The Application of Correlation Techniques to the Angular Spectrum of Scattered Radiation from Tokamak Plasmas¹

R. Nazikian

Princeton Plasma Physics Laboratory

Princeton, NJ 08543

Abstract

In the limit of the first Born approximation for a partially coherent secondary source, consisting of a spatially random plasma illuminated by a coherent plane wave, it is shown that the spectral coherence of the scattered radiation as measured on an arbitrary plane beyond the scatterer conveys information on the three dimensional intensity distribution of the random source. By defining a new two point statistical measure of the random field, closely related to the cross spectral density, we show that the fluctuation amplitude of the random source along the direction of the incident plane wave may be recovered from the measurement of the scattered radiation. The application of cross spectral techniques to fluctuation studies on tokamaks is considered.

1 Introduction

The theory of partial coherence as developed by Wolf² deals with the relationship between a planar radiation source and the spatial coherence of the radiated angular spectrum. It is only recently, however, that attention has turned to the three dimensional inverse scattering problem for random secondary anisotropic sources (i.e., coherent scattering from random refractive index fluctuations³). The key principle in the theory of partial coherence is that the state of coherence of the radiated angular spectrum carries information on the spatial distribution of the inhomogeneous radiator.

In standard scattering experiments on laboratory plasmas it is often assumed that only the spectral power density of the fluctuations, expressed in terms of the plasma form factor $S_n(K, \omega)$, contains useful information on the random medium.⁴ Unfortunately, the spectral power density alone contains no information on the spatial distribution of the random medium. As the poor spatial resolution of scattering systems for long wavelength fluctuations remains a major diagnostic limitation, it is appropriate to investigate whether the measurement of the angular coherence of the scattered radiation may be used to overcome this

limitation.

2 Spectral Correlation Analysis

We first consider a plane wave $u_0 = \exp(jk_0 z)$ incident on a bounded plasma density distribution $n_e(\mathbf{r})$. Under the first Born approximation the scattered angular spectrum $A_s(\mathbf{K}_\perp; z)$ (the two dimensional spatial Fourier transform in $\mathbf{r}_\perp$ of the scattered field $u_s(\mathbf{r}_\perp; z)$) in the parabolic approximation is given by ⁵

$$A_s(\mathbf{K}_\perp; z) = \mathcal{G}(\kappa_F; z) N(\mathbf{K}_F) \quad (1)$$

$$\mathcal{G}(\kappa_F; z) = -1/2 j k_0 \exp(j\kappa_F z) \quad (2)$$

where for the purpose of this paper we consider $N(\mathbf{K}_F)$ the three dimensional spatial Fourier transform of the fluctuating part of the normalized plasma density $n(\mathbf{r}) = (n_e(\mathbf{r}) - \langle n_e(\mathbf{r}) \rangle) / n_{cr}$, $n_{cr} = k_0^2 / 4\pi r_e$ (r_e is the classical electron radius), $\mathcal{G}$ is the propagator for the scattered radiation, $\kappa_F = (\mathbf{K}_\perp, \kappa_F)$ is the wave vector of the scattered field and $\mathbf{K}_F = (\mathbf{K}_\perp, K_F)$ is the corresponding wave vector in the plasma. To satisfy momentum conservation for the scattered field we require $K_F = \kappa_F - k_0 \simeq -K_\perp^2 / 2k_0$ in the parabolic approximation valid for forward an-

gle scattering. Given that in near field experiments it is more convenient to measure the scattered complex phase $\psi_s(\mathbf{r}_\perp; z)$ where the radiated field $u = u_0 \exp(\psi_s)$, the appropriate expression for the two dimensional Fourier transform $\Psi_s(\mathbf{K}_\perp; z)$ of the scattered complex phase in the first Rytov approximation is,⁵

$$\Psi_s(\mathbf{K}_\perp; z) = \mathcal{R}(K_F; z) N(\mathbf{K}_F), \quad (3)$$

$$\mathcal{R}(K_F; z) = -1/2 j k_0 \exp(j K_F z). \quad (4)$$

For plane wave illumination the angular spectrum A_s and the phase spectrum Ψ_s only differ by a constant phase for a fixed observer plane, hence for convenience, we deal for the most part with the complex phase. Given a sufficiently large set of independent observations of a homogeneous random medium, we may estimate the plasma form factor $S_n(\mathbf{K})$ by taking ensemble averages over either $A_s A_s^*$ or $\Psi_s \Psi_s^*$, where

$$\begin{aligned} \langle |A_s(\mathbf{K}_\perp; z)|^2 \rangle &= \langle |\Psi_s(\mathbf{K}_\perp; z)|^2 \rangle = 1/4 k_0^2 \langle |N(\mathbf{K}_F)|^2 \rangle \\ &= \lambda_0^2 r_e^2 \bar{n}_e V_p S_n(\mathbf{K}_F) \end{aligned} \quad (5)$$

where the plasma form factor is

$$S_n(\mathbf{K}_F) = \langle |N_e(\mathbf{K}_F)|^2 \rangle / \bar{n}_e V_p \quad (6)$$

and where $\bar{n}_e$ and V_p are the mean plasma density and plasma volume, respectively.

