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AUTHOR(S): George H. Pimbley

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ON A NONLINEAR SEMIGROUP THEORY FOR SYSTEMS

George H. Pimbley
Los Alamos National Laboratory
Los Alamos, New Mexico 87545

I. INTRODUCTION

Consider the abstract Cauchy problem in a B-space X :

$$\frac{dw}{dt} + Aw \ni 0, \quad w(0) = w_0, \quad (1)$$

where $A: D(A) \rightarrow X$, $w_0, w \in \overline{D(A)}$. By the Generation Theorem of Crandall and Liggett, [Refs. 3, p. 110; 4, pp. 266, 267, 276], if A is ω -accretive:

$$\|(v + \lambda[A + \omega I]v) - (y + \lambda[A + \omega I]y)\| \geq \|v - y\|, \quad \forall \lambda > 0, \quad \forall v, y \in D(A) \quad (2)$$

and if a "range condition":

$$R(I + \lambda A) \supseteq \overline{D(A)}, \quad \forall \lambda > 0, \quad \lambda\omega < 1, \quad (3)$$

is satisfied, then $\forall w_0 \in X$, the finite difference problem

$$\begin{aligned} \frac{w_\delta(t) - w_\delta(t - \delta)}{\delta} + Aw_\delta(t) &\ni 0, \quad t \geq 0 \\ w_\delta(t) &= w_0, \quad -\delta \leq t < 0 \end{aligned}$$

has a unique solution on $[0, \infty]$. If $w_0 \in \overline{D(A)}$, then $S(t)w_0 = \lim_{\delta \rightarrow 0} w_\delta(t)$ exists uniformly on bounded sets. $S(t)w_0 = \lim_{n \rightarrow \infty} (I + \frac{t}{n}A)^{-n}w_0$, $t \geq 0$, $w_0 \in \overline{D(A)}$. $S(t): \overline{D(A)} \rightarrow \overline{D(A)}$; $S(t)S(\tau) = S(t + \tau)$, $t, \tau \geq 0$; $S(0) = I$; $S(t)v$ is continuous in (t, v) , $t \geq 0$, $v \in \overline{D(A)}$; $\|S(t)v - S(t)w\| \leq e^{\omega t} \|v - w\|$, $t \geq 0$, $v, w \in D(A)$.

H. Brezis and A. Pazy have related methods applicable in Hilbert spaces [Refs. 1, 2, and 9].

These methods have power for nonlinear Cauchy problems, being particularly applicable when these problems involve only a single partial differential equation in a single unknown. The main problem is that of choosing a B-space in which the hypotheses can be satisfied. H. Brezis gives many examples [Ref. 2]. Again, B. Keyfitz noted phenomenologically that the solution of a single hydrodynamic conservation law is a semigroup in $L^1(R)$ [Ref. 7], and M. Crandall applied the above Crandall-Liggett theorem to generate this $L^1(R)$ semigroup in a general formulation [Ref. 3]. People have had trouble extending semigroup generation theory to Cauchy problems involving systems of nonlinear equations in several unknowns however, including systems of hydrodynamic conservation laws. It is curious why such a barrier should exist, because the older linear semigroup theory extends readily to systems, even hyperbolic systems [Ref. 6, Chap. XX, pp. 387-400].

In a conversation, G. Webb of Vanderbilt U. related some success with systems found in population modeling. It will be interesting to review his material.

Herein, the author attempts to extend the application of the Crandall-Liggett theorem to two conservation laws with linear coupling. In a conversation, J. Glimm of Rockefeller U. stated that this had not yet been done. Keeping things simple, we study Eq. (1) in $L^1(R) \times L^1(R)$, where A is given as the 2-vector:

$$A(\rho, u) = (\phi(\rho)_x + \sigma u_x, \sigma \rho_x + \psi(u)_x) \quad , \quad x \in R \quad , \quad \sigma > 0 \quad , \quad (4)$$

where $(\rho, u) \in D(A) \subset L^1(R) \times L^1(R)$, and ϕ, ψ are monotone increasing, with increasing but bounded derivatives, which vanish when $\rho = 0$ or $u = 0$. We

seek the solution $w = (\rho, u) \in \overline{D(A)}$ of (1), where $D(A)$ is to be determined.

