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ANALYTIC FORM FOR THE OFF-SHELL p - p T MATRIX

FOR THE GRAZ SEPARABLE INTERACTION

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ANALYTIC FORM FOR THE OFF-SHELL p-p T MATRIX
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Abstract

Analytic expressions are presented for the off-shell p-p T matrix for the Graz $\ell = 0$ and $\ell = 1$ separable potentials. Physical on-shell results are also obtained and used to compute the phase shifts for the 1S_0 , 3P_0 , 3P_1 and 3P_2 states. Improved fits are obtained to the experimental data. Analytic expressions are also presented for the effective range function and are used to compute the low-energy scattering parameters.

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1. Introduction

The process of including the Coulomb interaction in the computation of nucleon-nucleon scattering effects has always been one which involved numerical approximation techniques--either to evaluate integrals or to solve differential equations. In carrying out a study of the effects of the Coulomb interaction on the off-shell nucleon-nucleon T matrix, we had a need for a complete description of the off-shell p-p T matrix. The recent work of van Haeringen and van Wageningen (1975) demonstrated how one could obtain analytic T matrices for separable potentials plus the Coulomb interaction. Using their results, we have obtained expressions for the p-p T matrix for the separable potentials of Crepinsek, et al. (1975) (referred to as the Graz potentials) in the 1S_0 , 3P_0 , 3P_1 and 3P_2 states.

The phase shifts for the Graz potentials have also been re-evaluated using the physical on-shell form of the T matrix. We have found that the Graz parameters do not give a good fit in the 1S_0 case, and we have performed a parameter search to improve the fit. Adjustments have also been made to the parameters in the other states as well. We suspect that numerical errors involved in evaluating the integrals required in the p-p case lead to this discrepancy.

In addition, we have been able to compute analytic expressions for the effective range function in a few states, and have used these to compute the low-energy scattering parameters.

2. Formalism

This section outlines the formalism required to compute the partial-wave T operator in the presence of the Coulomb interaction for the separable potential described by Crepinsek, et al. (1975). The Coulomb potential is described by

$$V_c(r) = 2k\gamma/r \equiv -2s/r \quad (2.1)$$

where γ is the energy-independent Sommerfeld parameter. Adopting the usual convention $\hbar = 2m = 1$ where m is the reduced mass, we find that

$$|s|^{-1} = \text{the Bohr radius.} \quad (2.2)$$

It is well known that the scattering amplitude $f(\theta)$ for the sum of the Coulomb potential V_c and a short range potential V_s may be expressed as

$$f(\theta) = f_c(\theta) + f_{cs}(\theta) \quad (2.3)$$

where $f_c(\theta)$ is the Coulomb amplitude. The partial-wave expansion of $f_{cs}(\theta)$ is given by

$$f_{cs}(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) e^{2i\sigma_{\ell}} e^{i\delta_{\ell}^c} \sin \delta_{\ell}^c P_{\ell}(\cos \theta) \quad (2.4)$$

where the σ_{ℓ} are the standard Coulomb phases and the δ_{ℓ}^c are called the Coulomb-modified nuclear phase shifts.

One may also separate the total T operator into two terms

$$T = T_c + T_{cs} \quad (2.5)$$

where T_c is the T operator corresponding to V_c ,

$$T_c(E) = V_c + V_c G_0(E) T_c(E) \quad (2.6)$$

and T_{cs} is given by

$$T_{cs}(E) = [1 + T_c(E) G_0(E)] \tilde{T}_{cs}(E) [1 + G_0(E) T_c(E)] \quad (2.7)$$

where $\tilde{T}_{cs}(E)$ satisfies the operator equation

$$\tilde{T}_{cs}(E) = V_s + V_s G_c(E) \tilde{T}_{cs}(E). \quad (2.8)$$

The resolvents G_0 and G_c are defined by

$$G_0(E) = (E - H_0)^{-1} \quad (2.9)$$

$$G_c(E) = (E - H_0 - V_c)^{-1} \quad (2.10)$$

with H_0 representing the kinetic energy operator.

