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SATURATION OF DRIFT INSTABILITIES BY ELECTRON DYNAMICS

By

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SATURATION OF DRIFT INSTABILITIES BY ELECTRON DYNAMICS

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An analytical explanation is provided for phenomena observed in gyrokinetic particle simulations, which were used to study the nonlinear evolution of the universal instability. The nonlinear $\mathbf{E} \times \mathbf{B}$ advection of nearly resonant electrons is related to the deactivation of the instability mechanism, leading to oscillations of the perturbation amplitude around a nonzero saturated level. An equivalent description is given in terms of nonresonant mode coupling. A self-consistent evolution equation for the amplitude is obtained for the case where this mechanism predominates, and is successfully compared to the gyrokinetic simulations.

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I. INTRODUCTION

The investigation of finite amplitude, collisionless drift waves in magnetized plasma is a prerequisite for a complete understanding of confinement in laboratory plasma experiments. It is important to examine the full host of physical processes which are involved in these fluctuations in order to have confidence in a theory of confinement based on them. In this article we describe a saturation mechanism, related to electrostatic trapping of electrons, which has been overlooked in earlier studies of the drift instability.

Recent gyrokinetic particle simulations of drift instabilities^{1,2} allow us to assess the relative importance of various nonlinear terms in the saturation process; in particular, we find that electron dynamics often play a dominant role. It is the goal of this study to present details of that role; specifically, we elucidate a mechanism by which electrons alone, through the advective $\mathbf{E} \times \mathbf{B}$ nonlinearity, can cause saturation of collisionless drift instabilities. We present a simple analytic account and an unambiguous physical interpretation of this process, which, according to the numerical study, leads to saturation at realistic levels of density and potential fluctuations. These saturation levels are proportional to the linear growth rate, and are smaller than earlier predictions in many cases of interest. This study extends several concepts introduced by Sato³ to a kinetic instability, so that some of those concepts can be interpreted in terms of particle trajectories.

Various effects of the advective nonlinearity have been mentioned in previous discussions of broadband drift wave turbulence. They include spatial trapping, and the closely related phenomenon of perpendicular (configuration space) resonance broadening⁴⁻⁶; turbulent radial diffusion, which may modify shear stabilization processes^{7,8}; and granulation

of the phase space of trapped electrons.⁹ The mechanism described in the present work is novel in several respects. First, in the parameter range under consideration, although shear is neglected, the electron nonlinearities dominate, whereas previous studies have emphasized ion nonlinearities. Second, the saturation mechanism involves self-consistent collective motions of both spatially trapped and untrapped electrons. Third, the resonance in the parallel velocity is not modified by the nonlinear effect which we consider and is, thus, identical to that in the linear theory. Fourth, the perturbation consists of a small number of isolated, discrete modes with coherent particle responses, so there is no need for a statistical description, with its associated difficulties. By treating the interactions between the electrons and the potential fluctuations self-consistently, so that energy is conserved, our treatment allows for a dynamic saturation (nonlinear intensity oscillations about a nonzero mean intensity), and we are able to give a model evolution equation for, as well as a quantitative prediction of, the mean intensity as well as the amplitude and period of the nonlinear oscillations. We may expect this electron effect to contribute to saturation even when other processes, such as nonlinear ion Compton scattering,^{10,11} become significant. The completeness of description available in this case makes it an encouraging paradigm for future work in the nonlinear theory of wave-particle interaction.

The nonlinear evolution of drift instabilities in the absence of magnetic shear has been addressed by other investigators (e.g. Refs. 6,12-13), who have described several processes relevant to saturation for various parameter ranges. Let us briefly review some of this earlier work. Quasilinear relaxation of the electron distribution function in velocity space has been discussed by Sagdeev and Galeev¹³ and seems to be important in the case of strong linear instability. An analysis based entirely on resonant mode coupling (a weak turbulence theory) leads to predictions of unphysically large density perturbations (comparable in magnitude to the equilibrium density),^{6,14} an inconsistency which has motivated various so-called strong coupling theories, including the present work. (The particles are

strongly coupled to the wave in the sense that their trajectories are distorted so much as to preclude expansion in powers of a small coupling parameter.) Phase space trapping of the electrons occurs because of the parallel velocity space nonlinearity.¹⁵ This process disturbs the resonance,¹³ and may also be relevant to some of the results of Ref. 1, although it is not an effective stabilization mechanism in the case studied here.⁶ Phase space trapping can also destabilize large amplitude drift waves in combination with collisions.^{16,17} Dupree and others⁴⁻¹² have suggested a saturation mechanism involving kinetic ions: the ions are subject to spatial trapping in an electrostatic field, and the consequent Doppler shift can enhance the ion Landau damping through a resonance-broadening effect. This damping is related to stochastic ion heating in broadband drift wave turbulence.^{10,11} However, the particle simulations which inspired our analysis show that in many cases of interest the ions may be treated as a gyrokinetic fluid without altering the details of the saturated state; no ion heating is observable. Krylov¹⁸ has exhibited a saturated state involving spatially trapped electrons, but his detailed calculations are based on an ansatz, related to the spatial structure of the perturbation, which disagrees with the results of the particle simulations. In particular, Fourier analysis of the nonlinear gyrokinetic equations and of simulation results shows that an important process involves coupling to modes which violate that ansatz, in that they do not vary in the drift direction. The mechanism which we propose for the saturation of the universal mode is similar to processes discussed by Manheimer¹⁹ and by Drake and Huba²⁰ with reference to beam instabilities and trapped electron modes, respectively. However, their treatments are not completely self-consistent, inasmuch as they describe only static saturation. (The particular dependence of the $\mathbf{E} \times \mathbf{B}$ nonlinearity on the potential facilitates a nonperturbative treatment, akin to a wave-kinetic description, in the present case; *v. i.* Sec. IV.) Sato³ has given a description of a dynamic saturation process similar to ours, but his work employs a fluid description and applies to

the case where gradient relaxation is allowed, whereas our approach is based on kinetic theory and describes saturation in the presence of an imposed background gradient.