The scalar parameter σ appearing in operator A [see (4)] is a measure of the amount of linear coupling. The case $\sigma = 0$ can be regarded as already treated by Crandall (cf [Ref. 3]), though herein we restrict to one space variable. $\sigma = 0$ gives the uncoupled case.

II. ACCRETIVE PROPERTIES OF A

Preparatory to the introduction of a set on which A , defined in (4), can have an accretiveness property, let us define the following set (which is not a linear set):

$$V_M^{(1)} = \{\rho(x) \in L^1(R) \mid \rho \text{ absolutely continuous, } TV\{\rho\} \leq M \|\rho\|_1\},$$

where $TV\{\rho\}$ is the total variation of $\rho \in L^1(R)$, and $\|\rho\|_1 = \int_R |\rho| dx$ is the norm of ρ in $L^1(R)$. We write $V_M = V_M^{(1)} \times V_M^{(1)}$ for the Cartesian product of $V_M^{(1)}$ with itself.

One can easily find functions in $V_M^{(1)}$ with sum or difference not in $V_M^{(1)}$. By inspection, however, it is seen that $V_M^{(1)}$ is invariant to multiplication by a scalar. Hence, $V_M^{(1)}$ contains linear subspaces. By Zorn's lemma, $V_M^{(1)}$ contains in fact a maximal linear subspace, which is infinite dimensional because it contains the following infinite linearly independent sequence of functions.

$$\phi_n^{(\alpha)}(x) = \begin{cases} 0 & , \quad |x| \leq \alpha n \\ (x - \alpha n)(\alpha n + \alpha - x) & , \quad \alpha n < |x| < \alpha(n+1) \\ 0 & , \quad \alpha(n+1) \leq |x| \quad , \quad \alpha > 0 \end{cases} \quad (5)$$

provided $M \geq 3/\alpha$. Let us denote this maximal linear subspace of $V_M^{(1)}$ by $L_M^{(1)}$. Again, $L_M = L_M^{(1)} \times L_M^{(1)}$.

The non-negative elements of $V_M^{(1)}$, which do not belong to $L_M^{(1)}$, belong to a cone. Suppose $\rho_1, \rho_2 \in V_M^{(1)}$ and $\rho_1(x), \rho_2(x) \geq 0$. Then,

$$\begin{aligned} TV\{\rho_1(x) + \rho_2(x)\} &\leq TV\{\rho_1(x)\} + TV\{\rho_2(x)\} \\ &\leq M\|\rho_1(x)\|_1 + M\|\rho_2(x)\|_1 \\ &= M\left[\int_R \rho_1(x) dx + \int_R \rho_2(x) dx\right] = M\|\rho_1(x) + \rho_2(x)\|_1. \end{aligned}$$

We denote this positive cone, and its negative reflection, by $\pm K_M^{(1)}$.

Whether or not there are any elements of $V_M^{(1)}$ that are not in $\pm K_M^{(1)} \cup L_M^{(1)}$ is problematic. The author has not found any. The question will be seen to be unimportant. Provisionally we set $V_M^{(1)} = \pm K_M^{(1)} \cup L_M^{(1)}$. $L_M^{(1)}$ intersects $K_M^{(1)}$ since both contain sequence (5).

The intersection of $K_M^{(1)}$ with the unit sphere $\mathcal{S}^{(1)}$ of $L^1(R)$ is convex. Moreover, $-K_M^{(1)} \cap \mathcal{S}^{(1)}$ is a similar convex set. These two convex sets are symmetric with respect to the origin, and the elements ρ within each of these sets have $TV\{\rho\} \leq M$. Consider the cylinder $C_M^{(1)}$ filled out by all possible lines $\theta x + (1 - \theta)y$, $0 \leq \theta \leq 1$, $x \in K_M^{(1)} \cap \mathcal{S}^{(1)}$ and $y \in -K_M^{(1)} \cap \mathcal{S}^{(1)}$. Within this cylinder we have $TV\{\rho\} \leq M$ for every element ρ since $TV\{x\} \leq M$, $TV\{y\} \leq M$ in the above construction. Here we use the subadditive property of the total variation.

