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COMBUSTION

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TRAVELING-WAVE SOLUTIONS TO
REACTION-DIFFUSION SYSTEMS MODELING COMBUSTION

by

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ABSTRACT

We consider the deflagration-wave problem for a compressible reacting gas, with species involved in a single-step chemical reaction. In the limit of small Mach numbers, the one-dimensional traveling-wave problem reduces to a system of reaction-diffusion equations. Thermomechanical coefficients are temperature-dependent. Existence is proved by first considering the problem in a bounded domain, and taking an infinite-domain limit. In the singular limit of high activation energy in the Arrhenius exponential term, we prove strong convergence to a limiting free-boundary problem (discontinuity of the derivatives on the free boundary).

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0. INTRODUCTION

We consider the deflagration wave problem for a compressible reacting gas, with one reactant involved in a single step chemical reaction. In the limit of small Mach numbers, the one-dimensional traveling wave problem reduces to a system of two reaction-diffusion equations (cf. Section 1 for review of the basic flame equations). We assume both heat conductivity and diffusion coefficients temperature-dependent. For sake of clarity, we will first develop our methods on a simpler scalar case (corresponding to a Lewis number equal to one, cf. Section 1.4). The renormalized model is:

$$\begin{cases} -(k(u)u')' + cu' = g(u) & \text{on } \mathbb{R} \\ u(-\infty) = 0 & u(+\infty) = 1 \end{cases} \quad (0.1)$$

where u is the renormalized temperature, $0 < u \leq 1$, $k(u)$ is a C^1 function of u , which is strictly positive; $g(u)$ is a renormalized reaction term such that $g(u) = 0$ on $[0, \theta)$ and $g(u) > 0$ on $(\theta, 1)$ where θ is some ignition temperature ($0 < \theta < 1$). Moreover, $g(1) = 0$. c is the unknown mass flux of the wave. Next, we will investigate the system

$$\begin{cases} -u'' + cu' = f(u)v & \text{on } \mathbb{R} \\ -\Lambda v'' + cu' = -f(u)v & \text{on } \mathbb{R} \\ u(-\infty) = 0 & u(+\infty) = 1 \\ v(-\infty) = 1 & v(+\infty) = 0 \end{cases}, \quad (0.2)$$

where u is again the renormalized temperature ($0 < u < 1$), v is a

renormalized reactant concentration ($0 \leq v \leq 1$) and $f(u)$ as the same properties as $g(u)$, except that now $f(1) > 0$. In (0.2), Λ is taken constant again for simplicity.

However, our methods trivially extends to the full system:

$$\begin{cases} -(\lambda(u)u')' + c \kappa(u)u' = Q f(u)v \\ -(\mu(u)v')' + c v' = -f(u)v \end{cases} \quad (0.3)$$

where $\lambda(u)$ is the heat conductivity, $\mu(u)$ is the reactant diffusion coefficient and $\kappa(u)$ the variable specific heat of the mixture. Finally, Q is the heat of reaction and is a given positive constant.

The above equations (0.1), (0.2), (0.3) are nonlinear eigenvalue problems for c . Existence for the systems is proved by first considering the problem on a bounded domain. This allows the reduction of the corresponding problem to a fixed point formulation w.r.t. the triplet (u, v, c) . Then the usual Leray Schauder degree gives the existence in a bounded domain. Taking an infinite domain limit completes the proof. Obtaining strictly positive lower and upper bounds for c , independent of the size of the domain, is crucial for the above.

Considerable amount of work has been performed on the above equations in the asymptotic limit of infinite activation energy in the Arrhenius reaction term (see [4] and the bibliography there). Equivalently, f in (0.2) is now allowed to depend on ε (proportional to the reciprocal of the activation energy); $f_\varepsilon(u)v$, in (0.2), formally behaves as $\delta(u-1)$ when $\varepsilon \rightarrow 0$, where δ is the Dirac function centered at

zero. Such formal asymptotic limits have not been rigorously established from a mathematical point of view. In this paper, we prove strong convergence of the traveling waves to singular limit free boundary problems, with discontinuous derivatives. The plan of the paper is the following:

1. The basic equations of the premixed 1-dimensional laminar flame.
2. The scalar equation case.
3. The case of the system of equations.

This communication is a shorter version of a full paper to appear [2] with several proofs shortened or deleted. Our methods are also generic to the case of two reactants (that is, a system of 3 equations) and multistep branching reactions. In the latter case, detailed asymptotic analysis has been done by P. C. Fife and B. Nicolaenko [7,8].