An upper bound on saturation amplitudes can be obtained from an estimate of the free energy available to drive the instability. This estimate yields the criterion that the gradient of the density perturbation at saturation is comparable to the equilibrium gradient,²¹⁻²⁴ or, equivalently, that the amplitude of the fluctuating $\mathbf{E} \times \mathbf{B}$ velocity becomes comparable to the diamagnetic velocity. This criterion should be understood as implying a balance between a generalized force due to the background and some average, effective force due to perturbations, and it may also be discovered by balancing terms in the dynamical equations, as in Dupree's theory of resonance broadening by electrostatically trapped particles.⁶ The criterion must be distinguished from the sourceless case of global ("quasilinear") relaxation of the density gradient. Our analysis includes a calculation of the effective (dynamically and self-consistently varying) perturbation gradient which must be used to turn this phenomenological bound into a useful, quantitative expression. We find that the nonlinear advection alters the spatial contribution to the exchange of energy between the electrons and the electrostatic field, so that the advective term competes with the dissipative resonance term in the drift kinetic equation. Earlier discussions of this competition have been rather heuristic and have not referred explicitly to the mechanism detailed below. Furthermore, since the present discussion deals explicitly with the dissipation, we are able to refine the above-mentioned upper bound to include an additional factor of the (usually small) ratio of the linear growth rate to the wave frequency.

The detailed calculations in the remainder of the paper apply to electrostatic drift instabilities in a slab without magnetic shear. Although this restriction is severe with regard to practical applications, we believe it is justifiable since our principal goal here is to elucidate a fundamental physical mechanism. The same point of view has been adopted

in much of the prior work on the nonlinear physics of drift instabilities (e. g., Refs. 6,12,13). Moreover, the approximations involved may be quite appropriate to experiments in which the magnetic field is nearly shearless and a small number of drift modes coexist, such as the Advanced Concepts Torus experiment, ACT-I.²⁵ Further work will be needed to determine the effects of nonuniformity and curvature of magnetic fields on the mechanism presented here. However, since the linear instability mechanism of the toroidally induced universal mode²⁶ is identical to that of the slab mode, our investigations may be relevant to the toroidal problem as well. Also, ongoing research confirms that electron advection is important in the nonlinear evolution of collisional drift instabilities.²⁷ In Ref. 1 we discussed the application of the nonlinear $\mathbf{E} \times \mathbf{B}$ and $E_{\parallel} \partial f / \partial v_{\parallel}$ terms to particle diffusion; we found that the process described in this article is not a direct cause of significant anomalous diffusion, although it determines the field intensity which leads to diffusion by other mechanisms.

The remainder of this article is organized as follows. In Sec. II we present the details of the nonlinear system under consideration, and make some simplifications suggested by the numerical results. In Sec. III we relate the mechanism for saturation to the orbits of individual electrons in the electrostatic wave. We describe and solve in Sec. IV a simplified mode coupling model which shows how the finite amplitude potential reacts back on its instability mechanism and which predicts details of the saturated state in reasonable agreement with the simulations. In Sec. V we discuss some conclusions and implications. In the appendix we demonstrate that the simplified model presented in Sec. II conserves energy.

II. NONLINEAR EVOLUTION EQUATIONS

The system under investigation has been described in an earlier article, Ref. 1. We consider a gyrokinetic plasma^{2,28, 29} in a uniformly magnetized ($\mathbf{B} \equiv B\hat{e}_x$) slab with a density gradient ($\nabla \ln n_0 \equiv -\kappa\hat{e}_x$) and a three-dimensional electrostatic perturbation. We work in the local approximation: radial eigenmodes are modeled by forcing periodicity along the density gradient, and application to resonant surfaces is anticipated by fixing the ratio of wave numbers in the directions of the field and of the drift, $\theta := k_{\parallel}/k_y$. The gyrokinetic description accounts for the nonzero gyroradius of ions via coefficients $\Gamma(\mathbf{k}) := I_0(k_{\perp}^2 \rho_i^2) \exp(-k_{\perp}^2 \rho_i^2)$, where $\rho_i := v_{ti}/\omega_{ci}$ is the ion gyroradius for thermal velocity $v_{ti} := (T_i/m_i)^{1/2}$ and Larmor frequency $\omega_{ci} := eB/m_i c$. A difference between the uniform species temperatures is allowed, and measured by $\tau := T_e/T_i$. The displacement of ion charges from drifting guiding centers sustains an electrostatic drift wave, and resonant electrons moving against the density gradient destabilize the wave, since the electron response is out of phase with the electrostatic potential. The equations describing this instability are¹

$$\begin{aligned} \frac{\partial}{\partial t} f_e(\mathbf{k}) + ik_{\parallel} v_{\parallel} f_e(\mathbf{k}) + i(\omega_{*e} - k_{\parallel} v_{\parallel}) \phi(\mathbf{k}) \\ + \sum' \left(\mathbf{k}' \times \hat{e}_x \cdot \mathbf{k}'' + i \frac{m_i}{m_e} k'_{\parallel} \frac{\partial}{\partial v_{\parallel}} \right) \phi(\mathbf{k}') f_e(\mathbf{k}'') = 0, \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{\partial}{\partial t} f_i(\mathbf{k}) + ik_{\parallel} v_{\parallel} f_i(\mathbf{k}) + i(\omega_{*e} - \tau k_{\parallel} v_{\parallel}) \psi(\mathbf{k}) \\ + \sum' \left(\mathbf{k}' \times \hat{e}_x \cdot \mathbf{k}'' + ik'_{\parallel} \frac{\partial}{\partial v_{\parallel}} \right) \psi(\mathbf{k}') f_i(\mathbf{k}'') = 0, \end{aligned} \quad (2)$$

and

$$\tau(1 - \Gamma)\phi(\mathbf{k}) + \sum' (k_{\perp}''^2 + \mathbf{k}_{\perp}' \cdot \mathbf{k}_{\perp}'') \phi(\mathbf{k}') \int dv_{\parallel} f_i(\mathbf{k}) F_{Mi}$$

$$= \int dv_{\parallel} (f_i(\mathbf{k}) \Gamma F_{Mi} - f_e(\mathbf{k}) F_{Me}). \quad (3)$$

