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THEORY OF DISSIPATIVE DRIFT
INSTABILITIES IN SHEARED MAGNETIC
FIELDS

BY

LIU CHEN, P. N. GUZDAR, J. Y. HSU,
P. K. KAW, C. OBERMAN, AND R. WHITE

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LIU CHIA, J. N. TATAR, J. E. RICE, G. E. FAY, J. BERNAL
AND R. WHITE

Plasma Physics Laboratory, Princeton University
Princeton, New Jersey 08540

ABSTRACT

By using several different techniques, we have investigated the stability of electrostatic drift wave eigenmodes in a resistive plasma with finite magnetic shear. It is found that in the slab approximation, where usual shear damping is operative, resistivity contributes to an enhancement of this damping and the enhancement factor increases with the electron-ion collision frequency ν_{ei} . Thus no unstable eigenmodes result. If the shear damping is nullified, either by introducing a strong spatial variation of the density gradient, or by working in toroidal geometry with strong toroidal coupling effects, then unstable eigenmodes with growth rates increasing with ν_{ei} are recovered. A perturbation calculation shows that retention of electrostatic and the fluctuations associated with the mode and inclusion of temperature gradients does not alter these conclusions. Extensive numerical calculations and relation to published experimental results are also presented.

[†] Present Address: Gulf General Atomic, San Diego, CA

I. INTRODUCTION

It is generally recognized that low-frequency drift wave fluctuations, excited in confined plasmas by the release of expansion free energy through wave-particle ("collisionless") or collisional ("dissipative") interaction, may very well be contributing to anomalous plasma transport across magnetic field lines [1]. In toroidal confinement devices like tokamaks, stellarators, multipoles etc., one therefore uses magnetic shear to stabilize these waves [2]. In a theoretical description of such systems, the usual local approximations break down and one instead carries out a stability analysis for the relevant eigenmodes [3]. For collisionless drift waves driven by density gradients, such an eigenmode analysis was first carried out in the slab approximation by Pearlstein and Berk [4]. Using a perturbation technique, they purported to show that unstable drift wave eigenmodes may result, provided the shear is weaker than a critical value. These modes are peaked around the mode-rational surface (defined by $\underline{k} \cdot \underline{B} = 0$) and have outward propagating energy far away. To our knowledge, such an analysis for dissipative drift waves has not been carried out in any detail, although such waves may be playing an important role in certain regions of parameter and configuration space, even for high-temperature plasmas such as in tokamaks. The only previous works somewhat related to this topic that we have been able to find in the literature are a study of convective amplification of resistive drift waves without inclusion of the ion-sound term [5], and some crude attempts at estimating

the shear stabilization criterion for these waves [3,6].

In this paper we have examined the stability of resistive drift wave eigenmodes in the electrostatic approximation. Using a variety of methods, we come to the surprising conclusion that in a slab plasma with finite magnetic shear, drift wave eigenmodes are never unstable. The inclusion of resistivity is shown to further enhance the shear damping of drift wave eigenmodes by a factor which increases with the electron-ion collision frequency ν_{ei} . We find further that if the shear damping effects are nullified, either by localizing the mode energy with a strong spatial variation of the density gradient [7,8] or by using strong toroidal coupling effects [9], then unstable eigenmodes are recovered and the growth rates increase with ν_{ei} . Very recent work with collisionless drift waves [10-12] has shown that similar conclusions hold there and that the earlier results of Pearlstein and Berk and others [4,13] were in error because of the use of some questionable approximations. The general conclusion then, is that shear damping is rather effective in stabilizing drift wave eigenmodes and that the so called universal instability is actually non-existent. It should be emphasised, however, that even when there are no unstable eigenmodes, the possibility of a substantial convective amplification of propagating wave packets is not ruled out [14,15].

This paper is organized as follows: In Sec. 2, the physical problem is set up and the basic equations are derived. The eigenmode problem is solved by a matched asymptotic treatment in Sec. 3. In Sec. 4 we describe a WKB analysis of the problem.

This is valid in certain regions of parameter space and entirely agrees with the matched asymptotic treatment in the appropriate limit. Section 5 gives the details of a first order perturbation theory which again recovers a limiting case of the asymptotic treatment of Sec. 3. In this section, we also extend the perturbation theory to include (a) the effect of finite electron thermal conductivity along the magnetic field, and (b) any ambient normal or reverse electron temperature gradients. We conclude that the stability results obtained previously, remain essentially unchanged. Finally, since most of the above techniques are valid only in restricted regions of parameter space, we further corroborate and extend the results by a direct numerical integration of the differential equation describing the eigenmode. These results are presented in Sec. 6.