§1. THE BASIC EQUATIONS OF THE PREMIXED ONE-DIMENSIONAL LAMINAR FLAME

1.1. REACTIVE FLOW EQUATIONS IN ONE-DIMENSION.

We summarize the equations of a chemically reacting mixture; this is essentially a compressible, heat-conducting viscous fluid with the added complexity of species diffusion and source terms representing the chemical reaction. Let ρ be the total mass density, T the temperature, p the hydrostatic pressure and v the mass-average velocity. The reacting mixture is considered to be made of N fluids whose separate densities are ρY_i ($i = 1, 2, \dots, N$); here the Y_i are

mass fractions of species i , with molecular mass m_i :

$$\sum_{i=1}^N Y_i = 1 \quad (1.1)$$

In what follows, we will investigate in detail the case of a single reactant A which yields a global product P , in a one-dimensional geometry:

The total mass density satisfies the equation

$$\frac{D\rho}{Dt} + \rho \frac{\partial v}{\partial x} = 0 \quad , \quad (1.3)$$

where $D/Dt \equiv \partial/\partial t + v \frac{\partial}{\partial x}$ is the convective derivative.

The balance law for momentum is the same as for nonreactive flows, that is:

$$\rho \frac{Dv}{Dt} + \frac{\partial p}{\partial x} = \frac{4}{3} \frac{\partial}{\partial x} \left(\kappa \frac{\partial v}{\partial x} \right) \quad , \quad (1.4)$$

where κ is the dynamic viscosity coefficient and external forces on the mixture have been neglected. Energy conservation is expressed by:

$$\rho c_p \frac{DT}{Dt} - \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x} \right) = Q \omega + \frac{Dp}{Dt} + \frac{4}{3} \kappa \left(\frac{\partial v}{\partial x} \right)^2 \quad , \quad (1.5)$$

where $c_p(T)$ is the specific heat at constant pressure, $\lambda(T)$ the coefficient of thermal conductivity, Q the chemical heat release of

the reaction (1.2); ω measures the rate at which reaction (1.2) is proceeding:

$$\omega = B(T) \frac{\rho Y}{m} \exp\left(-\frac{E}{RT}\right) , \quad (1.6)$$

where E is the activation energy of the reaction (a constant); in some sense, E/R is the temperature below which the reaction is negligible; R is the perfect gas constant, and $B(T)$ has a weak dependence on T . (1.6) encompasses both the law of mass action and Arrhenius kinetics [4].

The continuity equation for the mass fraction Y of reactant A with molecular mass m is:

$$\rho \frac{DY}{Dt} - \frac{\partial}{\partial x} \left(\mu \frac{\partial Y}{\partial x} \right) = -m \omega , \quad (1.7)$$

where $\mu(T, Y)$ is the diffusion coefficient. From (1.1), the mass concentration of the combustion product P is $1-Y$, which enables its elimination. Finally, the equation of state for a perfect gas yields a supplementary constitutive law:

$$p = R \rho T . \quad (1.8)$$

1.2. APPROXIMATION OF COMBUSTION.

A flame is a low-speed wave whose Mach number M_0 is $\ll 1$. As an immediate consequence the variations in pressure are also small, i.e.:

$$p = p_c + \delta p , \quad \delta p = O(M_0^2) ,$$

so that we may set

$$p = p_c = \text{Const}$$

everywhere except in the momentum equation (1.4). The momentum equation now only controls small "flow-induced" variations δp in pressure [4] and uncouples from the remaining equations. Similarly, in the energy equation (1.5), one can neglect $\frac{Dp}{Dt} \equiv \frac{D}{Dt} \delta p$ and $(\frac{\partial v}{\partial x})^2 = O(M_0^2)$. Finally, in the combustion approximation, the Eqs. (1.3)-(1.7) reduce to:

$$\begin{cases} \frac{D\rho}{Dt} + \rho \frac{\partial v}{\partial x} = 0, \\ \rho c_p \frac{DT}{Dt} - \frac{\partial}{\partial x} (\lambda \frac{\partial T}{\partial x}) = +Q_w, \\ \rho \frac{DY}{Dt} - \frac{\partial}{\partial x} (\mu \frac{\partial Y}{\partial x}) = -m\omega, \end{cases} \quad (1.9)$$

together with the equation of state (1.8). Note that the combustion approximation does not imply a constant density approximation in one dimension.

1.3. THE FLAME FRONT EQUATIONS FOR A SINGLE REACTANT.

To study a one-dimensional flame moving with constant velocity $V_0 > 0$ to the left, it is appropriate to write (1.9) in the frame of reference of an observer moving at this same speed. Let $\xi = x + V_0 t$ be the observer's space variable, and prime denote the differentiation with respect to ξ , then (1.9) becomes:

$$V_0 \rho' + (\rho v)' = 0, \quad (1.10a)$$

$$c_p(\rho V_o + \rho v)T' - (\lambda T')' = Qw \quad , \quad (1.10b)$$

$$(\rho V_o + \rho v)Y' - (\mu Y')' = -m\omega \quad ; \quad (1.10c)$$

(1.10a) integrates to yield:

$$\rho(V_o + v) = \text{Const} \equiv c \quad , \quad (1.11a)$$

where c is the mass flux; then the energy and species equations uncouple from the rest of the system:

$$c c_p T' - (\lambda T')' = +Qw \quad (1.11b)$$

$$c Y' - (\mu Y')' = -m\omega \quad ; \quad (1.11c)$$

the following boundary conditions apply to (1.11): at $\xi = -\infty$, the mixture is cold and unburned:

$$\begin{aligned} T &= T(-\infty) \\ Y &= Y(-\infty) \quad ; \end{aligned} \quad (1.12a)$$

at $\xi = +\infty$ the reactant is burned out and

$$Y(+\infty) = 0 \quad ; \quad (1.12b)$$

$T(+\infty)$ is determined through Rankine-Hugoniot-like conditions obtained by integrating (1.11) from $-\infty$ to $+\infty$ and injecting (1.12a-b):

$$T(+\infty) = T(-\infty) + \frac{Q}{c_p m} Y(-\infty) \quad , \quad (1.12c)$$

where we have assumed c_p independent from T.