Here

$$\psi(\mathbf{k}) := \phi(\mathbf{k}) \Gamma(k_{\perp}) + \frac{1}{2} \sum' k'_{\perp} \cdot k''_{\perp} \phi(\mathbf{k}') \phi(\mathbf{k}'') \quad (4)$$

and $F_M := n_0 \exp(-v_{\parallel}^2/2v_t^2)/\sqrt{2\pi} v_t$ is the Maxwellian distribution [so that the total distribution function for species α is $F_{M\alpha} + \int d\mathbf{k} (2\pi)^{-3} \exp(i\mathbf{k} \cdot \mathbf{x}) f_{\alpha}(\mathbf{k})$], $\omega_{se}(\mathbf{k}) := \kappa k_y \rho_s^2 \omega_{ci}$ is the drift frequency, the prime on $\sum$ indicates summation over triads such that $\mathbf{k}' + \mathbf{k}'' = \mathbf{k}$, and the dimensions have been eliminated by rescaling:

$$\begin{aligned} \frac{e\phi}{T_e} &\rightarrow \phi, & k\rho_s &\rightarrow k, & \frac{v}{c_s} &\rightarrow v, \\ \frac{\omega_{se}}{\omega_{ci}} &\rightarrow \omega_{se}, & t\omega_{ci} &\rightarrow t, & \frac{f}{F_M} &\rightarrow f, \end{aligned}$$

where $c_s := (T_e/m_i)^{1/2}$ and $\rho_s := c_s/\omega_{ci}$. In the linear approximation, the terms involving the convolution sums are neglected. The dispersion relation which results is

$$1 + \tau + \left(1 - \frac{\omega_{se}}{\omega}\right) \zeta_e Z_e + \left(\tau - \frac{\omega_{se}}{\omega}\right) \zeta_i Z_i \Gamma = 0,$$

where $\zeta_{\alpha} := \omega/\sqrt{2}k_{\parallel}v_{t\alpha}$ and Z_{α} is the plasma dispersion function³⁰ of argument ζ_{α} . In the limit of small growth rate, the complex frequency is $\omega = \Omega_l + i\gamma_l$, where

$$\Omega_l = \frac{\omega_{se}\Gamma(k_{\perp})}{1 + \tau(1 - \Gamma(k_{\perp}))} \quad (5a)$$

and

$$\gamma_l = \frac{\pi\Omega_l^2(\omega_{se} - \Omega_l)}{k_{\parallel}\omega_{se}} F_M \left(\frac{\Omega_l}{k_{\parallel}} \right), \quad (5b)$$

which is the well-known universal mode due to the effects of finite ion gyroradius and destabilized by the electron Landau resonance.

Gyrokinetic particle simulations based on the nonlinear system of Eqs. (1–4) suggest several simplifications. First, there is no evidence of ion Landau damping, and the ion

distribution function remains nearly Maxwellian, so that ion kinetic effects [other than the ion polarization term in Eq. (3)] can be neglected without a noticeable change in the evolution of the fluctuations. We consider a finite box so that $\mathbf{k}_\perp = (2\pi n_x/L_x, 2\pi n_y/L_y)$ for integer (n_x, n_y) and write $\phi(n_x, n_y)$ for $\phi(\mathbf{k})$. Then, to a first approximation, we can represent the spatial mode structure by a three-harmonic truncation, keeping the fastest growing modes allowed by the boundary conditions,

$$\phi(1, \pm 1) \propto \exp[ik_x x \pm ik_y(y + \theta z)],$$

as well as their convolution harmonic $f(2, 0) \propto \exp(2ik_x x)$ (which is induced by the advective nonlinearity $\mathbf{v}_E \cdot \nabla \delta f_e$). Nonlinear ion advection can also be ignored in the case of a weak instability (e.g., $\theta = 0.01$), which is the case we shall study in detail below. However, when the instability is strong (e.g., $\theta = 0.002$), we find that both ion advection and the second nonlinear electron term, $E_\parallel \partial \delta f_e / \partial v_\parallel$, have noticeable effects. The parallel electron term allows for the exchange of energy between modes, via nonlinear Landau damping and phase space trapping,^{13,15} and leads to particle diffusion.¹ This process is so slow and so weak that it does not affect the overall saturation amplitude of the weakly unstable case. The potential corresponding to $f(2, 0)$ has no y or z dependence, so it does not constitute a drift wave, and does not evolve independently; neglecting it has no appreciable effect on the simulation results.