II. BASIC EQUATIONS

Consider first a plasma of plane slab geometry with both, the density variation and magnetic shear in x i.e., $N \equiv N(x), B_0(x) = B_0(\hat{e}_z + x\hat{e}_y/L_s)$. Here, L_s is the shear scale length. In this section, we shall assume the parallel electron thermal conductivity to be infinite and treat the electron dynamics as isothermal. (The effects of finite electron temperature fluctuations are treated in Sec. 5.) Ions will throughout be treated as a cold species. Using a fluid description for the dynamics of electrons as well as ions and assuming potential perturbations of the form $\phi(x,t) = \phi(x)\exp[i(k_y y - \omega t)]$, we may write the linearized equations of

continuity and motion as

$$\omega n_j + i \nabla_{\perp} \cdot (n(x) \tilde{v}_{\perp})_j - k_{\parallel} v_{\parallel j} = 0 \quad (2.1)$$

and

$$\omega v_{\parallel j} = k_{\parallel} \left[\frac{g}{m} \right]_j \phi - i v_{jj}' (v_{\parallel j} - v_{\parallel j}') + k_{\parallel} \left[\frac{T}{mN} \right]_j n_j . \quad (2.2)$$

Here, $j, j' = e, i$, $j \neq j'$, $k_{\parallel} = k_y x / L_s \equiv k_{\parallel}' x$,

$\tilde{v}_{\perp} = i k_y \tilde{e}_y + \tilde{e}_x d/dx$, $v_{ie} = -v_{ei} m_e / m_i$, $T_i = 0$,

$\tilde{v}_{ie} = \tilde{v}_E + \tilde{v}_{de}$, $\tilde{v}_{ii} = \tilde{v}_E + \tilde{v}_{pi}$,

$$\tilde{v}_E = -c(\nabla_{\perp} \phi \times \underline{B}_0) / B_0^2 ,$$

$$\tilde{v}_{de} = - (c T_e / e B_0^2 N) (\underline{B}_0 \times \nabla N) , \quad (2.3)$$

and

$$\tilde{v}_{pi} = - (c \omega / B_0 \omega_{ci}) \nabla_{\perp} \phi .$$

Note that ion parallel (to $\underline{B}_0$) motion is retained here so that shear damping effects due to ion-sound waves are present [4]. Note also that Eqs. (2.1)-(2.3) can be obtained using drift-kinetic equations with density conserving Krook collisional operators and working in the limit $v_{ei} > k_{\parallel} v_e$, ω .

Equations (2.1)-(2.3) may be used to express n_e and n_i in terms of ϕ . The quasi-neutrality condition then yields the following eigenmode equation

$$\rho_s^2 \left(\frac{d^2}{dx^2} - k_y^2 \right) \phi - \frac{x^2}{x^2 - ix_R^2} \left[1 - \frac{\omega_{*e}(x)}{\omega} - \frac{x^2}{x_s^2} \right] \phi = 0 \quad (2.4)$$

where $x_R^2 = \omega v_{ei}/k_{||}^2 v_{Te}^2$, $v_{Te}^2 = T_e/m_e$, $x_s^2 = \omega^2/k_{||}^2 C_s^2$,

$C_s^2 = T_e/m_i$, $\rho_s^2 = C_s^2/\omega_{ci}^2$, $\omega_{*e}(x) = k_y \rho_s C_s/L_n(x)$ and

$[L_n(x)]^{-1} = |d \ln N(x)/dx|$. The first term in Eq. (2.4) comes from the ion polarization drift, the coefficient $x^2/(x^2 - ix_R^2)$ from the resistive parallel electron motion and the (x^2/x_s^2) term from parallel ion motion.

In considering the problem with a spatial variation of ω_* , we treat the simple case in which ω_* is peaked at the mode-rational surface at $x = 0$ and has a parabolic profile viz:

$$\omega_{*e}(x) \cong \omega_{*0} \left(1 - \frac{x^2}{L_*^2} \right) \quad (2.5)$$

Equation (2.4) with ω_{*e} constant or a function of x [given by Eq. (2.5)] is the basic differential equation for the resistive drift mode in the slab approximation. For $x_s \rightarrow \infty$, $x \ll x_R$ and ω_{*e} constant, Coppi, et al., [5] demonstrated that no absolutely unstable eigenmodes exist and only convective amplification may occur. Moiseev and Sagdeev [8] analysed the

eigenvalue problem for Eq. (2.4) with $k_{||} = k_{||}' x = \text{constant}$ (no shear), ω_{*e} given by Eq. (2.5), $x_s \rightarrow \infty$ and $x \ll x_R$.

In the subsequent sections we shall address ourselves to the general eigenvalue problem posed by Eqs. (2.4) and (2.5). Since the equations are symmetric with respect to x , the solutions obviously exhibit parities. This fixes one boundary condition, e.g., we may have even solutions with $\phi' = 0$ at $x = 0$ or odd solutions with $\phi = 0$ at $x = 0$. The second boundary condition is for large $|x|$, where we look for outgoing waves. (This is the same as well-behaved solutions at $|x| \rightarrow \infty$ for $\text{Im } \omega > 0$). The latter condition is identical to the one proposed by Pearlstein and Berk [4] for the collisionless problem.

Before proceeding with a solution of the eigenvalue problem, we wish to derive the corresponding differential equation for resistive drift waves in an axisymmetric toroidal plasma with concentric, circular magnetic surfaces. In terms of the usual (r, θ, ξ) coordinates, we express the perturbations in the following form

$$\phi(r, \theta, \xi, t) = \exp[i(n\xi - m_0\theta - \omega t)] \tilde{\phi}(\theta, r), \quad (2.6)$$

and

$$\tilde{\phi}(\theta, r) = \sum_{\ell} \phi(\ell, s) \exp(-i\ell\theta). \quad (2.7)$$

Here r , θ , and ξ correspond, respectively, to the (minor) radial, poloidal and toroidal coordinates. n is the toroidal mode number. The θ dependence of ϕ arises because the toroidal magnetic field has θ dependence, viz.

