OCTOBER 1980

PPPL-1718

UC-20b,c,d

ANALYTICAL STUDY OF THE EXTERNAL
FIELD FOR NON-CIRCULAR TOKAMAK
WITH MULTIPOLE MOMENT
EXPANSION APPROACH

BY

O. OKADA, J. DELUCIA,
AND M. OKABAYASHI

MASTER
PLASMA PHYSICS
LABORATORY

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

This work was supported by the U.S. Department of Energy
Contract No. DE-AC02-76-CH0 3073. Reproduction, transla-
tion, publication, use and disposal, in whole or in part,
by or for the United States government is permitted.

With Multipole Moment Expansion Approach

C. Okada

Central Research Laboratory, Hitachi Ltd.

Kokubunji, Tokyo 185, JAPAN

J. DeLucia and M. Okabayashi

Plasma Physics Laboratory, Princeton University

Princeton, New Jersey 08544, U.S.A.

Abstract

An analytical study is made of the external field required to produce non-circular toroidal MHD equilibria. Here the external magnetic flux pattern is formulated with a series of multipole moments expanded around the magnetic axis. The present approach provides a common description of the external field characteristics of various devices rather than specifying location of poloidal coils. Furthermore, the preconceptual design of noncircular devices can be simplified since the arrangement of poloidal coil location is decoupled from the physics requirement.

DISCLAIMER

[illegible]

... ..

I. INTRODUCTION

The external magnetic field which is required for noncircular shell-less tokamaks has been studied mostly in terms of the virtual casing principle [1-6]. Here we propose to utilize the multipole moment expansion method [2] to obtain an analytic expression for the external flux and then relate the moment coefficients to the plasma shape. This method makes it convenient to discuss plasma equilibria independent of specific coil arrangements so that we can compare the machine capabilities of the poloidal coil system in a systematic manner. Furthermore, in the preconceptual design stage of noncircular tokamak the requirement of the other engineering consideration can be optimized separately from the physics. Also, this method leads us to discuss arbitrary elliptic equilibria in terms of the actual field pattern rather than with given fixed boundary conditions.

As discussed in Sec. III we carried out the present analysis with the same approach as previously reported in Ref. 3. The plasma poloidal magnetic surfaces are decomposed in the poloidal angle and the deformation coefficients are ordered in the inverse aspect ratio. The Grad-Shafranov equation is then solved by an inverse aspect ratio expansion. The equilibrium analysis inside the plasma [3] is analytically continued into the vacuum region. After the poloidal magnetic flux due to the plasma current is subtracted from the total poloidal magnetic flux, the resulting poloidal flux due to the current in the external coil is decomposed into the multipole moments. In order to investigate the applicability of the present approach to the actual device, the comparison with numerical calculation was performed for the Poloidal Divertor Experiment device (PDX) at Princeton (Sec. III).

II. ANALYSIS

The analysis is based on the ideal hydromagnetic model expressed in the non-orthogonal coordinate system (ρ_0, θ, ν) shown in Fig. 1. The magnetic surface is expressed as

$$\rho(\theta) = \rho_0 + \sum_n a_n(\rho_0) \cos n\theta, \quad (1)$$

where the surface label ρ_0 is the average minor radius of the surface, ρ is the local minor radius measured from the magnetic axis, and the summation is over the Fourier components of the deformation. The solution of the Grad-Shafranov equation [3] is continued into the vacuum region on the basis of the Laplace equation. The flux function and deformation coefficients in the vacuum region are

$$\psi(\rho_0) = \frac{\pi a^2 R_0 \mu j_0}{2} (1 + 2 \ln r_0) [1 + O(\epsilon^2)], \quad (2)$$

$$j_0 = -2\pi R_0 \frac{d}{d\psi} \left(P + \frac{B_0^2}{2\mu} \right), \quad (3)$$

$$\frac{a_1(\rho_0)}{a} = \left[-\frac{\epsilon}{2} r_0^2 \ln r_0 + \frac{a_1'}{2} r_0^2 + \frac{\epsilon}{4} (r_0^2 - 1) \right] [1 + O(\epsilon)], \quad (4)$$

$$m \geq 2$$

$$\frac{a_m(\rho_0)}{a} = \epsilon_m \left(\frac{m-1}{m} r_0^{m+1} + \frac{1}{m} r_0^{-m+1} \right) [1 + O(\epsilon)], \quad (5)$$

where ϵ is the inverse aspect ratio a/R_0 , r_0 is ρ_0/a and a_{10}' is $-(1 + 4\beta_p)\epsilon/4$. The plasma current density on the axis is given by j_0 . ϵ_m is the m -th deformation coefficient. Full expressions are given in the Appendix. We assume the same ordering that is used in Ref. 3. P and μ_0^2 are taken to be linear in ψ . The deformation coefficients normalized to the minor radius, a_m/a , are assumed to be of order a/R_0 .