Clearly, there exists a number $\hat{M} > M$ such that $C_{\hat{M}}^{(1)} \supseteq \mathcal{S}^{(1)} \cap S_M^{(1)}$, where, of course, $\mathcal{S}^{(1)}$ was the unit sphere of $L^1(R)$, and $S_M^{(1)}$ is the linear hull of $V_M^{(1)}$. This is because increasing M causes $V_M^{(1)}$ to expand, which in turn expands the convex set $C_M^{(1)}$. The resulting $C_{\hat{M}}^{(1)}$ consists of elements ρ for which $TV\{\rho\} \leq \hat{M}$, and $C_{\hat{M}}^{(1)}$ encloses $\mathcal{S}^{(1)} \cap S_M^{(1)}$. Thus $S_M^{(1)}$, the linear hull of $V_M^{(1)}$, has the property that if $\rho \in S_M^{(1)}$, then $TV\{\rho\} \leq \hat{M}\|\rho\|_1$. In fact

$S_M^{(1)} \subseteq L_M^{(1)}$, which is the maximal linear subspace of $V_M^{(1)}$. By choosing the minimal M , we can have $S_M^{(1)} = L_M^{(1)}$, and the codimension of $V_M^{(1)}$ with respect to $L_M^{(1)}$ is unity.

We infer that the positive cone $K_M^{(1)}$ of $L_M^{(1)}$ is of codimension unity with respect to $L_M^{(1)}$, and that given any positive $M > 0$, there exist positive numbers $M < M' < M'' < \dots$, such that $V_M^{(1)} \subset V_{M'}^{(1)} \subset V_{M''}^{(1)} \subset \dots$, etc., where $V_{M'}^{(1)} = \pm K_{M'}^{(1)} \cup L_{M'}^{(1)}$, etc., and where $K_M^{(1)}$ is the positive cone of $L_M^{(1)}$, $K_{M'}^{(1)}$ is the positive cone of $L_{M'}^{(1)}$, etc.

Using the above considerations as background, we can prove the following result.

Theorem 1: Let us restrict the operator A defined in (4) to the set $L_M = L_M^{(1)} \times L_M^{(1)}$. Thus $D(A) = L_M$. Then A is ω -accretive, with $\omega = \sigma M$.

Proof: Clearly A is defined on L_M since the vectors of L_M have absolutely continuous components in $L^1(R)$, and the image set is in $L^1(R) \times L^1(R)$. We take as the norm of $L^1(R) \times L^1(R)$ the function $\|\rho\|_1 + \|u\|_1$, where $\rho, u \in L^1(R)$. Using this norm, and any two vectors $(\rho, u), (\hat{\rho}, \hat{u}) \in L^1(R) \times L^1(R)$, we start out with the left side of inequality (2) as follows:

$$\begin{aligned} & \int_R |\rho - \hat{\rho} + \lambda[(\phi(\rho) - \phi(\hat{\rho}))_x + \sigma(u - \hat{u})_x] + \lambda M \sigma(\rho - \hat{\rho})| dx \\ & + \int_R |u - \hat{u} + \lambda[\sigma(\rho - \hat{\rho})_x + (\psi(u) - \psi(\hat{u}))_x] + \lambda M \sigma(u - \hat{u})| dx \\ & \geq \left| \int_R \text{sign}(\rho - \hat{\rho}) \{(\rho - \hat{\rho}) + \lambda[(\phi(\rho) - \phi(\hat{\rho}))_x + \sigma(u - \hat{u})_x] + \lambda M \sigma(\rho - \hat{\rho})\} dx \right| \\ & + \left| \int_R \text{sign}(u - \hat{u}) \{(u - \hat{u}) + \lambda[\sigma(\rho - \hat{\rho})_x + (\psi(u) - \psi(\hat{u}))_x] + \lambda M \sigma(u - \hat{u})\} dx \right| \\ & \geq \int_R |\rho - \hat{\rho}| dx + \lambda \int_R \text{sign}(\rho - \hat{\rho}) [(\phi(\rho) - \phi(\hat{\rho}))_x + \sigma(u - \hat{u})_x] dx + \lambda \sigma M \int_R |\rho - \hat{\rho}| dx \\ & + \int_R |u - \hat{u}| dx + \lambda \int_R \text{sign}(u - \hat{u}) [\sigma(\rho - \hat{\rho})_x + (\psi(u) - \psi(\hat{u}))_x] dx \\ & + \lambda \sigma M \int_R |u - \hat{u}| dx. \end{aligned} \quad (6)$$