$$\vec{B}_0 = B_T \vec{e}_\xi + B_\theta \vec{e}_\theta, \quad (2.8)$$

and

$$B_\theta = \bar{B}_0 \left(1 - \frac{r}{R} \cos \theta\right). \quad (2.9)$$

We assume $|m_0| \gg |\ell|$. The $\phi(\ell, s)$'s then correspond to poloidal "sidebands" coupled together by the toroidal effects. Furthermore, we have defined $s = (r - r_0)/\Delta r_s$, $|r - r_0| \ll r_0$, r_0 as the reference mode-rational surface where $m_0 = nq(r_0)$, $q(r) = rB_T/RB_\theta$ and $(\Delta r_s)^{-1} = m_0 q'/q$. In the present treatment, toroidal coupling enters through the spatial gradients of B_θ . Thus the terms $n_0 \vec{v} \cdot \vec{v}_E$ in electron and ion continuity equations are finite and given by

$$n_0 \vec{v} \cdot \vec{v}_E \approx -in_0 \omega_{de} \left[\cos \theta + i \frac{r_0}{m_0} (\sin \theta) \frac{\partial}{\partial r} \right] \left(\frac{e\phi}{T_e} \right) \quad (2.10)$$

where $\omega_{de} = (cT_e m_0 / eB_0 R r_0)$. In the electron continuity equation, we also have the following additional term

$$\vec{v} \cdot (n \vec{v}_{de}) = in_0 \omega_{de} \left[\cos \theta + i \frac{r_0}{m_0} (\sin \theta) \frac{\partial}{\partial r} \right] \left(\frac{n}{n_0} \right) \quad (2.11)$$

One can show using the drift-kinetic approach that these couplings are equivalent to those arising by retention of the magnetic-gradient drifts on the zeroth-order particle trajectories. Following a procedure, similar to that used above in the slab problem one can derive the two-dimensional eigen-mode equation given below.

$$\begin{aligned} & \left[1 - i \frac{\omega v_{ei}}{k_{||}^2 v_{Te}} \right] \rho_s^2 \left[\left(\Delta r_s \right)^{-2} \frac{\partial^2}{\partial s^2} - k_{yo}^2 \right] \left[\phi(\ell, s) \right] \\ &= \left[1 - \frac{\omega_{*e}}{\omega} - \frac{k_{||}^2 C_s^2}{\omega^2} + \frac{\omega_{de}}{\omega} \left[1 - i \frac{\omega_{*e} v_{ei}}{k_{||}^2 v_{Te}} \right] C \right] \phi(\ell, s) \end{aligned} \quad (2.12)$$

where $k_{yo} = m_o/r_o$, $k_{||} = (s - \ell)/qR$, and C is the toroidal coupling operator defined as

$$\begin{aligned} C\phi(\ell, s) &= \frac{1}{2} \left\{ \phi(\ell + 1, s) + \phi(\ell - 1, s) \right. \\ &\quad \left. + \frac{q'r_o}{q} \frac{\partial}{\partial s} \left[\phi(\ell + 1, s) - \phi(\ell - 1, s) \right] \right\}. \end{aligned} \quad (2.13)$$

In deriving Eq. (2.12), we have assumed $|\omega_{de}/\omega| < 1$.

To make further analytical progress, we reduce the above equation to a one dimensional form by using Taylor's strong coupling approximation [9] ($|\ell| > 1$). We thus seek ballooning-mode-like solutions [9, 16] in terms of a single variable $z = \ell - s$ and approximate

$$C\phi(\ell, s) \approx \left[1 + \left(\frac{1}{2} - \frac{q'r_o}{q} \right) \frac{d^2}{dz^2} \right] \phi(z) \quad (2.14)$$

Equation (2.12) now takes the one-dimensional form;

$$\left[D_1 \left(1 - iD_3 \frac{z_R^2}{z^2} \right) \frac{d^2}{dz^2} + D_2 - \frac{z^2}{z_s^2} - iD_4 \frac{z_R^2}{z^2} \right] \phi(z) = 0, \quad (2.15)$$

where

$$D_1 = \frac{\omega_{de}}{\omega} \left(\frac{1}{2} - \frac{q'r_o}{q} \right) - \frac{\rho_s^2}{\Delta r_s^2}, \quad (2.16)$$

$$D_2 = 1 - \frac{\omega_{*e}}{\omega} + k_{yo}^2 \rho_s^2 + \frac{\omega_{de}}{\omega}, \quad (2.17)$$

$$D_3 = \left[\frac{\omega_{de} \omega_{*e}}{\omega^2} \left(\frac{1}{2} - \frac{q'r_o}{q} \right) - \frac{\rho_s^2}{\Delta r_s^2} \right] / D_1, \quad (2.18)$$

$$D_4 = \frac{\omega_{de} \omega_{*e}}{\omega^2} + k_{yo}^2 \rho_s^2, \quad (2.19)$$

$$z_R^2 = \omega v_{ei} q^2 R^2 / v_{Te}^2, \quad (2.20)$$

and

$$z_s^2 = (qR\omega/C_s)^2. \quad (2.21)$$

Equation (2.15) is the relevant eigenmode equation to be solved for investigating the effects of strong toroidal couplings. For $\omega_{de} = 0$, it reduces to the equation of the slab case, Eq. (2.4).