The dominant nonlinear effect, then, involves the advective term $2k_x f(2, 0)k_y \phi(1, \pm 1)$, which describes the back-reaction of the gradient of the perturbed density on the system. We shall study the weakly unstable case, and work with the Fourier-transformed system corresponding to these approximations, viz.

$$\left(\frac{\partial}{\partial t} \pm ik_\parallel v\right)f_e(1, \pm 1) = \mp(\omega_{*e} - ik_\parallel v)\phi(1, \pm 1) \mp 2k_x k_y f_e^*(2, 0)\phi^*(1, \mp 1), \quad (6)$$

$$\frac{\partial}{\partial t} f_e(2, 0) = 2k_x k_y [f_e^*(1, 1)\phi^*(1, -1) - f_e^*(1, -1)\phi^*(1, 1)], \quad (7)$$

$$\frac{\partial}{\partial t} n_i(1, \pm 1) = \mp i \omega_{se} \Gamma \phi(1, \pm 1), \quad (8)$$

$$\tau(1 - \Gamma)\phi(1, \pm 1) = n_i(1, \pm 1) - \int dv F_M f_e(1, \pm 1). \quad (9)$$

(For brevity, we now write simply v in lieu of $v_{\parallel}$.) In the appendix we discuss the conservation of energy by the system of Eqs. (6-9). This system has been integrated numerically,¹ and the results closely resemble the full gyrokinetic simulations for the cases studied here, where $\theta = 0.01$, $\tau = 4$, $k_y \rho_s = k_x \rho_s = 0.843$, and $\omega_{se}/\omega_{ci} = 0.185$. Our goal is to give a physical explanation for the processes described by these equations, thereby showing how electron dynamics alone, in particular the advective nonlinearity, can account for the saturation and oscillation of fluctuation levels as seen in the simulations.

III. SPATIAL TRAPPING AND RELATED EFFECTS

Before examining the implications of the self-consistent mode coupling equations derived above, let us recall the behavior of electron guiding centers in a specified stationary, three-dimensional, electrostatic potential, without regard to self-consistency. In particular, consider a potential with the structure

$$\begin{aligned} \phi(\mathbf{x}, t) &= \phi_0 \cos(k_x x) \cos(k_y y + k_{\parallel} z - \omega t) \\ &= \phi_0 \cos(k_x x) \cos(k_y y_w), \end{aligned} \quad (10)$$

where $k_y y_w := k_y y + k_{\parallel} z - \omega t$ and ϕ_0 is a constant.

We note that the drift wave is destabilized by resonant electrons which experience the potential in the stationary wave frame and achieve an $\mathbf{E} \times \mathbf{B}$ velocity

$$\frac{dx}{dt} = \frac{ck_y \phi_0}{B} \cos(k_x x) \sin(k_y y_w), \quad (11)$$

$$\frac{dy_w}{dt} = -\frac{ck_x \phi_0}{B} \sin(k_x x) \cos(k_y y_w) - \left(\frac{\omega - k_{\parallel} v}{k_y} \right). \quad (12)$$

The coupled equations (11,12) constitute a Hamiltonian system where

$$H(x, y_w) = \frac{c\phi_0}{B} \cos(k_x x) \cos(k_y y_w) - \left(\frac{\omega - k_{\parallel} v}{k_y} \right) x, \quad (13)$$

and x plays the role of a momentum conjugate to y_w .

These trajectories have been studied in some detail by Ching,⁴ Horton,³¹ and Hirshman,⁵ but their investigations emphasized possible Doppler broadening of the ion — rather than the electron — resonance. For particles near resonance, such that

$$|\omega - k_{\parallel} v| \lesssim \left| \frac{ck_x k_y \phi_0}{B} \right|,$$

there is severe perturbation of the orbits in the direction of the density gradient. Indeed, some particles are confined to closed orbits in the x - y_w plane. Exactly resonant ($\omega = k_{\parallel} v_{\parallel}$) particle trajectories can be integrated in terms of Jacobi elliptic functions, with a bounce frequency of

$$\omega_b = \frac{2\pi}{4K(1 - (BH/c\phi_0)^2)} \frac{ck_x k_y \phi_0}{B}, \quad (14)$$

where K is the complete elliptic integral of the second kind.

Initially, motion along the field induces an accumulation of electrons in wells of the potential, tending to increase its amplitude. But as the perturbed resonant population ceases to move coherently with the potential, the loss of energy to the wave by this process can no longer predominate over Landau damping, and the wave will become stable. In particle simulations, this process is observed as a change in relative phase between density and potential.^{1,3,32} However, the period for oscillations along the gradient in a potential at the observed saturation amplitudes is comparable to the exponentiation time of the instability, so the approximation of a stationary potential is rather crude. In principle, one could perturb the resonant orbits by slow variation of the amplitude and include particles slightly off resonance to find the power loss from particles to the wave and, thus, to obtain

a nonlinear growth rate, but the complexity of such a program motivates us to undertake the approximation scheme discussed in Sec. IV below.

Nevertheless, we can draw some heuristic conclusions regarding the self-consistent wave-particle interaction by analogy to the situation of parallel phase space trapping. First, we note that the saturation amplitude ϕ_s can be estimated from the criterion that $\bar{\omega}_b(\phi_s) \approx \gamma_i$, where $\bar{\omega}_b(\phi)$ is an average bounce frequency in the static potential with amplitude ϕ , following arguments given by Manheimer.¹⁹ In the present case, Eq. (14) indicates that $\bar{\omega}_b(\phi) \approx \frac{4}{5}(k_x/\kappa)(e\phi/T_e)\Omega_i$, so the saturation level is expected to be $\delta n/n \approx (\kappa/k_x)(\gamma_i/\Omega_i)$. This saturation amplitude is smaller than that predicted by ion mechanisms in many cases of interest because the electron mechanism saturates when power transfer (which scales with growth rate) equilibrates, whereas the ion Doppler shift mechanisms saturate when free energy (which scales with Ω_i) equilibrates, therefore predicting $\delta n/n \approx \kappa/k_x$.^{6,21} Furthermore, analogy to the self-consistent theory of beam-plasma interaction³³ suggests that (a) the typical bounce frequency is increased somewhat by temporal evolution of the potential (by a factor of about $\frac{5}{3}$ in the present case), and (b) the intensity of the wave oscillates with the frequency of this bounce motion. These quantitative predictions are consistent with the results of numerical integration of the coupled-mode version of the gyrokinetic system.¹