To continue with the demonstration of inequality (2), we need the following lower estimates of the middle terms:

$$\begin{aligned}
 \int_R \text{sign}(\rho - \hat{\rho}) [\phi(\rho(x))_x - \phi(\hat{\rho}(x))_x + \sigma(u - \hat{u})_x] dx &= \sigma \int_R \text{sign}(\rho - \hat{\rho}) (u(x) - \hat{u}(x))_x dx \\
 &\geq -\sigma \int_R |(u(x) - \hat{u}(x))_x| dx = -\sigma \text{TV}\{u(x) - \hat{u}(x)\} \\
 \int_R \text{sign}(u - \hat{u}) [\sigma(\rho - \hat{\rho})_x + (\psi(u(x))_x - \psi(\hat{u}(x))_x)] dx &= \sigma \int_R \text{sign}(u - \hat{u}) (\rho(x) - \hat{\rho}(x))_x dx \\
 &\geq -\sigma \int_R |(\rho(x) - \hat{\rho}(x))_x| dx = -\sigma \text{TV}\{\rho(x) - \hat{\rho}(x)\} \quad , \quad (7)
 \end{aligned}$$

where we have utilized the vanishing of the two integrals:

$$\int_R \text{sign}(\rho - \hat{\rho}) [\phi(\rho(x))_x - \phi(\hat{\rho}(x))_x] dx = \int_R \text{sign}(u - \hat{u}) [(\psi(u(x))_x - \psi(\hat{u}(x))_x)] dx = 0 \quad .$$

Now if we assume that $(\rho, u), (\hat{\rho}, \hat{u}) \in L_M = L_M^{(1)} \times L_M^{(1)}$, then we see from inequalities (6) and (7) that

$$\begin{aligned}
 &\int_R |\rho - \hat{\rho} + \lambda[(\phi(\rho) - \phi(\hat{\rho}))_x + \sigma(u - \hat{u})_x] + \lambda M \sigma(\rho - \hat{\rho})| dx \\
 &\quad + \int_R |u - \hat{u} + \lambda[\sigma(\rho - \hat{\rho})_x + (\psi(u) - \psi(\hat{u}))_x] + \lambda M \sigma(u - \hat{u})| dx \\
 &\geq \int_R |\rho - \hat{\rho}| - \lambda \sigma M \int_R |u - \hat{u}| dx + \lambda \sigma M \int_R |\rho - \hat{\rho}| dx + \int_R |u - \hat{u}| dx - \lambda \sigma M \int_R |\rho - \hat{\rho}| dx + \lambda \sigma M \int_R |u - \hat{u}| dx \\
 &= \int_R |\rho - \hat{\rho}| dx + \int_R |u - \hat{u}| dx \quad ,
 \end{aligned}$$

since $\text{TV}\{u(x) - \hat{u}(x)\} \leq M \|u - \hat{u}\|_1$ and $\text{TV}\{\rho(x) - \hat{\rho}(x)\} \leq M \|\rho - \hat{\rho}\|_1$. This proves the theorem.

III. THE RANGE CONDITION

The next requirement in the generation theory, namely relation (3), is expressed concretely as the following system:

$$\begin{aligned}
 \rho + \lambda(\phi(\rho)_x + \sigma u_x) &= h_1 \\
 u + \lambda(\sigma \rho_x + \psi(u)_x) &= h_2 \quad , \quad \lambda > 0 \quad . \quad (8)
 \end{aligned}$$

Equations (8) are to be solved for the vector $(\rho, u) \in L_M = L_M^{(1)} \times L_M^{(1)}$, assuming that $(h_1, h_2) \in L_M$, in satisfaction of this condition. $M > 0$ is a constant that has not been fixed thus far.