In Section 3 of the paper, we shall consider a renormalized version of (1.11-1.12):

$$\begin{cases} c u' - u'' = v f(u) \\ c v' - \lambda v'' = -v f(u) \end{cases} \quad (1.13)$$

where u is the renormalized temperature, v the renormalized concentration of reactant, and $f(u)$ the renormalized formulation of w/Y . For simplification, λ , μ and c_p are assumed constant. The renormalized boundary conditions are:

$$\begin{aligned} u(-\infty) &= 0, \quad v(-\infty) = 1, \\ u(+\infty) &= 1, \quad v(+\infty) = 0. \end{aligned}$$

1.4. THE CASE OF LEWIS NUMBER EQUAL TO 1.

Consider again the (unnormalized) single reactant system (1.11). If the Lewis number

$$Le = \frac{\lambda}{\mu c_p} \quad (1.14)$$

is such that

$$Le = 1, \quad \forall T,$$

we can eliminate the concentration Y , through the use of the

"Shvab-Zeldovich variable" [4]:

$$Z = Q \frac{Y}{m} + c_p T ; \quad (1.15)$$

Z satisfies a purely convective diffusive equation

$$c Z' - (\mu Z')' = 0 ,$$

whose only bounded solution is

$$Z = \text{Constant} = Z(+\infty) ; \quad (1.17a)$$

since the reactant Y is depleted at $+\infty$, $Y(+\infty) = 0$ and

$$Z \equiv c_p T(+\infty) , \quad (1.17b)$$

where $T(+\infty)$ is the adiabatic flame temperature:

$$T(+\infty) \equiv \frac{Q}{c_p} \frac{Y(-\infty)}{m} + T(-\infty) . \quad (1.18)$$

Injecting the identity

$$Y \equiv \frac{mc_p}{Q} (T(+\infty) - T) \quad (1.19)$$

into equation (1.11b) we obtain the reduced scalar equation:

$$c T' - \left(\frac{\lambda}{c_p} T' \right)' = \rho B(T) (T(+\infty) - T) \exp \left(-\frac{E}{RT} \right) . \quad (1.20)$$

In Section 2, we will investigate the normalized version of (1.20)

$$c u' - (k(u)u')' = g(u) , \quad (1.21)$$

where

$$g(u) \equiv (1-u) f(u) ,$$

$f(u)$ defined in (1.13).

1.5. CONSIDERATIONS ON THE IGNITION TEMPERATURE.

Consider the flame equations (1.11). If we fix boundary conditions at $\xi = -\infty$, i.e., $T(-\infty)$ and $Y(-\infty)$, then the problem is improperly posed, since

$$\exp\left(\frac{-E}{RT(-\infty)}\right) \neq 0$$

and there exist no bounded solutions to (1.11). The origin of the difficulty is clear: this formulation requires the mixture to react all the way in from $\xi = -\infty$, so that by the time finite ξ is reached, the combustion would be complete [4]; this is the "cold-boundary" difficulty, on which a considerable amount of ingenuity has been spent [9,11]. To resolve it, we classically modify the reaction term ω through the introduction of an ignition temperature θ^c , such that $\omega \equiv 0 \forall T < \theta^c$; this is equivalent to replacing ω by $H(T - \theta^c) \omega$, where H is the Heaviside function. It has been proven [9] that if one takes a sequence θ_i^c such that $\theta_i^c \rightarrow T(-\infty)$, c_i converges to some limit c_∞ from below. However, this lack of universal significance for c disappears as the activation energy E becomes large; the high activation energy asymptotics developed in Section 2 completely circumvent this difficulty by yielding limiting formulas for c independent of θ^c [4].

§2. THE SCALAR EQUATION CASE

2.1. EXISTENCE AND UNIQUENESS RESULT.

The goal of this section is to study the problem:

Find a function $u: \mathbb{R} \rightarrow (0,1)$ and $c \in \mathbb{R}$ satisfying

$$\begin{cases} -(k(u)u')' + cu' = g(u) \text{ on } \mathbb{R} \\ u(-\infty) = 0 \quad u(+\infty) = 1 \end{cases} \quad (2.1)$$