The magnetic flux due to the plasma current which flows in the noncircular cross section obtained by expanding the general solution of the Laplace equation in powers of inverse aspect ratio [4],

$$\begin{aligned} \psi_{\text{plasma}}(\rho, \theta) = & \frac{\pi a^2 R_0 \mu j_0}{2} \left\{ -2 \ln \frac{8 R_0}{\rho} + 4 - \frac{\rho \cos \theta}{R_0} \left(\ln \frac{8 R_0}{\rho} - 1 \right) \right. \\ & \left. - \left(2 \ln \frac{8 R_0}{\rho} - 4 \right) \frac{C_0^1}{C_0^0} + \sum_{n \geq 1} \left[\frac{2 C_n^0}{C_0^0} \left(\frac{2 R_0}{\rho} \right)^n \cos n \theta \times \frac{n-1}{n+1} \frac{2m}{2m-1} \right] \right\} \quad (6) \end{aligned}$$

where current shape coefficients C_n are expanded in terms of ϵ , as $C_n = C_n^0 + C_n^1 + \dots$ and C_n^0 is of order $C_0^0 \cdot \epsilon^{n+1}$. This expansion differs from that of [4] in that $C_{nm+1}/C_{nm} \sim \epsilon$ and $C_{n+1}^0/C_n^0 \sim \epsilon$ except for C_0^0 . This is due to the ordering that the deformation coefficients are of order ϵ . The magnetic flux due to the external coils is obtained by subtracting Eq. (6) from Eq. (2) and

then expanded into the multipole moment expression,

$$\begin{aligned}\psi_{\text{ext}} &= \frac{B_D}{2} (R^2 - R_O^2) \\ &+ \frac{B_Q}{4R_O^2} [4R^2 Z^2 - (R^2 - R_O^2)^2] \\ &+ \frac{B_H}{6R_O^4} [8R^2 Z^4 - 12R^2 Z^2 (R^2 - R_O^2) + (R^2 - R_O^2)^3] + \dots, \quad (7)\end{aligned}$$

where B_D , B_Q , B_H and so on, represent the dipole, quadrupole, and hexapole components, and are given as

$$B_D = \frac{\pi R_O \mu j_O}{2} \epsilon^2 \left(\ln \frac{B}{\epsilon} - \frac{3}{2} - \frac{a_{10}'}{\epsilon} \right) [1 + O(\epsilon)], \quad (8)$$

$$B_Q = \frac{\pi R_O \mu j_O}{2} \epsilon_2 [1 + O(\epsilon)], \quad (9)$$

$$B_H = -\frac{\pi R_O \mu j_O}{2} \frac{\epsilon_3}{\epsilon} [1 + O(\epsilon)],$$

$$B_{2N} = -\frac{\pi R_O \mu j_O}{2} \left(-\frac{R_O}{2a} \right)^{N-2} (N-1) \epsilon_n [1 + O(\epsilon)]. \quad (10)$$

The full expressions and c_m^n are again given in the Appendix. c_m^n are determined so as p -dependent terms in multipole moment coefficients to be vanished.

The vertical magnetic field is related to B_D by $B_V = B_D/(2\pi)$, and agrees with the usual expression [5]. To the lowest order in a/R_0 , B_Q and B_H are proportional to ϵ_2 and ϵ_3 . This is consistent with the suggestion that ellipticity and triangularity can be controlled by B_Q and B_H respectively [5]. The leading terms of the multipole moment coefficients are in descending powers of a/R_0 , and proportional to the respective deformation coefficients which are of the order of a/R_0 . This ordering shows that the higher order multipole moment coefficient must be of the lower order in ϵ to realize a deformation of order ϵ .