Partial-wave projections of these operators may be formed by following the prescription

$$\langle \mathbf{x}' | A_\ell | \mathbf{x} \rangle = \int d\hat{\mathbf{x}} P_\ell(\hat{\mathbf{x}}' \cdot \hat{\mathbf{x}}) \langle \hat{\mathbf{x}}' | A | \hat{\mathbf{x}} \rangle \quad (2.11)$$

with $\mathbf{x}$ representing $\mathbf{r}$ or $\mathbf{p}$. Thus

$$\langle \hat{\mathbf{x}}' | A | \hat{\mathbf{x}} \rangle = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} P_\ell(\hat{\mathbf{x}}' \cdot \hat{\mathbf{x}}) \langle \mathbf{x}' | A_\ell | \mathbf{x} \rangle. \quad (2.12)$$

In this way we define the operators $T_{c,\ell}$, $V_{c,\ell}$, $T_{cs,\ell}$ and $T_{cs,\ell}^\gamma$. They satisfy the same operator equations as the corresponding full operator.

For p-p scattering, the potential employed by Crepinsek, et al. (1975), referred to as the Graz potential, has the form

$$V_{s,\ell} = \sum_{i=1}^2 |g_{i,\ell}\rangle \lambda_{i,\ell} \langle g_{i,\ell}| \quad (2.13)$$

with

$$\langle p | g_{1,\ell} \rangle = p^\ell (p^2 + \alpha^2)^{-\ell-1} \quad (2.14)$$

$$\langle p | g_{2,\ell} \rangle = p^{\ell+2} (p^2 + \beta^2)^{-\ell-2} \quad (2.15)$$

where the parameters have been determined for the 1S_0 , 3P_0 , 3P_1 , 3P_2 and 1D_2 states.* The operator T_s corresponding to V_s is also separable. For the Coulomb-modified operator T_{cs} , one can define Coulomb-modified form factors (van Haeringen and van Wageningen, 1975; hereafter referred to as VHW1) $|g_{i,\ell}^c(E)\rangle$

* Note that we have adopted a different sign convention for $V_{s,\ell}$ than that used in VHW1. Furthermore, VHW1 included a $\sqrt{2/\pi}$ factor in their $\langle p | g_{i,\ell} \rangle$ expressions.

by

$$|g_{i,l}^c(E)\rangle = (1 + T_c(E) G_0(E)) |g_{i,l}\rangle \quad (2.16)$$

in terms of which one can express $T_{cs,l}$,

$$T_{cs,l}(E) = \sum_{i,j=1}^2 |g_{i,l}^c(E)\rangle \tau_{ij,l}^c(E) \langle g_{j,l}^c(E^*)| \quad (2.17)$$

with

$$(\tau^c(E))_{ij,l}^{-1} = (\Lambda^{-1})_{ij} + \langle g_{i,l} | G_c(E) | g_{j,l} \rangle. \quad (2.18)$$

Here Λ is a diagonal matrix with elements $\lambda_i \delta_{ij}$.

In order to obtain $T_{cs,l}$ one needs to evaluate the form factors $\langle p | g_{i,l}^c(E) \rangle$ and the matrix elements $\langle g_{i,l} | G_c(E) | g_{j,l} \rangle$. Methods for evaluating these have been outlined in VHW1. Results have been calculated for the basic form factors $\langle p | g_{1,l=0} \rangle$ as well as for $\langle p | \bar{g}_{1,l=0} \rangle = (p^2 + \alpha^2)^{-2}$. These calculations have been extended to the $l = 1$ state (van Haeringen, 1977; hereafter referred to as VH2). We have used these results in deriving the complete expressions for theGraz p-p interactions.

