MASTER

GEOMETRY OF WAVE ELECTROMAGNETICS*

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Geometry of Wave Electromagnetics

It has been known now for over a century that light is an electromagnetic phenomenon; and that light propagates in empty space as a wave according to Maxwell's equations. In direct confirmation of this wave nature are the observation and quantitative explanation of interference and diffraction; and, in evidence of its transverse nature, of polarization and double refraction. There is no substitute for the wave theory of light.

On the other hand, there are a number of simple situations where a ray picture of light is useful. These include the formation of shadows of images by the pinhole camera, and reflection and refraction at simple surfaces like planes and conics. We have an adequate idea of the conditions under which this obtains as an approximation: it is most simply seen as a consequence of the extremely small wavelength of light.¹ We even know how to compute corrections to the ray approximations. We have, for example, the typical fluting of the shadow in diffraction at a straight edge. It has therefore been thought that the ray picture is always an approximation and should be transcended, since we know the wave theory to be accurate and adequate albeit clumsy.

We present a challenge to this commonly held view and construct an exact realization of light as generalized pencils of rays.² It is important to stress that we must use pencils of rays rather than single rays. We shall display exact equations of motion for the rays in the pencil, and we shall see that these rays tend to travel in straight lines in empty space (not too near the edge of a beam).

The same kind of question had arisen once before in wave optics but this was in relation to the question to what extent quantum optics had classical pictures.³ It was known that the correspondence principle suggested an approximation to quantum theory by classical phase space distributions: high photon number states could be related to intense classical wave fields. But we had shown that there was an exact correspondence between the classical and quantum pictures.⁴ The complete expression is embodied in the first and second fundamental theorems of quantum optics.⁵

While the problem of ray picture of wave electromagnetics is distinct from the above, it is logically of the same type. What we wish to show is that the exact correspondence reproduces all the known features of wave and ray optics but provides a clear insight into the scaling of interference and diffraction patterns. These patterns in the Franhofer region scale as if they are shadows cast by rectilinear rays.²

My interest in this work stems from a discussion I had with Emil Wolf exactly two decades ago! It was rekindled by Jacob Kuriyan who pointed out the immediate relevance of the work of Wolf⁶ and the previous work of Walther⁷ on radiative transfer. The astute reader

will see the germs of the ideas discussed here in their work, but also see that there are definite points of departure.

Rays in Wave Fields

We shall deal with statistical wave fields exclusively; the pure wave field can be considered as a special case. We may realize the wave field in terms of the two-point correlations of the electric field over the ensemble:

$$\Gamma_{jk}(r_1, r_2, t_1, t_2) = \langle E_j^\dagger(r_1, t_1) E_k(r_2, t_2) \rangle \quad (1)$$

The ensemble averages so obtained obey a wave equation in the variables r_1, t_1 as well as in the variables r_2, t_2 . It is known that quite rigorously we may show complete one-to-one correspondence between the quantum mechanical expectation values for second quantized field operators and the classical ensemble averages for classical (analytic signal) wave fields: this is the second fundamental theorem of quantum optics.⁵ We shall therefore not distinguish the two theories, but deal directly with the correlation functions themselves. For a stationary ensemble the correlation function does not depend individually on the times t_1, t_2 but only on their differences. In this case we can define the specific correlation function:

$$\Gamma_{jk}(r_1, r_2; \nu) = (1/2\pi) \int_{-\infty}^{\infty} d\tau e^{-i\nu\tau} \Gamma_{jk}(r_1, r_2, t+\tau, t) \quad (2)$$

which is independent of time. To avoid inessential complications, we shall deal exclusively with stationary ensembles and time-independent specific correlation functions.

Various physical quantities of interest can be expressed in terms of the specific correlation tensors. The energy density $u(\underline{r})$ is given by

$$u(\underline{r}) = (1/4\pi) \sum_j \int d\nu \Gamma_{jj}(\underline{r}, \underline{r}; \nu). \quad (3)$$