It has been shown, in Section 1.4, that (2.1) corresponds to a reduction to a scalar equation if the Lewis number is one. The more general situation where $k(u) \neq 1$ is investigated here as a model for the more complex systems of the following section. In (2.1), c is an unknown of the problem (representing the mass flux), so we are looking for a nonlinear eigenvalue problem. Many earlier works have assessed the question of existence for problems of the type (2.1). A good survey of the question with a discussion of the literature can be found in the notes of P. C. Fife [5,6]. We should mention, specifically devoted to combustion, the works of Ya. I. Kanel' [10], W. Johnson and W. Nachbar [9]. In the context of population genetics, the paper of D. G. Aronson and H. F. Weinberger [1] is also relevant. All these papers use a "phase plane" approach to solve (2.1). Here, using a more analytical approach, namely a "shooting method", we solve (2.1) with more general hypotheses than the former works. Precisely, the function g will satisfy the following conditions.

$$g : [0,1] \rightarrow \mathbb{R}^+ \quad (\text{i.e., } g \geq 0) \quad (2.2)$$

There is some θ , (ignition temepature) $0 < \theta < 1$, for which

$$g \equiv 0 \text{ on } [0, \theta) \text{ and } g > 0 \text{ on } (\theta, 1). \text{ Moreover } g(1) = 0. \quad (2.3a)$$

$$g \text{ is Lipschitz continuous on } [\theta, 1]. \quad (2.3b)$$

The nonlinear coefficient k will satisfy:

$$k : [0, 1] \rightarrow \mathbb{R} \text{ is a } C^1 \text{ function and } k(s) \geq \alpha > 0, \quad 0 \leq s \leq 1. \quad (2.4)$$

The smoothness assumption on k is the minimal one in order to work with classical solutions. Indeed, u will be of class C^2 on $\mathbb{R}$, excepting the points where $u = 0$. We are now ready to state the

Theorem 2.1. Assuming the condition (2.2) to (2.4), there exists $u: \mathbb{R} \rightarrow [0, 1]$ and $c > 0$ solution of (2.1). The function u is of class C^1 and strictly monotone increasing on $\mathbb{R}$, of class C^2 on $\mathbb{R} - \{x_0\}$ with $u(x_0) = 0$. Furthermore, u and c are unique (up to a translation of the origin).

Remark 2.2. It is also possible to precise the (exponential) rate of decay of $u(x)$ as x goes to infinity. 11

The existence part of this theorem will not be proved here (see [2]).

We just notice that a shooting argument is possible due to the following:

Lemma 2.3. Suppose (u, c) is a solution of (2.1). Then, $c > 0$, $0 < u < 1$, $u' > 0$ and $\lim_{x \rightarrow +\infty} u'(x) = 0$.

Indeed, to solve (2.1) is equivalent to solving a problem on $\mathbb{R}^+$:

$$\begin{cases} -(k(u)u')' + cu' = g(u) \text{ on } \mathbb{R}^+ \\ u(0) = 0 \quad u'(0) = c\theta \, k(\theta)^{-1} \\ u(+\infty) = 1 \end{cases} \quad (2.5)$$

2.2. PROOF OF UNIQUENESS.

It suffices to prove the uniqueness for the problem (2.5), equivalent to (2.1) (Lemma 2.3). We proceed by contradiction, introducing a "hodograph variable." Precisely, u (solution of (2.5)) being strictly increasing, we define a function $x:[0,1) \rightarrow \mathbb{R}^+$ by

$$x(s) = u^{-1}(s) \quad (2.6)$$

Clearly, this function is continuous, strictly increasing and $x(0) = 0$, $x(1) = +\infty$. Then, set

$$z(s) = \frac{d}{ds} x(s) \quad .$$

A simple calculation shows that the problem (2.5) becomes (now $\frac{d}{ds} = ')$:

$$\begin{cases} -\left(\frac{k(s)}{z(s)}\right)' + c = g(s) \, z(s) \quad , \quad 0 < s < 1 \\ z(0) = \frac{k(0)}{c\theta} \quad z(1) = +\infty \end{cases} \quad (2.7)$$

noting that $z(s) > 0 \quad 0 < s < 1$.

So, suppose (u_1, c_1) and (u_2, c_2) are two solutions of (2.5) with $c_1 < c_2$. The corresponding functions necessarily satisfy $z_1 > z_2$ on $[0, 1]$. If not, for some σ , $0 < \sigma < 1$, we have:

$$z_1(s) > z_2(s) \quad , \quad \forall s \in [0, \sigma) \text{ and } z_1(\sigma) = z_2(\sigma) \quad . \quad (2.8)$$

Hence, using the equation (2.7), we obtain:

$$-\left(\frac{k(\sigma)}{z_1(\sigma)} - \frac{k(\sigma)}{z_2(\sigma)}\right) = c_2 - c_1 > 0 \quad .$$

Thus letting $w(s) = \frac{k(s)}{z_1(s)} - \frac{k(s)}{z_2(s)}$, we get $w'(\sigma) < 0$ which contradicts

$w(s) < 0$, $\forall s \in [0, \sigma)$ and $w(\sigma) = 0$ consequence of (2.8). Now the integration of (2.7) over $[0, 1]$ for z_1 and z_2 yields:

$$c_1 - c_2 = \int_0^1 g(s)(z_1(s) - z_2(s)) \, ds$$

But $c_1 < c_2$, $z_1 > z_2$ on $[0, 1]$, and $g(s) > 0$ on $(0, 1)$ by (2.3). Thus (2.9) is impossible and $c_1 < c_2$; $u_1 = u_2$ follows from the uniqueness of the initial value problem associated to (2.5) ($0 < u(x) < 1$ on $\mathbb{R}^+$).

