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**Linear and Nonlinear Symmetrically Loaded
Shells of Revolution Approximated with
the Finite Element Method**

University of California

LOS ALAMOS SCIENTIFIC LABORATORY

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Linear and Nonlinear Symmetrically Loaded Shells of Revolution Approximated with the Finite Element Method

William A. Cook

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PREFACE

Nuclear Material shipping containers have shells of revolution as a basic structural component. Analytically modeling the response of these containers to severe accident impact conditions requires a nonlinear shell-of-revolution model that accounts for both geometric and material nonlinearities. Present models are limited to large displacements, small rotations, and nonlinear materials.

This report discusses a first approach to developing a finite element nonlinear shell of revolution model that accounts for these nonlinear geometric effects. The approach developed accounts for large strains as well as large displacements but is limited to small rotations. It also allows the modeling of nonlinear materials.

A later report will present a nonlinear shell-of-revolution finite element model that will account for large displacements, large strains, large rotations, and nonlinear materials.

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NOMENCLATURE

LATIN

- B - Boundary for shell (ends)
- $\underline{B}$ - B matrix (see Appendix A and B) strain displacement approximation
- c - Curvature (1/R)

C_s
 $C_{s\theta}$
 C_θ
 D_s
 $D_{s\theta}$
 D_θ

}

Shell material properties (see Chapter II)

- h - Two-and three-nodal point shape functions for shells (see Appendix E)
- I - Mixed variational function (see Appendix C)
- L - Length of shell
- L_e - Length of shell finite element
- M_s - Meridional bending moment
- $\bar{M}_s$ - Applied meridional bending moment
- M_θ - Circumferential (hoop) bending moment
- m - Number of nodal points in shell element
- N_s - Meridional stress resultant
- $\bar{N}_s$ - Applied meridional stress resultant
- N_θ - Circumferential (hoop) stress resultant
- $\underline{N}$ - Matrix of stress resultants and bending moments
- P - Potential energy functional
- p - Pressure on shell
- Q - Shear stress resultant

- $\bar{Q}$ - Applied shear stress resultant
- R - Radius of curvature
- r - Radial coordinate
- S - Shear on shell
- s - Meridional coordinate
- U - Shell strain energy functional
- u_n - Normal displacement
- u_r - Radial displacement
- u_s - Meridional displacement
- u_z - Axial displacement
- z - Axial coordinate

SUBSCRIPTS

- i - Initial membrane stress or bending moment
Also used as an index counter
- j
- $j+1$
- } Increment (or iteration)
- o - Designates that this quantity refers to the undeformed shell configuration

SUPERSCRIPTS

- 2, 3, and m - The number of nodal points per element. Identifies the shape function to be used (see Appendix E).

GREEK

- β - Rotation
- $\bar{\beta}$ - Applied rotation
- δ - Variation operator
- δ_{ik} - Kronecker delta
- Δ - Incremental operator

- ϵ_s - Meridional membrane strain
- ϵ_θ - Circumferential (hoop) membrane strain
- $\underline{\epsilon}$ - Membrane and bending strain matrix
- κ_s - Meridional bending strain
- κ_θ - Circumferential (hoop) bending strain
- ξ - Normal coordinate for shell element (see Fig. E-1)
- ϕ - Azimuthal angle for shell configuration (see Fig. F-1)
- ψ - $\frac{du_n}{d\xi}$ (see Appendix E)

LINEAR AND NONLINEAR SYMMETRICALLY LOADED SHELLS OF REVOLUTION
APPROXIMATED WITH THE FINITE ELEMENT METHOD

by

William A. Cook

ABSTRACT

The development of a finite element nonlinear shell model is described. The approach uses incremental loads and a linear shell model with equilibrium iterations. Sixteen linear models are developed, eight using the potential energy variational principle and eight using a mixed variational principle. Four of these are suitable for extension to nonlinear shell theory. A nonlinear shell theory is derived, and a computational technique used in its solution is presented.

I. INTRODUCTION AND RESULTS

This report describes the development of a finite element nonlinear shell model for symmetrically loaded shells of revolution. This shell model is to be used with finite element continuum models as they exist in the NONSAP code.^{1,2} NONSAP has four- to eight-node axisymmetric isoparametric elements. To have a compatible number of common nodes, the shell model must have either two- or three-node elements. NONSAP solves nonlinear continuum problems by incrementing loads and then iterating linear continuum models to satisfy equilibrium. Similarly, linear shell models must be studied when developing a nonlinear shell model.

In 1963 Grafton and Strome reported on the use of finite elements in the solution of symmetrically loaded shells.³ They introduced the two-node conical element, a very reliable element that is still used in many codes. Many researchers wanted a curved element, because many shells of revolution are spherical or elliptical in shape. Nickell and Sato reported on a two-node curved element in 1970.⁴ One of the objections to these elements is that the displacement function assumed for the shell is of higher order than the displacement

functions assumed for continuum elements. Thus, when these elements adjoin one another, they are incompatible. To overcome this difficulty, Herrman, Taylor, and Green introduced a two-node conical element that was compatible with linear displacement continuum elements.⁵ A mixed variational principle was used to accomplish this. Both the Grafton and Strome and the Nickell and Sato elements use two displacements and a meridional rotation as the generalized variables in the finite element approach. Herrman, Taylor, and Green use two displacements and the meridional moment.

The following linear shell models were developed (summarized in Table I).

- Potential Energy Model - Linear meridional displacement, cubic normal displacement, and quadratic rotation approximations.
 1. Two-node conical element.
 2. Two-node curved element with curvature data required.
- Potential Energy Model - Quadratic meridional displacement, quintic normal displacement, and quadratic rotation approximations.
 3. Three-node curved element with curvature data required.
 4. Three-node curved element with curvature data calculated from the three nodes.
- Mixed Model - Linear approximations for meridional displacement, normal displacement, and meridional moment.
 5. Two-node conical element.
 6. Two-node curved element with curvature data required.
- Mixed Model - Quadratic approximations for meridional displacement, normal displacement, and meridional moment.
 7. Three-node curved element with curvature data required.
 8. Three-node curved element with curvature data calculated from the three nodes.
- Potential Energy Model - Linear meridional displacement, cubic normal displacement, and quadratic rotation approximations. Also, for the hoop strain, a linear approximation for the radial displacement.
 9. Two-node conical element.
 10. Two-node curved element with curvature data required.
- Potential Energy Model - Quadratic meridional displacement, quintic normal displacement, and quartic rotation approximation. Also, for the hoop strain, a quadratic approximation for the radial displacement.

11. Three-node curved element with curvature data required.
12. Three-node curved element with curvature data calculated from the three nodes.
- Mixed Model - Linear approximation for meridional displacement, and meridional moment. Also, for the hoop strain, a linear approximation for the radial displacement.
13. Two-node conical element.
14. Two-node curved element with curvature data required.
- Mixed Model - Quadratic approximations for meridional displacement, normal displacement, and meridional moment. Also, for hoop strain, a quadratic approximation for the radial displacement.
15. Three-node curved element with curvature data required.
16. Three-node curved element with curvature data calculated from the three nodes.

The difference between the first eight models and the last eight models is the way the hoop strain is approximated. This results in different B matrices (strain displacement approximations), which are described in Appendixes A and B. The first eight models are simpler from a theoretical basis. However, the last eight models are simpler from a programming and calculational basis.

Several good references, such as Ref. 6, discuss the finite element method and how stiffness matrices and force vectors are accumulated. In this report the discussion of finite element formulations will be restricted to single element stiffnesses and force vector modifications caused by initial stress components. Models 1-4 and 9-12 are derived in Sec. II, and models 5-8 and 13-16 are derived in Sec. III.

To evaluate these shell models, several problems were analyzed. First, Ref. 3 gives three example problems: a plate with a shear load, a cylindrical shell with a shear load, and a hemispherical shell with an applied moment. The nodal spacing used is shown in Ref. 3, and shell models 1, 4, 5, 8, 9, 12, 13, and 16 all solved these problems very accurately. The remaining shell models all require additional geometric data and are more accurate models because continuities and discontinuities can be modeled exactly between elements. Second, a hemisphere with pressure loading was analyzed using a mesh with coarse nodal spacing. The pressure loads were calculated to be compatible with two- and three-node continuum isoparametric element boundaries, thus making it possible to evaluate the response of the shell with a continuum. Shell models 1-4 and

TABLE I
LINEAR FINITE ELEMENT SHELL MODELS

Model	Formulation	Nodes per Element	Curvature Required Input	Radial Disp. Assumption
1	P. E.	2	No	No
2	↓	2	Yes	↓
3	↓	3	Yes	↓
4	↓	3	No	↓
5	Mixed	2	No	↓
6	↓	2	Yes	↓
7	↓	3	Yes	↓
8	↓	3	No	↓
9	P. E.	2	No	Yes
10	↓	2	Yes	↓
11	↓	3	Yes	↓
12	↓	3	No	↓
13	Mixed	2	No	↓
14	↓	2	Yes	↓
15	↓	3	Yes	↓
16	↓	3	No	↓

9-12 (potential energy formulation) are incompatible with the pressure loads, and the normal displacements oscillate about the exact result. The mixed formulation shell models 5-8 and 13-16 are compatible with the pressure loads and are more accurate. In summary, 88 example problems have been studied and are bound for reference.* The conclusion is that the mixed formulations with three

nodes and curvature data supplied are the most accurate (shell models 7 and 15). However, all shell models are satisfactory if enough elements are used. The mixed variational principle, as presented in Appendix C, is restricted to small rotation theory. The models requiring that curvature data be supplied are not acceptable for nonlinear models that will be incremented because the results of one calculation will alter the curvature of a following calculation. Thus, the shell models to be used for nonlinear shell analysis are 1, 4, 9, and 12. Though these models were very accurate, they were the least accurate of the 16 shell models developed.

*Available from author on request.

The computational technique used in Sec. IV to develop a nonlinear shell model consists of the following five steps.

1. Use a variational principle that satisfies equilibrium. This report uses the one described in Appendix D. This potential energy functional is a Eulerian formulation where the geometric quantities, the shell quantities, and the loads all refer to the deformed configuration of the shell.
2. Modify the variational principle for incremental loadings.
3. Linearize the variational principle such that it may be solved directly.
4. Use the finite element method to approximate the linearized incremental variational principle. This step is identical to solving a linear problem.
5. By iteration approximate the original nonlinear variational principle. This iteration is necessary because the variational principle was linearized, and iteration ensures that equilibrium is satisfied. The iteration step is identical to the increment step, except the applied loads do not change. Also, this iteration step makes it possible to include nonlinear materials.