We now define the Wolf tensor

$$W_{jk}(\underline{r}, \underline{p}; \nu) = (1/2\pi) \nu^{-1} \int d^3q e^{+ip \cdot q} \Gamma_{jk}(r + \frac{1}{2}q, r - \frac{1}{2}q; \nu) \quad (4)$$

which will be presently associated with the density of polarizable rays with wave number $\underline{p}$ in the direction of $\underline{p}$ at the location $\underline{r}$. We note that while the sum on the right hand side of (3) runs over

three indices, by virtue of Gauss's theorem contained in Maxwells' equations it really reduces to two polarizations. One may go to the helicity basis appropriate for description of circularly polarized photons and introduce the two valued helicity index σ :

$$u(\underline{r}) = (1/4\pi) \sum_{\sigma=1}^2 \int d\nu \Gamma(\underline{r}, \underline{r}; \nu, \sigma). \quad (5)$$

The Wolf tensor W is not diagonal in the helicity σ in general but we may introduce in place of (4) the Wolf matrix

$$W(\underline{r}, \underline{p}; \nu, \sigma_1, \sigma_2) \quad (6)$$

adapted to the helicity basis.

We now identify $W(\underline{r}, \underline{p})$ as the density of light rays in the pencil passing through the point $\underline{r}$ in the direction $\underline{p}$. For the model of stationary ensembles that we choose to discuss we note that there is a specific frequency ν for each ray; and there is a wave number which is $(p^2)^{1/2}$, not necessarily coinciding with ν . There is also a polarization label for the rays; one may take the diagonal elements of the Wolf matrix $W(\underline{r}, \underline{p}; \nu, \sigma, \sigma)$ as giving the density of rays with polarization σ at $\underline{r}$ in the direction $\underline{p}$ with wave number $(p^2)^{1/2}$ and frequency ν . We observe that W so defined is real (which is not true of the correlation tensor) but not necessarily positive.

Light Rays: Bright and Dark

Since a negative density of rays might appear unsatisfactory and unfamiliar we digress to point out that it is not unphysical. The measurable quantities of the theory are dynamical quantities like the energy density in the form of the photometric intensity of illumination at any location; the total flux of light from a source; or the flux of momentum across a surface as determined by the light pressure. Wave theory of light cautions as against the attempt to make measurements on localized pencils of light which are too narrow despite their formal use in consideration of radiative transfer. On the other hand, a little reflection shows that the redistribution of light implicit in the phenomenon of "light added to light producing darkness" in the double slit interference pattern most clearly shows that the ray density W cannot always be nonnegative. In fact, it is precisely the indefinite sign of the ray density that makes it possible to have an exact ray picture for arbitrary wave fields. We shall, therefore, not be surprised to have indefinite signs for the ray density in a number of contexts where wave optical effects make their appearance. (See, however (17) and the following discussion.)

We note, in passing, that Babinet's Principle is automatically

satisfied.

Since $W(\underline{r}, \underline{p}; \nu)$ is interpreted as the density of rays, the density of energy and the density of momentum are, respectively

$$\epsilon(\underline{r}) = \int d\nu \nu \int d^3 p W(\underline{r}, \underline{p}; \nu) \quad (7)$$

and

$$p_j(\underline{r}) = \int d\nu \int d^3 p p_j W(\underline{r}, \underline{p}; \nu). \quad (8)$$

The first of these terms coincide with the energy density (5) but the second does not quite coincide with the Poynting vector, which in our present notation may be written

$$s_j(\underline{r}) = \nu^{-1} \left\{ + \frac{\partial}{\partial q_k} \Gamma_{jk}(\underline{r} + \frac{1}{2}\underline{q}, \underline{r} - \frac{1}{2}\underline{q}) - \frac{\partial}{\partial q_j} \Gamma_{kk}(\underline{r} + \frac{1}{2}\underline{q}, \underline{r} - \frac{1}{2}\underline{q}) \right\}_{\underline{q}=0} \quad (9)$$

The second term coincides with (8) but there is an additional term which may be expressed as a curl. Consequently, it does not contribute to net fluxes. We conclude that the Poynting vector includes, in addition to the density of momentum associated with the pencil of rays a circulating local momentum current. In general, it is a very small quantity and can be estimated to be of the order of

$$c/\nu L$$

where c is the velocity of light and L the linear dimension of the beam across which there is an appreciable variation of the intensity. For most practical purposes this difference may be neglected.

We still have to deal with the problem of both wave number $(p^2)^{1/2}$ and frequency ν being defined for the pencil of rays without a relation of the form

$$\nu = c(p^2)^{1/2}.$$

This happens in those cases we have great departure from locally plane waves; and when we have many waves, the wave numbers are averages of pairs of wave numbers of the waves. Accordingly the wave numbers satisfy the inequality

$$(p^2) \leq \nu^2/c^2. \quad (10)$$

It is as if we have "dispersion in free space" but remember that this is only appearance since the wave number we define is to be

interpreted as the "momentum of a localized photon".