For the 1S_0 state,

$$\langle p | g_{1,l=0}^c(k^2) \rangle = \frac{1}{p^2 + \alpha^2} - \frac{1}{k^2 + \alpha^2} \frac{k}{p} [F_{i\gamma}(Aa) - F_{i\gamma}(A/a)] \quad (2.19)$$

and

$$\langle p | g_{2,l=0}^c(k^2) \rangle = \frac{p^2}{(p^2 + \beta^2)^2} - \frac{k\gamma\beta}{(p^2 + \beta^2)(k^2 + \beta^2)} + \frac{k(\gamma\beta - k)}{(k^2 + \beta^2)^2} \frac{k}{p} [F_{i\gamma}(Ba) - F_{i\gamma}(B/a)]. \quad (2.20)$$

Here the following notations have been employed:

$$A = (\alpha + ik)/(\alpha - ik)$$

$$B = (\beta + ik)/(\beta - ik)$$

$$a = (p - k)/(p + k) \quad (2.21)$$

$$F_{i\gamma}(z) = {}_2F_1(1, i\gamma; 1 + i\gamma; z). \quad (2.22)$$

The three corresponding matrix elements $\langle g_{i,l} | G_c(E) | g_{j,l} \rangle$ are also readily found using the results of VHW1.

$$\langle g_{1,\ell=0} | G_c | g_{1,\ell=0} \rangle = -\frac{1}{2\alpha(\alpha+ik)^2} - \frac{2ik}{(\alpha^2+k^2)^2} F_{i\gamma}(A^2) \quad (2.23)$$

$$\begin{aligned} \langle g_{1,\ell=0} | G_c | g_{2,\ell=0} \rangle &= \frac{-\beta}{2(\alpha+\beta)^2(\beta^2+k^2)} + \frac{k(\beta\gamma-k)}{(\alpha+\beta)(\alpha+ik)(\beta+ik)(\beta^2+k^2)} \\ &+ \frac{2ik^2(\beta\gamma-k)}{(\alpha^2+k^2)(\beta^2+k^2)^2} F_{i\gamma}(AB) \end{aligned} \quad (2.24)$$

$$\begin{aligned} \langle g_{2,\ell=0} | G_c | g_{2,\ell=0} \rangle &= -\frac{(\beta^2+3k^2-2\beta k\gamma)}{16\beta(\beta^2+k^2)^2} - \frac{k^2(k-\beta\gamma)^2}{2\beta(\beta+ik)^2(\beta^2+k^2)^2} \\ &- \frac{2ik^3(k-\beta\gamma)^2}{(\beta^2+k^2)^4} F_{i\gamma}(B^2). \end{aligned} \quad (2.25)$$

In order to evaluate the Coulomb-modified phase shift δ_ℓ^c , we must derive expressions for the physical on-shell T matrices, i.e., using the appropriate asymptotic states as defined by van Haeringen (1976). The physical on-shell T matrix is given by

$$\begin{aligned} t_{cs,\ell}(k) &= \langle k\infty - | T_{cs,\ell}(k^2) | k\infty \rangle \\ &= \sum_{i,j=1}^2 \langle k\infty - | g_{i,\ell}^c \rangle \tau_{ij}^c \langle g_{j,\ell}^c | k\infty \rangle \end{aligned} \quad (2.26)$$

with τ^c given previously in equation (2.18). The method of deriving the form factors $\langle g_{i,\ell}^c | k\infty \rangle$ has been described by van Haeringen (1976). Following this technique, we find that

$$\langle g_{1,\ell=0}^c | k\infty \rangle = (\alpha^2+k^2)^{-1} \Gamma(1+i\gamma) e^{-\pi\gamma/2} A^{-i\gamma}, \quad (2.27)$$

$$\langle g_{2,\ell=0}^c | k\infty \rangle = k(k-\beta\gamma)(k^2+\beta^2)^{-2} \Gamma(1+i\gamma) e^{-\pi\gamma/2} B^{-i\gamma}. \quad (2.28)$$