The proof is complete.

2.3. ASYMPTOTIC ANALYSIS

Now we assume that g depends on a parameter $\varepsilon > 0$ (essentially the inverse of the activation energy) and we let $g = g_\varepsilon$. The

corresponding solutions of (2.1) will be denoted by $(u_\varepsilon, c_\varepsilon)$. We also set:

$$K_\varepsilon(u) = \int_0^u k(s) g_\varepsilon(s) ds \quad (2.10)$$

The function g_ε depends on ε in the following way:

For each $\varepsilon > 0$, g_ε verifies conditions (2.2) - (2.4) with a fixed θ , $0 < \theta < 1$. (2.11)

There exists θ_ε , $\theta \leq \theta_\varepsilon \leq 1$ such that $\theta_\varepsilon \nearrow 1$ when $\varepsilon \searrow 0$ and (2.12)

$$\lim_{\varepsilon \searrow 0} \max_{0 < s \leq \theta_\varepsilon} g_\varepsilon(s) = 0$$

$$\lim_{\varepsilon \searrow 0} K_\varepsilon(1) = m > 0 \text{ and } m < +\infty \quad (2.13)$$

These hypotheses include, in particular, the Arrhenius law described in the first section. We state:

Theorem 2.4. Assuming the conditions (2.11)-(2.13), the unique solution $(u_\varepsilon, c_\varepsilon)$ to (2.1) such that $u_\varepsilon(0) = 0$, when $\varepsilon \searrow 0$, converges strongly in $H^1(\mathbb{R}) \times \mathbb{R}$ and $C^0(\mathbb{R}) \times \mathbb{R}$ to (u_0, c_0) . Moreover, $c_0 = \sqrt{2m}$ and u_0 is the unique (continuous) solution of the problem:

$$-(k(u)u')' + c_0 u' = c_0 \delta_{x=0} \quad (2.14)$$

$$u(-\infty) = 0 \quad u(+\infty) = 1 \quad u(0) = 0,$$

where $\delta_{\bar{x}=\bar{x}}$ is the Dirac measure at the point $\bar{x}$ and $\bar{x}$ is uniquely determined by the condition $u(o) = 0$.

Remark 2.5. The theorem and its proof give a rigorous justification for the heuristic "internal layer analysis" by which this type of asymptotic limit was usually obtained. We now sketch the proof of the theorem which consists of five parts.

Proof:

1) A priori estimates.

Multiplying the equation in (2.1) successively by 1, u_L , $k(u_L)u_L'$ and integrating from 0 to x we obtain:

$$-k(u_L(x)) u_L'(x) + c_L u_L(x) = \int_0^x g_L(u_L) dx \quad (2.15)$$

$$-k(u_L(x)) u_L(x) u_L'(x) + \frac{1}{2} c_L (u_L(x)^2 + 0^2) + \int_0^x k(u_L) u_L'^2 dx =$$

$$\int_0^x g_L(u_L) u_L dx \quad (2.16)$$

$$-\frac{1}{2} k(u_L(x))^2 u_L'(x)^2 + \frac{1}{2} c_L^2 0^2 + c_L \int_0^x k(u_L) u_L'^2 dx = K_L(u_L(x)) \quad (2.17)$$

Moreover, since $0 < u_\varepsilon < 1$, $u_\varepsilon(+\infty) = 1$ and $g_\varepsilon(u_\varepsilon) \geq 0$, letting $x \rightarrow +\infty$ in these identities, we deduce easily upper and lower bounds for c_ε , independently of ε (hypothesis 2.13):

$$\sqrt{2K_\varepsilon(1)} \leq c_\varepsilon \leq \frac{1}{\theta} \sqrt{2K_\varepsilon(1)} . \quad (2.18)$$

2) Let us define $x_\varepsilon > 0$ by $u_\varepsilon(x_\varepsilon) = \theta_\varepsilon$, for each $\varepsilon > 0$. We claim that:

$$x_\varepsilon < \frac{1-\theta}{\alpha} \equiv x_0 , \quad \alpha > 0 . \quad (2.19)$$

Since

$$\int_0^{x_\varepsilon} u'_\varepsilon \, dx = u_\varepsilon(x_\varepsilon) - u_\varepsilon(0) \leq 1 - \theta ,$$

it is sufficient to prove $u'_\varepsilon(x) \geq \alpha > 0$ on $[0, x_\varepsilon]$. From (2.17), since K_ε is nondecreasing, we get, if $\beta = \max_{0 \leq s \leq 1} k(s)$:

$$-\frac{\beta^2}{2} u'_\varepsilon(x)^2 + \frac{1}{2} c_\varepsilon^2 \theta^2 \leq K_\varepsilon(\theta_\varepsilon) , \quad 0 < x < x_\varepsilon .$$

