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NONLINEAR MAGNETOHYDRODYNAMICS

Progress Report

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ABSTRACT

Incompressible MHD turbulence is considered for both 2D and 3D plasmas in cylindrical geometry. It is found that for virtually all initial conditions (including quiescent ones) the plasma is nonlinearly unstable in that mean square turbulent velocity fields develop. However, there is a unique stable state of extremal magnetic helicity/energy ratio for which no turbulent fields develop [in 2D with $B_z = \text{const.}$, it is the state of extremal mean square vector potential/energy]. It is force free and is just the Taylor state. A conjecture can be put forward (based on a dual cascade argument for resistive MHD) to explain Taylor's hypothesis. In spherical geometry, the stable axisymmetric state is the spheromak.

For an incompressible cylindrical plasma, information theory is used to construct most probable MHD states by maximizing the probability measure for the toroidal field and current subject to some global constraints. The resulting states are not force-free and are appropriate for Tokamaks and reversed field pinches.

Experimental phase information from X-ray detectors acting to sense the instantaneous location of $m=2$ island structures and from a B_θ detector sensitive to these same structures but from outside the plasma have been compared with a theory which relates such information to relative motion between the perturbation structures and the plasma. The results indicate very low relative velocities.

PREFACE

There were three topics studied under contract EY-76-S-05-5260 with the U. D. Department of Energy:

1. MHD Turbulences in Confinement Geometry
2. Most Probable MHD Equilibrium States
3. Perturbed Magnetic Field Phase Slips for Tokamaks

The research was partially performed with David Montgomery, Leaf Turner (LASL) Linda Vahala, John Ambrosiano (Ph.D. candidate, William and Mary) together with colleagues at ORNL.

1. MHD Turbulence in Confinement Geometry

The standard approach to plasma stability has been to first consider some equilibrium, then to perform a linear normal mode stability analysis on this equilibrium. If the equilibrium is (linearly) unstable then the standard treatment is to consider the nonlinear problem in the language and concepts of the linearized theory (nonlinear dispersion relation, nonlinear growth rates...). However, it is of some importance to break away from this approach on at least two grounds: (i) the breakup of the problem into an equilibrium and then the stability of small perturbations about this equilibrium is a theoretical expedient quite removed from the actual experimental situation, and (ii) the turbulence can be sufficiently strong so as to render the linear concepts irrelevant and inappropriate in considering the nonlinear problem.

In our research, we shall apply the methods of statistical mechanics in considering the nonlinear dynamics and evolution of an incompressible axisymmetric turbulent plasma in cylindrical geometry. This approach was proposed by Burgers¹ and Lee² and extended by Kraichnan³ to finding the absolute equilibrium states of a model problem for incompressible Navier-Stokes fluid turbulence. Since the, Kraichnan's theory has been verified by numerical simulation^{4,5} and applied (with numerical verification) to truncated, non-dissipative models of two and three dimensional magnetohydrodynamic turbulence⁶⁻¹⁰. All these calculations were performed assuming periodic boundary conditions, thus excluding any possibility of a toroidal current.

Recently, the theory has been generalized so as to allow non-zero net toroidal current by considering the plasma to be confined by perfectly conducting cylindrical walls, ¹¹⁻¹⁴ and thus allowing possible application to problems of interest in controlled fusion research. While truncated non-dissipative models of turbulence omit effects of non-zero viscosity and resistivity, and that these effects alter the predicted steady states for turbulent cascade processes and the spectral transfer functions, nevertheless, these models are useful in examining any tendency of the plasma or fluid to form coherent macroscopic structures (even in the dissipative case)³⁻¹⁰. Moreover, even in the case of dissipation the turbulent fields continually try to relax to these model states and they would relax to them were it not for the dissipation.