The physical on-shell part of T_{cs} for the Graz interaction can now be expressed in the following familiar form,

$$\begin{aligned} t_{cs,\ell}(k) &= \{ \lambda_1 [1 - \lambda_2 H_\ell(k^2)] g_{1,\ell}^2(k) + \lambda_2 [1 - \lambda_1 G_\ell(k^2)] g_{2,\ell}^2(k) \\ &+ 2\lambda_1 \lambda_2 M_\ell(k^2) g_{1,\ell}(k) g_{2,\ell}(k) \} / D_\ell(k^2) \end{aligned}$$

with

$$D_\ell(k^2) = (1 - \lambda_1 G_\ell(k^2))(1 - \lambda_2 H_\ell(k^2)) - \lambda_1 \lambda_2 M_\ell^2(k^2) \quad (2.29)$$

and where

$$\begin{aligned} g_{1,\ell}(k) &\equiv \langle g_{1,\ell}^c | k \infty \rangle \\ G_\ell(k^2) &\equiv \frac{\pi}{2} \langle g_{1,\ell} | G_c(k^2) | g_{1,\ell} \rangle \\ H_\ell(k^2) &\equiv \frac{\pi}{2} \langle g_{2,\ell} | G_c(k^2) | g_{2,\ell} \rangle \\ M_\ell(k^2) &\equiv \frac{\pi}{2} \langle g_{1,\ell} | G_c(k^2) | g_{2,\ell} \rangle. \end{aligned} \quad (2.30)$$

The phase shift δ_ℓ^c may then be evaluated using

$$k \cot \delta_\ell^c = -\frac{2}{\pi} \operatorname{Re}[e^{2i\sigma_\ell}/t_{cs,\ell}]. \quad (2.31)$$

For the 3P_1 and 3P_2 states, a one-term separable interaction was employed by Crepinsek, et al. (1975) for which the necessary calculations have been done by VH2. Here

$$\langle g_{1,\ell=1}^c | k \infty \rangle = k(\alpha^2 + k^2)^{-2} \Gamma(2+i\gamma) e^{-\pi\gamma/2} A^{-i\gamma}. \quad (2.32)$$

The expressions for $\langle p | g_{1,\ell=1}^c \rangle$ and $\langle g_{1,\ell=1} | G_c | g_{1,\ell=1} \rangle$ are given in equations (2.3) and (2.4) of VH2.

A rank two interaction has been employed for the 3P_0 state. Extending the results of VH2 we find that

$$\begin{aligned} \langle p | g_{2,\ell=1}^c(k^2) \rangle &= \frac{p^3}{(p^2 + \beta^2)^3} + \frac{i\gamma k^2(\beta^2 - k^2)}{p(\beta^2 + k^2)^3} + \frac{\beta k \gamma (p^2 - k^2)(5p^2 k^2 + \beta^2 p^2 + 3\beta^2 k^2 - \beta^4)}{4p(\beta^2 + p^2)^2(\beta^2 + k^2)^3} \\ &+ \frac{k^3(2k - \beta\gamma)}{2p(\beta^2 + k^2)^3} \left[\left(i\gamma - \frac{p^2 + k^2}{2pk} \right) F_{i\gamma}(Ba) + \left(i\gamma + \frac{p^2 + k^2}{2pk} \right) F_{i\gamma}(B/a) \right] \\ &+ \frac{\beta \gamma k^3}{2p(\beta^2 + k^2)^3} \left[\left(i\gamma - \frac{p^2 + k^2}{2pk} \right) \frac{1}{1 - Ba} + \left(i\gamma + \frac{p^2 + k^2}{2pk} \right) \frac{1}{1 - B/a} \right] \end{aligned} \quad (2.33)$$

with the associated asymptotic form,

$$\langle g_{2,\ell=1}^c | k \infty \rangle = \frac{(2k - \beta\gamma)}{2(\beta^2 + k^2)^3} k^2 \Gamma(2 + i\gamma) e^{-\pi\gamma/2} B^{-i\gamma}. \quad (2.34)$$

The lengthy expressions for the corresponding matrix elements $\langle g_{i,\ell=1} | G_c | g_{j,\ell=1} \rangle$ have been deferred to the appendix.

