quad (31)$$

For semi-infinite fluid layers, it may be seen from (27) that ν vanishes. Thus the system cannot be stabilized by the finite amplitude effects for semi-infinite fluids layers up to this order*.

III. Discussion on Stability Criterion.

In order that the condition (28) is fulfilled, it is necessary that $\nu > 0$. Since the vapor density $\rho^{(1)}$ is usually much smaller than the liquid density $\rho^{(2)}$, we have approximately for $\rho^{(1)} \ll \rho^{(2)}$:

* In Pl, due to an error in (A-11), erroneous conclusions were obtained.

$$v = \frac{\alpha^2 k}{\rho(1)} (\coth kh_1 \coth 2kh_1 - 1) [N + \frac{\alpha_2}{k} - 2 \coth kh_1], \quad (32)$$

where

$$N = \frac{\alpha^2 (1 + \coth^2 kh_1)}{6\rho(1)\sigma k_c^2}. \quad (33)$$

In these expressions, k and k_c are essentially the same.

If we neglect α_2 , then the criterion $v > 0$ is equivalent to $N > 2 \coth kh_1$. Let us consider the limiting cases of large and small h_1 :

$$(i) \quad kh_1 \gg 1.$$

Then we have $\coth kh_1 \approx 1$, hence the criterion $v > 0$ becomes

$$\frac{\alpha^2}{\rho(1)} > 6\sigma k_c^3. \quad (34)$$

$$(ii) \quad kh_1 \ll 1.$$

Then we have $\coth kh_1 \approx \frac{1}{kh_1}$, and the criterion $v > 0$ becomes

$$\frac{\alpha^2}{\rho(1)} > 12\sigma h_1 k_c^4. \quad (35)$$

From the criterions (28), (34) and (35), it is seen that a relevant non-dimensional parameter is

$$\delta = \frac{\alpha^2}{\rho(1)\sigma k_c^3}.$$

When the parameter δ is large enough then the system can be stabilized due to the combined effects of the finite amplitude and the heat and mass transfer. Since α is proportional to the heat flux of the system, thus when the heat flux is sufficiently intense, the system can be stabilized. Moreover, with the same the heat flux, the thinner the vapor layer the easier the system can be stabilized.

Appendix

Order ϵ^n , $n = 1, 2, 3$

$$\frac{\partial^2 \phi_n^{(j)}}{\partial x^2} + \frac{\partial^2 \phi_n^{(j)}}{\partial y^2} = 0, \quad j = 1, 2; \quad (A-1)$$

$$\frac{\partial \phi_n^{(1)}}{\partial y} = 0, \quad \text{on } y = -h_1; \quad (A-2)$$

$$\frac{\partial \phi_n^{(2)}}{\partial y} = 0, \quad \text{on } y = h_2. \quad (A-3)$$

The interfacial conditions are given on $y = 0$ as follows:

Order ϵ

$$\rho^{(1)} \left[\frac{\partial \eta_1}{\partial T_0} - \frac{\partial \phi_1^{(1)}}{\partial y} \right] = \{2\}, \quad (A-4)$$

$$= -\alpha \eta; \quad (A-5)$$

$$\rho^{(1)} \left[\frac{\partial \phi_1^{(1)}}{\partial T_0} + g \eta_1 \right] = \{2\} + \sigma \frac{\partial^2 \eta_1}{\partial x^2}. \quad (A-6)$$

Order ϵ^2

$$\rho^{(1)} \left(\frac{\partial \eta_2}{\partial T_0} - \frac{\partial \phi_2^{(1)}}{\partial y} + \left(\frac{\partial \eta_1}{\partial T_1} - \frac{\partial \phi_1^{(1)}}{\partial x} \right) \frac{\partial \eta_1}{\partial x} - \eta_1 \frac{\partial \phi_1^{(1)}}{\partial y^2} \right) = \{2\}, \quad (A-7)$$

$$= -\alpha (\eta_2 + \alpha_2 \eta_1^2), \quad (A-8)$$

$$\rho^{(1)} \left\{ \frac{\partial \phi_2^{(1)}}{\partial T_0} + g \eta_2 + \left[\frac{\partial \phi_1^{(1)}}{\partial T_1} + \frac{\partial^2 \phi_1^{(1)}}{\partial T_0 \partial y} \eta_1 + \frac{1}{2} \left(\frac{\partial \phi_1^{(1)}}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi_1^{(1)}}{\partial x} \right)^2 - \frac{\partial \phi_1^{(1)}}{\partial y} \left(\frac{\partial \phi_1^{(1)}}{\partial y} - \frac{\partial \eta_1}{\partial T_0} \right) \right] \right\} = \{2\} + \sigma \frac{\partial^2 \eta_2}{\partial x^2}, \quad (A-9)$$