To evaluate the nonlinear shell models (1, 4, 9, and 12), several problems were analyzed. First, the plate, the cylindrical, and the hemispherical problems described in Ref. 3 were analyzed in the nonlinear mode. Second, a cylindrical problem was analyzed with a large axial load. The axial displacement at the point of the load was nine times the length of the original shell. Third, a hemisphere with an internal pressure was analyzed. This pressure was large enough that the radial displacement was as large as the original radial coordinate. These models solved these problems accurately, and the solutions are bound for reference. Fourth, a cylindrical shell was loaded such that it would deform into a spherical shape. These nonlinear shell models would not solve this problem since the shape functions restrict shell motion to small rotation.

II. LINEAR SHELLS USING POTENTIAL ENERGY VARIATIONAL PRINCIPLE

The potential energy functional described in Eqs. (D-1) and (D-2) can be written as

$$\delta P(u_s, u_n, \beta) = \delta U(u_n, u_n, \beta) - \int_L r \{ S \delta u_s + p \delta u_n \} ds \quad (1)$$

$$- r \left[\bar{N}_s \delta u_s + \bar{Q} \delta u_n + \bar{M}_s \delta \beta \right]_B, \quad (1)$$

where

$$\begin{aligned} \delta U(u_s, u_n, \beta) = \int_L r \left\{ N_s \delta \epsilon_s + N_\theta \delta \epsilon_\theta + M_s \delta \kappa_s + M_\theta \delta \kappa_\theta \right. \\ \left. + Q \left(\delta \beta - \frac{\delta u_s}{R} + \frac{d(\delta u_n)}{ds} \right) \right\} ds. \end{aligned} \quad (2)$$

The strains ϵ_s , ϵ_θ , κ_s , κ_θ are defined as

$$\epsilon_s = \frac{du_s}{ds} + \frac{u_n}{R},$$

$$\epsilon_\theta = \frac{\cos \phi u_s - \sin \phi u_n}{r} = \frac{u_r}{r},$$

$$\left. \begin{aligned} \kappa_s &= \frac{d\beta}{ds}, \text{ and} \\ \kappa_\theta &= \frac{\beta \cos \phi}{r} \end{aligned} \right\} \quad \text{where } \beta = \frac{u_s}{R} - \frac{du_n}{ds}.$$

For constitutive relations we used the following,

$$N_s = C_s \epsilon_s + C_{s\theta} \epsilon_\theta + N_{si}$$

$$N_{\theta} = C_{s\theta} \epsilon_s + C_{\theta} \epsilon_{\theta} + N_{\theta i}$$

$$M_s = D_s \kappa_s + D_{s\theta} \kappa_{\theta} + M_{si} \quad (4)$$

$$M_{\theta} = D_{s\theta} \kappa_s + D_{\theta} \kappa_{\theta} + M_{\theta i},$$

$$\begin{aligned} \delta U(u_s, u_n, \beta) &= \int_L r \left\{ (C_s \epsilon_s + C_{s\theta} \epsilon_{\theta}) \delta \epsilon_s + (C_{s\theta} \epsilon_s + C_{\theta} \epsilon_{\theta}) \delta \epsilon_{\theta} \right. \\ &\quad \left. + (D_s \kappa_s + D_{s\theta} \kappa_{\theta}) \delta \kappa_s + (D_{s\theta} \kappa_s + D_{\theta} \kappa_{\theta}) \delta \kappa_{\theta} \right\} ds \\ &\quad + \int_L r \left\{ N_{si} \delta \epsilon_s + N_{\theta i} \delta \epsilon_{\theta} + M_{si} \delta \kappa_s + M_{\theta i} \delta \kappa_{\theta} \right\} ds \\ &= \int_L r \underline{\underline{\epsilon}}^T \underline{\underline{D}} \delta \underline{\underline{\epsilon}} ds + \int_L r \underline{\underline{N}}_i^T \delta \underline{\underline{\epsilon}} ds, \end{aligned} \quad (5)$$

where

$$\underline{\underline{\epsilon}}^T \equiv (\epsilon_s \quad \epsilon_{\theta} \quad \kappa_s \quad \kappa_{\theta})$$

$$\underline{\underline{N}}^T \equiv (N_{si} \quad N_{\theta i} \quad M_{si} \quad M_{\theta i})$$

and

$$\underline{\underline{D}} = \begin{bmatrix} C_s & C_{s\theta} & 0 & 0 \\ C_{s\theta} & C_{\theta} & 0 & 0 \\ 0 & 0 & D_s & D_{s\theta} \\ 0 & 0 & D_{s\theta} & D_{\theta} \end{bmatrix}. \quad (6)$$

From Appendix E, Eqs. (E-1)-(E-3), the assumed approximations for the displacements and rotations over each element are

$$u_s(\xi) = \sum_{i=1}^m h_{si}^m u_{si},$$

$$u_n(\xi) = \sum_{i=1}^m \left\{ h_{\beta i}^m q_i c_i u_{si} + h_{ni}^m u_{ni} - h_{\beta i}^m q_i \beta_i \right\}, \text{ and}$$

$$\begin{aligned} \beta(\xi) = \sum_{i=1}^m \left\{ \left[c(\xi) h_{si}^m - \frac{q_i c_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \right] u_{si} - \frac{1}{q(\xi)} \frac{dh_{ni}^m}{d\xi} u_{ni} \right. \\ \left. + \frac{q_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \beta_i \right\}. \end{aligned} \quad (7)$$

When these assumed displacements and rotations are used to approximate the strain ϵ_s , ϵ_θ , κ_s , and κ_θ .

$$\epsilon_s = \frac{du_s}{ds} + \frac{u_n}{R} = \sum_{i=1}^m \left\{ \left[\frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} + q_i c_i h_{\beta i}^m c(\xi) \right] u_{si} \right.$$

$$\left. + c(\xi) h_{ni}^m u_{ni} - c(\xi) h_{\beta i}^m q_i \beta_i \right\};$$

$$\epsilon_\theta = \frac{u_s \cos\phi + u_n \sin\phi}{r} = \sum_{i=1}^m \left\{ \left[\frac{\cos\phi(\xi)}{r(\xi)} h_{si}^m + \frac{\sin\phi(\xi)}{r(\xi)} h_{\beta i}^m q_i c_i \right] u_{si} \right.$$

$$\left. + \frac{\sin\phi(\xi)}{r(\xi)} h_{ni}^m u_{ni} - \frac{\sin\phi(\xi)}{r(\xi)} q_i h_{\beta i}^m \beta_i \right\},$$

or a simplifying assumption for this equation may be

$$u_r(\xi) = \sum_{i=1}^m h_{si}^m u_{ri}, \quad \text{and then} \quad \epsilon_\theta = \sum_{i=1}^m \frac{h_{si}^m}{r(\xi)} u_{ri}; \quad (8)$$

$$\begin{aligned} \kappa_s = \frac{d\beta}{ds} = \sum_{i=1}^m \left\{ \left[\frac{h_{si}^m}{q(\xi)} \frac{dc(\xi)}{d\xi} + \frac{c(\xi)}{q(\xi)} \frac{dh_{si}^m}{d\xi} - \frac{q_i c_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) \right. \right. \\ \left. \left. - \frac{q_i c_i}{(q(\xi))^2} \frac{d^2 h_{\beta i}^m}{d\xi^2} \right] u_{si} - \left[\frac{1}{q(\xi)} \frac{dh_{ni}^m}{d\xi} \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) + \frac{1}{(q(\xi))^2} \frac{d^2 h_{ni}^m}{d\xi^2} \right] u_{ni} \right. \\ \left. + \left[\frac{q_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) + \frac{q_i}{(q(\xi))^2} \frac{d^2 h_{\beta i}^m}{d\xi^2} \right] \beta_i \right\}; \text{ and} \\ \kappa_\theta = \frac{\beta \cos \phi}{r} = \sum_{i=1}^m \left\{ \left[\left(c(\xi) h_{si}^m - \frac{q_i c_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \right) \frac{\cos \phi(\xi)}{r(\xi)} \right] u_{si} \right. \\ \left. - \left[\frac{dh_{ni}^m}{d\xi} \frac{\cos \phi(\xi)}{q(\xi) r(\xi)} \right] u_{ni} + \left[\frac{q_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \frac{\cos \phi(\xi)}{r(\xi)} \right] \beta_i \right\}. \end{aligned}$$

At this point transform the displacements from u_{si} , u_{ni} to u_{ri} , u_{zi} . This is accomplished with the following transformation equations,

$$\begin{aligned} u_{si} &= \cos \phi_i u_{ri} - \sin \phi_i u_{zi} \text{ and} \\ u_{ni} &= \sin \phi_i u_{ri} + \cos \phi_i u_{zi}, \end{aligned} \quad (9)$$

where $\phi_i = \phi(\xi_i)$, ξ_i being the value of ξ at nodal point i .

The relation between the strains and the displacements is taken to be

$$\underline{\epsilon} = \underline{B} \underline{u} \quad (10)$$

where

$$\underline{u}^T = (u_{r1}, u_{z1}, \epsilon_1, \dots, \epsilon_m) \quad \text{and}$$

$$\underline{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,3m} \\ b_{2,1} & b_{2,2} & \dots & b_{2,3m} \\ b_{3,1} & b_{3,2} & & b_{3,3m} \\ b_{4,1} & b_{4,2} & & b_{4,3m} \end{bmatrix}.$$

The $b_{i,j}$ terms are tabulated in Appendix A.

When Eq. (10) is substituted into Eq. (5), we obtain

$$\delta U = \underline{u}^T \int_{L_c} \underline{r} \underline{B}^T \underline{D} \underline{B} \, ds \, \delta \underline{u} + \int_{L_c} \underline{r} \underline{N}_1 \underline{B} \, ds \, \delta \underline{u}. \quad (11)$$

Thus the stiffness matrix is

$$\int_{-1}^1 \underline{r}(\xi) \, q(\xi) \, \underline{B}^T \underline{D} \underline{B} \, d\xi, \quad (12)$$

because $ds = q(\xi) \, d\xi$, and the modification to the force vector due to initial stress is

$$\int_{-1}^1 \underline{r}(\xi) \, q(\xi) \, \underline{N}_1^T \underline{B} \, d\xi. \quad (13)$$

The resulting stiffness matrix equation approximates the equilibrium equations in Appendix E. Equations (12) and (13) can be calculated numerically provided the following quantities are known:

- (1) The shape functions h_{si}^m , h_{ni}^m , and $h_{\beta i}^m$ and the first and second derivatives of the shape functions with respect to ξ (see Appendix E).
- (2) The radial and axial coordinates $r(\xi)$, $z(\xi)$.
- (3) The normalized meridional length $q(\xi)$ and $q_i \equiv q(\xi_i)$. Also the derivative $\frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right)$,
- (4) The azimuthal angle $\phi(\xi)$ and $\phi_i \equiv \phi(\xi_i)$, and
- (5) The curvature $c(\xi) = \frac{1}{R(\xi)}$ where $c_i \equiv c(\xi_i)$, and also, the derivative $\frac{dc(\xi)}{d\xi}$.