III. MATCHED ASYMPTOTIC TREATMENT

We first examine the eigenvalue problem for the slab case with $L_* = \infty$, i.e., Eq. (2.4) with ω_{*e} independent of x . After changing variables and rearranging the terms Eq. (2.4) may be rewritten as

$$\left\{ \frac{d^2}{d\tau^2} + \gamma - \frac{\tau^2}{4} - \frac{\lambda}{\tau^2 + \tau_R^2} \right\} \phi(\tau) = 0, \quad (3.1)$$

where $\tau = x \exp(i\pi/4)/\lambda$,

$$\lambda^2 = \rho_s x_s / 2, \quad (3.2)$$

$$\gamma = i \frac{\lambda^2}{\rho_s} \left[1 - \frac{\omega_{*e}}{\omega} + k_y^2 \rho_s^2 - i \frac{x_R^2}{x_s^2} \right] \quad (3.3)$$

$$\tau_R^2 = x_R^2 / \lambda^2, \quad (3.4)$$

and

$$\Lambda = i \left(\frac{x_R}{\rho_s} \right)^2 \left\{ 1 - \frac{\omega_{*e}}{\omega} - i \frac{x_R^2}{x_s^2} \right\} . \quad (3.5)$$

We solve the eigenvalue problem as posed by Eq. (3.1) by the following matching procedure. In order that the matching procedure be valid, we require

$$\tau_R^2 = 2 \left(\frac{v_{ei}}{\omega_{*e}} \right) \left(\frac{m_e}{m_i} \right) \left(\frac{L_s}{L_n} \right) \ll 1 , \quad (3.6)$$

In the outer region $|\tau^2| \gg |\tau_R^2|$, we have Eq. (3.1) as

$$\left\{ \frac{d^2}{d\tau^2} + \gamma - \frac{\tau^2}{4} - \frac{\Lambda}{\tau^2} \right\} \phi_0(\tau) = 0 .$$

The solution which properly decays away for $|x| \rightarrow \infty$ and $\text{Im}\omega > 0$ is given by

$$\phi_0(\tau) = A_0 (\tau^2/2)^{(2\alpha + 1)/4} U(a, b, \tau^2/2) \exp [- \tau^2/4] \quad (3.7)$$

where

$$\alpha = - (1 + 4\Lambda)^{1/2}/2 , \quad (3.8)$$

$$b = 1 + \alpha , \quad (3.9)$$

$$a = (b - \gamma)/2 , \quad (3.10)$$

and U is the Kummer's Confluent Hypergeometric function.
Meanwhile, in the inner region, $|\tau| \sim |\tau_R|$, we have

$$\left(\frac{d^2}{d\tau^2} - \frac{\Lambda}{\tau^2 + \tau_R^2} \right) \phi_I(\tau) = 0 ,$$

and the solution is

$$\phi_I(\tau) = (\tau_I^2 - 1)^{1/2} [A_I P_v^1(\tau_I) + B_I Q_v^1(\tau_I)] , \quad (3.11)$$

where

$$v = - (\alpha + 1/2) , \quad (3.12)$$

$\tau_I = \tau/i\tau_R = x \exp(-i\pi/4)/x_R$, P_v^1 and Q_v^1 are the Associated Legendre functions. The ratio B_I/A_I is decided by the parity condition at $x = 0$. We have for even- ϕ modes,

$$\left(B_I / A_I \right)_e = i \frac{2}{\pi} \cos \left(\frac{\nu \pi}{2} \right) \exp \left(- i \frac{\nu \pi}{2} \right) , \quad (3.13)$$

and, for odd- ϕ modes,

$$\left(B_I / A_I \right)_o = - \frac{2}{\pi} \sin \left(\frac{\nu \pi}{2} \right) \exp \left(- i \frac{\nu \pi}{2} \right) . \quad (3.14)$$

We note that in deriving Eqs. (3.13) and (3.14), we have only considered the $x \geq 0$ half-space and assumed

$$\text{Im}[x_R \exp(i\pi/4)] > 0 . \quad (3.15)$$

We now proceed to match ϕ_0 and ϕ_I . First, for $|\tau^2/2| \ll 1$, we have ϕ_0 as

$$\phi_0^<(\tau) = \frac{A_o}{\sin \pi b} \left[\frac{(\tau^2/2)^{-\nu/2}}{\Gamma(1+a-b)\Gamma(b)} - \frac{(\tau^2/2)^{(1+\nu)/2}}{\Gamma(a)\Gamma(2-b)} \right] \quad (3.16)$$

where Γ denotes the usual Gamma function. Meanwhile, for $|\tau_I| = |\tau/\tau_R| \gg 1$, ϕ_I becomes

$$\phi_I^>(\tau) = \frac{A_I 2^\nu}{\pi^{1/2}} \left[\tau_I^\nu + \frac{\Gamma(\nu+1)}{\Gamma(\nu)} + 2^{-(1+2\nu)} \tau_I^{-\nu} \right. \\ \left. \frac{\Gamma(-\nu-1/2)}{\Gamma(-1-\nu)} \left(1 + \pi \frac{B_I}{A_I} \cot \pi \nu \right) \right] . \quad (3.17)$$

Matching $\phi_0^<$ and $\phi_I^>$ then yields the following eigenvalue condition

$$\frac{\Gamma(a)}{\Gamma(a+v+1/2)} = - \left[\frac{i\tau_R}{2^{3/2}} \right]^{1+2v} \frac{\Gamma(v)\Gamma(-v-1/2)\Gamma(1/2-v)}{\Gamma(-1-v)\Gamma(v+1/2)\Gamma(3/2+v)} \cdot \left[1 + \pi \frac{B_I}{A_I} \cot \pi v \right] . \quad (3.18)$$

Note the definitions of various symbols here from Eqs. (3.2)-(3.5), (3.8)-(3.10) and (3.12)-(3.14).