First we propose the easier task of solving Eqs. (8) for $(\rho, u) \in L^1(R) \times L^1(R)$, assuming that $(h_1, h_2) \in L^1(R) \times L^1(R)$.

Theorem 2: For every $\lambda > 0$, and every pair $h_1, h_2 \in L^1(R)$, Eqs. (8) possess a weak solution (ρ, u) with $\rho, u \in L^1(R)$, provided $0 \leq \sigma < \sigma_0$, where σ_0 is a singular value.

Proof: First we assume that $h_1, h_2 \in C_0^1(R)$, i.e., of compact support. In fact, let $h_1 \equiv h_2 \equiv 0$ for $x \leq a, x \geq b$. Suppose that Eqs. (8) have a piecewise smooth solution pair ρ, u . Then by defining $a_{11}(x) = \phi(\rho(x))/\rho(x)$ and $a_{22}(x) = \psi(u(x))/u(x)$, we can write Eqs. (8) in "linear disguise":

$$\begin{aligned} \rho + \lambda(a_{11}(x)\rho + \sigma u)_x &= h_1, \\ u + \lambda(\sigma\rho + a_{22}(x)u)_x &= h_2, \end{aligned} \quad (9)$$

where we can suppose that $0 < a_{11}(x) \leq M_1, 0 < a_{22}(x) \leq M_1, M_1$ some constant.

Equations (9) are placed in standard form by putting $v = a_{11}\rho + \sigma u, w = \sigma\rho + a_{22}u$:

$$\begin{pmatrix} v \\ w \end{pmatrix}_x + \frac{\lambda^{-1}}{\Delta(x)} \begin{pmatrix} a_{22}(x) & -\sigma \\ -\sigma & a_{11}(x) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad (10)$$

where $\Delta(x) = a_{11}(x)a_{22}(x) - \sigma^2 \geq \phi'(0)\psi'(0) - \sigma^2 > 0$ if $\sigma < \sigma_0$.

By using a triangularization result of Diliberto [Ref. 5, Theorem 1], there exists an orthogonal matrix $B_\lambda(x)$ with entries $b_{ij}(x) \in C^{(1)}(R)$, such that if we put $\begin{pmatrix} v \\ w \end{pmatrix} = B_\lambda(x) \begin{pmatrix} y \\ z \end{pmatrix}$ then y, z will satisfy the triangular differential equations

$$\begin{pmatrix} y \\ z \end{pmatrix}_x + \lambda^{-1} \begin{pmatrix} C_{11}(x) & C_{12}(x) \\ 0 & C_{22}(x) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \lambda^{-1} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \lambda^{-1} B_\lambda^{-1}(x) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad (11)$$

where $|C_{ij}(x)| \leq C_1$, and $C_{11}(x) \rightarrow \mu_1^{(0)}$ as $x \rightarrow \pm\infty$, where $\mu_i^{(0)}, i = 1, 2$ are the positive roots of $(\phi'(0) - \mu)(\psi'(0) - \mu) - \sigma^2 = 0, 0 \leq \sigma < \sigma_0$.

Equation (11) represents two coupled differential equations. Assuming that $y \equiv z \equiv 0$ for $x \leq a$, and therefore that $v \equiv w \equiv 0$ for $x \leq a$, it is an elementary exercise to solve successively Eq. (11) for z and y in $a \leq x \leq b$, and then to produce an *a priori* bound for the nonlinear system

$$\begin{pmatrix} v \\ w \end{pmatrix}_x + \lambda^{-1} \begin{pmatrix} \phi(v,w) \\ \psi(v,w) \end{pmatrix} = \lambda^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad (12)$$

valid for $-\infty < x \leq b$. System (12) is the standard form of Eqs. (8) obtained by putting $v = \phi(\rho) + \sigma u$, $w = \sigma \rho + \psi(u)$. The expressions $\rho = \phi(v,w)$, $u = \psi(v,w)$ exist because the Jacobian $\phi'(\rho)\psi'(u) - \sigma^2 > \phi'(0)\psi'(0) - \sigma^2 > 0$ for all values $-\infty < \rho < \infty$, $-\infty < u < \infty$.