Propagation of Pencils of Rays

The two wave equations satisfied by $\Gamma_{jk}(r_1, r_2; \nu)$ can be displayed in a manner reminiscent of the eikonal equation for a classical wave. To start with, we have

$$\begin{aligned} (\nabla_1^2 + \nu^2/c^2) \Gamma_{jk}(\underline{r}_1, \underline{r}_2; \nu) &= 0; \\ (\nabla_2^2 + \nu^2/c^2) \Gamma_{jk}(\underline{r}_1, \underline{r}_2; \nu) &= 0; \end{aligned} \quad (11)$$

which can be rewritten

$$\left\{ \left(\frac{1}{2} \nabla \pm i \underline{p} \right)^2 + \nu^2/c^2 \right\} W_{jk}(\underline{r}, \underline{p}; \nu) = 0. \quad (12)$$

Consequently,

$$\underline{p} \cdot \nabla W_{jk}(\underline{r}, \underline{p}; \nu) = 0 \quad (13)$$

and

$$\left(p^2 - \frac{1}{4} \nabla^2 \right) W_{jk}(\underline{r}, \underline{p}; \nu) = \frac{\nu^2}{c^2} W_{jk}(\underline{r}, \underline{p}; \nu). \quad (14)$$

The inequality of p^2 and ν^2/c^2 is evident from (14); they are equal in those locations where there is no local variation of the Wolf tensor and hence of the energy density. But in the edge domain of a light beam, we have bending of the light rays and the wave number p then no longer equals ν/c . The wave vector $\underline{p}$ must bend according to (13). So as to be along the level surfaces of $W_{jk}(\underline{r})$. One notes that while the specific correlation tensors were divergence-free, this translates into the polarization being transverse to the wave vector only in the region of local constancy. More generally we have

$$\nabla_{1j} \Gamma_{jk}(\underline{r}_1, \underline{r}_2; \nu) = \nabla_{2k} \Gamma_{jk}(\underline{r}_1, \underline{r}_2; \nu) = 0$$

and hence

$$\begin{aligned} \left(p_j - \frac{i}{2} \nabla_j \right) W_{jk}(\underline{r}, \underline{p}; \nu) &= 0. \\ \left(p_k + \frac{i}{2} \nabla_k \right) W_{jk}(\underline{r}, \underline{p}; \nu) &= 0. \end{aligned} \quad (15)$$

Hence the polarization will be purely transverse as long as the variation of the Wolf tensor vanishes locally. These are immediate consequences of the equations of propagation (13) and (14).

Rewriting (14) in the form of a wave equation for the Wolf tensor

$$\left\{ \nabla^2 + 4 \left[\frac{v^2}{c^2} - p^2 \right] \right\} W_{jk}(\underline{r}, p; v) = 0. \quad (16)$$

We see that the "wave effects" in the propagation of the pencil of rays is confined to the departure of the wave number p from v/c (apart from the possibility of the Wolf tensor!). By virtue of the inequality (10) we see that those components with

$$v^2/c^2 - p^2 = \mu^2/4 > 0$$

alternate rapidly with a typical Yukawa falloff $e^{-\mu r/r}$ from the source. At large distances only the "mass shell" contributions with $v = cp$ are available.

Pencils of Rays Forming Interference and Diffraction Patterns

To illustrate the use of the ray pencil picture of wave electromagnetics and display the scaling law, we simplify life (for the author and for the reader!) by replacing transverse vector light by "scalar light". The wave equations (11) are now scalar equations for a scalar two-point correlation function $\Gamma(r_1, r_2; v)$ and the Wolf function (4) is now a scalar function $W(\underline{r}, p; v)$. It is still not necessarily point wise positive, but instead satisfies the positivity condition

$$\int d^3 r \int d^3 p W(\underline{r}, p; v) M(\underline{r}, p) \geq 0 \quad (17)$$

where $M(\underline{r}, p)$ is any "testing phase space density"

$$M(\underline{r}, p) = \int d^3 q f^*(\underline{r} + \frac{1}{2} \underline{q}) f(\underline{r} - \frac{1}{2} \underline{q}). \quad (18)$$

So we may say that while the ray pencil contains tamasic (dark) rays and pradipa (shining) rays⁸, any reasonable test will produce non-negative illumination.