The eigenvalue condition, Eq. (3.18), is in general difficult to solve analytically for the complex frequency ω . However, approximate results may be obtained in certain interesting limits:

$$A. \quad |\Lambda| \approx k^2 |x_R|^2 < 1/4 .$$

In this weakly collisional limit, we have approximately, from Eqs. (3.8) and (3.12)-(3.14),

$$\alpha \approx - (1/2) - \Lambda , \quad v \approx \Lambda , \quad (B_I/A_I)_e \approx 2i/\pi , \quad (B_I/A_I)_o = -v(1 - iv\pi/2) .$$

For even- ϕ modes, Eq. (3.18) thus becomes

$$\Gamma(a)/\Gamma(a + \Lambda + \frac{1}{2}) \approx - 2^{3/2} (\tau_R/\pi\Lambda) . \quad (3.19)$$

Making the further approximation

$\Lambda < \tau_R$ (i.e., $k^2 |x_R|^2 < x_R (\rho_s x_s)^{-1/2} < 1$), Eq. (3.19) gives

$$a \approx -n + (-1)^{n+1} 2^{-3/2} (\pi \Lambda) / [\tau_R \Gamma(-n + \frac{1}{2})] \quad (3.20)$$

Concentrating on the lowest $n = 0$ even eigenmode and using

Eqs. (3.3), (3.9) and (3.10), we obtain the dispersion

relation

$$1 - \frac{\omega_{*e}}{\omega} = - \left[\frac{L_n}{i L_s} + k_y^2 \rho_s^2 \right] \left[1 + \left(\frac{v_{ei}}{\omega_{*e}} \frac{m_e}{m_i} \frac{L_s}{L_n} \right)^{1/2} \right] \quad (3.21)$$

Equation (3.21) leads to the surprising conclusion that in a sheared geometry resistivity only contributes to an enhancement of the damping of drift waves. A detailed analysis shows that in this limit, the expected growth terms are proportional to v_{ei} and always subdominant to the resistivity induced enhanced damping, proportional to $v_{ei}^{1/2}$. Thus no unstable eigenmode exists.

As we go to higher collisionalities and higher $k|x_R|$, such that $|\tau_R| < |\Lambda| < 1/4$, we have from eq. (3.19)

$$a \approx -n - \frac{1}{2} - \Lambda + (-1)^{n+1} 2^{3/2} \frac{\tau_R}{\pi \Lambda \Gamma(-n - 1/2)}; \quad n = 0, 1, \dots \quad (3.21)$$

The dispersion relation for the $n = 0$ even eigenmode in this limit then becomes

$$1 - \frac{\omega_{*e}}{\omega} = - \left(k_Y^2 \rho_s^2 + i 3 \frac{L_n}{L_s} \right) - 2 \left(\frac{v_{ei}}{\omega_{*e}} \frac{m_e}{m_i} \frac{L_s}{L_n} \right) \left(k_Y^2 \rho_s^2 + i \frac{5}{2} \frac{L_n}{L_s} \right) - 8 \left(\frac{L_n}{L_s} \right)^2 \pi^{-3/2} \left(k_Y^2 \rho_s^2 + i 3 \frac{L_n}{L_s} \right)^{-1} \left(\frac{v_{ei}}{\omega_{*e}} \frac{m_e}{m_i} \frac{L_s}{L_n} \right)^{-1/2} . \quad (3.23)$$

If we compare Eqs. (3.21) and (3.23), we see that the modes are further damped as the resistivity increases. Thus, again there is no unstable eigenmode.

Let us now turn to the case of odd- ϕ modes; Eq. (3.19) is replaced by

$$\frac{\Gamma(a)}{\Gamma(a + \Lambda + 1/2)} = - \frac{\pi \Lambda \tau_R}{2^{1/2}} . \quad (3.24)$$

Noting that $|\Lambda \tau_R| \propto v_{ei}^{3/2} \ll 1$, we have, from Eq. (3.2),

$$a \approx -m - \Lambda - 1/2 ; m = 0, 1, \dots \quad (3.25)$$

where we have neglected $O(v_{ei}^{3/2})$ corrections. The dispersion

relation corresponding to the lowest ($m = 0$) odd eigenmode then is

$$1 - \frac{u_{*e}}{\omega} \approx - \left(k_y^2 \rho_s^2 + i 3 \frac{L_n}{L_s} \right) - 2 \left(\frac{v e i}{\omega_{*e}} \frac{m_e}{m_i} \frac{L_s}{L_n} \right) \left(k_y^2 \rho_s^2 + i \frac{5}{2} \frac{L_n}{L_s} \right) . \quad (3.26)$$

Here, again, resistivity contributes to the damping of the waves and no unstable eigenmode exists.

$$B. \quad |\Lambda| \approx k^2 |x_R|^2 > 1 :$$

In this strongly collisional limit, we have $\alpha \approx -\Lambda^{1/2}$ and $v \approx \Lambda^{1/2} - 1/2$. Assuming $\text{Im } \Lambda^{1/2} < 0$, we have with $|\Lambda| > 1$, $\text{Im } v \approx \text{Im } \Lambda^{1/2} < -1$. Equation (3.18), then becomes

$$\frac{\Gamma(a)}{\Gamma(a + v + 1/2)} = \pm i \left(\frac{\tau_R}{2^{3/2}} \right)^{1 + 2v} \frac{\Gamma(v) \Gamma(-v - 1/2) \Gamma(1/2 - v)}{\Gamma(-1 - v) \Gamma(v + 1/2) \Gamma(3/2 + v)} . \quad (3.27)$$