III. COMPARISON WITH NUMERICAL CALCULATION

The applicability of this analysis is examined by comparing it with the numerical results of the Equilibrium Code for PDX device. The comparison procedure is shown in Fig. 2. The numerical calculation of the free boundary problem starts from the given set of currents in the poloidal coils and the plasma current. The analytical calculation predicts cross-sectional shape from the given multipole moment coefficients which are obtained from the vacuum field due to the current in the external coils. To do this, Eqs. (8) through (10) are solved for j_0 , ϵ_2 , ϵ_3 ... given B_D , B_Q , B_H The plasma current, the major radius, and β_p are also given initially.

In Fig. 3 are shown the plasma flux contours for a numerically calculated PDX equilibrium together with the corresponding analytical result. The top half shows the numerical results for a plasma current of 500 kA which is distributed such that $dP/d\psi$ and $dB_0^2/d\psi$ are independent of ψ and $\beta_p = 0.13$. The bottom half is the analytical result. The agreement of surface shape is excellent except in the plasma-vacuum boundary region. This is due to the existence of the divertor which causes the increase of higher order multipole moment field in this region.

The vacuum flux lines are shown in Fig. 4 where the top half is the numerical result and the bottom half is the analytical result reconstructed from the multipole moments of the vacuum field. The plasma cross section is indicated by a dotted line. The agreement is quite good near the magnetic axis, while near the boundary, especially near the divertor, the agreement deteriorates. The field pattern due to higher moments is compared with the actual field in Fig. 5. The external field due to the divertor coils (which produces higher moments) is shown on top while field pattern approximated with B_Q and B_H components is shown on bottom. Again the region where the plasma

current flows is indicated by a dotted line. The local effect of the divertor is obvious around the divertor throat area.

The slight disagreement in the cross section in Fig. 3 is due to the difference in the higher multipole moments. Therefore, better agreement could be obtained if we consider higher moments such as a dodecapole moment and so on in the analytical calculation. Also the excellent agreement in the magnetic surface shape in the central region in spite of the slight difference in the higher moments, shows that the magnetic surface shape in the central region is determined mostly by B_0 and B_H .

IV. EQUILIBRIA WITH DISTRIBUTED PLASMA CURRENT

The present analysis is limited to uniform plasma current distribution, i.e., $dP/d\psi$, and $dB_0^2/d\psi$ independent of ψ . It might, however, be instructive to compare with the numerical results for a distributed plasma current.

In Fig. 6, the numerical result for a distributed plasma current profile ($dP/d\psi$ and $dB_0^2/d\psi$ proportional to $\psi^{1/2}$) is compared to the corresponding analytical result for a flat current profile. The numerical result is shown in the top half of the figure. Total plasma current is 500 kA and β_p is 0.09. The agreement near the axis is again quite satisfactory, while the cross section of the analytical result is a little smaller. A more concentrated current density (like the Bessel function with zero order) is used to get the numerical result on the top half of Fig. 7. Again, the plasma current is 500 kA, while β_p is 0.06. Satisfactory agreement is limited to the magnetic axis region. The cross section is much smaller than the numerical result.

The shrinkage of the boundary in the analytical result corresponds to the extent of the plasma current distribution, namely, when the toroidal current is highly concentrated, the analysis shows a smaller radius and larger current

density like Fig. 7; while the toroidal current is rather broader, the shrinkage is less.

V. CONCLUSION

The present analysis based on the multipole moment expansion of the poloidal field is in quite satisfactory agreement with the numerical result of the Equilibrium Code. The multipole moment coefficients of the external magnetic field are obtained analytically. The quadrupole moment coefficient is proportional to an ellipticity coefficient, the hexapole moment coefficient to a triangularity, and so on, in the lowest order of an inverse aspect ratio.

The multipole moment coefficients obtained here can be used to discuss plasma shapes and coil arrangements separately in the design phase of tokamaks. Namely, the physical discussions with respect to the cross-sectional shapes can be decoupled from the distribution of poloidal coils, while the analytical expressions of the external fields obtained here can be used to compare the experimental results of the different noncircular tokamaks.

ACKNOWLEDGMENT

One of the authors (O.O.) wishes to express his sincere thanks to Dr. H. Watanabe, the Executive Managing Director of Hitachi Ltd., who gave him the one year leave, and to Dr. D. M. Meade of Princeton Plasma Physics Laboratory, who accepted him to take part in the PDX group. Also we appreciate various comments of Drs. J. Johnson, J. Greene and R. Grimm. Discussions with members of the PDX group have been deeply appreciated.