Qualitatively, one may expect the complicated motion in the x - y_w plane to lead to overstabilization, in the sense that particles will continue to move along the gradient so as to damp the wave below the stable threshold, whereupon the linear instability mechanism is resurrected. This behavior is observed in the simulations. Both trapped and untrapped particles will exchange energy with the wave, so that nearly periodic variation of the amplitude, with details depending nontrivially on initial conditions, will persist until the phases of the individual particle trajectories are thoroughly mixed. A self-consistent

steady state may be reached after ergodization of trapped particles along level curves of the Hamiltonian.¹⁵ (This randomization could be hastened by collisions, which tend to disrupt the coherent relation between the electrostatic field and the electron response. The tendency of collisions to decorrelate particles from the wave suggests that they will increase the intensity required for saturation.²⁷) The results presented in Ref. 1 indicate that for large parallel wavelengths there is some nonlinear destabilization process, involving both ions and electrons, leading to saturation at larger amplitudes than those implied by spatial trapping alone. This process occurs only if the parallel nonlinear terms are retained, and probably involves phase space trapping of electrons; compare the collisional case discussed by Ott and Manheimer.¹⁶ We conjecture that the perpendicular resonance may interact with the phase space resonance through time variation of $v_{\parallel}$, leading to radial diffusion even in the steady-state potential. In the fully three-dimensional situation, we anticipate that the coexistence of modes with different phase velocities may lead to resonance overlap, so that a statistical theory would be appropriate. The presence of shear in the magnetic field will introduce more complicated radial dependence into the $\mathbf{E} \times \mathbf{B}$ velocity; we do not expect this to alter the basic physics of the orbit distortion theory.

Although the single particle picture is intuitively appealing, the wave and the instability mechanism are collective processes. That is, the density response is computed from a velocity integral over all particles, and the basic linear wave is most simply described in the Fourier domain, rather than in configuration space. The simulations confirm that the fully nonlinear evolution is well described by a small number of wave numbers, so the spatial Fourier transform is appropriate for the nonlinear theory as well.

The density of the trapped and nearly-trapped particles can be decomposed into harmonics of the linearly unstable perturbation, and we can identify these with the terms entering the nonlinear gyrokinetic description as convolutions. Of course, since the Fourier

transformation involves integrations over the x - y plane, and trajectories vary appreciably over this domain (even for a single value of $v_{\parallel}$), the details of trapping are smeared out in the mode coupling picture. However, the effect described above involves collective contributions of particles in large bands of phase space, so the physical processes are not obscured by this smearing.

The infinite set of coupled equations for the Fourier amplitudes is equivalent to the original kinetic equation in real space. The equivalence is lost, however, when the set of Fourier amplitudes is truncated to a finite number, as is necessary for numerical study. The adequacy of such a truncation is suggested by the collective nature of the response, and we expect that the low order Fourier harmonics will constitute a close approximation to the full description. This expectation is corroborated by the smoothness of typical electron trajectories described above (*cf.* figures in Ref. 4). In fact, the particle simulations show that the second harmonic $f(2,0)$ alone suffices to account for the nonlinear effects in the case of moderate $k_{\parallel}$. In order to confirm the predominant role of the second harmonic $f(2,0)$ in the particle trajectory description, we numerically integrated the orbits in the stationary potential [Eq. (10)] and Fourier-transformed the induced density response. As expected, in the linear growth phase, $n_e(1,1)$ and $n_e(1,-1)$ dominate, and $n_e(2,0)$ grows on the time scale of saturation, while the other harmonics $n_e(3,1)$, $n_e(4,0)$, etc. remain negligible for the observed times [Figs. (1-4)]. In the appendix, we show that the truncation does not vitiate the conservation of energy by the gyrokinetic description.

IV. MODEL EQUATION

The goal of this section is to quantify and model the role of the advective nonlinearity. As is common in the study of nonlinear phenomena, we choose a simple case where the effect

can be separated from other physical processes which usually complicate its appearance in practice. In the present case, we observe that the three-mode system [Eqs. (6-9)] conserves the equality of the modes $\phi(1, 1)$ and $\phi(1, -1)$ if this equality is imposed initially. Although the case of equal initial amplitudes is obviously restrictive, the behavior observed in the simulations is not sensitive to moderate differences in initial conditions. Therefore, for simplicity, we shall consider here the case $\phi(1, 1) = \phi(1, -1)$.

In order to obtain a quantitative description of the back reaction of the potential on its own growth mechanism, it is convenient to introduce a new set of variables which describe the slow evolution of the three-mode system. We eliminate the fast oscillation by removing the real part of the linear frequency, Ω_l [Eq. (5)]. Our new variables are a growth term, $\psi(t) := \ln[\phi(1, 1) \exp\{i\Omega_l t\}]$, which is proportional to time t in the linear theory; a nonadiabatic response coefficient, $\xi(v, t) := (f(1, 1) - \phi(1, 1))/\phi(1, 1)$; and the second density harmonic, $g(v, t) := -if(2, 0)$, which is a real quantity for this case of equal initial amplitudes. In terms of these variables, the three-mode system including the advective electron nonlinearity is

$$\frac{\partial \xi}{\partial t} - i \left(\Omega_l - k_{\parallel} v + i \frac{d\psi}{dt} \right) \xi = -i \left(\omega_{*e} - \Omega_l - i \frac{d\psi}{dt} - 2k_x k_y g \right), \quad (15)$$

$$\frac{\partial g}{\partial t} = -4k_x k_y \exp(2 \operatorname{Re} \psi) \operatorname{Im} \xi, \quad (16)$$

$$\frac{d\psi}{dt} = \frac{i}{\tau(1 - \Gamma)} (\omega_{*e} - \Omega_l - 2k_x k_y \langle g \rangle + \langle k_{\parallel} v \xi \rangle), \quad (17)$$

where $\langle \dots \rangle$ indicates $\int dv F_M \dots$. The role of the term proportional to g in Eq. (15) is consistent with the notion that the second harmonic, which approximates the perturbed particle motion, impairs the electron destabilization. [Compare the right-hand side of Eq. (15) with the numerator of Eq. (5b).] Upon noting that this term competes with the ω_{*e} term and that Ω_l itself is proportional to ω_{*e} , we conclude that the free energy of the density gradient can no longer drive the instability when this nonlinear, dynamic

correction to the effective gradient grows up. Our goal is to quantify this intuitive notion as much as possible.