Here, + and - correspond to the even- and odd- ϕ modes, respectively. Applying the Stirling's formula [17] to the Gamma functions, Eq. (3.27) can be further reduced to

$$\frac{\Gamma(a)}{\Gamma(a + \Lambda^{1/2})} = \mp i \left(\frac{e \tau_R}{i 2^{3/2} \Lambda^{1/2}} \right)^{2\Lambda^{1/2}} . \quad (3.28)$$

If we note that $|\tau_R^2/\Lambda| \approx (k^2 x_s \rho_s)^{-1} \ll 1$ and $\text{Re}\Lambda^{1/2} > 1$, Eq. (3.28) gives, for the lowest eigenmode and both parities,

$$a \approx -\Lambda^{1/2}. \quad (3.29)$$

The corresponding dispersion relation then is given by

$$1 - \frac{\omega_{*e}}{\omega} = -k_y \rho_s \left[k_y \rho_s + (1 + i) \left(2 \frac{v_{ei}}{\omega_{*e}} \frac{m_e}{m_i} \right)^{1/2} \right], \quad (3.30)$$

which clearly shows that the resistive damping effects persist in this limit and no unstable eigenmode exists.

In summary, we have shown that, as long as $\tau_R^2 \ll 1$, i.e. $(v_{ei} m_e L_s / \omega_{*e} m_i L_n) \ll 1$, electrostatic drift waves in slab geometries and sheared magnetic fields are, in addition to the usual shear-damping effects, further damped by the resistive effects and, thus, no absolutely unstable eigenmode exists. The waves, of course, can still be convectively unstable [5, 14, 15]. Now, a more careful examination of the above analyses reveals that these results are closely related to the fact that drift waves in slab geometries experience shear damping (i.e., the corresponding potential is an anti-well rather than a well). We therefore expect if shear damping can be nullified by, for example, taking ω_{*e} to be a localized function of x , the stability properties could be qualitatively different. This is what we explore next by solving Eqs. (2.4) and (2.5) with $L_* \neq \infty$.

After a change of variables and a rearrangement of terms, Eqs. (2.4) and (2.5) may be rewritten as

$$\left[\frac{d^2}{d\xi^2} + \gamma_0 - \frac{\xi^2}{4} - \frac{\Lambda_0}{\xi^2 - i\xi_R} \right] \phi(\xi) = 0, \quad (3.31)$$

where $\xi = x/\lambda_0$, $\lambda_0^2 = x_0 \rho_s / 2$,

$$x_0^{-2} = \frac{\omega_{*0}}{\omega} \frac{1}{L_*^2} - \frac{1}{x_s^2}, \quad (3.32)$$

and γ_0 , Λ_0 , ξ_R have expressions identical to those for γ , Λ and τ_R , Eqs. (3.2) - (3.5), with the substitutions λ_0 , ω_{*0} and x_0 for λ , ω_{*e} and x_s . To nullify shear damping, we must obviously require that $R_e(x_0^{-2}) > 0$; i.e., from Eq. (3.32),

$$L_*^2 < x_s^2 (\omega_{*0}/\omega). \quad (3.33)$$

If we assume $|\xi_R|^2 \ll 1$, we can use the same matching procedure as the one discussed above for $L_* = \infty$ equation. We can again obtain an eigenvalue condition which has interesting approximate forms in the limits $|\Lambda_0| < 1/4$ and $|\Lambda_0| > 1/4$. We now omit the detailed algebra and only present the results of the dispersion relations derived for the lowest eigenmodes.

For even- ϕ modes and with $|\Lambda_0| < |\xi_R| < 1$, we have

$$1 - \frac{\omega_{*0}}{\omega} \approx - \left[k_y^2 \rho_s^2 + \frac{\rho_s}{x_0} \right] \left[1 + \frac{\sqrt{\pi}}{2} (1 - i) \left(\frac{v_{ei}}{\omega_{*0}} \frac{m_e}{m_i} \frac{L_s^2}{L_n^2} \frac{\rho_s}{x_0} \right)^{1/2} \right] \quad (3.34)$$

Equation (3.34) shows that when $R_e x_0^{-2} > 0$, the shear damping is indeed nullified, as expected. Furthermore, resistivity now contributes to instability growth and we have unstable eigenmodes. The growth rate is proportional to $v_{ei}^{1/2}$. As the collisionality is increased, we go to the limit $|\xi_R| < |\Lambda_0| < 1$ and the new dispersion relation becomes

$$1 - \frac{\omega_{*0}}{\omega} \approx - \left[k_y^2 \rho_s^2 + \frac{\rho_s}{x_0} \right] + 2i \left[k_y^2 \rho_s^2 + \frac{5}{2} \frac{\rho_s}{x_0} \right] \left(\frac{m_e}{m_i} \frac{v_{ei}}{\omega_{*0}} \frac{L_s^2}{L_n^2} \frac{\rho_s}{x_0} \right) + \left(\frac{4}{\pi} \right)^{3/2} \left(\frac{1+i}{2} \right) \left(\frac{\rho_s}{x_0} \right)^2 \left[k_y^2 \rho_s^2 + \frac{\rho_s}{x_0} \right]^{-1} \left(\frac{m_e}{m_i} \frac{v_{ei}}{\omega_{*0}} \frac{L_s^2}{L_n^2} \frac{\rho_s}{x_0} \right)^{-1/2} \quad (3.35)$$

Equation (3.35) shows that as resistivity is increased, the instability growth rate is enhanced and unstable eigenmodes persist as long as $R_e x_0^{-2} > 0$.