*This work is supported in part by the United States Department of Energy Contract No. DE-AC02-76-CHO-3073.

REFERENCES

- 1 AZKHAROV, L.E., Nucl. Fusion 13, (1973) 595.
- 2 JOHNSON, J.L., et al., Princeton Plasma Physics Laboratory Report PPPL-1463 (1978).
- 3 OKADA, O., Phys. Fluids 19, (1976) 2034.
- 4 GREENE, J.M., et al., Phys. Fluids 14, (1971) 671.
- 5 MUKHAVATOV, V.S. and SHAFRANOV, V.D., Nucl. Fusion 11, (1971) 605.
- 6 SHAFRANOV, V.D., Nucl. Fusion 12, (1972) 599.

(PPPL-806478)

Fig. 1. The coordinate system (ρ_0, θ, ϕ) . The magnetic surfaces are labelled by ρ_0 , while θ and ϕ are the poloidal and toroidal angle, respectively.

(PPPL-796246)

Fig. 2. The procedure of comparison between the analytical and the numerical results. The comparison with the experiment is also included in the diagram.

(PPPL-796244)

Fig. 3. Comparison between the analytical and numerical result of the PDx tokamak. The top half is the numerical, and the bottom half the analytical result. This configuration will be applied to the following figures. Plasma current is 500 kA and the distribution is uniform. β_p is 0.13.

(PPPL-806480)

Fig. 4. Comparison of the vacuum flux lines. The analytical results are reconstructed from the moment up to decapole. Plasma boundary is shown by a dotted line.

(PPPL-806479)

Fig. 5. Comparison of vacuum field lines after the subtraction of the vertical (dipole) component at the axis. Plasma boundary is again shown by a dotted line.

(PPPL-796242)

Fig. 6. Comparison with the numerical equilibrium with a fat plasma current distribution. Plasma current is 500 kA and β_p is 0.09 .

(PPPL-792645)

Fig. 7. Comparison with the peaked current equilibrium (roughly proportional to the zero order Bessel function). Plasma current is 500 kA and β_p is 0.06 .

APPENDIX

Full expression of Eqs. (2), (4) and $a_m(\rho_0)$ for $m \geq 2$ are,

$$\psi(\rho_0) = \frac{\pi a^2 R_0 \mu_j}{2} \left[1 + 2 \ln r_0 \right]$$

$$- \frac{1}{n} \frac{1}{2} \epsilon_n^2 \left\{ \frac{(n-1)^2 (2n+1)}{2n^2} (r_0^{2n} - 1) - \frac{2n-1}{2n^2} (r_0^{-2n} - 1) + \frac{2n-1}{2} - \ln r_0 \right\}$$

$$- \frac{3}{8} \epsilon^2 r_0^2 (\ln r_0)^2 + \left(\frac{3}{4} a_{10}^1 \epsilon r_0^2 + \frac{3}{8} \epsilon^2 r_0^2 \right) \ln r_0$$

$$- \frac{3}{8} a_{10}^1{}^2 (r_0^2 - 1) - \frac{3}{8} a_{10}^1 \epsilon (r_0^2 - 1) - \frac{15}{32} \epsilon^2 (r_0^2 - 1)$$

$$+ \frac{\epsilon^2}{32} \left(\frac{1}{r_0^2} - 1 \right) - \frac{a_{10}^1{}^2}{2} - \frac{3 - 4\beta_p}{32} a_{10}^1 \epsilon - \frac{1 - 2\beta_p}{16} \epsilon^2$$

$$- \left\{ \frac{a_{10}^1{}^2}{4} + \frac{3 - 4\beta_p}{16} a_{10}^1 \epsilon - \frac{2 + \beta_p}{8} \epsilon^2 \right\} \ln r_0 \Big] , \quad (2a)$$

$$\frac{a_1(\rho_o)}{a} = -\frac{\epsilon}{2} r_o^2 \ln r_o + \frac{a'_{10}}{2} r_o^2 + \frac{\epsilon}{4} (r_o^2 - 1)$$

$$+ \sum_{n=2}^{\infty} \epsilon_n \epsilon_{n+1} \left\{ (n-1) (r_o^{2n+2} - r_o^2) - \frac{r_o^{-2n} - r_o^2}{n+1} - r_o^2 r_{n+1} \right\}$$