This result is very familiar and forms the basis for coherent scattering techniques on tokamak plasmas. However, the fluctuation volume is seldom known *a priori* and is usually assumed to be the intersection of the transmitter and receiver patterns with the plasma volume assuming the random medium is homogeneous. To obtain further information on the distribution of the fluctuations within the scattering volume, it is necessary to determine the coherence of the angular spectrum of scattered radiation.

In a generalization of the spectral density, the cross spectral density $\mathcal{W}_s$ is defined as

$$\mathcal{W}_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z) = \langle \Psi_s^*(\mathbf{K}_\perp; z) \Psi_s(\mathbf{K}'_\perp; z) \rangle, \quad (7)$$

which reduces to the usual spectral power density for $\mathbf{K}_\perp = \mathbf{K}'_\perp$. Also, given the two point density correlation function $C_n(\mathbf{r}_1, \mathbf{r}_2) = \langle n(\mathbf{r}_1)n(\mathbf{r}_2) \rangle$, we readily obtain the result,

$$\mathcal{W}_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z) = 1/4 k_0^2 \exp[j(K'_F - K_F)z] C_n(-\mathbf{K}_F, \mathbf{K}'_F) \quad (8)$$

where $C_n(\mathbf{K}_F, \mathbf{K}'_F)$ is the six dimensional Fourier transform of the spatial correlation function evaluated in the parabolic approximation on the surface $\mathbf{K}_F = (\mathbf{K}_\perp, K_F)$ in wavenumber

space and where use is made of the Hermitian symmetry of the Fourier transform of the normalized density, $N^*(-\mathbf{K}) = N(\mathbf{K})$. The coherence of the angular spectrum of scattered radiation is given by the *normalized cross spectral density*,

$$\mu_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z) = \frac{\mathcal{W}_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z)}{[\mathcal{W}_s(\mathbf{K}_\perp, \mathbf{K}_\perp; z)]^{1/2} [\mathcal{W}_s(\mathbf{K}'_\perp, \mathbf{K}'_\perp; z)]^{1/2}} \quad (9)$$

where upon substitution of Eq. (8) into Eq. (9) we obtain the important relation

$$\begin{aligned} \mu_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z) &= \exp[j(K'_F - K_F)z] \\ &\times \frac{C_n(-\mathbf{K}_F, \mathbf{K}'_F)}{[C_n(-\mathbf{K}_F, \mathbf{K}_F)]^{1/2} [C_n(-\mathbf{K}'_F, \mathbf{K}'_F)]^{1/2}}. \end{aligned} \quad (10)$$

The normalized cross spectral density has been studied at great length with respect to planar sources, however, for weak scattering from extended three dimensional random media, we need to introduce a very useful new quantity, the *cross spectral correlation*,

$$\tilde{\mathcal{W}}_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z) = \langle \Psi_s(\mathbf{K}_\perp; z) \Psi_s(\mathbf{K}'_\perp; z) \rangle, \quad (11)$$

and proceeding in a similar fashion as above, we may define the *normalized cross spectral correlation* of the field,

$$\tilde{\mu}_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z) = \frac{\tilde{\mathcal{W}}_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z)}{[\mathcal{W}_s(\mathbf{K}_\perp, \mathbf{K}_\perp; z)]^{1/2} [\mathcal{W}_s(\mathbf{K}'_\perp, \mathbf{K}'_\perp; z)]^{1/2}} \quad (12)$$

which in terms of the plasma two point density correlation becomes

$$\begin{aligned} \tilde{\mu}_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z) = & -\exp[j(\mathbf{K}_F + \mathbf{K}'_F)z] \\ & \times \frac{C_n(\mathbf{K}_F, \mathbf{K}'_F)}{[C_n(-\mathbf{K}_F, \mathbf{K}_F)]^{1/2}[C_n(-\mathbf{K}'_F, \mathbf{K}'_F)]^{1/2}}. \end{aligned} \quad (13)$$