Again, assuming $v \equiv w \equiv 0$, $x \leq a$, nonlinear Eqs. (12) can be integrated stepwise up to $x = b$ for any fixed $\lambda > 0$, using a local existence theorem. For small λ , the steps are small, but existence is had in principle for any $\lambda > 0$.

For $x > b$, Eqs. (12) are autonomous. Since we have assumed that ϕ, ψ have bounded increasing derivatives (so-called "superlinear" behavior, see p. 2), a standard-phase plane analysis shows that $v \equiv w \equiv 0$ is an attractive node for $0 \leq \sigma < \sigma_0$. Hence $v \rightarrow 0$, $w \rightarrow 0$ as $x \rightarrow \infty$. With this, we now have a bounded weak solution of Eqs. (12), valid for $-\infty < x < \infty$, and therefore we also have a solution of Eqs. (8). Using it, we can form the linear disguise problem, Eq. (10), again, and pass once more to Diliberto's triangularization, Eq. (11). Again we solve successively Eqs. (11), and recall that $C_{ij}(x) \rightarrow \mu_i^{(0)}$ as $x \rightarrow \infty$, where $\mu_i^{(0)}$, $i = 1, 2$, are positive for $0 \leq \sigma < \sigma_0$. It is now an elementary exercise to show that $y, z \in L^1(\mathbb{R})$ and therefore that $v, w \in L^1(\mathbb{R})$. Finally we achieve that $\rho, u \in L^1(\mathbb{R})$.

All of the above was based on the assumption that $(h_1, h_2) \in C_0^{(1)}(R) \times C_0^{(1)}(R)$. $C_0^{(1)}(R)$ is dense in $L^1(R)$ however, and $C_0^{(1)}(R) \times C_0^{(1)}(R)$ is dense in $L^1(R) \times L^1(R)$. Hence the theorem is proved for $(h_1, h_2) \in L^1(R) \times L^1(R)$, and for any fixed σ in the interval $[0, \sigma_0]$.

For $\sigma_0 < \sigma$, problem (8) has no solution in $L^1(R) \times L^1(R)$ unless $h_1 \equiv h_2 \equiv 0$. When $\sigma = \sigma_0$, $v \equiv w \equiv 0$ is yet an attractive node as $x \rightarrow \infty$, but v, w is not in $L^1(R)$. For $\sigma_0 < \sigma$, saddle point behavior sets in at $v \equiv w \equiv 0$, and no solution is expected for problem (8) unless $h_1 \equiv h_2 \equiv 0$.

The existence result of Theorem 2 is essential in proving the range condition. If we assume as required that $(h_1, h_2) \in L_M = L_M^{(1)} \times L_M^{(1)} \subset L^1(R) \times L^1(R)$ for some $M > 0$, then by Theorem 2, at least there exists a vector $(\rho, u) \in L^1(R) \times L^1(R)$ satisfying problem (8). The big question is whether we have $(\rho, u) \in L_M$ also. If the problem is linear, i.e., if $A(\rho, u) = (\phi'(0)\rho_x + \sigma u_x, \sigma \rho_x + \psi'(0)u_x)$, and if Eq. (12) is therefore written

$$\begin{pmatrix} v \\ w \end{pmatrix}_x + \frac{\lambda^{-1}}{\Delta} \begin{pmatrix} \psi'(0) & -\sigma \\ -\sigma & \phi'(0) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \quad \begin{matrix} h_1, h_2 \in L_M^{(1)}, \lambda > 0 \\ \Delta = \phi'(0)\psi'(0) - \sigma^2 > 0 \end{matrix} \quad (13)$$

the proof of the following result is not difficult.