$$+ \epsilon_2 \left[-\frac{\epsilon}{2} r_o^4 \ln r_o + \frac{a'_{10}}{2} (r_o^4 - 2r_o^2 + 1) - \epsilon r_o^2 \beta_p + \frac{\epsilon}{8} (2r_o^4 - \right.$$

$$\left. r_o^2 - 2 + r_o^{-2} \right) \Big],$$

(4a)

$$\frac{a_2(\rho_o)}{a} = \epsilon_2 \left(\frac{r_o^3}{2} + \frac{1}{2r_o} \right)$$

$$+ \sum_{n=2}^{\infty} \epsilon_n \epsilon_{n+2} \left[\frac{(n+1)(n-1)(2n+3)}{2n(n+2)} (r_o^{2n+3} - r_o^3) \right.$$

$$\left. - \frac{(2n+1)}{2n(n+2)} (r_o^{2n-1} - r_o^3) \right]$$

$$- \left\{ \frac{(n-1)(2n+1)}{4} - \frac{(n+1)(n-1)(2n-3)}{4(n+2)} - \frac{2n+1}{n} \right\} (r_o^{-1} - r_o^3)$$

$$+ \epsilon_3 \left[-\frac{5}{6} \epsilon r_o^5 \ln r_o + \frac{1}{12} \frac{\epsilon}{r_o} \ln r_o + \left(\frac{5}{6} a'_{10} + \frac{5}{12} \epsilon \right) (r_o^5 - r_o^3) \right]$$

$$+ \frac{\epsilon}{8} (r_o^{-3} - r_o^3)$$

$$+ \left\{ \frac{\epsilon}{48} - \frac{13}{24} a'_{10} - \frac{5-4\beta_p}{48} \epsilon - \frac{3}{16} \epsilon + \frac{5}{12} a'_{10} \right\} (r_o^{-1} - r_o^3) \Big]$$

$$+ \left(\frac{3}{16} \epsilon^2 \ln^2 r_o - \frac{3}{8} \epsilon a'_{10} \ln r_o + \frac{\epsilon^2}{16 r_o^2} \ln r_o \right) r_o^3 + \frac{\epsilon}{16} (a'_{10} + \frac{\epsilon}{2}) \left(\frac{1}{r_o} - r_o \right)$$

$$- \left(\frac{11}{96} a'^2_{10} + \frac{1-4\beta_p}{96} \epsilon a'_{10} + \frac{1-2\beta_p}{48} \epsilon^2 - \frac{7}{32} \epsilon a'_{10} \right) \left(\frac{1}{r_o} - r_o^3 \right), \quad (5a)$$

$$\frac{a_3(\rho_o)}{a} = \epsilon_3 \left(\frac{2}{3} r_o^4 + \frac{1}{3} r_o^{-2} \right)$$

$$+ \sum_{n=2}^{\infty} \epsilon_n \epsilon_{n+3} \left[\frac{(n+2)^2 (n-1)}{n(n+3)} (r_o^{2n+4} - r_o^4) - \frac{n+1}{n(n+3)} \right.$$

$$\left. (r_o^{-2n-2} - r_o^4) \right.$$

$$+ \left\{ \frac{(n+2)^2 (n-1)}{3(n+3)} - \frac{n^2-1}{3} + \frac{n+1}{3n} \right\} (r_o^{-2} - r_o^4) \Big] +$$

$$\epsilon_4 \left(a'_{10} \left(\frac{9}{8} (r_o^6 - r_o^4) \right. \right.$$

$$+ \frac{3}{8} (r_o^{-2} - r_o^4) \Big) + \epsilon \left\{ \left(-\frac{9}{8} r_o^6 + \frac{1}{8} r_o^{-2} \right) \ln r_o + \frac{9}{16} (r_o^6 - r_o^4) \right.$$

$$+ \frac{1}{48} (r_o^{-2} - r_o^4) + \frac{1}{8} (r_o^{-4} - r_o^4) - \left(\frac{11}{24} \frac{a'_{10}}{\epsilon} + \frac{7-4\beta_0}{96} + \frac{1}{6} \right) \times$$

$$\times (r_o^{-2} - r_o^{-4}) \Big] \Big]$$

$$+ \epsilon_2 \left[\epsilon - \frac{r_o^4}{2} \ln r_o - \frac{1}{8} (r_o^2 - r_o^4) - \frac{1}{24} (r_o^{-2} - r_o^4) \right] , \quad (5b)$$