With the above results, one has an analytic expression for the off-shell T matrix $\langle p | T_\ell(k^2) | p' \rangle$ for the $\ell = 0$ and $\ell = 1$ partial waves of the Graz p-p potential.

3. Phase Shifts

Employing the results of the previous section, we have computed the phase shifts for the Graz potentials in the 1S_0 , 3P_0 , 3P_1 and 3P_2 states. In the 1S_0 case, the Graz parameters yield a phase shift which is too small at low energies. This error is possibly due to the difficulty of accurately evaluating numerically the integrals involved with the Coulomb interaction. We have modified the parameter sets to significantly improve the fit to the data. Figure 1 shows the different fits for the 1S_0 case. The parameters are presented in Table 1.

In re-examining the fits for the 3P_0 and 3P_1 partial waves, we have chosen to use the low-energy phase shift data derived from the analysis of Arndt et al. (1974) by Kermode and McKerrel (1977). This data is claimed to give a better χ^2 fit than the Livermore results, MAW-x (MacGregor et al. 1969). The resulting potential parameters are also listed in Table 1. In these cases, and in the 3P_2 case, there is only a small change in the phase shift.

4. The Effective Range Function

In the theory of scattering by a short-range potential, the effective range function defined by

$$K_\ell(k^2) = k^{2\ell+1} \cot \delta_\ell(k) \quad (4.1)$$

contains useful information about the interaction. Subject to certain constraints on the potential form, this function is real analytic at $k^2 = 0$, with the following expansion,

$$K_\ell(k^2) \approx -a_\ell^{-1} + \frac{1}{2} r_\ell k^2 - P r_\ell^3 k^4 + Q r_\ell^5 k^6 \quad (4.2)$$

used to define the low-energy scattering parameters.

If the potential is the sum of a short-range potential V_s and a Coulomb potential V_c , then the appropriate form of the effective range function becomes

$$K_{cs,\ell}(k^2) = k^{2\ell+1} \begin{pmatrix} \ell+i\gamma \\ \ell \end{pmatrix} \begin{pmatrix} \ell-i\gamma \\ \ell \end{pmatrix} \times \left[2\gamma H(\gamma) + e^{-\pi\gamma} \Gamma(1+i\gamma) \Gamma(1-i\gamma) (\cot \delta_\ell^c - i) \right] \quad (4.3)$$

with the function $H(\gamma)$ defined by

$$H(\gamma) \equiv \psi(i\gamma) + (2i\gamma)^{-1} - \ln [-i\gamma \operatorname{sgn}(s)]. \quad (4.4)$$

Here $\psi(i\gamma)$ is the digamma function. This form is valid for both repulsive ($s < 0$) and attractive ($s > 0$) Coulomb potentials.

By expressing $\cot \delta_\ell^c$ in terms of the Coulomb phase σ_ℓ , and the physical on-shell t matrix,

$$t_{cs,\ell} = i (\pi k)^{-1} e^{2i\sigma_\ell} \left[e^{2i\delta_\ell^c} - 1 \right] \quad (4.5)$$

by means of the relation,

$$\cot \delta_\ell^c - i = -2e^{2i\sigma_\ell} / \pi k t_{cs,\ell} \quad (4.6)$$

we can obtain a closed expression for $K_{cs,l}(k^2)$.

$$K_{cs,l}(k^2) = k^{2l+1} \binom{l+i\gamma}{l} \binom{l-i\gamma}{l} \times \left[2\gamma H(\gamma) - 2e^{-\pi\gamma} \Gamma(1+i\gamma) \Gamma(1-i\gamma) e^{2i\sigma_l/\pi k} t_{cs,l} \right] \quad (4.7)$$

Hence, by using our previous expression (2.29) for $t_{cs,l}$, we are able to evaluate $K_{cs,l}$ directly.