To this point we have only dealt with various old and new definitions. We now assume an analytic model for the two point density correlation C_n in which the mean square amplitude of fluctuations $\langle n^2 \rangle$ varies slowly compared to the fluctuation correlation length. This is known as the quasi-homogeneous approximation introduced by Wolf,²

$$C_n(\mathbf{r}, \mathbf{r}') = I(1/2(\mathbf{r} + \mathbf{r}'))\rho(\mathbf{r} - \mathbf{r}') \quad (14)$$

where $I(\mathbf{r}) \simeq \langle n^2(\mathbf{r}) \rangle$ and for a homogeneous medium, $I(\mathbf{r})$ is just a constant. It can then be shown that the six dimensional Fourier transform $\mathcal{C}_n$ of C_n may be expressed in the form

$$\mathcal{C}_n(\mathbf{K}, \mathbf{K}') = \Sigma(\mathbf{K} + \mathbf{K}')S(1/2(\mathbf{K} - \mathbf{K}')), \quad (15)$$

where $\Sigma(\mathbf{K})$ is the Fourier transform of $I(\mathbf{r})$ and $S(\mathbf{K})$ is the Fourier transform of $\rho(\mathbf{r})$. To interpret this expression, we have for a homogeneous medium,

$$\mathcal{C}_n(\mathbf{K}, -\mathbf{K}') = \delta(\mathbf{K} - \mathbf{K}')S(1/2(\mathbf{K} + \mathbf{K}')), \quad (16)$$

which is simply proportional to the form factor for $\mathbf{K} = \mathbf{K}'$, and is zero otherwise. We note that in the quasi-homogeneous approximation, the correlation length [functional width of $\rho(\mathbf{r})$] is considerably narrower than the scale length for mean square amplitude variation in the medium [variations in $I(\mathbf{r})$] and so by the nature of Fourier transforms, the function $S(\mathbf{K})$ is uniform over the region of spectral coherence determined by the function $\Sigma(\mathbf{K})$.⁶

Substituting Eq. (15) into Eq. (13) and evaluating the normalized spectral correlation for opposite scattering vectors, $\mathbf{K}_F = (\mathbf{K}_\perp, K_F)$ and $\mathbf{K}'_F = (-\mathbf{K}_\perp, K_F)$ (c.f. Fig. 1) we obtain,

$$\begin{aligned} \bar{\mu}_s(\mathbf{K}_\perp, -\mathbf{K}_\perp; z) = & -\exp(j2K_F z) \\ & \times \frac{\Sigma(0, 2K_F)S(\mathbf{K}_\perp, 0)}{\Sigma(0, 0)S^{1/2}(\mathbf{K}_\perp, K_F)S^{1/2}(\mathbf{K}_\perp, -K_F)} \end{aligned} \quad (17)$$

where use is made of the symmetry of $S(\mathbf{K})$. By quasi-homogeneity and within the parabolic approximation, $|\mathbf{K}_F| \simeq |\mathbf{K}_\perp|$ and $S(\mathbf{K}_\perp, K_F) \simeq S(\mathbf{K}_\perp, 0)$ for $\Sigma(0, 2K_F)/\Sigma(0, 0) \sim 1$. Hence the normalized cross spectral correlation is approximately given by

$$\bar{\mu}_s(\mathbf{K}_\perp, -\mathbf{K}_\perp; z) = -\exp(j2K_F z)\Sigma(2K_F)/I_0 \quad (18)$$

where we have assumed the random medium is homogeneous in the direction perpendicular to the incident plane wave, (valid in

narrow beam geometries) and where

$$I_0 = \int_{-\infty}^{\infty} I(z) dz. \quad (19)$$

The homogeneity condition for fluctuations perpendicular to the beam is not necessary, but it is convenient for the present analysis.