Upon solving Eq. (15) formally by use of a Green function, we find

$$\xi(t) = G(t; 0)\xi(0) - i \int_0^t dt' G(t; t') \left(\omega_{se} - \Omega_l - 2k_x k_y g(t') - i \frac{d\psi(t')}{dt'} \right), \quad (18)$$

where $G(t; t') := \exp[i(\Omega_l - k_{\parallel} v)(t - t') - \psi(t) + \psi(t')]$. Since we are interested in evolution of the system for relatively long times, we may temporally coarse-grain over the short phase mixing time and ignore the initial conditions (as in the linear theory) when evaluating a velocity moment of Eq. (18):

$$\begin{aligned} \langle k_{\parallel} v \xi \rangle \approx & -i \int_0^t dt' \left\langle \exp\{i\Omega_l(t - t') - ik_{\parallel} vt - \psi(t) + \psi(t')\} \right. \\ & \times \left. \left(\omega_{se} - \Omega_l - 2k_x k_y g(t') - i \frac{d\psi(t')}{dt'} \right) \frac{\partial}{\partial t'} (e^{ik_{\parallel} vt'}) \right\rangle. \end{aligned} \quad (19)$$

(This approximation exaggerates the rate of approach to a steady state, according to the account given in Sec. III.) We integrate Eq. (19) by parts and substitute the result into Eq. (17):

$$\frac{d\psi}{dt} \approx \frac{\Omega_l}{1 + \tau(1 - \Gamma)} \int_0^t dt' \left\langle G(t; t') (\omega_{se} - \Omega_l - 2k_x k_y g(t')) \right\rangle + O(\gamma_l / \Omega_l). \quad (20)$$

Our use of a formal expansion in orders of γ_l / Ω_l is consistent with the assertion that only nearly resonant particles are involved in the exchange of energy, and that the modes described by the linear theory remain well defined and only mildly distorted when the nonlinear terms are retained. The adequacy of such a weak-coupling approximation²³ for the present problem has been confirmed numerically. We proceed by adopting a stationary approximation for the Green function,

$$G(t; t') \rightarrow G_s(t - t') := \exp \left\{ \left[i(\Omega_l - k_{\parallel} v) - \frac{d\psi(t)}{dt} \right] (t - t') \right\},$$

since G occurs under velocity integrals and we are interested in a time scale longer than that for phase mixing. Integration of Eq. (20) yields

$$\begin{aligned}\delta\psi(t) &:= \psi(t) - \psi(0) \\ &\approx \frac{\Omega_l}{1 + \tau(1 - \Gamma)} \int_0^t dt \int_0^{t'} dt'' \langle G_s(t' - t'') (\omega_{se} - \Omega_l - 2k_x k_y g(t'')) \rangle\end{aligned}\quad (21)$$

$$= \frac{\Omega_l}{1 + \tau(1 - \Gamma)} \int_0^t dt'' \langle (\omega_{se} - \Omega_l - 2k_x k_y g(t'')) R(t - t'') \rangle, \quad (22)$$

where $R(t) := \int_0^t d\tau G_s(\tau)$ yields the usual resonant denominator in the linear regime. In a similar fashion, we approximate the density harmonic:

$$\begin{aligned}g(t) &= -4k_x k_y \int_0^t dt' \Phi(t') \operatorname{Im} \xi(t') \\ &\approx 4k_x k_y \int_0^t dt' \Phi(t') \int_0^{t'} dt'' \langle \omega_{se} - \Omega_l - 2k_x k_y g(t'') \rangle \operatorname{Re} G_s(t' - t''),\end{aligned}\quad (23)$$

$$\begin{aligned}&\approx 4k_x k_y (\omega_{se} - \Omega_l) \int_0^t dt' \Phi(t') R(t') \\ &\quad - 8(k_x k_y)^2 \int_0^t dt' \Phi(t') \int_0^{t'} dt'' g(t'') \operatorname{Re} G_s(t' - t''),\end{aligned}\quad (24)$$

where $\Phi := |\phi|^2 = \exp(2\operatorname{Re} \psi)$ is proportional to the (slowly varying) perturbation energy.