Two approaches are described in Appendix G for determining items (2) through (5).

III. LINEAR SHELLS USING MIXED VARIATION PRINCIPLE

The functional described by Eqs. (C-1-C-3) can be written as

$$\begin{aligned} \delta I(u_s, u_n, M_s) = & \delta U(u_s, u_n, M_s) - \int_L r \left\{ S \delta u_s + p \delta u_n \right\} ds \\ & - r \left[\bar{N}_s \delta u_s + \bar{Q} \delta u_n + \bar{\beta} \delta M_s \right]_B, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \delta U(u_s, u_n, M_s) = & \int_L \left\{ r N_s \delta \epsilon_s + r N_\theta \delta \epsilon_\theta - r \left[\frac{M_s}{D_s} + \left(1 - \frac{D_{s\theta}}{D_s} \right) \kappa_\theta \right] \delta M_s \right. \\ & \left. - r \beta \frac{d(\delta M_s)}{ds} - r Q(M_s, \kappa_\theta) \left[\frac{\delta u_s}{R} - \frac{d(\delta u_n)}{ds} \right] \right\} ds \end{aligned} \quad (15)$$

and

$$Q(M_s, \kappa_\theta) = \frac{dM_s}{ds} + \left(1 - \frac{D_{s\theta}}{D_s}\right) \frac{\cos\phi}{r} M_s - \left(D_\theta - \frac{D_{s\theta}^2}{D_s}\right) \kappa_\theta \frac{\cos\phi}{r} . \quad (16)$$

The strains ϵ_s , ϵ_θ , and κ_θ are

$$\begin{aligned} \epsilon_s &= \frac{du_s}{ds} + \frac{u_n}{R} , \\ \epsilon_\theta &= \frac{\cos\phi u_s + \sin\phi u_n}{r} = \frac{u_r}{r} , \text{ and} \\ \kappa_\theta &= \beta(u_s, u_n) \frac{\cos\phi}{r} = \left[\frac{u_s}{R} - \frac{du_n}{ds} \right] \frac{\cos\phi}{r} . \end{aligned} \quad (17)$$

For constitutive relations, we have

$$N_s = C_s \epsilon_s + C_{s\theta} \epsilon_\theta + N_{si}$$

$$N_\theta = C_{s\theta} \epsilon_s + C_\theta \epsilon_\theta + N_{\theta i}$$

$$\left. \begin{aligned} M_\theta &= D_s \kappa_s + D_{s\theta} \kappa_\theta \\ M_s &= D_{s\theta} \kappa_s + D_\theta \kappa_\theta \end{aligned} \right\} \text{ where } \kappa_s = \frac{d\beta(u_s, u_n)}{ds} , \quad (18)$$

Eq. (15) can be written as

$$\delta U(u_s, u_n, M_s) = \int_L r \underline{\epsilon}^T \underline{D} \delta \underline{\epsilon} ds + \int_L r \underline{N}_i^T \underline{D} \delta \underline{\epsilon} ds , \quad (19)$$

where

$$\underline{\underline{e}}^T = \left(\epsilon_s, \epsilon_\theta, M_s, \kappa_\theta, \frac{dM}{ds} \frac{s}{D_s}, \beta(u_s, u_n) \right)$$

$$D = \begin{bmatrix} C_s & C_{s\theta} & 0 & 0 & 0 & 0 \\ C_{s\theta} & C_\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{D_s} & -1 + \frac{D_{s\theta}}{D_s} & 0 & 0 \\ 0 & 0 & -1 + \frac{D_{s\theta}}{D_s} & D_\theta - \frac{D_{s\theta}^2}{D_s} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad (20)$$

$$\underline{\underline{N}}_i^T = (N_{si}, N_{\theta i}, 0, 0).$$

Let both displacements and the moment be approximated over each element with the same shape functions. The displacements and moments are given by

$$u_s(\xi) = \sum_{i=1}^m h_{si}^m u_{si}$$

$$u_n(\xi) = \sum_{i=1}^m h_{si}^m u_{ni} \quad (21)$$

$$M_s(\xi) = \sum_{i=1}^m h_{si}^m M_{si}$$

When these assumed displacements are substituted into the $\underline{\epsilon}$ quantities one gets

$$\epsilon_s = \sum_{i=1}^m \left\{ \frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} u_{si} + c(\xi) h_{si}^m u_{ni} \right\} ; \text{ and}$$

$$\epsilon_\theta = \sum_{i=1}^m \left\{ h_{si}^m \frac{\cos\phi(\xi)}{r(\xi)} u_{si} + h_{si}^m \frac{\sin\phi(\xi)}{r(\xi)} u_{ni} \right\}$$

$$\text{or, assuming } u_r(\xi) = \sum_{i=1}^m h_{si}^m u_{ri} ,$$

$$\epsilon_\theta = \sum_{i=1}^m \frac{h_{si}^m}{r(\xi)} u_{ri} ;$$

$$M_s = \sum_{i=1}^m h_{si}^m M_{si} ;$$

(22)

$$\kappa_\theta = \sum_{i=1}^m \left\{ c(\xi) h_{si}^m \frac{\cos\phi(\xi)}{r(\xi)} u_{si} - \frac{\cos\phi(\xi)}{q(\xi) r(\xi)} \frac{dh_{si}^m}{d\xi} u_{ni} \right\} ;$$

$$\frac{dM_s}{ds} = \sum_{i=1}^m \frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} M_{si} ; \text{ and}$$

$$\beta(u_s, u_n) = \sum_{i=1}^m \left\{ c(\xi) h_{si}^m u_{si} - \frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} u_{ni} \right\} .$$

At this point transform the displacements u_i, u_{ni} , to u_{ri}, u_{zi} . This is accomplished with the following transformation equations:

$$\begin{aligned} u_{si} &= \cos\phi_i u_{ri} - \sin\phi_i u_{zi} \text{ and} \\ u_{ni} &= \sin\phi_i u_{ri} + \cos\phi_i u_{zi}, \end{aligned} \quad (23)$$

where $\phi_i = \phi(\xi)$, ξ_i being the value of ξ at nodal point i .
Thus

$$\underline{u} = \underline{B} \underline{u} \quad (24)$$

where

$$\underline{u}^T = (u_{r1}, u_{z1}, M_{s1}, \dots, M_{sm})$$

$$\underline{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdot & \cdot & \cdot & b_{1,3m} \\ b_{2,1} & b_{2,2} & \cdot & \cdot & \cdot & b_{2,3m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{6,1} & b_{6,2} & \cdot & \cdot & \cdot & b_{6,3m} \end{bmatrix}.$$

The $b_{i,j}$ terms are tabulated in Appendix B. When Eq. (24) is substituted into Eq. (19), one obtains

$$\delta U = \underline{u}^T \int_{L_e} r \underline{B}^T \underline{D} \underline{B} \, ds \, \delta \underline{u} + \int_{L_e} r \underline{N}_i^T \underline{B} \, ds \, \delta \underline{u}. \quad (25)$$

Thus this element stiffness matrix is

$$\int_{-1}^1 r(\xi) q(\xi) \underline{\underline{B}}^T \underline{\underline{D}} \underline{\underline{B}} d\xi \quad (26)$$

since $ds = q(\xi) d\xi$

and the modification to the force vector from initial stresses is

$$\int_{-1}^1 r(\xi) q(\xi) \underline{\underline{N}}_i^T \underline{\underline{B}} d\xi . \quad (27)$$

Equations (26) and (27) can be calculated numerically provided the following quantities are known:

- (1) the shape functions h_{si}^m and the first derivative of the shape function $\frac{dh_{si}^m}{d\xi}$ (see Appendix E),
- (2) the radial and axial coordinates $r(\xi)$ and $z(\xi)$,
- (3) the normalized meridional length $q(\xi)$,
- (4) the azimuthal angle $\phi(\xi)$ and $\phi_i \equiv \phi(\xi_i)$, and
- (5) the curvature $c(\xi) \equiv \frac{1}{R(\xi)}$.

Two approaches are described in Appendix G for determining items (2) through (5).

IV. NONLINEAR SHELLS USING POTENTIAL ENERGY VARIATIONAL PRINCIPLE

The derivation of the shell equilibrium equations in Appendix F and the potential energy functional in Appendix D are both valid for large strain conditions when the geometric quantities r , z , ϕ , s , R , L , and B ; the shell quantities N_s , N_θ , M_s , and M_θ ; and the loads per area p and S all refer to the deformed configuration of the shell. However, the shape functions from Appendix E limit this approach to small rotations. Thus

$$\begin{aligned} \delta P(u_s, u_n, \beta) &= U(u_s, u_n, \beta) - \int_L r \left\{ S \delta u_s + p \delta u_n \right\} ds \\ - r \left[\overline{N}_s \delta u_s + \overline{Q} \delta u_n + \overline{M}_s \delta \beta \right]_B &= 0 \end{aligned} \quad (28)$$

where

$$\begin{aligned} \delta U(u_s, u_n, \beta) = \int_L r \left\{ N_s \left(\frac{d(\delta u_s)}{ds} + \frac{\delta u_n}{R} \right) + N_\theta \frac{\delta u_r}{r} \right. \\ \left. + M_s \frac{d(\delta \beta)}{ds} + M_\theta \frac{\cos \phi}{r} \delta \beta + Q \left(\frac{d(\delta u_n)}{ds} + \delta \beta - \frac{\delta u_s}{R} \right) \right\} ds. \end{aligned} \quad (29)$$

The membrane and bending strains associated with nonlinear shells are defined in Appendix H.