In the $l = 0$ case, with the Graz p-p interaction, it is straightforward to substitute our expression for $t_{cs,l=0}$, and directly evaluate $K_{cs,l=0}$. We have obtained an expression for the effective range function in this case in terms of a function $W(\gamma; \mu, \nu)$ defined by van Haeringen (1977),

$$W(\gamma; \mu, \nu) = (i\gamma)^{-1} A^{i\gamma} B^{i\gamma} \left[F_{i\gamma}(AB) - \frac{1}{2} \right] + H(\gamma) \quad (4.8)$$

and depending on $\mu = k\gamma/\alpha$ and $\nu = k\gamma/\beta$. $W(\gamma; \mu, \nu)$ is independent of l and the form of the potential employed. Furthermore, it is a real analytic function of γ^{-2} at $\gamma^{-2} = 0$, and can be expanded in a power series in terms of γ^{-2} ,

$$W(\gamma; \mu, \nu) = \sum_{n=0}^{\infty} \omega_{2n}(\mu, \nu) (i\gamma)^{-2n} \quad (4.9)$$

The real coefficients $\omega_{2n}(\mu, \nu)$ are symmetric in μ and ν and are given in VH2. Using this expansion it is a simple matter to deduce the scattering parameters from $K_{cs,l=0}$.

We have found that

$$K_{cs,l=0}(k^2) = 2k\gamma \left\{ \left[R_\alpha + W(\gamma; \mu, \mu) \right] \left[R_\beta + W(\gamma; \nu, \nu) \right] - \left[R_{\alpha\beta} + \left[W(\gamma; \mu, \nu) \right]^2 \right] / \left[R_\alpha + R_\beta - 2R_{\alpha\beta} \right] + W(\gamma; \mu, \mu) + W(\gamma; \nu, \nu) - 2W(\gamma; \nu, \nu) \right\} \quad (4.10)$$

where

$$\begin{aligned}
 R_\alpha &= -\frac{A^{2i\gamma}}{\gamma k} \left[\frac{(\alpha^2 + k^2)^2}{\pi \lambda_1} + \frac{(\alpha^2 - k^2)}{4\alpha} \right] \\
 R_\beta &= -\frac{B^{2i\gamma}}{\gamma k} \left[\frac{(\beta^2 + k^2)^4}{\pi \lambda_2 k^2 (k - \beta\gamma)^2} + \frac{(\beta^2 + k^2)^2 (\beta^2 + 3k^2 - 2\beta k\gamma)}{32\beta k^2 (k - \beta\gamma)^2} \right. \\
 &\quad \left. + \frac{(\beta^2 - k^2)}{4\beta} \right] \\
 R_{\alpha\beta} &= -\frac{A^{i\gamma} B^{i\gamma}}{2\gamma k} \left[\frac{\beta (\alpha^2 + k^2) (\beta^2 + k^2)}{2k (\alpha + \beta)^2 (k - \beta\gamma)} + \frac{(\alpha\beta - k^2)}{(\alpha + \beta)} \right] \quad (4.11)
 \end{aligned}$$

Employing the expansion (4.9) together with power series expansions for R_α , R_β , and $R_{\alpha\beta}$, we have evaluated the first three scattering parameters. These are presented in Table 2.

For the $\ell = 1$ case, we have obtained an analytic expression only for the one-term potentials, 3P_1 and 3P_2 . In these cases

$$K_{cs, \ell=1}(k^2) = R_1^{(1)} + R_1^{(2)} W(\gamma; \mu, \mu) \quad (4.12)$$

with

$$\begin{aligned}
 R_1^{(1)} &= -A^{2i\gamma} \left[\frac{2}{\pi \lambda} (\alpha^2 + k^2)^4 + \frac{k^2 \gamma^2}{2\alpha} (\alpha^2 - k^2) \right. \\
 &\quad \left. - \frac{k\gamma}{8\alpha^2} (\alpha^2 + k^2)^2 - \frac{1}{16\alpha^3} (k^6 + 9\alpha^2 k^4 - 9\alpha^4 k^2 - \alpha^6) \right] \\
 R_1^{(2)} &= 2\gamma k^3 (1 + \gamma^2) \quad (4.13)
 \end{aligned}$$