It should also be noted that in hydrodynamics for the interesting case of high Reynolds number $R \gg 1$, the equilibrium flow profile is linearly highly unstable but no trace of the results of a linear normal-mode analysis survives in the properties of the resulting turbulent flows. Here $R = UL/\nu$, where U is a typical mean flow speed, L is a characteristic length such as the diameter of the pipe, and ν is the kinematic viscosity of the fluid. These results are crudely independent of such details as the cross-section of the pipe. One could suspect the existence of similarly general quantitative results for the development of magnetohydrodynamic turbulence for a plasma carrying an electric current. Navier-Stokes flow is, after all a special case of magnetohydrodynamics, and it is not unreasonable to imagine qualitative similarities between the mechanical and magnetic aspects of the general problem. For example, one might conjecture that magnetohydrodynamic turbulence would result when the Reynolds number $\tilde{R} \equiv c_A L / (c^2 / 4\pi\sigma)$ was large. (Here, c_A is the mean Alfvén speed associated with the spatially-varying, plasma-current-supported magnetic field, L is the character-

istic length associated with that spatial variation, σ is the conductivity in C.G.S. units, and c is the speed of light. $c^2/4\pi\sigma$ is in effect a resistivity with the same dimensions as kinematic viscosity.) $\tilde{R}$ may in fact cross about the same ranges of values for tokamak plasmas on the threshold of disruptions as R does for pipe flow near the onset of fully-developed Navier-Stokes turbulence: values of the order of a thousand or so.

The basic procedure for finding the absolute equilibrium states of a plasma is to: (i) expand the velocity and magnetic fields in an appropriately chosen eigenfunction basis, (ii) truncate these infinite series expansions and consider the phase space consisting of the generalized Fourier coefficients, (iii) identify those (quadratic) invariants of the motion which remain invariant under any finite level of truncation in the series expansions (the so-called "rugged invariants"), and, (iv) construct the equilibrium Gibbs canonical ensemble (with the rugged invariants and any externally imposed constraints in the exponent).

From the MHD action principle¹⁵, time and gauge invariance lead to the following quadratic integrals of the motion which are the rugged invariants: total energy E

$$E = \int d^3x (\vec{v}^2 + \vec{B}^2) \quad (1)$$

the magnetic helicity H_M

$$H_M = \int d^3x \vec{A} \cdot \vec{B} \quad (2)$$

where $\vec{A}$ is the magnetic vector potential ($\nabla \times \vec{A} = \vec{B}$), and the cross helicity H_c

$$H_c = \int d^3x \vec{v} \cdot \vec{B} \quad (3)$$

For external constraints we usually chose that of constant toroidal current (typical of Tokamak operation)

$$I_z = \int d^3x J_z = \text{const.} \quad (4)$$

If we consider the low β Tokamak case¹² of constraining the toroidal magnetic field B_z such that

$$B_z = \text{const.} \quad (5)$$

then H_M , Eq.(2), is no longer a (quadratic) rugged invariant. The choice of $B_z = \text{const.}$ introduces another symmetry invariance which leads to a new rugged invariant

$$\int d^2x A_z^2 = \text{const.}, \quad (6)$$

on assuming toroidal symmetry.

The two major items we have examined in these turbulence models are (1) for what choices of the global constraints will the initially quiescent plasma ($\vec{v} \approx 0$) remain nonlinearly stable, i.e. not develop a finite mean kinetic energy with an equipartition spectrum in the velocity modes, and, (2) conjectured dual cascade possibilities for resistive MHD.

In our first paper¹¹, we considered 3D MHD cylindrical turbulence. Since the magnetic helicity is a rugged invariant, it is convenient to expand the magnetic and velocity vector potentials in the eigenfunction basis of the curl operator with radially homogeneous boundary conditions. All the eigenvalues are thus determined geometrically except one, which must be determined by the externally imposed constraints e.g. the ratio of toroidal current to toroidal flux, I_z/ψ_T . For sufficiently small values of I_z/ψ_T , the unique stable state is one with maximal ratio of H_M/E . It is a force-free, purely magnetic mode with

$$\vec{B} = B_0 [J_1(\lambda r) \hat{\theta} + J_0(\lambda r) \hat{z}]$$

where λ is determined by the value I_z/ψ_T . This is precisely Taylor's reversed field relaxed axisymmetric state. As we increase I_z/ψ_T the toroidal field shows reversal until I_z/ψ_T gets so large that the lowest eigenvalue now is that deter-

mined geometrically. The resulting state is unstable since now $\langle \vec{v}^2 \rangle \neq 0$. Also in this paper¹¹ we gave a conjectured relaxation mechanism to explain the significance of Taylor's assumption of finding those states which allow the decay of energy to a minimum value compatible with conserved magnetic helicity (Taylor does not specify the decay process): from the spectral forms of E and H_M one can conjecture a dual cascade process in a dissipative medium with E being cascaded to large λ 's where dissipative effects are dominant. H_M cascades to lower λ 's, and so should be little affected by dissipative effects.