To increment for a nonlinear solution, consider Eqs. (28) and (29) for the $j + 1$ increment. The variation of the potential energy functional is

$$\begin{aligned} \delta U(u_{s_{j+1}}, u_{n_{j+1}}, \beta_{j+1}) - \int_{L_{j+1}}^{r_{j+1}} \left\{ S_{j+1} \delta u_{s_{j+1}} + p_{j+1} \delta u_{n_{j+1}} \right\} ds_{j+1} \\ = 0 \end{aligned} \quad (30)$$

$$- r_{j+1} \left\{ \bar{N}_{s_{j+1}} \delta u_{s_{j+1}} + \bar{Q}_{j+1} \delta u_{n_{j+1}} + \bar{M}_{s_{j+1}} \delta \beta_{j+1} \right\}_{B_{j+1}} = 0$$

where

$$\begin{aligned} \delta U(u_{s_{j+1}}, u_{n_{j+1}}, \beta_{j+1}) = \int_{L_{j+1}}^{r_{j+1}} \left\{ N_{s_{j+1}} \left(\frac{d(\delta u_{s_{j+1}})}{ds_{j+1}} + \frac{\delta u_{n_{j+1}}}{R_{j+1}} \right) \right. \\ \left. + N_{\theta_{j+1}} \frac{\delta u_{r_{j+1}}}{r_{j+1}} + M_{s_{j+1}} \frac{d(\delta \beta_{j+1})}{ds_{j+1}} + M_{\theta_{j+1}} \frac{\cos \phi_{j+1}}{r_{j+1}} \right. \\ \left. + Q_{j+1} \left(\frac{d(\delta u_{n_{j+1}})}{ds_{j+1}} + \delta \beta_{j+1} - \frac{\delta u_{s_{j+1}}}{R_{j+1}} \right) \right\} ds_{j+1}. \end{aligned} \quad (31)$$

$$\delta \beta_{j+1} + Q_{j+1} \left(\frac{d(\delta u_{n_{j+1}})}{ds_{j+1}} + \delta \beta_{j+1} - \frac{\delta u_{s_{j+1}}}{R_{j+1}} \right) ds_{j+1}.$$

Make the following assumptions.

- o The incremental displacements, rotations, membrane and bending strains, stress results, and bending moments are as described in Appendixes I and J.
- o The variations of displacements and rotation are

$$\delta u_{s_{j+1}} = \delta \left(\Delta u_{s_j} \right) ,$$

$$\delta u_{n_{j+1}} = \delta \left(\Delta u_{n_j} \right) , \text{ and}$$

$$\delta \beta_{j+1} = \delta \left(\Delta \beta_j \right) .$$

- o The following quantities are approximately equal

$$R_{j+1} \approx R_j ,$$

$$ds_{j+1} \approx ds_j ,$$

$$L_{j+1} \approx L_j ,$$

$$B_{j+1} \approx B_j ,$$

$$r_{j+1} \approx r_j , \text{ and}$$

$$\phi_{j+1} \approx \phi_j .$$

With these assumptions

$$\delta U \left(\Delta u_{s_j} , \Delta u_{n_j} , \Delta \beta_j \right) - \int_{L_j}^{r_j} \left\{ s_{j+1} \delta (\Delta u_{s_j}) + p_{j+1} \delta (\Delta u_{n_j}) \right\} ds_j \quad (32)$$

$$- r_j \left\{ \bar{N}_{s_{j+1}} \delta(\Delta u_{s_j}) + \bar{Q}_{j+1} \delta(\Delta u_{n_j}) + \bar{M}_{s_{j+1}} \delta(\Delta \beta_j) \right\}_{B_j} = 0$$

or

$$\delta U (\Delta u_{s_j}, \Delta u_{n_j}, \Delta \beta_j) = \int_{L_j} r_j \underline{\varepsilon}_j^T \underline{D}_j \delta \underline{\varepsilon}_j ds_j + \int_{L_j} r_j (\underline{N}_i)_j^T \delta \underline{\varepsilon}_j ds_j, \quad (33)$$

where

$$\underline{\varepsilon}_j^T \equiv (\Delta \varepsilon_{s_j}, \Delta \varepsilon_{\theta_j}, \Delta \kappa_{s_j}, \Delta \kappa_{\theta_j})$$

$$(\underline{N}_i)_j^T \equiv \left(N_{s_j} + (N_{s_i})_{j+1}, N_{\theta_j} + (N_{\theta_i})_{j+1}, M_{s_j} + (M_{s_i})_{j+1}, \right.$$

$$\left. M_{\theta_j} + (M_{\theta_i})_{j+1} \right)$$

(34)

$$\underline{D}_j \equiv \begin{bmatrix} C_{s_j} & C_{s\theta_j} & 0 & 0 \\ C_{s\theta_j} & C_{\theta_j} & 0 & 0 \\ 0 & 0 & D_{s_j} & D_{s\theta_j} \\ 0 & 0 & D_{s\theta_j} & D_{\theta_j} \end{bmatrix}$$

$\underline{D}_j$ is the tangential material property matrix. See Appendix K for details concerning nonlinear material properties.

Equations (32), (33), and (34) are identical to Eqs. (1), (5), and (6). Thus, the solution approach described in Section II applies to each increment and iteration for nonlinear shell models. Again, the iteration step is the same as the increment step except the applied loads do not change. These are

$$S_{j+1} = S_j,$$

$$P_{j+1} = P_j,$$

$$\bar{N}_{s_{j+1}} = \bar{N}_{s_j},$$

$$\bar{Q}_{j+1} = \bar{Q}_j,$$

$$\bar{M}_{s_{n+1}} = \bar{M}_{s_j},$$

$$\left(N_{s_i} \right)_{j+1} = \left(N_{s_i} \right)_j,$$

$$\left(N_{\theta_i} \right)_{j+1} = \left(N_{\theta_i} \right)_j,$$

$$\left(M_{s_i} \right)_{j+1} = \left(M_{s_i} \right)_j, \text{ and}$$

$$\left(M_{\theta_1} \right)_{j+1} = \left(M_{\theta_1} \right)_j.$$

V. RECOMMENDATIONS

From this study several recommendations can be made.

1. Develop additional example problems to check out and improve the nonlinear shell models.
2. Study methods of improving the convergence criteria and the convergence rate for the nonlinear shell models.
3. Investigate variational methods, for example

- a. A Lagrangian (undeformed geometric configuration) approach, which would be compatible with the NONSAP code.
- b. A mixed formulation using either the Lagrangian (undeformed geometric configuration) approach or the Eulerian (deformed geometric configuration) approach.

With one of these variational methods better finite element shell models could be used for nonlinear analysis.

4. Develop an improved shell model by adding shear deformation and rotatory inertia. Also consider changes in shell thickness as the shell stretches or is compressed. Such changes would have to be consistent with the shell theory assumption.
5. Investigate other methods of solving the nonlinear equations besides iterating.
6. Eliminate the small rotation restrictions.

$$+ \frac{\sin \phi(\xi) r(\xi)}{h_m^{n_1} \cos \phi_1} \text{ , and } p_{ue}$$

$$b_{2, 31-1} = - \left[\frac{\cos \phi(\xi) r(\xi)}{h_m^{s_1}} + \frac{\sin \phi(\xi) r(\xi)}{h_m^{n_1} \cos \phi_1} q_1 c_1 \right] \sin \phi_1$$

$$+ \frac{\sin \phi(\xi) r(\xi)}{h_m^{n_1} \sin \phi_1} \text{ ,}$$

$$b_{2, 31-2} = \left[\frac{\cos \phi(\xi) r(\xi)}{h_m^{s_1}} + \frac{\sin \phi(\xi) r(\xi)}{h_m^{n_1} \cos \phi_1} q_1 c_1 \right] \cos \phi_1$$

$$b_{1, 31} = - \cos(\xi) h_m^{g_1} q_1 \text{ ;}$$

$$b_{1, 31-1} = - \left[\frac{1}{h_m^{s_1}} \frac{dp}{d\xi} + q_1 c_1 \cos(\xi) h_m^{g_1} \sin \phi_1 + c(\xi) h_m^{n_1} \cos \phi_1 \right] \text{ ,}$$

$$b_{1, 31-2} = \left[\frac{1}{h_m^{s_1}} \frac{dp}{d\xi} + q_1 c_1 \cos(\xi) h_m^{g_1} \cos \phi_1 + c(\xi) h_m^{n_1} \sin \phi_1 \right] \text{ ,}$$

B MATRIX FOR DISPLACEMENTS AND ROTATION

APPENDIX A

$$b_{2, 3i} = - \frac{\sin\phi(\xi)}{r(\xi)} q_i h_{\beta i}^m ,$$

Or, using the simplifying assumption $u_r(\xi) = \sum_{i=1}^m h_{si}^m u_{ri}$,

$$b_{2, 3i-2} = - \frac{h_{si}^m}{r(\xi)} ,$$

$$b_{2, 3i-1} = 0 ,$$

$$b_{2, 3i} = 0 ;$$

$$\begin{aligned} b_{3, 3i-2} = & \left[\frac{h_{si}^m}{q(\xi)} \frac{dc(\xi)}{d\xi} + \frac{c(\xi)}{q(\xi)} \frac{dh_{si}^m}{d\xi} - \frac{q_i c_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) \right. \\ & \left. - \frac{q_i c_i}{(q(\xi))^2} \frac{d^2 h_{\beta i}^m}{d\xi^2} \right] \cos\phi_i - \left[\frac{1}{q(\xi)} \frac{dh_{ni}^m}{d\xi} \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) \right. \\ & \left. + \frac{1}{(q(\xi))^2} \frac{d^2 h_{ni}^m}{d\xi^2} \right] \sin\phi_i , \end{aligned}$$

$$\begin{aligned}
b_{3, 3i-1} = & - \left[\frac{h_{si}^m}{q(\xi)} \frac{dc(\xi)}{d\xi} + \frac{c(\xi)}{q(\xi)} \frac{dh_{si}^m}{d\xi} - \frac{q_i c_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) \right. \\
& \left. - \frac{q_i c_i}{(q(\xi))^2} \frac{d^2 h_{\beta i}^m}{d\xi^2} \right] \sin \phi_i - \left[\frac{1}{q(\xi)} \frac{dh_{ni}^m}{d\xi} \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) \right. \\
& \left. + \frac{1}{(q(\xi))^2} \frac{d^2 h_{ni}^m}{d\xi^2} \right] \cos \phi_i ,
\end{aligned}$$

$$b_{3, 3i} = \frac{q_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) + \frac{q_i}{(q(\xi))^2} \frac{d^2 h_{\beta i}^m}{d\xi^2} ;$$

$$\begin{aligned}
b_{4, 3i-2} = & \left[c(\xi) h_{si}^m - \frac{q_i c_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \right] \frac{\cos \phi(\xi)}{r(\xi)} \cos \phi_i \\
& - \frac{dh_{ni}^m}{d\xi} \frac{\cos \phi(\xi)}{q(\xi) r(\xi)} \sin \phi_i ,
\end{aligned}$$

$$\begin{aligned}
b_{4, 3i-1} = & - \left[c(\xi) h_{si}^m - \frac{q_i c_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \right] \frac{\cos \phi(\xi)}{r(\xi)} \sin \phi_i \\
& - \frac{dh_{ni}^m}{d\xi} \frac{\cos \phi(\xi)}{q(\xi) r(\xi)} \cos \phi_i , \text{ and}
\end{aligned}$$

$$b_{4, 3i} = \frac{q_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \frac{\cos\phi(\xi)}{r(\xi)}$$

where $i = 1, \dots, m$.