Order ϵ^3

$$\begin{aligned} \rho^{(1)} \left\{ \frac{\partial \eta_3}{\partial T_0} - \frac{\partial \phi_3^{(1)}}{\partial y} + \left[\frac{\partial \eta_2}{\partial T_1} + \frac{\partial \eta_1}{\partial T_2} + \frac{\partial \phi_1^{(1)}}{\partial x} \frac{\partial \eta_2}{\partial x} + \frac{\partial \eta_1}{\partial x} \left(\frac{\partial \phi_2^{(1)}}{\partial x} + \frac{\partial^2 \phi_1^{(1)}}{\partial x \partial y} \eta_1 \right) - \left(\frac{\partial^2 \phi_1^{(1)}}{\partial y^2} \eta_2 + \frac{\partial^2 \phi_2^{(1)}}{\partial y^2} \eta_1 + \frac{\eta_1^2}{2} \frac{\partial^3 \phi_1^{(1)}}{\partial y^3} \right) \right] \right\} \\ = \{2\}, \end{aligned} \quad (A-10)$$

$$= -\alpha (\eta_3 + 2\alpha_2 \eta_1 \eta_2 + \alpha_3 \eta_1^3), \quad (A-11)$$

$$\begin{aligned} \rho^{(1)} \left\{ \frac{\partial \phi_3^{(1)}}{\partial T_0} + g \eta_3 + \left[\frac{\partial \phi_2^{(1)}}{\partial T_1} + \frac{\partial \phi_1^{(1)}}{\partial T_2} + \frac{\partial^2 \phi_1^{(1)}}{\partial T_0 \partial y} \eta_2 + \left(\frac{\partial^2 \phi_2^{(1)}}{\partial T_0 \partial y} + \frac{\partial^2 \phi_1^{(1)}}{\partial T_1 \partial y} \right) \eta_1 + \frac{1}{2} \frac{\partial^3 \phi_1^{(1)}}{\partial T_0 \partial y^2} \eta_1^2 \right] \right. \\ + \frac{1}{2} \left[2 \frac{\partial \phi_1^{(1)}}{\partial x} \frac{\partial \phi_2^{(1)}}{\partial x} + \eta_1 \frac{\partial}{\partial y} \left(\frac{\partial \phi_1^{(1)}}{\partial x} \right)^2 \right] \\ + \frac{1}{2} \left[2 \frac{\partial \phi_1^{(1)}}{\partial y} \frac{\partial \phi_2^{(1)}}{\partial y} + \eta_1 \frac{\partial}{\partial y} \left(\frac{\partial \phi_1^{(1)}}{\partial y} \right)^2 \right] \\ - \left(\frac{\partial \eta_1}{\partial T_0} - \frac{\partial \phi_1^{(1)}}{\partial y} \right) \left(\frac{\partial \phi_1^{(1)}}{\partial x} \frac{\partial \eta_1}{\partial x} - \frac{\partial \phi_2^{(1)}}{\partial y} \right) \\ \left. - \left(\frac{\partial \phi_1^{(1)}}{\partial y} \right) \left(\frac{\partial \eta_2}{\partial T_0} + \frac{\partial \eta_1}{\partial T_1} - \frac{\partial \phi_2^{(1)}}{\partial y} + \frac{\partial \phi_1^{(1)}}{\partial x} \frac{\partial \eta_1}{\partial x} \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + n_1 \frac{\partial}{\partial y} \left(\frac{\partial \phi_1^{(1)}}{\partial y} \left(\frac{\partial \phi_1^{(1)}}{\partial y} - \frac{\partial n_1}{\partial T_0} \right) \right) \\
& = \{2\} + \sigma \left[\frac{\partial^2 \eta_3}{\partial x^2} - \frac{3}{2} \left(\frac{\partial^2 \eta_1}{\partial x^2} \right) \left(\frac{\partial \eta_1}{\partial x} \right)^2 \right].
\end{aligned} \tag{A-12}$$

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2. D. Y. Hsieh: "Interfacial Stability with Mass and Heat Transfer". Report No. COO-4155-2, Division of Applied Mathematics. Brown University. (1977).