To simplify life further, we shall work in the Huyghen's approximation. It is possible to carry out the more elaborate exact calculations with transverse waves and without the use of the Huyghen's approximation. But here we present the simplified version in the Franhoufer region for the specific amplitude:

$$\phi_v(\underline{r}) \sim \frac{iv}{|\underline{r}-\underline{r}'|} \iint dx' dy' |\underline{r}-\underline{r}'|^{-1} \cdot \exp \left\{ i \frac{v}{c} |\underline{r}-\underline{r}'| \right\} \phi_v(\underline{r}'). \quad (19)$$

For a single slit at $x'=a$ we obtain the two-point function

$$\Gamma \left(x + \frac{\xi}{2}, y + \frac{\eta}{2}, z + \frac{\zeta}{2}; x - \frac{\xi}{2}, y - \frac{\eta}{2}, z - \frac{\zeta}{2} \right) \sim \exp \left\{ -i \left(\frac{v\zeta}{c} \right) \right\} \exp \left\{ i \frac{v}{c} (x-a) \xi / z \right\}$$

and the scalar Wolf function

$$W(x, y, z, p_1, p_2, p_3; v) \delta(p_2) \delta(p_3 - v) \cdot \delta \left(p_1 - \frac{v(x-a)}{c} \right) \quad (20)$$

apart from inessential constants. This expression is easily interpreted. It describes an elementary pencil originating at the slit $x'=a$ and propagating geometrically to the point x on the "screen"; we have been careless about quadratic terms and so the fact that $p^2 \neq v^2/c^2$ is not to be taken seriously. There is an inverse square law attenuation contained in (19), but apart from that there is local isotropy. The ray density given by the scalar Wolf function (20) is clearly nonnegative.

We now turn to the two-slit interference pattern. In this case we can still calculate by elementary methods the two-point function

$$\Gamma \sim \exp \left\{ -\frac{iv}{c} \right\} \left\{ \exp \frac{iv(x-a)\xi}{cz} + \exp i \frac{v(x+a)\xi}{cz} + 2 \exp \frac{iv\zeta}{cz} \cos \left(2 \frac{vax}{cz} + \phi \right) \right\} \quad (21)$$

and the Wolf function

$$W \sim \frac{c^2}{v^2 z^2} \delta(p_2) \delta(p_3 - v) \cdot \left\{ \delta \left(p_1 - \frac{v(x-a)}{cz} \right) + \delta \left(p_1 - \frac{v(x+a)}{cz} \right) - 2 \cos \left(\frac{vax}{cz} + \phi \right) \delta \left(p_1 - \frac{vx}{cz} \right) \right\}. \quad (22)$$

Here ϕ is the phase difference between the two slits at $x' = \pm a$. The pattern corresponds to three distinct pencils of rays. One from

each slit with elementary geometric propagation with only pradipa rays but the third pencil apparently originating at a location between the two slits contains both tamasic and pradipa rays. For the $\phi=0$ case the central ray from the third beam is pradipa, while for the $\phi=\pi$ case it is tamasic. The angular separation between the pradipa and tamasic rays is given by

$$\Delta\theta = \pi e/va = \pi\lambda/a \quad (23)$$

in accordance with the elementary result. The scaling law for the fringe pattern

$$\Delta x = z\Delta\theta = \pi\lambda z/a \quad (24)$$

is an immediate consequence of the geometric propagation law and (22). The pattern scales since light rays travel in straight lines even in the formation of interference patterns!

It is of interest to point out that the need for three rather than two beams for two (coherent) sources is an essential component of the present exact ray picture and originates in the linearity of ray patterns in the two-point function. We recognize that the third pencil with both tamasic and pradipa rays is directly dependent on the coherence of the two (virtual) sources given by the two slits as evidenced by the appearance of the phase angle ϕ . For incoherent virtual sources this pencil of rays vanishes: this can be most easily seen as an average over the phase angle.

As an additional bonus we now recognize that most traditional descriptions of photon interference, for example, by the Copenhagen school are incomplete and inadequate. They all depend on two pencils of photon rays.