Since Eq. (22) has the form of a convolution, a Fourier transform in time is indicated:

$$\begin{aligned}\tilde{\psi}(\omega) &:= \int_0^\infty dt e^{i\omega t} \delta\psi(t) \\ &\approx \frac{\Omega_l}{1 + \tau(1 - \Gamma)} \left\langle \left(\frac{\omega_{se} - \Omega_l}{-i\omega} - 2k_x k_y \tilde{g}(\omega) \right) \tilde{R}(\omega) \right\rangle.\end{aligned}\quad (25)$$

The fact that the term in Eq. (25) containing $\tilde{g}$ contributes only in the nonlinear regime, near saturation, leads us to approximate $\Phi(t)$ in Eq. (24) by a saturation value which we can obtain by an iterative treatment of Eqs. (22-23), viz.,

$$\begin{aligned}g_l(t) &\approx -4k_x k_y \int_0^t dt' \Phi(0) e^{2\gamma_l t'} \operatorname{Im} \left(\frac{\omega_{se} - \Omega_l}{\Omega_l - k_{\parallel} v + i\gamma_l} \right) \\ &\approx \frac{2k_x k_y}{\gamma_l} \Phi(t) (\omega_{se} - \Omega_l) \pi \delta(k_{\parallel} v - \Omega_l).\end{aligned}\quad (26)$$

Then the nonlinear growth rate is

$$\gamma_{nl} := \operatorname{Re} \frac{d\psi}{dt} \approx \gamma_l \left(1 - \frac{4(k_x k_y)^2 \Phi}{\gamma_l^2} \right). \quad (27)$$

Thus, the saturation value is roughly $\Phi_s := (\gamma_l/2k_x k_y)^2$. (This estimate is nearly identical to the bounce period calculation of the previous section.) The wave-kinetic equation associated with Eq. (27),

$$\frac{d\Phi}{dt} = 2\gamma_{nl}\Phi, \quad (28)$$

can be integrated numerically and predicts a time-dependent approach to saturation according to the relation

$$\Phi(t) = \Phi_s \left[1 + \left(\frac{\Phi_s}{\Phi_0} - 1 \right) \exp(-2\gamma_l t) \right]^{-1}. \quad (29)$$

An iterative scheme to extend this approach to intermediate times was proposed in Ref. 1, where we introduced a term *ad hoc* as a heuristic substitute for the more involved analysis which follows.

Equation (29) does not describe the intensity oscillations in the nonlinear stage which are expected on physical grounds and observed in the numerical study. We can recover that behavior by extending the Fourier analysis begun in the previous paragraph. To take into account the appearance of the intensity under integrations, we will approximate it by a time-averaged value, which because of the oscillations is roughly $\frac{1}{2}\Phi_s$. Then the Fourier transform of Eq. (24) is, after approximating R by its real part,

$$-i\omega \tilde{g}(\omega) \approx \frac{4k_x k_y (\omega_{se} - \Omega_l)}{1 + 4k_x^2 k_y^2 \Phi_s \tilde{R}(\omega)} \tilde{S}(\omega), \quad (30)$$

where $\tilde{S}(\omega)$ is the Fourier transform of $S(t) := \Phi(t)R(t)$. Hence Eq. (25) becomes

$$\tilde{\psi}(\omega) \approx \frac{\Omega_l}{1 + \tau(1 - \Gamma)} \left(\frac{\omega_{se} - \Omega_l}{-i\omega} \right) \left\langle \left(1 - \frac{8k_x^2 k_y^2 \tilde{S}(\omega)}{1 + 4k_x^2 k_y^2 \Phi_s \tilde{R}(\omega)} \right) \tilde{R}(\omega) \right\rangle. \quad (31)$$

The linear growth rate can be recovered by neglecting the term proportional to $\tilde{S}(\omega)$ and using a linear approximation for $\tilde{R}(\omega)$. To simplify the problem further we approximate $\tilde{S}(\omega)$ and $\tilde{R}(\omega)$. Using the average value for Φ in the expression for S gives $\tilde{S}(\omega) \approx \frac{1}{2} \Phi_s \tilde{R}(\omega)$. A quasilinear analysis of R yields

$$\begin{aligned} \tilde{R}(\omega) &\approx \int_0^\infty dt e^{i\omega t} \int_0^t dt' \exp(-\gamma t') \cos[(\Omega_l - k_{\parallel} v)t'] \\ &\approx -\left(\frac{1}{\omega}\right) \frac{\omega + i\gamma_l}{(\omega + i\gamma_l)^2 - (\Omega_l - k_{\parallel} v)^2}, \end{aligned} \quad (32)$$

so Eq. (31) is roughly equivalent to

$$\begin{aligned} \tilde{\psi}(\omega) &\approx \frac{\Omega_l}{1 + \tau(1 - \Gamma)} \left(\frac{\omega_{se} - \Omega_l}{-i\omega} \right) \left\langle \frac{\tilde{R}(\omega)}{1 + 4k_x^2 k_y^2 \Phi_s \tilde{R}(\omega)} \right\rangle \\ &\approx \frac{\Omega_l}{1 + \tau(1 - \Gamma)} \left(\frac{\omega_{se} - \Omega_l}{-i\omega} \right) \left\langle \frac{1}{(\omega + i0)^2 - (\Omega_l - k_{\parallel} v)^2 - 4k_x^2 k_y^2 \Phi_s} \right\rangle. \end{aligned} \quad (33)$$

The velocity integration may be estimated by noting that the evolution is slow, and hence dominated by small frequencies ω :

$$\begin{aligned} \left\langle \frac{1}{(\omega + i0)^2 - (\Omega_l - k_{\parallel} v)^2 - 4k_x^2 k_y^2 \Phi_s} \right\rangle &\approx \frac{1}{2} (\omega^2 - 4k_x^2 k_y^2 \Phi_s)^{-\frac{1}{2}} \\ &\times \left\langle \frac{1}{k_{\parallel} v - \Omega_l - (\omega^2 - 4k_x^2 k_y^2 \Phi_s)^{\frac{1}{2}} - i0 \operatorname{sgn} \omega} \right. \\ &\quad \left. - \frac{1}{k_{\parallel} v - \Omega_l + (\omega^2 - 4k_x^2 k_y^2 \Phi_s)^{\frac{1}{2}} + i0 \operatorname{sgn} \omega} \right\rangle \\ &\approx \frac{i\pi \operatorname{sgn} \omega}{2k_{\parallel}} (\omega^2 - 4k_x^2 k_y^2 \Phi_s)^{-\frac{1}{2}} \\ &\times \left[F_M \left(\frac{\Omega_l + (\omega^2 - 4k_x^2 k_y^2 \Phi_s)^{\frac{1}{2}}}{k_{\parallel}} \right) \right. \\ &\quad \left. + F_M \left(\frac{\Omega_l - (\omega^2 - 4k_x^2 k_y^2 \Phi_s)^{\frac{1}{2}}}{k_{\parallel}} \right) \right] \\ &\approx \frac{i\pi \operatorname{sgn} \omega}{k_{\parallel}} (\omega^2 - 4k_x^2 k_y^2 \Phi_s)^{-\frac{1}{2}} F_M \left(\frac{\Omega_l}{k_{\parallel}} \right). \end{aligned}$$