We now investigate the odd- ϕ modes. Here the dispersion

relation is given by

$$1 - \frac{\omega_{*O}}{\omega} \approx - \left(k_Y^2 \rho_s^2 + 3 \frac{\rho_s}{x_O} \right) + 2i \left(k_Y^2 \rho_s^2 + \frac{5}{2} \frac{\rho_s}{x_O} \right) \left(\frac{m_e}{m_i} \frac{v_{ei}}{\omega_{*O}} \frac{L_s^2}{L_n^2} \frac{\rho_s}{x_O} \right) . \quad (3.36)$$

where we have neglected corrections of order $v_{ei}^{3/2}$. Equation (3.36) shows that odd- ϕ unstable eigenmodes also exist.

Finally, we go to the strongly collisional limit, $|\Lambda_O| \approx k^2 |x_R|^2 \gg 1$. Taking $\text{Im} \Lambda_O^{1/2} < 0$, one can show that the corresponding dispersion relation for both parities takes the form

$$1 - \frac{\omega_{*O}}{\omega} \approx - k_Y^2 \rho_s^2 + (i - 1) k_Y \rho_s \left(\frac{\rho_s L_s}{x_O L_n} \right) \left(\frac{m_e}{m_i} \frac{v_{ei}}{\omega_{*O}} \right)^{1/2} . \quad (3.37)$$

Thus even for strong collisions, resistivity continues to play a destabilizing role and unstable eigenmodes persist.

From the above discussions it is clear that the existence of unstable eigenmodes in the slab approximation with a spatial variation of ω_{*e} is critically dependent on the condition Eq. (3.33). If Eq. (3.33) is obeyed, shear damping is nullified and resistivity contributes to unstable eigenmodes; otherwise

only damped eigenmodes exist. Condition (3.33) requires that the density gradient variation scale-length $L_* < \rho_s (L_s/L_n)$, which is rather a sharp variation of the density gradient.

As first pointed out by Taylor [9], shear damping could also be nullified by strong toroidal couplings. We thus expect that resistive drift waves could have unstable eigenmodes in toroidal geometries. This expectation is borne out by the following analysis. The eigenmode equation for the toroidally-coupled mode is given by Eq. (2.15); which, again, can be rewritten in the now familiar form

$$\left[\frac{d^2}{d\tau^2} + \gamma_1 - \frac{\tau^2}{4} - \frac{\Lambda_1}{\tau^2 - i\tau_R} \right] \phi(\tau) = 0, \quad (3.38)$$

where $\tau = z/\lambda_1$, $\lambda_1^2 = \omega q R D_1^{1/2} / 2C_s$,

$$\gamma_1 = (D_2 - iD_3 z_R^2 / z_s^2) \lambda_1^2 / D_1, \quad (3.39)$$

$$\Lambda_1 = iz_R^2 (D_4 - D_3 D_2), \quad (3.40)$$

and $\tau_R^2 = D_3 z_R^2 / \lambda_1^2$. The D's, z_R^2 and z_s^2 are defined in Eqs. (2.6) - (2.21). We note that the stability properties as posed by Eq. (3.38) depends critically on the sign of D_1 .

For $D_1 < 0$, Eq. (3.38) is qualitatively the same as that in a slab geometry; i.e., Eq. (2.4). The results are thus similar: i.e., shear damping is operative and the eigenmodes are further damped by the resistivity. We therefore concentrate only on the case with $D_1 > 0$; i.e., strong toroidal coupling limit where shear damping is nullified. Again, we assume $|\tau_R|^2 \ll 1$ and solve Eq. (3.38) as done for the slab case. We omit the details and present only the final dispersion relations.

In the weakly collisional limits, $|\Lambda_1| < |\tau_R| < 1$, we have for the even- ϕ modes

$$1 - \frac{\omega_* e}{\omega} \approx - (k_{yo}^2 \rho_s^2 + \epsilon_n) + \frac{\epsilon_n^{3/2}}{2^{1/2} q k_{yo} \rho_s} + 2^{-5/4} (i - 1) \epsilon_n^{1/4} \\ (k_{yo}^2 \rho_s^2 + \epsilon_n) (\pi q k_{yo} \rho_s \frac{v_{ei}}{s \omega_* e} \frac{m_e}{m_i})^{1/2} \quad (3.41)$$

Here, $\epsilon_n^{-1} = R d \ln [N(r)] / dr$ and we have taken $r_0 q' / q < 1/2$. Equation (3.41) clearly shows that, with $D_1 \approx \epsilon_n / 2 > 0$, shear damping is indeed nullified by the toroidal couplings and unstable eigenmodes do exist. The growth rates scale as $v_{ei}^{1/2}$. As v_{ei} increases such that $|\tau_R| < |\Lambda_1| < 1/4$, the

dispersion relation becomes

$$1 - \frac{\omega_{*e}}{\omega} \approx - (k_{yo}^2 \rho_s^2 + \epsilon_n) + \frac{3\epsilon_n^{3/2}}{2^{1/2} qk_{yo}\rho_s} + i \left(\frac{2}{\epsilon_n}\right)^{1/2} qk_{yo}\rho_s$$

$$(k_{yo}^2 \rho_s^2 + \epsilon_n) \frac{v_{ei}}{\omega_{*e}} \frac{m_e}{m_i} + \frac{2^{9/4}}{\pi^{3/2}} \frac{(i+1)}{(k_{yo}^2 \rho_s^2 + \epsilon_n)}$$

$$\frac{\epsilon_n^{11/4}}{(qk_{yo}\rho_s)^{5/2}} \left(\frac{\omega_{*e}}{v_{ei}} \frac{m_i}{m_e} \right)^{1/2} . \quad (3.42)$$

Thus, the eigenmodes continue to be unstable.