Theorem 3: If the A in (4) is linear (with constant coefficients) and if Eq. (13) represents the standard form of the corresponding linearized Eqs. (8), then the assumption that $(h_1, h_2) \in L_M$ (and $M > 0$) implies that a solution (ρ, u) is in L_M also. Thus the range condition is satisfied with $D(A) = L_M$ in the linear case.

Proof: The symmetric matrix in (13) is diagonalizable by a constant orthogonal transformation T . There results the separated system:

$$\begin{aligned} y_x + \lambda^{-1} \mu_1^{(0)} y &= \lambda^{-1} k_1 \\ z_x + \lambda^{-1} \mu_2^{(0)} z &= \lambda^{-1} k_2 \end{aligned} \quad , \quad \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = T^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \quad (14)$$

L_M being a linear vector space, $(h_1, h_2) \in L_M$ implies that $(k_1, k_2) \in L_M = L_M^{(1)} \times L_M^{(1)}$. Since $0 \leq \sigma \leq \sigma_0$, $\mu_1^{(0)}$, $\mu_2^{(0)}$ are positive.

Let us assume temporarily that h_1, h_2 and k_1, k_2 are in $L_M^{(1)} \cap H^{1,1}(R)$. Then we may also consider the differentiated equations:

$$\begin{aligned} y_{xx} + \lambda^{-1} \mu_1^{(0)} y_x &= \lambda^{-1} k_{1x} \\ z_{xx} + \lambda^{-1} \mu_2^{(0)} z_x &= \lambda^{-1} k_{2x} \end{aligned} \quad (15)$$

One solves Eqs. (15) for y_x, z_x using the same process as with solving Eqs (14) for y, z . Accordingly, from (15)

$$\begin{aligned} \int_R |y_x| dx &\leq \lambda^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^x e^{-\lambda^{-1} \mu_1^{(0)} (x-\hat{x})} |k_{1x}(\hat{x})| d\hat{x} \\ &= \lambda^{-1} \int_{-\infty}^{\infty} |k_{1x}(\hat{x})| d\hat{x} \int_{\hat{x}}^{\infty} e^{-\lambda^{-1} \mu_1^{(0)} (x-\hat{x})} dx = \int_R |k_{1x}| d\hat{x} \end{aligned}$$

Similarly, $\int_R |z_x| dx \leq \int_R |k_{2x}| dx$. Hence $TV\{y\} \leq TV\{k_1\}$, $TV\{z\} \leq TV\{k_2\}$. The same process, used with Eqs. (14), shows that $\|y\|_1 \leq \|k_1\|_1$, $\|z\|_1 \leq \|k_2\|_1$. Thus we collect the following relationships from Eqs. (14) and (15):

$$\begin{aligned} \|y\|_1 &\leq \|k_1\|_1, \quad \|z\|_1 \leq \|k_2\|_1 \\ TV\{y\} &\leq TV\{k_1\}, \quad TV\{z\} \leq TV\{k_2\} \end{aligned} \quad (16)$$

We note that if k_1, k_2 are in the positive cone $K_M^{(1)}$ of $L_M^{(1)}$ (cf a remark in Section II of this paper) then $\|y\|_1 = \|k_1\|_1$, $\|z\|_1 = \|k_2\|_1$.

Now suppose k_1, k_2 are indeed in the positive cone $K_M^{(1)}$ of $L_M^{(1)}$. We then write, using (16):

$$\begin{aligned} TV\{y\} &\leq TV\{k_1\} \leq \check{M} \|k_1\|_1 = \check{M} \|y\|_1 \\ TV\{z\} &\leq TV\{k_2\} \leq \check{M} \|k_2\|_1 = \check{M} \|z\|_1 \end{aligned}$$

whence $(y,z) \in V_M = V_M^{(1)} \times V_M^{(1)}$. Thus if $(k_1, k_2) \in K_M$, then $(y,z) \in V_M$. By taking the linear hull L_M of K_M , we therefore see that, because of the linearity of all of the operations involved, if $(k_1, k_2) \in L_M$, then $(y,z) \in L_M$.