The inversion of the Fourier transform can be carried out by contour integration. Comparison with the linear dispersion relation, Eq. (5), leads to

$$\gamma_{nl} = \gamma_l J_0(2k_x k_y \sqrt{\Phi_s} t), \quad (34)$$

where J_0 is the ordinary Bessel function of the first kind. The implication of Eq. (34) that the saturation process commences at $t = 0$ arises from our use of a quasistatic potential in evaluating $\tilde{S}(\omega)$. A better approximation, in the form of a self-consistent evolution equation for the intensity, can be obtained by replacing the estimated average value with a time integral, viz.,

$$\frac{d|\phi|}{dt} = |\phi| \gamma_l J_0\left(\frac{\gamma_l}{\sqrt{\Phi_s}} \int_0^t dt |\phi|\right). \quad (35)$$

The result of numerical integration of Eq. (35) is compared with that of Eqs. (6-9) in Figs. (5-6). Eq. (35) correctly predicts the linear growth as well as the dynamic saturation due to the complicated trajectories of electrons in the evolving potential field. The magnitude of oscillations of the nonlinear growth rate observed in the simulations is in reasonable agreement with the theoretical value. The time scale of these oscillations agrees well for the weak instability case ($\theta = 0.01$), but for the stronger instability ($\theta = 0.002$) several of the heuristic simplifications (notably the weak-coupling approximation) used in arriving at Eq. (35) are invalid, so the predicted time scale is too large. As mentioned above, our analysis overestimates the rate of approach to a static steady state.

V. CONCLUSIONS

This paper has described the saturation of the universal instability through the impairment of the destabilization mechanism (inverse electron Landau damping). We found that the electron $\mathbf{E} \times \mathbf{B}$ trajectories, which lead to energy loss to the electrostatic field

in the linear theory, also allow extraction of some of that energy from a finite amplitude potential. A nonresonant mode-coupling description allowed for nonperturbative analysis of this effect. We derived an approximate evolution equation for the electrostatic potential which agrees qualitatively and quantitatively with gyrokinetic particle simulations. These simulations showed that the stabilization mechanism presented here can predominate over all others in some cases. The details of cooperation between this mechanism and others, and the application to toroidal drift instabilities, remain to be investigated, but the similarity of the linear theory to the toroidally induced universal mode suggests that the electron mechanism may contribute to saturation there as well. The process discussed above is also important in the nonlinear evolution of collisional drift instabilities.²⁷

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APPENDIX

Here we establish that energy is conserved by the reductions leading to the system of Eqs. (6-9). A more complete discussion of energy conservation in kinetic theories (with extensions to statistical closures) will be found in Refs. 34 and 35. It suffices to show that the change in electrostatic energy is compensated by the energy flow to particles. Since parallel ion currents have been neglected, and the perpendicular current is proportional to the $\mathbf{E} \times \mathbf{B}$ velocity, this conservation is assured if the following expression vanishes:

$$\int d^3x \left(\frac{1}{2} \frac{\partial}{\partial t} E^2 - \mathbf{j}_e \cdot \mathbf{E} \right) =$$

$$\sum_k \phi_k^* \left(k^2 [1 + \tau(1 - \Gamma(k))] \frac{d\phi_k}{dt} - \int dv_{\parallel} v_{\parallel} f_k^e \right) [1 + \tau(1 - \Gamma(k))].$$

The factors of $[1 + \tau(1 - \Gamma(k))]$ arise because ϕ is the average potential around an ion gyroradius.²⁸ For the three-mode truncation, the sum over wave numbers reduces to

$$k^2 [1 + \tau(1 - \Gamma(k))] \frac{d}{dt} [\phi(1, 1) \phi^*(1, 1)] + i \int dv_{\parallel} k_{\parallel} v_{\parallel} (f_e(1, 1) \phi^*(1, 1) - f_e^*(1, 1) \phi(1, 1))$$

plus the analogous term with wave numbers $(1, -1)$. Each of these terms vanishes independently, as is easily seen by taking the time derivative of Eq. (9) and substituting from Eqs. (6) and (8).

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FIGURE CAPTIONS

Fig. 1. Density response to imposed potential $\phi \propto \cos(k_x x) \cos(k_y y)$, calculated by integration along characteristics given by Eqs. (11-12). Evolution of Fourier components of the density, $|\tilde{n}(lk_x, mk_y)|^2$, measured in arbitrary units, is shown for some nontrivial modes: (a) $l = 1, m = 1$; (b) $l = 3, m = 1$; (c) $l = 2, m = 0$; (d) $l = 4, m = 0$.

Fig. 2. Evolution of the field intensity $|\phi(1,1)|^2$ in the strongly unstable case of Ref. 1, $\theta = 0.002$: (a) integration of mode coupling equations [Eqs. (6-9)]; (b) integration of reduced equation [Eq. (35)].

Fig. 3. Evolution of field intensity in the weakly unstable case of Ref. 1, $\theta = 0.01$.

Fig. 1

Fig. 2

Fig. 3