We then considered a 2D version¹² of the above paper by assuming no z -variation and $B_z \equiv 0$. It was noted that a constant toroidal magnetic field could be included without any change in the dynamics of the problem. Since the magnetic helicity is now no longer a (quadratic) rugged invariant, we expand the z -component of the vector potentials in terms of the eigenfunctions of the Helmholtz scalar wave equation. All the eigenvalues are now geometrically determined. Because of the assumed symmetry, magnetic flux surfaces exist and it is found that the stable state has flux surfaces which are concentric circles while the unstable states have magnetic islands.

This paper was then generalized to allow for an arbitrary toroidal magnetic field B_z but still keeping the problem 2D.¹³ Now the magnetic helicity is again a rugged invariant (as in the 3D calculation¹¹) which necessitates the use of the eigenfunctions of the curl operator. Again one of the eigenvalues is not determined by the geometric boundary conditions and must be determined by the externally imposed values of I_z/ψ_T . One can now see how the magnetic flux surfaces deform as the external constraint I_z/ψ_T is increased till it passes the first geometrically determined eigenvalue - and the plasma then goes unstable. Again, magnetic island form, but somewhat more coherently than is the 2D case of $B_z = \text{const}$. Moreover, in this paper¹³ we pointed out that this nonlinear stabili-

ty result was in sharp contrast with both the results of a linearized δW analysis as well as that from the stability analysis of the linearized spectral coefficient evolution equations.

It has also been found¹⁴ that for spherical systems, one can also look for the stable states from the absolute equilibrium expertation values of the fields. The stable state is force-free and reduces to the Hill vortex structure for axisymmetric plasma - the spheromak^{17,18}.

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2. Most Probable MHD Equilibrium States

It has been conventional to discuss magnetohydrodynamic (MHD) configurations in terms of their linear stability properties. This has the advantage of leading to a large number of well-posed, soluble, linear eigenvalue problems. It also involves some major drawbacks: there are literally infinite numbers of configurations to investigate. The range of possible MHD stability behavior is, for example, far greater than arises in the case of the Orr-Sommerfeld equation of hydrodynamic stability theory. Moreover, the global information available from diagnostics currently in use and the experimental possibilities for manipulating profiles are usually far less than what is necessary for a complete determination of an MHD state. Given the prevalent assumption that information about single realizations of MHD states is desirable, what seems to be needed is some further systematic basis for choice among the enormous variety of imaginable profiles.

Our purpose here is to propose such a systematic basis, by identifying the "most probable" state of an MHD fluid, compatible with the existence of only a few known global constants or constraints (examples are: the energy, the toroidal magnetic flux, the total toroidal electric current, and the magnetic helicity). The concept of a "most probable" state is a common idea in the equilibrium statistical mechanics of systems of point particles. The idea was generalized some time ago by Joyce and Montgomery¹ in connection with the negative temperature states of an electrostatic guiding center plasma. This calculation, in turn, owed something to van Kampen's² treatment of the condensation of

a van der Waals gas, and to the "information theory" formulation of statistical mechanics (e.g. Katz³). To our knowledge, it is the first time these methods have been applied to a strictly continuum system, as contrasted with a mean-field limit of a system of point particles.

More specifically, assuming only radially-dependent profiles for a cylindrical incompressible plasma, we find most probable MHD states by maximizing the probability measure for the toroidal magnetic field $B_z(r)$ and current $J_z(r)$, subject to the global constraints of given energy, magnetic helicity, toroidal flux and total toroidal current. The basic procedure is to first subdivide the plasma cross sectional area into cells and then to introduce a discretized representation (via a set of occupation numbers) for the $J_z(r)$ and $B_z(r)$ fields. The standard Boltzmann combinatorial expression is then assumed for the a priori probability of the discretized representation. Using Stirling's formula, the logarithm of this probability is maximized subject to the discretized constraints. Finally, one passes back to the continuum limit and solves the resulting differential equations to obtain the "most probable" equilibria subject to the given imposed constraints. This is essentially the idea behind the information theory approach³ which assigns a probability functional $W[f_1(\vec{x}), f_2(\vec{x}), \dots]$ to a set of non-negative independent functions such that

$$\ln W = - \sum_i \int d^2\vec{x} f_i(\vec{x}) \ln f_i(\vec{x})$$

is to be maximized subject to given constraints.