APPENDIX B
B MATRIX FOR DISPLACEMENTS AND MERIDIONAL MOMENT

$$b_{1, 3i-2} = \frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} \cos\phi_i + c(\xi) h_{si}^m \sin\phi_i ,$$

$$b_{1, 3i-1} = \frac{-1}{q(\xi)} \frac{dh_{si}^m}{d\xi} \sin\phi_i + c(\xi) h_{si}^m \cos\phi_i ,$$

$$b_{1, 3i} = 0 ;$$

$$b_{2, 3i-2} = \frac{\cos\phi(\xi) h_{si}^m \cos\phi_i + \sin\phi(\xi) h_{si}^m \sin\phi_i}{r(\xi)} \quad \text{and}$$

$$b_{2, 3i-1} = \frac{\cos\phi(\xi) h_{si}^m \sin\phi_i + \sin\phi(\xi) h_{si}^m \cos\phi_i}{r(\xi)} ,$$

or, using the simplifying assumption $u_r(\xi) = \sum_{i=1}^m h_{si}^m u_{ri}$,

$$b_{2, 3i-2} = \frac{h_{si}^m}{r(\xi)} ,$$

$$b_{2, 3i-1} = 0 ,$$

$$b_{2, 3i} = 0 ;$$

$$b_{3, 3i-2} = 0 ,$$

$$b_{3, 3i-1} = 0 ,$$

$$b_{3, 3i} = h_{si}^m ;$$

$$b_{4, 3i-2} = \frac{\cos\phi(\xi)}{r(\xi)} \left\{ c(\xi) h_{si}^m \cos\phi_i - \frac{1}{q(\xi)} \frac{dh_{si}^i}{d\xi} \sin\phi_i \right\} ,$$

$$b_{4, 3i-1} = \frac{\cos\phi(\xi)}{r(\xi)} \left\{ - c(\xi) h_{si}^m \sin\phi_i - \frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} \cos\phi_i \right\} ,$$

$$b_{4, 3i} = 0 ;$$

$$b_{5, 3i-2} = 0 ,$$

$$b_{5, 3i-1} = 0 ,$$

$$b_{5, 3i} = \frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} ;$$

$$b_{6, 3i-2} = c(\xi) h_{si}^m \cos\phi_i - \frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} \sin\phi_i ,$$

$$b_{6, 3i-1} = - c(\xi) h_{si}^m \sin\phi_i - \frac{1}{q(\xi)} \frac{dh_{si}^m}{d\xi} \cos\phi_i , \text{ and}$$

$$b_{6, 3i} = 0$$

where $i = 1 \dots m$.

APPENDIX C

MIXED VARIATIONAL PRINCIPAL

The following variational principle, originally formulated by Hermann, Taylor, and Green in Ref. 5, is an expression for the first two equilibrium equations and the meridional moment constitutive relation.

$$\begin{aligned} \delta I(u_s, u_n, M_s) = & \delta U(u_s, u_n, M_s) - \int_L r \left\{ S \delta u_s + p \delta u_n \right\} ds \\ & - r \left[\bar{N}_s \delta u_s + \bar{Q} \delta u_n + \bar{\beta} \delta M_s \right]_B = 0 \end{aligned} \quad (C-1)$$

where

$$\begin{aligned} \delta U(u_s, u_n, M_s) = & \int_L \left\{ r N_s \left(\frac{d(\delta u_s)}{ds} + \frac{\delta u_n}{R} \right) + N_\theta \delta u_r \right. \\ & - r \left[\frac{M_s}{D_s} + \left(1 - \frac{D_{s\theta}}{D_s} \right) \kappa_\theta \right] \delta M_s - r \beta \frac{d(\delta M_s)}{ds} \\ & \left. - r Q(M_s, \kappa_\theta) \left[\frac{\delta u_s}{R} - \frac{d(\delta u_n)}{ds} \right] \right\} ds \quad \text{and} \end{aligned} \quad (C-2)$$

$$Q(M_s, \kappa_\theta) = \frac{dM_s}{ds} + \left[1 - \frac{D_{s\theta}}{D_s} \right] \frac{\cos \phi}{r} M_s - \left[D_\theta - \frac{D_{s\theta}^2}{D_s} \right] \kappa_\theta \frac{\cos \phi}{r}. \quad (C-3)$$

Equation (C-3) is obtained from the moment and shear equilibrium equation and the M_θ constitutive relation.

Since $\delta u_r = \cos\phi \delta u_s + \sin\phi \delta u_n$ and

$$\int_L r f \frac{dg}{ds} ds = [r f g]_B - \int g \frac{d(r f)}{ds} ds,$$

$\delta I(u_s, u_n, M_s)$ can be written as

$$\begin{aligned} \delta I(u_s, u_n, M_s) = & \int_L \left\{ \left[-\frac{d(r N_s)}{ds} + N_\theta \cos\phi - \frac{r Q}{R} - r S \right] \delta u_s \right. \\ & + \left[N_\theta \sin\phi - \frac{d(r Q)}{ds} + r \frac{N_s}{R} - r P \right] \delta u_n \\ & + \left. \left[r \left(-\frac{M_s}{D_s} + \frac{D_{s\theta}}{D_s} \kappa_\theta + \kappa_s \right) \right] \delta M_s \right\} ds \\ & + r \left[(N_s - \bar{N}_s) \delta u_s + (Q - \bar{Q}) \delta u_n + (\beta - \bar{\beta}) \delta M_s \right]_B = 0 \end{aligned} \tag{C-4}$$

The following relation was used:

$$\frac{d(r \beta)}{ds} = r \kappa_\theta + r \kappa_s. \tag{C-5}$$

Equation (C-4) is an expression of the meridional and normal equations of equilibrium, the meridional moment (M_s) constitutive relation, and boundary conditions.

Note that because the relation $\frac{\rho}{r} \frac{dr}{ds}$ was used for κ_θ in this development [for Eq. (C-5)] and because this is a small rotation relation, this section is restricted to small rotation theory.

APPENDIX D POTENTIAL ENERGY VARIATIONAL PRINCIPLE

The theorem of minimum potential energy may be expressed as follows:

$$\begin{aligned} \delta P(u_s, u_n, \beta) = & \delta U(u_s, u_n, \beta) - \int_L r \left\{ S \delta u_s + p \delta u_n \right\} ds \\ & - r \left[\bar{N}_s \delta u_s + \bar{Q} \delta u_n + \bar{M}_s \delta \beta \right]_B = 0 \end{aligned} \quad (D-1)$$

where

$$\begin{aligned} \delta U(u_s, u_n, \beta) = & \int_L r \left\{ N_s \left(\frac{d(\delta u_s)}{ds} + \frac{\delta u_n}{R} \right) + N_\theta \frac{\delta u_r}{r} + M_s \frac{d(\delta \beta)}{ds} \right. \\ & \left. + M_\theta \frac{c \cos \phi}{r} \delta \beta + Q \left(\frac{d(\delta u_n)}{ds} + \delta \beta - \frac{\delta u_s}{R} \right) \right\} ds . \end{aligned} \quad (D-2)$$

That Eq. (D-1) is an expression of equilibrium can be shown as follows:

$\delta u_r = \cos \phi \delta u_s + \sin \phi \delta u_n$, and by integration by parts

$$\int_L r f \frac{dg}{ds} ds = \left[r f g \right]_B - \int_L g \frac{d(r f)}{ds} ds \quad .$$

Making these substitutions into the strain energy expression

$$\delta U(u_s, u_n, \beta) \quad (\text{Eq. (D-2)}),$$

$$\begin{aligned} \delta U(u_s, u_n, \beta) = & \int_L \left\{ \left[- \frac{d(r N_s)}{ds} + N_\theta \cos \phi - Q \frac{r}{R} \right] \delta u_s \right. \\ & + \left[\frac{r N_s}{R} + N_\theta \sin \phi - \frac{d(r Q)}{ds} \right] \delta u_n \\ & + \left. \left[- \frac{d(r M_s)}{ds} + M_\theta \cos \phi + r Q \right] \delta \beta \right\} ds \\ & + r \left[N_s \delta u_s + Q \delta u_n + M_s \delta \beta \right]_B. \end{aligned} \quad (\text{D-3})$$

Substituting Eq. (D-3) into Eq. (D-1)

$$\begin{aligned}
\delta P(u_s, u_n, \beta) = & \int_L \left\{ \left[-\frac{d(r N_s)}{ds} + N_\theta \cos \phi - \frac{Q r}{R} - r S \right] \delta u_s \right. \\
& + \left[\frac{r N_s}{R} + N_\theta \sin \phi - \frac{d(r Q)}{ds} - r p \right] \delta u_n \\
& + \left. \left[-\frac{d(r M_s)}{ds} + M_\theta \cos \phi + r Q \right] \delta \beta \right\} ds \\
& + r \left[(N_s - \bar{N}_s) \delta u_s + (Q - \bar{Q}) \delta u_n \right. \\
& + \left. (M_s - \bar{M}_s) \delta \beta \right]_B = 0
\end{aligned}
\tag{D-4}$$

Equation(D-4) is an expression for equilibrium and boundary conditions.

APPENDIX E
SHAPE FUNCTIONS FOR SHELL ELEMENTS

Figure E-1 shows the normalized coordinate ξ . The ξ designates nodal point 1 when $\xi = 1$, nodal point 2 when $\xi = -1$, and nodal point 3 when $\xi = 0$.