So, with (2.18), we obtain

$$-\frac{\beta^2}{2} u'_\varepsilon(x)^2 \geq K_\varepsilon(1) \theta^2 - K_\varepsilon(\theta_\varepsilon)$$

and the lower bound follows, for ε small enough, from (2.12) and (2.13).

$$3) \quad \lim_{\varepsilon \downarrow 0} c_\varepsilon = \sqrt{2m} = c_0.$$

From (2.12) and (2.19) we get, as $\varepsilon \rightarrow 0$:

$$0 < \int_0^{x_\varepsilon} g_\varepsilon(u_\varepsilon) dx \leq \frac{1-\theta}{\alpha} \max_{\theta \leq s \leq \theta_\varepsilon} g_\varepsilon(s) \rightarrow 0.$$

Then (2.15), written at $x = x_\varepsilon$, implies:

$$\lim_{\varepsilon \downarrow 0} \{k(\theta_\varepsilon) u'_\varepsilon(x_\varepsilon) - c_\varepsilon \theta_\varepsilon\} = 0 \quad (2.20)$$

But, multiplying (2.1) by $k(u_\varepsilon)u'_\varepsilon$ then integrating from x_ε to ∞ , shows that:

$$\frac{1}{2}k(\theta_\varepsilon)^2 u'_\varepsilon(x_\varepsilon)^2 < K_\varepsilon(1) - K_\varepsilon(\theta_\varepsilon). \quad (2.21)$$

Combining (2.20) and (2.21) yields

$$\lim_{\varepsilon \downarrow 0} c_\varepsilon = \sqrt{2m}$$

and (2.18) yields

$$\lim_{\varepsilon \downarrow 0} c_\varepsilon = \sqrt{2m}.$$

Therefore,

$$\lim_{\varepsilon \downarrow 0} c_\varepsilon = \sqrt{2m} = c_0.$$

4) On $\mathbb{R}^-$, due to (2.3), the convergence of u_ε to u_0 solution of the problem:

$$-(k(u_0) u_0')' + c_0 u_0' = 0 \quad \text{on } \mathbb{R}^-$$

$$u_0(0) = \theta \quad u_0'(0) = \frac{c_0 \theta}{k(\theta)}$$

is straightforward. Moreover, on $[x_\varepsilon, +\infty)$, $\theta_\varepsilon \leq u_\varepsilon(x) < 1$; so (2.19) shows that $u_\varepsilon \rightarrow u_0$ in the C^0 norm on $[x_0, +\infty)$, where $u_0(x) \equiv 1$ $\forall x \geq x_0$.

5) It remains to study the limit of u_ε on the bounded interval $[0, x_0]$. Letting $x \rightarrow +\infty$ in (2.17) shows that $\int_0^{+\infty} u_\varepsilon'(x)^2 dx$ and thus the $H^1((0, x_0))$ - norm of u_ε remains bounded independently of ε . Using (2.19) and the compact embedding $H^1((0, x_0)) \hookrightarrow C^0([0, x_0])$ there exists a subsequence $\varepsilon_n \rightarrow 0$ such that

$$\begin{cases} x_{\varepsilon_n} \rightarrow \bar{x} < x_0 \\ u_{\varepsilon_n} \rightarrow u_0 \text{ in } C^0([0, x_0]) \end{cases} \quad (2.22)$$

Since $\theta_\varepsilon \neq 1$, we know that $u_0(x) \equiv 1$, $\forall x > \bar{x}$. But, by (2.12), $\kappa_{\varepsilon_n}(u_{\varepsilon_n}(x))$ converges uniformly to 0 on any interval $[0, \bar{x} - \delta]$ with $\delta > 0$. Therefore, the equation shows that u_0 is of class C^2 on

$\mathbb{R} - (\{\bar{x}\})$ and verifies:

$$\begin{cases} -(k(u_0)u_0')' + c_0 u_0' = 0 \text{ on } (-\infty, \bar{x}] \\ u_0(0) = 0 \quad u'(0) = \frac{c_0 \theta}{\bar{\kappa}(\theta)} \end{cases} \quad (2.23)$$

One checks easily, that $\bar{x}$ is uniquely determined by $u_0(0) = 0$. Thus u_0 is also unique and (2.22) holds for any subsequence. The $H^1(\mathbb{R})$ convergence is straightforward and (2.23), together with $u_0(x) \equiv 1$ for $x > \bar{x}$, implies (2.14). The proof is complete. $\square$

§3. THE CASE OF THE SYSTEM OF EQUATIONS

3.1. EXISTENCE RESULT ON A BOUNDED DOMAIN.

The goal of this section is to study the problem:

Find two functions $u, v: (-a, +a) \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ satisfying.