We now have the following chain of implications:

$$(h_1, h_2) \in L_M \Rightarrow (k_1, k_2) \in L_M \Rightarrow (y,z) \in L_M \Rightarrow (v,w) = T(y,z) \in L_M.$$

Finally, $\rho = \Delta^{-1}(\psi'(0)v - \sigma w)$ and $u = \Delta^{-1}(-\sigma v + \phi'(0)w)$. Then since $(v,w) \in L_M$, it is clear that $(\rho, u) \in L_M$ since (ρ, u) is just a linear combination, with constants, of elements of L_M . This proves Theorem 3 since $H^{1,1}(R) \cap L_M^{(1)}$ is dense in $L_M^{(1)}$.

In view of Theorems 1 and 3, the Crandall-Liggett generation theorem is thus applicable to linear systems, which is not surprising. The resulting semigroup $S(t)$ might have been produced in accord with Hille's now classical linear theory [Ref. 6, Chap. XII] as a uniformly bounded semigroup, as the spectrum of A is imaginary. The nonspecification of M may not be a drawback in linear theory. Admitting vector components of higher variation in domain and as initial vectors might be expected to produce wilder time-evolutionary behavior.

At this writing we have not yet dealt satisfactorily with the range condition in the nonlinear problem. By use of Diliberto's diagonalization theorem [Ref. 5, Theorem 2], one might pattern a proof after that of Theorem 3 of this paper. One would thereby diagonalize "disguised linear" Eq. (10). In using Diliberto's matrix transformations $B_\lambda(x), B_\lambda^{-1}(x)$, however, one increases total variations of vector components with each application. Hence one must be very careful about the ranges of $B_\lambda(x), B_\lambda^{-1}(x)$. Not enough is known expressly about $B_\lambda(x), B_\lambda^{-1}(x)$ yet to use sufficient care.

Using phase plane methods with Eqs. (8), it can be shown that if h_1, h_2 are sufficiently broad step functions of $K_M^{(1)}$, then ρ, u are in $V_M^{(1)}$. This is

a step in the right direction. Exploration of broken linear forms of ϕ, ψ has not been fruitful.

The author next intends to investigate homotopy as a means of inferring desired properties of nonlinear Eqs. (8) from similar known properties of linearized Eqs. (8).

It would be enough to show the existence of just one $M > 0$ such that solving Eqs. (8) maps L_M into itself. M would then be σ -dependent, as might be the case. Incidentally, taking $\sigma < 0$ does not produce anything new. Equations (8) again become singular at $-\sigma_0$.

Perhaps the range condition cannot be proved generally. This may be the point where semigroup theory goes wrong for systems. Then we can investigate how the theory might be modified to take into account our difficulties. Failing that, we are left with Glimm's stepwise method [Ref. 8, pp. 28-31], or with the invariant regions methods of Smoller and Conley (e.g., see [Ref. 10]) which require sometimes troublesome *a priori* estimates and oftentimes restricted initial data. Also, there is brute numerical force.

IV. CONCLUDING REMARKS

The author originally intended to study the possible application of a semigroup generation theorem to the problem of isentropic one-dimensional flow, which is the simplest physically interesting hydrodynamic problem. A generic form of these equations is:

$$\begin{aligned} \rho_t + \phi_1(u\rho)_x + \phi_2(u)_x &= 0 \\ u_t + \psi_1(\rho)_x + \psi_2(u)_x &= 0 \end{aligned} ,$$

in which $\phi_1, \phi_2, \psi_1 \in \uparrow$, but with $\psi_2 \notin \uparrow$, $\psi_2' \in \uparrow$. One seems always to have a saddle point behavior with Eqs. (8) in this problem, and accretiveness was not achieved. The author then reverted to the system

$$\rho_t + \phi_1(\rho)_x + \phi_2(u)_x = 0$$

$$u_t + \psi_1(\rho)_x + \psi_2(u)_x = 0 \quad ,$$

for which Eqs. (8) are soluble if $\phi_1, \phi_2, \psi_1, \psi_2 \in \uparrow$, and accretiveness is a little nearer. With linear coupling, the present paper results.

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