For two nodal point elements the mid-side nodal point (nodal point 3 in Fig. E-1) is deleted. Thus for assumed displacement functions let

$$u_s = a + b \xi \quad \text{and} \quad u_n = c + d\xi + e\xi^2 + f\xi^3.$$

a, b, c, d, e , and f can be solved for from the following conditions,

$$\begin{aligned} \xi = 1 & \quad u_s = u_{s1} , \\ \xi = -1 & \quad u_s = u_{s2} , \\ \xi = 1 & \quad u_n = u_{n1} , \\ \xi = -1 & \quad u_n = u_{n2} , \\ \xi = 1 & \quad \frac{du_n}{d\xi} = \psi_1 , \\ \xi = -1 & \quad \frac{du_n}{d\xi} = \psi_2 . \end{aligned}$$

$$\text{Thus } u_s = \sum_{i=1}^2 h_{si}^2 u_{si} \text{ and } u_n = \sum_{i=1}^2 h_{ni}^2 u_{ni} + h_{\beta i}^2 \psi_i$$

where the shape functions h_{si}^2 , h_{ni}^2 , and $h_{\beta i}^2$ are

$$\begin{aligned} h_{s1}^2 &= \frac{1}{2} (1 + \xi), & h_{n1}^2 &= \frac{1}{4} (2 + 3\xi - \xi^2), & h_{\beta 1}^2 &= \frac{1}{4} (-1 - \xi + \xi^2 + \xi^3) \\ h_{s2}^2 &= \frac{1}{2} (1 - \xi), & h_{n2}^2 &= \frac{1}{4} (2 - 3\xi + \xi^2), & h_{\beta 2}^2 &= \frac{1}{4} (1 - \xi - \xi^2 + \xi^3) \end{aligned}$$

Fig. B-1. Normalized coordinate (ξ) for symmetrically loaded shell of revolution model.

and the superscript designates the number of nodal points in the element.

For three nodal point elements the assumed displacement functions are

$$u_s = a + b\xi + c\xi^2 \quad \text{and} \quad u_n = d + e\xi + f\xi^2 + g\xi^3 + h\xi^4 + i\xi^5.$$

The constants $a, b, c, d, e, f, g, h,$ and i can be solved for from the following conditions:

$\xi = 1$	$u_s = u_{s1} ,$
$\xi = -1$	$u_s = u_{s2} ,$
$\xi = 0$	$u_s = u_{s3} ,$
$\xi = 1$	$u_n = u_{n1} ,$
$\xi = -1$	$u_n = u_{n2} ,$
$\xi = 0$	$u_n = u_{n3} ,$

$$\xi = 1 \quad \frac{du_n}{d\xi} = \psi_1 \quad ,$$

$$\xi = -1 \quad \frac{du_n}{d\xi} = \psi_2 \quad ,$$

$$\xi = 0 \quad \frac{du_n}{d\xi} = \psi_3 \quad .$$

$$\text{Thus } u_s = \sum_{i=1}^3 h_{si}^3 u_{si} \text{ and } u_n = \sum_{i=1}^3 \left(h_{ni}^3 u_{ni} + h_{\beta i}^3 \psi_i \right),$$

where the shape functions h_{si}^3 , h_{ni}^3 , and $h_{\beta i}^3$ are

$$h_{s1}^3 = \frac{1}{2} \xi (1 + \xi), \quad h_{n1}^3 = \frac{1}{4} \xi^2 (4 + 5 \xi - 2 \xi^2 - 3 \xi^3),$$

$$h_{s2}^3 = -\frac{1}{2} \xi (1 - \xi), \quad h_{n2}^3 = \frac{1}{4} \xi^2 (4 - 5 \xi - 2 \xi^2 + 3 \xi^3),$$

$$h_{s3}^3 = 1 - \xi^2, \quad h_{n3}^3 = (1 - \xi^2)^2,$$

$$h_{\beta 1}^3 = 1/4 \xi^2 (-1 - \xi + \xi^2 + \xi^3) ,$$

$$h_{\beta 2}^3 = 1/4 \xi^2 (1 - \xi - \xi^2 + \xi^3) ,$$

$$h_{\beta 3}^3 = \xi (1 - \xi^2)^2 ,$$

and the superscript designates the number of nodal points in element.

At this point the variable ψ_i should be replaced with the rotation β_i . The definition of rotation (for small rotations) is

$$\beta(\xi) = \frac{u_s(\xi)}{R(\xi)} - \frac{du_n(\xi)}{ds} .$$

Let $q(\xi) d\xi = ds$ and $c(\xi) = \frac{1}{R(\xi)}$; then

$$\beta(\xi) = c(\xi) u_s(\xi) - \frac{1}{q(\xi)} \frac{du_n(\xi)}{d\xi}.$$

Since

$$u_s(\xi) = \sum_{i=1}^m h_{si}^m u_{si} \text{ and } u_n(\xi) = \sum_{i=1}^m \left(h_{ni}^m u_{ni} + h_{\beta i}^m \psi_i \right),$$

$$\beta(\xi) = c(\xi) \sum_{i=1}^m h_{si}^m u_{si} - \frac{1}{q(\xi)} \sum_{i=1}^m \left(\frac{dh_{ni}^m}{d\xi} u_{ni} + \frac{dh_{\beta i}^m}{d\xi} \psi_i \right).$$

If ξ_k is the value of ξ at a nodal point then $h_{si}^m(\xi_k) = \delta_{ik}$,

$$\left. \frac{dh_{ni}^m}{d\xi} \right|_{\xi = \xi_k} = 0, \text{ and } \left. \frac{dh_{\beta i}^m}{d\xi} \right|_{\xi = \xi_k} = \delta_{ik}. \text{ Let } \beta(\xi_k) = \beta_k,$$

$$c(\xi_k) = c_k, \text{ and } q(\xi_k) = q_k.$$

Then

$$\beta_k = c_k u_{sk} - \frac{\psi_k}{q_k} \quad \text{and} \quad \psi_k = q_k (c_k u_{sk} - \beta_k).$$

The displacements and rotation may be approximated as

$$u_s(\xi) = \sum_{i=1}^m h_{si}^m u_{si},$$

$$u_n(\xi) = \sum_{i=1}^m \left\{ h_{\beta i}^m q_i c_i u_{si} + h_{ni}^m u_{ni} - h_{\beta i}^m q_i \beta_i \right\}, \quad (E-2)$$

$$\beta(\xi) = \sum_{i=1}^m \left\{ \left[c(\xi) h_{si}^m - \frac{1}{q(\xi)} q_i c_i \frac{dh_{\beta i}^m}{d\xi} \right] u_{si} - \frac{1}{q(\xi)} \frac{dh_{ni}^m}{d\xi} u_{ni} + \frac{q_i}{q(\xi)} \frac{dh_{\beta i}^m}{d\xi} \beta_i \right\}. \quad (E-3)$$

APPENDIX F EQUILIBRIUM EQUATIONS

There are three equilibrium equations for a symmetrically loaded shell of revolution as shown in Figs. F-1 - F-3. These are equilibrium along the meridional coordinate s , the normal coordinate n , and rotational equilibrium.

Equilibrium along the meridional coordinate

$$-N_s \cos(\Delta\phi/2) r \Delta\theta + \left(N_s + \Delta N_s \right) \cos(\Delta\phi/2) (r + \Delta r) \Delta\theta$$

$$-N_\theta \Delta\theta \Delta s \cos\phi + S (r + \Delta r/2) \Delta\theta \Delta s + (Q + \Delta Q) \sin(\Delta\phi/2) (r + \Delta r) \Delta\theta$$

$$+ Q \sin(\Delta\phi/2) r \Delta\theta = 0.$$

Fig. F-1. Geometry for symmetrically loaded shell of revolution.

Fig. F-2. Stress resultants, bending moments, and loads for a symmetrically loaded shell of revolution.

Fig. F-3. Geometry, stress resultants, bending moments, and loads for a symmetrically loaded shell of revolution.

As $\Delta\phi$ approaches zero, $\sin(\Delta\phi/2)$ approaches $\Delta\phi/2$ and $\cos(\Delta\phi/2)$ approaches the value 1. Also, since $R \Delta\phi = \Delta s$ and $\Delta\phi = \frac{\Delta s}{R}$,

$$\begin{aligned} & -N_s r \Delta\theta + N_s r \Delta\theta + \Delta N_s r \Delta\theta + N_s \Delta r \Delta\theta + \Delta N_s \Delta r \Delta\theta \\ & -N_\theta \Delta\theta \Delta s \cos\phi + S r \Delta\theta \Delta s + S \frac{\Delta r}{2} \Delta\theta \Delta s + Q \frac{\Delta s}{2R} r \Delta\theta + Q \frac{\Delta s}{2R} \Delta r \Delta\theta \\ & + \Delta Q \frac{\Delta s}{2R} r \Delta\theta + \Delta Q \frac{\Delta s}{2R} \Delta r \Delta\theta + Q \frac{\Delta s}{2R} r \Delta\theta = 0 . \end{aligned}$$

Simplifying and dividing this equation by $\Delta s \Delta\theta$,

$$\begin{aligned} & \frac{(\Delta N_s r + N_s \Delta r)}{\Delta s} - N_\theta \cos\phi + Q \frac{r}{R} + S r + \frac{\Delta N_s \Delta r}{\Delta s} + \frac{S}{2} \Delta r + \frac{Q}{2R} \Delta r + \frac{\Delta Q r}{2R} \\ & + \frac{\Delta Q \Delta r}{2R} = 0 . \end{aligned}$$

As Δs approaches zero and $\Delta\theta$ approaches zero,

$$\frac{\Delta N_s r + N_s \Delta r}{\Delta s} \rightarrow \frac{d(r N_s)}{ds}, \quad \frac{\Delta N_s \Delta r}{\Delta s} \rightarrow \frac{dN_s}{ds} \Delta r \rightarrow 0$$

$\Delta Q \rightarrow 0$ and $\Delta r \rightarrow 0$. Thus

$$\frac{d(r N_s)}{ds} - N_\theta \cos\phi + Q \frac{r}{R} + S r = C. \quad (F-1)$$

Equilibrium along the normal coordinate

$$\begin{aligned}
 & (Q + \Delta Q) \cos(\Delta\phi/2) (r + \Delta r) \Delta\theta - Q \cos(\Delta\phi/2) r \Delta\theta - N_\theta \Delta\theta \sin\phi \Delta s \\
 & - \left(N_s + \Delta N_s \right) \sin(\Delta\phi/2) (r + \Delta r) \Delta\theta - N_s \sin(\Delta\phi/2) r \Delta\theta \\
 & + p \Delta s (r + \Delta r/2) \Delta\theta = 0 .
 \end{aligned}$$

Simplifying as before ,

$$\begin{aligned}
 & \frac{Q \Delta r + r \Delta Q}{\Delta s} - N_\theta \sin\phi - \frac{N_s r}{R} + p r + \frac{\Delta Q}{\Delta s} \Delta r - \frac{N_s \Delta r}{2 R} - \frac{N_s}{2 R} (r + \Delta r) \\
 & + \frac{p}{2} \Delta r = 0 .
 \end{aligned}$$

In the limit

$$\frac{d(r Q)}{ds} - N_\theta \sin\phi - \frac{N_s r}{R} + p r = 0 . \quad (F-2)$$

Rotational equilibrium

$$\begin{aligned}
 & \left(M_s + \Delta M_s \right) (r + \Delta r) \Delta\theta - M_s r \Delta\theta - M_\theta \cos\phi \Delta s \Delta\theta \\
 & - (Q + \Delta Q) (r + \Delta r) \Delta\theta \frac{\Delta s}{2} - Q r \Delta\theta \frac{\Delta s}{2} = 0 .
 \end{aligned}$$

Simplifying as before,

$$\frac{M_s \Delta r + \Delta M_s r}{\Delta s} - M_\theta \cos \phi - Q r + \frac{\Delta M_s \Delta r}{\Delta s} - (Q + \Delta Q) \frac{\Delta r}{2} - \Delta Q \frac{r}{2} = 0 .$$

In the limit

$$\frac{d(r M_s)}{ds} - M_\theta \cos \phi - Q r = 0 \quad (F-3)$$

These equilibrium equations written in this form apply for a circular plate, cylinder, and any other general shell of revolution that is symmetrically loaded and has symmetrical boundary conditions. See Refs. 7 and 8 for additional discussion concerning these equilibrium equations.