$$\begin{cases} -u'' + cu' = f(u)v & \text{on } (-a, +a) \\ -\Lambda u'' + cu' = -f(u)v & \text{on } (-a, +a) \\ -u'(-a) + cu(-a) = 0 & u(a) = +1 \\ -\Lambda v'(-a) + cv(-a) = c & v(a) = 0 \\ u(0) = 0 \end{cases} \quad (3.1)$$

where $0 < \theta < 1$, $\Lambda > 0$ are given and $a > 0$. Letting $a \rightarrow +\infty$ will yield the needed existence result on $\mathbb{R}$ (see § 3.2); the problem is also useful for some numerical calculations. The assumptions on f are

similar to those on g (see § 2). Precisely:

$$f: [0,1] \rightarrow \mathbb{R}^+ \quad (3.2)$$

$$\text{There is some } \theta, 0 < \theta < 1, \text{ for which } f \equiv 0 \text{ on } [0, \theta) \text{ and } f > 0 \text{ on } (\theta, 1] \quad (3.3)$$

$$f \text{ is Lipschitz continuous on } [\theta, 1] \quad (3.4)$$

$$G(1) \equiv \int_{\theta}^1 f(s)(1-s) \, ds < +\infty \quad (3.5)$$

Theorem 3.1. Assuming conditions (3.2) to (3.5), there exists $u, v : (-a, +a) \rightarrow (0, 1)$ and $c > 0$ solution of (3.1). The function u (resp. v) is of class C^1 and strictly monotone increasing (resp. decreasing).

We will not give the complete proof of this theorem (see [2]), but just show how to reduce the problem (3.1) to a fixed point equation then solvable using Leray-Schauder degree. Let us emphasize again that c is part of the unknowns in (3.1). The setting for the fixed point equation is as follows. Let us set $I_a = (-a, +a)$ and define $X = C^1(\bar{I}_a)^2 \times \mathbb{R}$, equipped with the natural norm. For (u, v, c) given in X , and $\tau \in [0, 1]$, let (U, V) be the solution of the problem:

$$\begin{cases} -U'' + cU' = \tau f(u)v & \text{on } I_a \\ -\lambda V'' + cV' = -\tau f(u)v & \text{on } I_a \\ -U'(-a) + cU(-a) = 0 & U(a) = +1 \\ -\lambda V'(-a) + cV(-a) = c & V(a) = 0 \end{cases} \quad (3.6)$$

One easily checks that U, V are in $C^1(\bar{I}_a)$ and that the application:

$$K_\tau : X \rightarrow X : (u, v, c) \rightarrow (U, V, c - u(0) + \theta) \quad (3.7)$$

is compact for any $\tau \in [0, 1]$. Then, setting $F_\tau = I - K_\tau$, to solve the problem (3.1) is equivalent to finding a fixed point of K_τ or a zero of F_τ . The main step in the proof of the theorem 3.1 is the:

Proposition 3.2: Assuming (3.2) to (3.5), there exists positive and finite constants $R, \underline{c}, \bar{c}$ such that:

$$F_\tau(\partial\Omega) = (I - K_\tau)(\partial\Omega) \neq 0 \quad 0 \leq \tau \leq 1$$

where K_τ is defined by (3.7) and

$$\Omega = \{(u, v, c) \in X \mid \|u\|_{C^1(\bar{I}_a)} < R, \|v\|_{C^1(\bar{I}_a)} < R, \underline{c} < c < \bar{c}\}.$$

Moreover $\deg(F_\tau, \Omega, 0) = -1$, for $0 \leq \tau \leq 1$.

Remark 3.3. The solutions u, v of (3.1) are in fact in $W^{2,\infty}(I_a)$. $\square$

Remark 3.4. Moreover, we show the existence of $R, \underline{c}, \bar{c}$ independent of a , which will allow to pass to the limit $a = +\infty$ (see § 3.2). $\square$

3.2. EXISTENCE RESULT ON $\mathbb{R}$.

We now consider the problem:

Find two functions $u, v: \mathbb{R} \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ satisfying $(0 < \theta < 1$ and

$\lambda > 0$ are given):

$$\left\{ \begin{array}{ll} -u'' + cu' = f(u)v & \text{on } \mathbb{R} \\ -\lambda u'' + cv' = -f(u)v & \text{on } \mathbb{R} \\ u(-\infty) = 0 \quad u(+\infty) = 1 \\ v(-\infty) = 1 \quad v(+\infty) = 0 \\ u(0) = 0 \end{array} \right. \quad (3.8)$$

Theorem 3.5. Assuming the conditions (3.2) to (3.5), there exist $u, v: \mathbb{R} \rightarrow (0, 1)$ and $c > 0$ solutions of (3.3). Moreover the functions u, v are in $W^{2, \infty}(\mathbb{R})$ and u (resp. v) is strictly monotone increasing (resp. decreasing).

Remark 3.6. We conjecture the uniqueness of the solution of (3.8). $\square$

A solution (u, v, c) of (3.8) is obtained as a limit, as $a \rightarrow +\infty$, of a solution of (3.1). This supposes adequate a priori estimates independent of a , for any solution, noted (u_a, v_a, c_a) of (3.1).

We refer to [2] for details. Here we just sketch the proof of the crucial step, that is the bounds on c_a .