As a second illustration we consider the diffraction of (scalar) light by a thin slit. The two-point function and the Wolf function can be calculated by elementary techniques. The latter has the form

$$\begin{aligned} W \sim & \frac{c^2}{v^2 z^2} \delta(p_2) \delta(p_3 - v) \\ & \cdot \left\{ \delta\left(p_1 - \frac{v(x-a)}{cz}\right) + \delta\left(p_1 - \frac{v(x+a)}{cz}\right) \right. \\ & \left. - 2 \cos \frac{vax}{cz} \delta\left(p_1 - \frac{xv}{cz}\right) \right\}. \end{aligned} \quad (25)$$

In this case, again, we see three pencils of rays, the first two are very similar to the case of two slits but the third one originates in the center of the slit. The third pencil contains both kinds of rays.

We note, that in all these cases the total contribution of the three pencils at any field point add up to a nonnegative value by virtue of the Schwartz inequality. Hence the photometric intensity at any field point is nonnegative.

As a third example of some interest in its own right, let us consider the van Cittert-Zernike-Wolf theorem⁹ on the propagation of light emerging from a primary radiating surface and the development of partial coherence by propagation. A primary radiating surface is a surface from which light emerges outward, no two distinct points of the surface having a mutual coherence. In accordance with an obvious generalization of (22) we then see that the illumination field will be simply represented in the ray pictures. Every point on the primary radiating surface becomes the source of rectilinear rays. Consequently in the "far zone" the rays of light will all be practically parallel over areas which subtend small solid angles at the source. By reading (4) from right to left, we see this to imply a sizable coherence patch in the far field. This is the van Cittert-Zernike-Wolf theorem.

Discussion

In this paper we have discussed the construction of pencils of rays which exactly represent any given wave field. This is not a high frequency limit in the sense of the geometric theory of Keller and others¹; nor is it an asymptotic treatment. It is an exact picture, the wave aspects making themselves felt in the possible "dispersion in free space" and the curved light rays in empty space (13) and (14); and in the possible nonpositivity of the Wolf function (4). We consider these effects to be real physical effects rather than defects of the theory.

The phase space description of quantum theory was introduced into physics by Wigner¹⁰ using the characteristic function defined as the expectation values of the Weyl unitary operators. Moyal¹¹ showed that quantum dynamics can be reformulated in terms of it using the Moyal sine brackets. Related distribution functions were introduced by Bargman and by Segal from the point of view of the Hilbert space of entire functions¹² and by Klauder from the point of view of continuous representations.¹³ The distribution functions which correctly give expectation values of normal ordered quantum electrodynamical variables was introduced by Glauber and by Sudarshan.^{3,4} But the application to radiative transfer and rays in electromagnetic wave fields began with the work of Walther⁷ and of Wolf.⁶ The present theory was worked out by the author during the past few years; and we note that Bastiaans¹⁴ has made several applications of the theory. Lohman¹⁵ has pointed out that the traditional musical notation is an early application of such phase space distributions!

It may not be amiss to clarify a point of some confusion in the literature. The Wigner-Moyal density function is appropriate to a normalized quantum mechanical state vector (or more generally,

a density matrix). As such the double integral of this distribution function is always unity; and all eigenvalues of the Wigner-Moyal distribution considered as a kernel would be of absolute value less than unity but summing to unity. The Wolf tensors, on the other hand, do not have any such constraints. This stems from their being expectation values in a field, and it could have any non-negative trace whatever. The modes⁵ are identifiable with wave functions but they have nontrivial multiplier factors. This is, of course, immediately obvious from elementary optics!

As a matter of some historical interest, it may not be amiss to recall two earlier works. About three decades ago Bohm¹⁶ published two papers on "Quantum Theory in Terms of Hidden Variables" which made use of a "quantum-mechanical force". This work which gave birth to a considerable amount of discussion was modeled on the eikonal theory¹; but the quantum-mechanical force is reminiscent of the curved light rays in empty space of the present theory. Bohm did not examine the ray density function but instead concentrated on the discussion of the equation for the path and, consequently, the possible nonpositivity of the ray density function was not recognized. Secondly, we have shown that dynamics in terms of phase space pictures obtained by Moyal¹¹ is essentially the only one available for any linear phase space formulation.¹⁷

Finally, since the picture in terms of ray pencils is an exact reformulation of the wave theory it follows that given the pencils of rays, the wave field can be obtained by inverting (4). This correspondence obtains for more general situations involving arbitrary passive linear media¹⁸ which could be suitably nonuniform.

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