When r , z , ϕ , s , and R refer to the deformed configuration, the equilibrium Eqs. (F-1) - (F-3) are valid for nonlinear geometric shells.

APPENDIX G

GEOMETRIC QUANTITIES REQUIRED FOR STIFFNESS

AND FORCE MODIFICATION CALCULATIONS

Two approaches are described for determining the geometric quantities used for the shell stiffness and force modification calculations. (See pgs. 16-22 of text). These are

Approach 1

Item 2: Input the coordinates r and z at each nodal point and assume

$$r(\xi) = \sum_{i=1}^m h_{si}^m r_i \quad \text{and}$$

$$z(\xi) = \sum_{i=1}^m h_{si}^m z_i ,$$

then

$$\frac{dr(\xi)}{d\xi} = \sum_{i=1}^m \frac{dh_{si}^m}{d\xi} r_i \quad \text{and}$$

$$\frac{dz(\xi)}{d\xi} = \sum_{i=1}^m \frac{dh_{si}^m}{d\xi} z_i ,$$

and

$$\frac{d^2 r(\xi)}{d\xi^2} = \sum_{i=1}^m \frac{d^2 h_{si}^m}{d\xi^2} r_i \quad \text{and}$$

$$\frac{d^2 z(\xi)}{d\xi^2} = \sum_{i=1}^m \frac{d^2 h_{si}^m}{d\xi^2} z_i .$$

Item 3: The normalized meridional length $q(\xi)$ is defined as follows:

$$q(\xi) \equiv \frac{ds}{d\xi} = \sqrt{\left(\frac{dr(\xi)}{d\xi}\right)^2 + \left(\frac{dz(\xi)}{d\xi}\right)^2}$$

and

$$\frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) = \frac{-\frac{d^2 z(\xi)}{d\xi^2} \frac{dz(\xi)}{d\xi} - \frac{d^2 r(\xi)}{d\xi^2} \frac{dr(\xi)}{d\xi}}{(q(\xi))^3}$$

Item 4: The azimuthal angle $\phi(\xi)$ is determined as

$$\cos\phi(\xi) = \frac{dr}{ds} = \frac{1}{q(\xi)} \frac{dr(\xi)}{d\xi} \quad \text{and}$$

$$\sin\phi(\xi) = -\frac{dz}{ds} = -\frac{1}{q(\xi)} \frac{dz(\xi)}{d\xi}$$

$$\text{Item 5: The curvature } c(\xi) \equiv \frac{1}{R(\xi)} = \frac{\frac{d^2 z(\xi)}{d\xi^2} \frac{dr(\xi)}{d\xi} + \frac{d^2 r(\xi)}{d\xi^2} \frac{dz(\xi)}{d\xi}}{(q(\xi))^3}$$

For the derivative of the curvature assume

$$c(\xi) = \sum_{i=1}^m h_{si}^m c_i$$

where $c_i = c(\xi_i)$,

then
$$\frac{dc(\xi)}{d\xi} = \sum_{i=1}^m \frac{dh_{si}^m}{d\xi} c_i .$$

Approach 2

Item 2: Same as approach 1

Item 4: Input the azimuthal angle ϕ at each nodal point (ϕ_i), then assume

$$\phi(\xi) = \sum_{i=1}^m h_{si}^m \phi_i ,$$

then

$$\frac{d\phi(\xi)}{d\xi} = \sum_{i=1}^m \frac{dh_{si}^m}{d\xi} \phi_i \quad \text{and} \quad \frac{d^2\phi(\xi)}{d\xi^2} = \sum_{i=1}^m \frac{d^2h_{si}^m}{d\xi^2} \phi_i .$$

Item 5: Input the curvature c at each nodal point (c_i) then assume

$$c(\xi) = \sum_{i=1}^m h_{si}^m c_i ,$$

then

$$\frac{dc(\xi)}{d\xi} = \sum_{i=1}^m \frac{dh_{si}^m}{d\xi} c_i .$$

Item 3: The normalized meridional length $q(\xi)$ is defined as

$$q(\xi) d\xi \equiv R(\xi) d\phi ;$$

thus

$$q(\xi) = \frac{1}{c(\xi)} \frac{d\phi(\xi)}{d\xi} , \text{ and}$$

$$\frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) = \frac{\frac{dc(\xi)}{d\xi} \frac{d\phi(\xi)}{d\xi} - c(\xi) \frac{d^2\phi(\xi)}{d\xi^2}}{\left(\frac{d\phi(\xi)}{d\xi} \right)^2} .$$

For plates and cylinders $c(\xi) = 0$. $\phi(\xi) = 0$ for a plate and $\phi(\xi) = \pi/2$ for a cylinder. In these cases $q(\xi)$ is input for each nodal point of the element (q_i), and then assume

$$\frac{1}{q(\xi)} = \sum_{i=1}^m h_{si}^m \frac{1}{q_i} \quad \text{then} \quad \frac{d}{d\xi} \left(\frac{1}{q(\xi)} \right) = \sum_{i=1}^m \frac{dh_{si}^m}{d\xi} \frac{1}{q_i} .$$

APPENDIX II

GEOMETRIC NONLINEAR SHELL QUANTITIES

One of the basic assumptions used in geometric linear shell theory is that the initial and final geometry are essentially the same. Thus, for geometric nonlinear shell theory, the geometric and shell quantities must be identified as to whether they refer to deformed or undeformed geometry. All quantities in this report will refer to deformed geometry unless they are subscripted with a "o" in which case they will refer to undeformed geometry.

Therefore, the geometric quantities used to describe a shell are

- the coordinates r , z and r_o , z_o ;
- the angle from the axis of revolution ϕ and ϕ_o ;
- the meridional coordinate s and s_o ;
- the normalized meridional length increment q and q_o ; and
- the radius of curvature R and R_o .

The shell quantities are

- the membrane strains ϵ_s , ϵ_θ and ϵ_{s_o} , ϵ_{θ_o} ;
- the bending strains κ_s , κ_θ and κ_{s_o} , κ_{θ_o} ;
- the stress resultants, N_s , N_θ and N_{s_o} , N_{θ_o} ; and
- the bending moments M_s , M_θ and M_{s_o} , M_{θ_o} .

Let the membrane and bending strains be defined as follows (see Ref. 9):

$$\begin{aligned} \bullet \quad \epsilon_s &= \frac{q - q_o}{q}, & \epsilon_\theta &= \frac{r - r_o}{r} \\ \bullet \quad \kappa_s &= \frac{d(\phi - \phi_o)}{ds}, & \kappa_\theta &= \frac{\sin\phi - \sin\phi_o}{r} \end{aligned}$$

and

$$\bullet \quad \epsilon_{s_0} = \frac{q - q_0}{q_0}, \quad \epsilon_{\theta_0} = \frac{r - r_0}{r_0}$$

$$\bullet \quad \kappa_{s_0} = \frac{d(\phi - \phi_0)}{ds_0}, \quad \kappa_{\theta_0} = \frac{\sin\phi - \sin\phi_0}{r_0}$$

(as r, r_0 approach zero; see Appendix L.)

Some interesting transformations between the deformed and undeformed quantities are:

$$ds = \frac{q}{q_0} ds_0$$

$$ds_0 = \frac{q_0}{q} ds$$

$$ds = (1 + \epsilon_{s_0}) ds_0$$

$$ds_0 = (1 - \epsilon_s) ds$$

$$q = q_0 (1 + \epsilon_{s_0})$$

$$q_0 = q (1 - \epsilon_s)$$

$$r = r_0 (1 + \epsilon_{\theta_0})$$

$$r_0 = r (1 - \epsilon_\theta)$$

$$\epsilon_s = \frac{\epsilon_{s_0}}{1 + \epsilon_{s_0}}$$

$$\epsilon_{s_0} = \frac{\epsilon_s}{1 - \epsilon_s}$$

$$\epsilon_{\theta} = \frac{\epsilon_{\theta_0}}{1 + \epsilon_{\theta_0}}$$

$$\epsilon_{\theta_0} = \frac{\epsilon_{\theta}}{1 - \epsilon_{\theta}}$$

$$\kappa_s = \frac{\kappa_{s_0}}{1 + \epsilon_{s_0}}$$

$$\kappa_{s_0} = \frac{\kappa_s}{1 - \epsilon_s}$$

$$\kappa_{\theta} = \frac{\kappa_{\theta_0}}{1 + \epsilon_{\theta_0}}$$

$$\kappa_{\theta_0} = \frac{\kappa_{\theta}}{1 - \epsilon_{\theta}}$$

Differential relations

$$ds = R \, d\phi = q \, d\xi,$$

$$ds_0 = R_0 \, d\phi_0 = q_0 \, d\xi.$$

APPENDIX I

INCREMENTING AND LINEARIZING OF ϵ_s , ϵ_{θ} , κ_s , AND κ_{θ}
FOR NONLINEAR SHELLS

Incrementing

Since $\epsilon_s = \frac{q - q_0}{q}$ (see Appendix H), for the $j+1$ increment

$$\epsilon_{s_{j+1}} = \frac{q_{j+1} - q_0}{q_{j+1}} = \frac{q_j}{q_{j+1}} \left\{ \frac{q_j - q_0}{q_j} + \frac{q_{j+1} - q_j}{q_j} \right\}$$

$$= \frac{1}{1 + \Delta\epsilon_{s_j}} \left\{ \epsilon_{s_j} + \Delta\epsilon_{s_j} \right\}, \quad \Delta\epsilon_{s_j} = \frac{q_{j+1} - q_j}{q_j}.$$