Proposition 3.7. Let $a_0 > 0$ fixed. Then, if (u_a, v_a, c_a) is any solution, c_a satisfies :

$$\left\{ \frac{2G(1)}{|\lambda-1|+1} \right\}^{\frac{1}{2}} \leq c_a \leq \text{Max} \left\{ \text{Max} \left\{ \sqrt{\frac{2M}{0}}, 2M \right\}, -\frac{\text{Log } \theta}{a_0} \right\}, \quad (3.9)$$

for any $a \geq a_0$. Here $M \equiv \sup_{0 \leq s \leq 1} f(s)$.

Proof: Adding the two equations in (3.1) one easily shows the pointwise inequalities $|u + v - 1| \leq |\Lambda - 1|v$ and $|u + v - 1| \leq \frac{|\Lambda - 1|}{\Lambda} (1 - u)$ from which one deduces $\frac{1}{|\Lambda - 1| + 1} (1 - u) \leq v$. Now, from the equation $-u'' + cu' = f(u)v$, multiplied successively by 1, u , u' and integrated from 0 to $+a$, one gets (using $f(u)u \geq 0$ and $u \leq 1$) after some calculations:

$$\frac{c_a^2}{2} \geq \int_0^a f(u)vu' \, dx.$$

The lower bound on c_a follows then from the lower bound on v since $f(u)u' \geq 0$. For the upper bound, one introduces the unique solution $\bar{u}$ of the problem:

$$\begin{cases} -\bar{u}'' + c\bar{u}' = H & \text{on } (-a, +a) \\ -\bar{u}'(-a) + c\bar{u}(-a) = 0 & \bar{u}(a) = 1 \end{cases}$$

where H is the Heaviside function at $x = 0$. It is easy to show $u_a(x) \leq \bar{u}(x)$ on $(-a, +a)$. In particular $\theta = u_a(0) \leq \bar{u}(0)$. The upper bound follows then from an explicit calculation of $\bar{u}(0)$. $\square$

3.3 ASYMPTOTIC ANALYSIS.

We are now looking for an analysis similar to the one of Section 2.3. So we assume f depending on a parameter $\varepsilon > 0$ and set $f = f_\varepsilon$. The dependency on ε is as follows:

For each $\varepsilon > 0$, f_ε verifies conditions (3.2) -- (3.4)

with a fixed θ , $0 < \theta < 1$.

(3.10)

$\exists \theta_\varepsilon$, $\theta \leq \theta_\varepsilon \leq 1$ such that $\theta_\varepsilon \uparrow 1$ when $\varepsilon \downarrow 0$ and

(3.11)

$$\lim_{\varepsilon \downarrow 0} \max_{\theta \leq s \leq \theta_\varepsilon} f_\varepsilon(s)(1-s) = 0$$

$$\lim_{\varepsilon \downarrow 0} \int_{\theta}^1 f_\varepsilon(s)(1-s) ds \equiv \lim_{\varepsilon \downarrow 0} G_\varepsilon(1) \equiv m < +\infty. \quad (3.12)$$

Remark 3.8. The function $s \mapsto f_\varepsilon(s)(1-s)$ will play the role of g_ε in Section 2.3. (1)

In what follows, for each $\varepsilon > 0$, $(u_\varepsilon, v_\varepsilon, c_\varepsilon)$ will be a solution of (3.8), where $f = f_\varepsilon$.

Theorem 3.9. Assuming the conditions (3.10) to (3.12), there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ such that $(u_{\varepsilon_n}, v_{\varepsilon_n}, c_{\varepsilon_n})$ converge strongly in $H^1(\mathbb{R})^2 \times \mathbb{R}$ and $C^0(\mathbb{R})^2 \times \mathbb{R}$ to (u_0, v_0, c_0) . Moreover, u_0, v_0 are solutions of the problem

$$\begin{cases} -u'' + c_0 u' = c_0 \delta_{x=\bar{x}} \\ -\Delta v'' + c_0 v' = -c_0 \delta_{x=\bar{x}} \\ u(-\infty) = 0 \quad u(+\infty) = 1 \\ v(-\infty) = 1 \quad v(+\infty) = 0 \end{cases}$$

where $\bar{x}$ is uniquely determined by the condition $u(0) = 0$. The proof generalizes the proof of the Theorem 2.4, and we refer to [2].

Conclusion: We have just given here a brief account of the methods used in the work [2]. These methods are fairly flexible and apply to more complex systems than (3.8) also of interest in the study of deflagration waves. For example, one can introduce nonlinear diffusion coefficients in (3.8), as for problem (2.1). One can also consider reaction terms in (3.8) of the form $f(u)v^n$ (this corresponds to a simple n^{th} order reaction). Finally, one can also study reactive flow associated to a reaction $A + B \rightarrow P$. The system of interest is now:

$$-(D(U)U')' + c U' = F(U) K, \quad ,$$

on $\mathbb{R}$ where $U: \mathbb{R} \rightarrow \mathbb{R}^3$, $F: \mathbb{R}^3 \rightarrow \mathbb{R}$ and K is a given vector of $\mathbb{R}^3$. These problems are worked out in [2]. More general systems are studied in [3].

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