Parameter values computed using these expressions have also been included in Table 2. Analytic expressions for the scattering length and effective range in this case are readily extracted.

$$-a^{-1} = 2\mu^3 \alpha^3 \operatorname{Re} \Gamma(0, 4\mu) - \frac{1}{16} \alpha^3 e^{-4\mu} (32\alpha^5/\pi\lambda + 1 - 2\mu + 8\mu^2) \quad (4.14)$$

$$\frac{1}{2} r = 2\mu\alpha R \Gamma(0, 4\mu) + \frac{1}{6} \mu\alpha - e^{-4\mu} [9\alpha/16 + (4 + 4\mu/3) 2\alpha^6/\pi\lambda] \quad (4.15)$$

The scattering parameter values computed using the Graz parameters in the 1S_0 state are in good agreement with the values deduced from experiment, whereas our improved fit to the phase shift data has resulted in a scattering length which is too large. This change is easy to understand. The potential parameters were determined by fitting the experimental data starting at $E(\text{lab}) = 1.0$ MeV. The phase shift at this energy has already risen to approximately 33° . The detailed behavior of the phase shift below 1.0 MeV has a dramatic influence in the value of the scattering length. If one performs a least-squares polynomial fit to the phase shift data in order to deduce the scattering parameters, one readily finds how sensitive these parameters are to very small changes in the values of $\delta(E)$ near zero energy. Of course this is due to the way the scattering parameters are defined. Our improved fit to the phase shifts above 1.0 MeV has resulted in a potential with a relatively poor scattering length.

Appendix

The expressions for the matrix elements $\langle g_{i,l=1} | G_c | g_{j,l=1} \rangle$ are given here

$$\begin{aligned} \langle g_{i,l=1} | G_c(k^2) | g_{i,l=1} \rangle &= \frac{2ik^3}{(\alpha^2 + k^2)^4} [1 - (1 + \gamma^2) F_{i\gamma}(AB)] \\ &+ \frac{ik}{8\alpha^2(\alpha - ik)^4} - \frac{1}{16\alpha^3(\alpha - ik)^2} + \frac{k\gamma}{8\alpha^2(\alpha^2 + k^2)^2} \\ &- \frac{(k\gamma)^2}{2\alpha(\alpha + ik)^2(\alpha^2 + k^2)^2} \end{aligned} \quad (A.1)$$

$$\begin{aligned} \langle g_{i,l=1} | G_c(k^2) | g_{2,l=1} \rangle &= \frac{ik^4}{(\alpha^2 + k^2)^2(\beta^2 + k^2)^3} [2k \\ &+ (\beta\gamma - 2ik)(1 + \gamma^2) F_{i\gamma}(AB)] + \frac{k(2\alpha k + \beta k + i\alpha k)}{4(\alpha + \beta)^3(\alpha - ik)^2(\beta - ik)^3} \\ &- \frac{(5\alpha\beta - 4ika - ik\beta)}{8(\alpha + \beta)^4(\alpha - ik)(\beta - ik)^2} + \frac{k\gamma(\alpha\beta^2 + 2\alpha k^2 + \beta k^2)}{4(\alpha + \beta)^3(\alpha^2 + k^2)(\beta^2 + k^2)^2} \\ &+ \frac{\beta k^2 \gamma [\beta\gamma + k(1 + \gamma^2)]}{2(\alpha + \beta)(\alpha^2 + k^2)(\beta^2 + k^2)^2(\alpha + ik)(\beta + ik)} \\ &- \frac{k^2 \gamma^2 (2\beta^2 + 3\alpha\beta + 3i\beta k + 4iak)}{4(\alpha + \beta)^2(\alpha + ik)(\beta + ik)^2(\alpha^2 + k^2)(\beta^2 + k^2)} \end{aligned} \quad (A.2)$$