Similarly, since $\epsilon_\theta = \frac{r - r_0}{r}$, for the $j+1$ increment,

$$\begin{aligned}\epsilon_{\theta_{j+1}} &= \frac{r_{j+1} - r_0}{r_{j+1}} = \frac{r_j}{r_{j+1}} \left\{ \frac{r_j - r_0}{r_j} + \frac{r_{j+1} - r_j}{r_j} \right\} \\ &= \frac{1}{1 + \Delta\epsilon_\theta} \left\{ \epsilon_{\theta_j} + \Delta\epsilon_{\theta_j} \right\}, \quad \Delta\epsilon_{\theta_j} = \frac{r_{j+1} - r_j}{r_j}.\end{aligned}$$

Also, since $\kappa_s = \frac{d(\phi_j - \phi_0)}{ds} = \frac{1}{q} \frac{d(\phi - \phi_0)}{d\xi}$,

$$\begin{aligned}\kappa_{s_{j+1}} &= \frac{q_j}{q_{j+1}} \left\{ \frac{1}{q_j} \frac{d(\phi_j - \phi_0)}{d\xi} + \frac{1}{q_j} \frac{d(\phi_{j+1} - \phi_j)}{d\xi} \right\} \\ &= \frac{1}{1 + \Delta\epsilon_{s_j}} \left\{ \kappa_{s_j} + \Delta\kappa_{s_j} \right\}, \quad \Delta\kappa_{s_j} = \frac{1}{q_j} \frac{d(\phi_{j+1} - \phi_j)}{d\xi}.\end{aligned}$$

Again, since $\kappa_\theta = \frac{\sin\phi - \sin\phi_0}{r}$,

$$\begin{aligned}
\kappa_{\theta_{j+1}} &= \frac{r_j}{r_{j+1}} \left\{ \frac{\sin \phi_j \cos (\phi_{j+1} - \phi_j) - \sin \phi_0}{r_j} \right. \\
&\quad \left. + \frac{\sin (\phi_{j+1} - \phi_j) \cos \phi_j}{r_j} \right\} \\
&= \frac{1}{1 + \Delta \epsilon_{\theta}} \left\{ \frac{\sin \phi_j \cos (\phi_{j+1} - \phi_j) - \sin \phi_0}{r_j} \right. \\
&\quad \left. + \frac{\sin (\phi_{j+1} - \phi_j) \cos \phi_j}{r_j} \right\} .
\end{aligned}$$

Let $\Delta u_{r_j} \equiv r_{j+1} - r_j$, $\Delta u_{z_j} \equiv z_{j+1} - z_j$,

$\Delta \beta_j \equiv \phi_{j+1} - \phi_j$. Then

$\Delta u_{n_j} = \sin \phi \Delta u_{r_j} + \cos \phi \Delta u_{z_j}$ and

$\Delta u_{s_j} = \cos \phi \Delta u_{r_j} - \sin \phi \Delta u_{z_j}$.

Linearizing

The linearizing assumptions are:

- $\Delta \epsilon_{s_j}$ and $\Delta \epsilon_{\theta_j}$ are small compared to 1 ,

$$\text{or } \frac{r_j}{r_{j+1}} \cong 1 \quad \text{and} \quad \frac{q_j}{q_{j+1}} \cong 1 ;$$

- $\Delta \beta_j \equiv \phi_{j+1} - \phi_j$, is a small angle such that $\cos(\Delta \beta_j) \cong 1$ and

$$\sin(\Delta \beta_j) \cong \Delta \beta_j ; \quad \text{and}$$

- $q_{j+1} = (R_j + \Delta u_{n_j}) \frac{d\phi_j}{d\xi} + \frac{d(\Delta u_{s_j})}{d\xi}$ (see Ref. 9).

Thus

$$\epsilon_{s_{j+1}} \cong \epsilon_{s_j} + \Delta \epsilon_{s_j} , \quad \text{where} \quad \Delta \epsilon_{s_j} = \frac{\Delta u_{n_j}}{R_j} + \frac{d(\Delta u_{s_j})}{ds_j} ;$$

$$\epsilon_{\theta_{j+1}} \cong \epsilon_{\theta_j} + \Delta \epsilon_{\theta_j} , \quad \text{where} \quad \Delta \epsilon_{\theta_j} = \frac{\Delta u_{r_j}}{r_j} ;$$

$$\kappa_{s_{j+1}} \cong \kappa_{s_j} + \Delta \kappa_{s_j} , \quad \text{where} \quad \Delta \kappa_{s_j} = \frac{d(\Delta \beta_j)}{ds_j} ; \quad \text{and}$$

$$\kappa_{\theta_{j+1}} \cong \kappa_{\theta_j} + \Delta \kappa_{\theta_j} , \quad \text{where} \quad \Delta \kappa_{\theta_j} = \frac{\Delta \beta_j \cos \phi_j}{r_j} .$$

APPENDIX J

INCREMENTAL N_s , N_θ , M_s , AND M_θ

Consider the following constitutive relations:

$$N_s = C_s \epsilon_s + C_{s\theta} \epsilon_\theta + N_{s_i}$$

$$N_\theta = C_{s\theta} \epsilon_s + C_\theta \epsilon_\theta + N_{\theta_i}$$

$$M_s = D_s \kappa_s + D_{s\theta} \kappa_\theta + M_{s_i}$$

$$M_\theta = D_{s\theta} \kappa_s + D_\theta \kappa_\theta + M_{\theta_i}$$

The $j+1$ incremental form of these equations is

$$\begin{aligned} N_{s_{j+1}} &= C_s \epsilon_{s_{j+1}} + C_{s\theta} \epsilon_{\theta_{j+1}} + (N_{s_i})_{j+1} \\ &= C_s (\epsilon_{s_j} + \Delta\epsilon_{s_j}) + C_{s\theta} (\epsilon_{\theta_j} + \Delta\epsilon_{\theta_j}) + (N_{s_i})_{j+1} \\ &= N_{s_j} + C_s \Delta\epsilon_{s_j} + C_{s\theta} \Delta\epsilon_{\theta_j} + (N_{s_i})_{j+1} \\ &= N_{s_j} + (N_{s_i})_{j+1} + \Delta N_{s_j}, \end{aligned}$$

$$\text{where } \Delta N_{s_j} = C_s \Delta \epsilon_{s_j} + C_{s\theta} \Delta \epsilon_{\theta_j} .$$

Similarly,

$$N_{\theta_{j+1}} = N_{\theta_j} + (N_{\theta_i})_{j+1} + \Delta N_{\theta_j} ,$$

$$\text{where } \Delta N_{\theta_j} = C_{s\theta} \Delta \epsilon_{s_j} + C_{\theta} \Delta \epsilon_{\theta_j} .$$

$$M_{s_{j+1}} = M_{s_j} + (M_{s_i})_{j+1} + \Delta M_{s_j} ,$$

$$\text{where } \Delta M_{s_j} = D_s \Delta \kappa_{s_j} + D_{s\theta} \Delta \kappa_{\theta_j} .$$

$$M_{\theta_{j+1}} = M_{\theta_j} + (M_{\theta_i})_{j+1} + \Delta M_{\theta_j} ,$$

$$\text{where } \Delta M_{\theta_j} = D_{s\theta} \Delta \kappa_{s_j} + D_{\theta} \Delta \kappa_{\theta_j} .$$

APPENDIX K

NONLINEAR MATERIAL PROPERTIES

Consider the nonlinear material shown in Fig. K-1. The stress resultant N_s refers to the deformed shell configuration, while the membrane strain ϵ_{s_0} refers to the undeformed shell configuration. The stress resultants and bending

Fig. K-1. Nonlinear material (N_s vs ϵ_{s_0}).

Fig. K-2. Nonlinear material (N_s vs ϵ_s)

moments that refer to the deformed shell configuration and the membrane and bending strains that refer to the undeformed shell configuration are the ones usually associated with experimental work. In this report the membrane and bending strains used are those referring to the deformed shell configuration. Thus, using the transformation shown in Appendix H, the nonlinear material properties can be transformed to relate membrane and bending strains to stress resultants and bending moments where all quantities refer to the deformed shell configuration. Using the transformation from ϵ_{s_0} to ϵ_s , Fig. K-2 is generated.

From these two figures it is apparent that N_{sj} may be obtained using the secant modulus shown in either Fig. K-1 or K-2, while for ΔN_{sj} and $\Delta \epsilon_{sj}$ the tangential modulus shown in Fig. K-2 must be used. These same conclusions can be made for all material properties. Thus, given the membrane and bending strains in either the deformed or undeformed shell configuration, the deformed shell configuration membrane stresses and bending moments may be obtained using the corresponding secant material properties. However, for incremental membrane and bending strains the tangential material properties for the deformed shell configuration quantities are required.

APPENDIX L

ϵ_θ AND κ_θ FOR CLOSED ($r = 0$) SHELLS

When a shell of revolution is closed ($r = 0$), special considerations are required to determine the hoop strain ϵ_θ and the change of curvature κ_θ . Since $\frac{dr}{ds} = \cos\phi$, the equilibrium Eqs. (F-1) and (F-3) become

$$N_s = N_\theta \quad \text{and}$$

$$M_s = M_\theta$$

as r approaches zero. Thus for the constitutive relation of

$$N_s = C_s \epsilon_s + C_{s\theta} \epsilon_\theta + N_{si}$$

$$N_\theta = C_{s\theta} \epsilon_s + C_\theta \epsilon_\theta + N_{\theta i}$$

$$M_s = D_s \kappa_s + D_{s\theta} \kappa_\theta + M_{si}$$

$$M_\theta = D_{s\theta} \kappa_s + D_\theta \kappa_\theta + M_{\theta i}$$

$$\epsilon_\theta = \epsilon_s \left(\frac{C_s - C_{s\theta}}{C_\theta - C_{s\theta}} \right) + \frac{N_{si} - N_{\theta i}}{C_\theta - C_{s\theta}} \quad \text{and}$$

$$\kappa_\theta = \kappa_s \left(\frac{D_s - D_{s\theta}}{D_\theta - D_{s\theta}} \right) + \frac{M_{si} - M_{\theta i}}{D_\theta - D_{s\theta}}$$

as r approaches zero.

These relations are valid for both linear and nonlinear shells.

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