We now define a new function, the *spectral coherence*

$$\hat{I}(2K_F; z_0) = -\tilde{\mu}_s(\mathbf{K}_\perp, -\mathbf{K}_\perp; z_0) \quad (20)$$

which is Hermitian symmetric in K_F and where the measurement is performed on the plane $z = z_0$. Experimentally, the spectral coherence may be determined from the measured complex phase only for $K_F = -|K_F|$, however, as the form of Eq. (18) must be Hermitian symmetric in K_F [due to the Hermitian symmetry of $\Sigma(\mathbf{K})$], we may simply extend the measured coherence into $K_F = +|K_F|$. We note at this point that apart from the normalizing factor I_0 , $\hat{I}(K_F; z_0)$ is in fact the Fourier transform of $I(z')$ where $z' = z + z_0$. Multiplying $\hat{I}(K_F; z_0)$ by the inverse Fourier kernel $\exp[jK_F(z - z_0)]$ and integrating over K_F we obtain, (given K_0 is the highest resolved wavenumber in the plasma),

$$\hat{I}(z) = \int_{-K_0^2/k_0}^{K_0^2/k_0} \exp[jK(z - z_0)] \hat{I}(K; z_0) dK / 2\pi$$

$$\begin{aligned}
&\simeq \int_{-\infty}^{\infty} \exp[jK(z - z_0)] \hat{I}(K; z_0) dK / 2\pi \\
&= I(z) / I_0
\end{aligned} \tag{21}$$

provided $\hat{I}(K_F; z_0)$ is fully determined (i.e., for large enough K_0). The essential result is that by determining the normalized cross spectral correlation from the complex phase of the scattered radiation, both the absolute location of the scatterer as well as the intensity distribution of the secondary radiation source in the direction of beam propagation may be determined. Further, apart from the quasi-homogeneity constraint, the result obtained is independent of the degree of spatial isotropy of the scatterer. In comparison, for the plane wave illumination of a random refractive medium, the normalized cross spectral density μ_s of the scattered field provides little information on the distribution of the random medium in the direction of the incident wave.

For the integration in Eq. (21) to approximate an inverse Fourier transform, a condition on the minimum useful probing wavelength λ_0 must be satisfied. Namely, we require $K_0^2/k_0 \geq \Delta K_z \sim 2\pi/L_z$ where L_z is the thickness of the fluctuating region. Hence, we require $\lambda_0 \geq \Lambda^2/L_z$ where Λ is the shortest resolved wavelength. For fusion plasmas, taking $\Lambda = 1\text{cm}$ and $L_z \sim 1\text{m}$, then

the optimum wavelength λ_0 is around $100\mu\text{m}$. We point out that even if the absolute phase of the scattered field for all wavenumbers cannot be determined such as in standard microwave scattering experiments, the normalized correlation $|\bar{\mu}_s|$ may still be measured by heterodyne detection techniques commonly used in plasma fluctuation studies. It can be shown that for receivers sampling approximately the same wavenumber, the coherence between two detector signals is related to $|\mu_s(\mathbf{K}_\perp, \mathbf{K}'_\perp; z)|$ (the normalized cross spectral density), from which the solid angle of the scatterer at the receivers may be determined. However, if the detector signals sampling opposite wavevectors are correlated, the coherence is related to the normalized spectral correlation $|\bar{\mu}_s(\mathbf{K}_\perp, -\mathbf{K}_\perp; z)|$.

Finally, if we introduce a Gaussian envelope

$$I(z) = n_0 \exp(-(z - z_1)^2/L_z^2) \quad (22)$$

over an otherwise homogeneous random medium, we then obtain for the normalized cross spectral correlation,

$$|\bar{\mu}_s(\mathbf{K}_\perp, -\mathbf{K}_\perp; z)| = \exp(-Q^2) \quad (23)$$

where $Q = |K_F|L_z$ is simply the Klein-Cook parameter,⁷ a parameter derived elsewhere in the context of scattering from non-

random sinusoidal phase screens of finite extent.

3 Conclusion

By the introduction of a new two point statistical measure over the angular spectrum of scattered radiation from random plasma fluctuations, the local intensity of the scatterer along the direction of the incident plane wave may be recovered. Experimentally, for large magnetically confined plasmas, a single beam interferometer operating at $\sim 100\mu\text{m}$ is sufficient to recover the local intensity of plasma fluctuations given that the quasi-homogeneity constraint is a reasonable approximation to the plasma fluctuations.

4 Acknowledgments

I wish to acknowledge the efforts of Dr. John Howard for carefully reading the manuscript and for providing various suggestions. This work was supported by the U.S. Department of Energy Contract No. DE-AC02-76CH03073.

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Figure Captions

Fig.1

Scattered radiation from random plasma fluctuations is projected onto the Fourier transform plane of a lens. Correlations between various points in the Fourier transform plane may be used to recover the three dimensional mean square distribution of density fluctuations within the scattering volume. The mapping from spatial to spectral coordinates in the Fourier transform plane of the lens and within the parabolic approximation is given by $\mathbf{K}_\perp = \mathbf{r}_\perp k_0/f$ where f is the focal length of the lens.⁵

FIG. 1