and for $m \geq 4$

$$\frac{a_m(\rho_o)}{a} = \epsilon_m \left(\frac{m-1}{m} r_o^{m+1} + \frac{1}{m} r_o^{-m+1} \right)$$

$$+ \sum_{n=2}^{\infty} \epsilon_n \epsilon_{n+m} \left[\frac{(n+m-1)(n-1)(2n+m+1)}{2n(n+m)} (r_o^{2n+m+1} \right.$$

$$\left. - r_o^{m+1} \right)$$

$$- \frac{2n+m-1}{2n(n+m)} (r_o^{-2n-m+1} - r_o^{m+1}) + \left\{ - \frac{(n-1)(m+2n-1)}{2m} \right.$$

$$\left. + \frac{(n+m-1)(n-1)(2n+m+1)}{2m(n+m)} \right\}$$

$$+ \frac{2n+m-1}{2mn} \} (r_o^{-m+1} - r_o^{m+1})] + \epsilon_{m+1} \left[-\frac{m(m+3)}{4(m+1)} \epsilon r_o^{m+3} \ln r_o \right.$$

$$+ \frac{1}{4} \epsilon r_o^{-m+1} \frac{m-1}{m+1} \ln r_o + \left\{ \frac{m(m+3)}{4(m+1)} (a'_{10} + \frac{\epsilon}{2}) (r_o^{m+3} - r_o^{m+1}) \right\}$$

$$+ \left\{ -\frac{m^2 + 4m + 1}{4m(m+1)} a'_{10} - \frac{2m+1-4\beta_p}{8m(m+1)} \epsilon - \frac{m+1}{8m} \epsilon + \frac{m-1}{8m(m+1)} \epsilon \right.$$

$$+ \frac{m+3}{4(m+1)} a'_{10} \} (r_o^{-m+1} - r_o^{m+1}) + \frac{\epsilon}{8} (r_o^{-m-1} - r_o^{m+1})]$$

$$+ \epsilon_{m-1} \left[-\frac{(m-2)(m+1)}{4(m-1)} \epsilon r_o^{m+1} \ln r_o \right.$$

$$+ \frac{m-3}{4(m-1)} \epsilon r_o^{-m+3} \ln r_o$$

$$+ \left\{ -\frac{m-3}{4(m-1)} (r_o^{-m+3} - r_o^{m+1}) \right.$$

$$+ \frac{m-3}{4m} (r_o^{-m+1} - r_o^{m+1}) \} a'_{10}$$

$$+ \{ -\frac{m-2}{8} (r_o^m - 1 - r_o^{m+1}) - \frac{m-3}{8(m-1)} (r_o^{-m+3} - r_o^{m+1})$$

$$+ \frac{m-4}{8m} (r_o^{-m+1} - r_o^{m+1}) \} \varepsilon]$$

$$+ \frac{m-2}{n} \sum_{n=2}^{\infty} \varepsilon_n \varepsilon_{m-n} \left[\frac{(n-1)(2n-m+1)}{4n(m-n)} (r_o^{2n-m+1} - r_o^{m+1}) \right] +$$

$$\frac{(n-m+1)(2n-m-1)}{2n(m-n)} (r_o^{-2n+m+1} - r_o^{m+1})$$

$$+ \left\{ \frac{m-1}{4m} - \frac{(n-1)(2n-m+1)}{4mn} - \frac{(n-m+1)(2n-m-1)}{4m(m-n)} \right\}$$

$$(r_o^{-m+1} - r_o^{m+1}) \} .$$

(5c)

The first lines of each expression are the lowest order terms given in the main text.

An expression of $\psi_{\text{plasma}}(\rho, \theta)$ which includes terms of the order of ϵ^2 is given as