The resulting ordinary differential equations have to be solved numerically except for the case when the magnetic helicity is not a specified constraint when an analytic treatment is possible. In this case⁴, $B_z = \text{const.}$, and a single parameter (which basically characterizes the axial current channel width) family of fields is obtained which is appropriate for low β -Tokamaks both

with skin currents and for currents peaked on axis. If one retains the magnetic helicity as a constraint $B_z = \text{const.}$ Preliminary results⁵ indicate that reversal field states are readily found. These states are not force-free.

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3. Perturbed Magnetic Field Phase Slip for Tokamaks¹

Coherent helical structures have been observed in Tokamaks, and these structures appear to rotate within the plasma. The typical experimental diagnostics are (i) magnetic field pickup loops located near the vacuum wall and (ii) arrays of collimated soft X-ray detectors scanning along chords passing through the plasma. For a give mode (in this case an $m = 2$ mode) both the magnetic loops and the X-ray detectors observe persistent coherent oscillations --the Mirnov oscillations--in the frequency range 1-30 kHz at exactly the same frequency.

It is believed that the Mirnov oscillations are due to helically perturbed current structures corresponding to magnetic islands within the plasma and the soft X-ray emission is due primarily to electron temperature perturbations coincident with the same magnetic islands. The oscillation in the observed signal is apparently associated with the rotation of the island structure. Since the electron temperature and the current perturbations are evidently associated with the same island structure, the oscillation frequency of the observed X-ray and magnetic field perturbations should be (and is) identical.

The idea behind this calculation is the following: if the highly conducting plasma between the $m = 2$ island and the magnetic loops is not rotating with the same angular velocity as the island structure then there should be an experimentally measurable phase difference between the observed X-ray and the magnetic field oscillations. That is, there should be a relationship between these phase differences and the relationship between the mode and plasma

fluid rotations. The development of the mode-fluid rotation relationship during the Tokamak discharge may significantly effect plasma confinement and stability.

In this paper¹ we set up a very simple model giving this phase difference Δ as a function of σS , where σ is the electrical conductivity of the plasma and S is the slip between the frequency of the signal, ω , and the poloidal and/or toroidal rotational frequency of the plasma

$$S = \omega - \frac{mV_\theta}{r} - kV_z$$

S may be non-zero because the magnetic islands may have a different apparent rotation velocity from the surrounding plasma because of diamagnetic and other effects in the dispersion relation for resistive tearing modes. Alternatively, the angular velocity of the intervening plasma may not be uniform, since there is not a prior reason to expect the plasma to rotate as a rigid body.

In this model we wish to determine the effects of fluid rotation on the magnetic fields produced by a $m = 2$ magnetic island assuming the magnetic fields in the absence of rotation are known. In practice, we will use the (unrotated) saturated $m = 2$ magnetic island current as found from the nonlinear computer codes of White, Monticello, Rosenbluth and Waddell.²

We assume helical symmetry in cylindrical geometry and ignore the vacuum region between the plasma and the wall. Rosenbluth's reduced MHD equations²⁻⁴ are generalized so as to include equilibrium flow velocities and the Hall effect, using the standard large aspect ratio expansion in the low β Tokamak ordering. To avoid having to solve the resulting nonlinear partial differential equations by nontrivially extending the nonlinear computer codes of White, et.al., (which are very complex and heavy users of machine time) we

ignore all coupling effects of the rotation on the island saturation mechanism. This is a reasonable assumption for low β , and allows us to dispense with the force balance equation. The resulting set of equations are just ordinary differential equations which are quite simple and speedily solved on the computer.

The chief results are (using typical ORMAK parameters and conductivity profiles): (i) as $S \rightarrow 0$ a jump of π occurs in the phase difference Δ as we cross the zero of the island current with Δ becoming insensitive to the location of the X-ray diode (i.e., all diodes scanning positions to the right of the island zero have $\Delta = \pi/2$, while all those to the left have $\Delta = -\pi/2$). However, for even small slips ($S \geq 20$) Δ starts to exhibit a strong dependence on the radial scanning location of the diode. (ii) for peaked conductivity profiles, Δ is observed to saturate and then decrease at large values of S ($S \geq 2000$).

The experimental results on ORMAK indicate that during the flat-topped current interval, the zero of the island current occurs around $r/a \approx 0.5$ with $S \approx 0$. That is, we have essentially locked electron fluid and mode motions.

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