$$\begin{aligned} \langle g_{2,l=1} | G_c(k^2) | g_{2,l=1} \rangle &= \frac{ik^3}{2(\beta^2 + k^2)^6} [4k^4 \\ &+ (1 + \gamma^2)(8\beta^4 + 16\beta^2 k^2 - \beta^2 k^2 \gamma^2 + 4\beta k^3 \gamma + 4k^4) F_{i\gamma}(\beta^2)] \\ &+ \frac{ik(3\beta^2 - 14i\beta k - 19k^2)}{256\beta^2(\beta - ik)^6} + \frac{(7k^2 + 9i\beta k - 3\beta^2)}{256\beta^3(\beta - ik)^4} \end{aligned}$$

$$\begin{aligned}
& + \frac{k\gamma(3\beta^4 + 14\beta^2 k^2 + 19k^4)}{256\beta^2(\beta^2 + k^2)^4} - (1 + \gamma^2) \frac{\gamma k^3 \beta(2\beta + k\gamma)}{8(\beta + ik)^2(\beta^2 + k^2)^4} \\
& + \frac{k^2 \gamma^2 (-19\beta^4 + 14\beta^2 k^2 - 44i\beta^3 k - 36i\beta k^3 + 25k^4)}{64\beta(\beta + ik)^4(\beta^2 + k^2)^3} \\
& + \frac{k^2 \gamma [\beta\gamma + k(1 + \gamma^2)] (9\beta^3 + 11i\beta^2 k + 13\beta k^2 + 15ik^3)}{32(\beta + ik)^3(\beta^2 + k^2)^4}
\end{aligned} \tag{A.3}$$

Figure Captions

Figure 1(a). 1S_0 p-p phase shift. -- Graz Set I parameters. ---- Improved fit, Set I. $^{\circ}$ Experimental data (MacGregor et al. 1969).

Figure 2(a). 1S_0 p-p phase shift. -- Graz Set II. parameters. ---- Improved fit, Set. II. $^{\circ}$ Experimental data (MacGregor et al. 1969).

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Table 1. Parameters of the Graz potential for the p-p partial waves.*

		Coupling constants [MeV fm ^{-(2l+1)}]		Inverse ranges [fm ⁻¹]	
		$\lambda_{1,l}$	$\lambda_{2,l}$	α	β
1S_0	Set I	-107.386	∞	1.254	3.2992
		-172.695	∞	1.4161	3.0335
	Set II	-99.315	2407.4	1.244	2.3601
		-185.224	9920.37	1.4582	2.4628
3P_0	Set I	-87.428	∞	1.04658	3.52030
		-156.576	∞	1.15875	3.43069
	Set II	-102.007	39465.8	1.07102	2.33805
		-167.309	473656.9	1.17018	2.92517
3P_1		3181.95		1.79536	
		4147.20		1.88763	
3P_2		-10299.5		2.54070	
		-10300.0		2.54090	

*The parameters of Crepinsek et al. (1975) are given on the first line of each parameter set, whereas those obtained in the present study are on the second line.

Table 2. Low-energy scattering parameters.*

		a (fm)	r (fm)	P
1S_0	Set I	-7.60	2.58	-7.09
		-9.16	2.33	-18.78
	Set II	-7.61	2.58	-7.30
		-9.15	2.32	-26.63
	Exp. data ⁽¹⁾	-7.79	2.84	
	One-term Yamaguchi	-7.51	2.80	
3P_1		0.831	-3.91	-0.325
		0.721	-4.09	-0.290
	AHR ⁽³⁾	1.99	-8.00	0.000
3P_2		-0.299	2.807	0.980
		-0.298	2.809	0.978

* Same format as in Table 1.

(1) Slobodrian, 1968.

(2) van Haeringen, 1975.

(3) Kesmode and McKerrel, 1977.

Fig. 5 (b)
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