$$\begin{aligned}
 \psi_{\text{plasma}}(\rho, \theta) = & \frac{\pi a^2 R_o \mu j_o}{2} \left[-2 \ln \frac{8R_o}{\rho} + 4 - \frac{\rho \cos \theta}{R_o} \left(\ln \frac{8R_o}{\rho} - 1 \right) \right. \\
 & - \left(2 \ln \frac{8R_o}{\rho} - 4 \right) \left(\frac{C_o^1}{C_o} + \frac{C_o^2}{C_o} \right) - \left\{ 2 \ln \frac{8R_o}{\rho} + 1 + \right. \\
 & \left. \left(2 - \ln \frac{8R_o}{\rho} \right) \cos 2\theta \right\} \frac{\rho^2}{8R_o^2} \\
 & - \frac{\rho \cos \theta}{R_o} \left(\ln \frac{8R_o}{\rho} - 1 \right) \frac{C_o^1}{C_o} + \frac{C_o^1}{C_o} \frac{4R_o}{\rho} \left(\cos \theta + \frac{3\rho}{4R_o} + \frac{\rho \cos 2\theta}{rR_o} + \frac{C_o^1}{C_o} \frac{4R_o}{\rho} \cos \theta \right. \\
 & \left. + \frac{C_o^2}{C_o} \left(\frac{2R_o}{\rho} \right)^2 \left(\cos 2\theta + \frac{5\rho}{4R_o} \cos \theta + \frac{\rho}{4R_o} \cos 3\theta \right) \times 4 + \right.
 \end{aligned}$$

$$\frac{C_2^1}{C_0^0} \left(\frac{2R_0}{\rho} \right)^2 4 \cos 2\theta$$

$$+ \dots + \sum \left\{ \frac{C_n^0}{C_0^0} \left(\frac{2R_0}{\rho} \right)^n \left(\cos n\theta + \frac{2n+1}{4} \frac{\rho}{R_0} \cos(n-1)\theta + \frac{\rho}{4R_0} \right. \right.$$

$$\left. \cos(n+1)\theta \right\} \times$$

$$\times 2 \sum_{m=1}^{n-1} \frac{2m}{2m-1} + \frac{C_n^1}{C_0^0} \left(\frac{2R_0}{\rho} \right)^n 2 \cos n\theta$$

$$\left. \sum_{m=1}^{n-1} \frac{2m}{2m-1} \right\}] . \quad (6a)$$

Full expressions of multipole moment coefficients B_D and B_Q are

$$B_D = \frac{\pi R_0^4 \mu_j}{2} \left[\epsilon^2 \left(\ln \frac{8}{\epsilon} - \frac{3}{2} - \frac{a_{10}'}{\epsilon} \right) + \epsilon \sum_{n=2}^{\infty} \epsilon_n \epsilon_{n+1} \right]$$

$$2n + \epsilon \epsilon_2 (2\epsilon \beta_p + 2a'_{10}) \quad , \quad (8a)$$

$$B_Q = \frac{\pi R_o u_j^2}{2} \left[\epsilon_2 - n \frac{\epsilon}{2} \epsilon_n \epsilon_{n+2} \frac{(4n+1)(n+3)}{2n} \right.$$

$$+ \epsilon_3 \left\{ -\frac{5}{2} a'_{10} - \frac{5}{16} (1 + 4\beta_p) \epsilon \right\}$$

$$- \frac{1 - 8\beta_p}{96} \epsilon^2 - \frac{5 + \beta_p}{12} \epsilon a'_{10} - \frac{7}{48} a_{10}'^2$$

$$+ \frac{3}{8} \epsilon^2 \left(\ln \frac{8}{\epsilon} - \frac{3}{2} - \frac{a'_{10}}{\epsilon} \right) \quad , \quad (9a)$$

Current shape coefficient C_m^n are,

$$\frac{C_1^0}{C_0^0} = \frac{a^2}{8R_o^2}$$

$$\frac{C_2^0}{C_0^0} = -\frac{a^2}{16R_0^2} \epsilon_2$$

$$\frac{C_3^0}{C_0^0} = -\frac{a^3}{64R_0^3} \epsilon_3$$

$$\frac{C_0^1}{C_0^0} = 0$$

$$\frac{C_1^1}{C_0^0} = -\frac{\epsilon}{2} n \sum_{n=2}^{\infty} \epsilon_n \epsilon_{n+1} (n+1) + \frac{\epsilon \epsilon_2}{4} \left(\frac{3}{2} \epsilon - a'_{10} \right)$$

$$\frac{C_2^1}{C_0^0} = -\frac{\epsilon^2}{16} n \sum_{n=2}^{\infty} \epsilon_n \epsilon_{n+2} \frac{3(3n+1)}{2n} - \frac{\epsilon_2 \epsilon_3}{16} \left(a'_{10} - \frac{19-20\beta}{16} \epsilon \right)$$

$$- \frac{9}{256} a'_{10} \epsilon^3 - \frac{11 - 8\beta_p - 80\beta_p^2}{256 \times 16} \epsilon^4 .$$