As for the odd- ϕ modes, we obtain, in this weakly collisional limit $|\Lambda_1| < 1/4$, the following dispersion relation

$$1 - \frac{\omega_{*e}}{\omega} \approx - (k_{yo}^2 \rho_s^2 + \epsilon_n) + \frac{3}{2^{1/2}} \frac{\epsilon_n^{3/2}}{qk_{yo}\rho_s} + iq \left(\frac{2}{\epsilon_n}\right)^{1/2} qk_{yo}\rho_s$$

$$(k_{yo}^2 \rho_s^2 + \epsilon_n) \frac{v_{ei}}{\omega_{*e}} \frac{m_e}{m_i} , \quad (3.43)$$

which indicates that odd- ϕ modes also have unstable eigenmodes.

We now discuss the collisional limit, $|\Lambda_1| > 1$. In this limit the dispersion relation for both parities is given by

$$1 - \frac{\omega_{*e}}{\omega} \approx - (k_{yo}^2 \rho_s^2 + \epsilon_n) + (i+1) (k_{yo}^2 \rho_s^2 + \epsilon_n)^{1/2} \left(\epsilon_n \frac{v_{ei}}{n\omega_{*e}} \frac{m_e}{m_i} \right)^{1/2} .$$

(3.44)

Thus, unstable eigenmodes persist into this strongly collisional regime and the growth rates increases with v_{ei} as $v_{ei}^{1/2}$.

We thus have shown that, in torodial geometries, toroidal-coupling effects can qualitatively modify the stability properties of resistive drift-wave eigenmodes. If the toroidal couplings are strong enough so as to nullify shear damping ($D_1 > 0$), then unstable eigenmodes do exist and the growth rates increase with v_{ei} . We note that $D_1 > 0$ requires $r_0 q'/q < 1/2$ and $(k_{y0} \rho_s r_0 q'/q)^2 < \epsilon_n (1/2 - r_0 q'/q)$. Furthermore, the above results are limited to $|\tau_R|^2 \ll 1$ or

$$\frac{v_{ei}}{\omega_{*e}} \frac{m_e}{m_i} \ll \frac{(\epsilon/2)^{3/2}}{q k_{y0} \rho_s} \quad (3.45)$$

Finally, Taylor's strong-coupling approximations [9] requires $\lambda_1^2 > 1$ or $q k_{y0} \rho_s > 2(2\epsilon)^{1/2}$.

IV. WKB TREATMENT

Let us consider the eigenvalue problem described by the differential equation

$$\frac{d^2 \phi}{dx^2} + G(\omega, x) \phi = 0, \quad (4.1)$$

and the boundary conditions $\phi(\phi') = 0$ at $x = 0$ for odd (even)

modes and well-behavedness as $|x| \rightarrow \infty$ for $\text{Im } \omega > 0$. We may use standard phase integral methods (WKB method) to solve this problem if $G'/G^2 \ll 1$ for the entire domain of x . For Eq. (2.4), which is of interest to us, the appropriate conditions are satisfied if $k_y^2 |x_R|^2 \gg 1$. Note that this not only overlaps with the $|\Lambda| > 1$ approximation of the matched asymptotic treatment in Sec. 3, but indeed extends the analysis to the domain $|\tau_R|^2 \gtrsim 1$, where the matched asymptotic treatment breaks down.

The WKB solutions of Eq. (4.1) are

$$\phi_{\pm} = G^{-1/4} \exp[\pm i \int G^{1/2}(\omega, x) dx] . \quad (4.2)$$

A general solution may thus be written as

$$\phi \approx a_+ \phi_+ + a_- \phi_- \quad (4.3)$$

The coefficients a_+ , a_- are not fixed over the whole complex z plane ($z = x + iy$) and the WKB method consists in relating, for a given solution of Eq. (4.1), the WKB approximation in one region of the z -plane to that in another. A guide to these connection formulae is obtained by mapping out all the zeroes and poles of the function $G(z, \omega)$ in the complex z plane. This is done by drawing the Stokes diagram for the function $G(z, \omega)$, in the standard manner [18, 19]. Stokes and Anti-Stokes lines emanating from the zeroes of $G(z, \omega)$ describe the

global properties of WKB solutions. Along the Anti-Stokes lines, the solutions $\phi_{\pm}$ oscillate i.e., have constant amplitude. Having obtained, by examination of the Anti-Stokes structure, a pair of zeroes appearing to be associated with a mode, one uses standard connection formulae [18] for getting the eigenvalue condition. Notice however that the zeroes of G and indeed the entire Stokes structure itself depends on ω , which has to be finally derived from the eigenvalue condition. The problem is therefore often not amenable to analytical treatment. Iterative numerical methods however work very well and a code for carrying out such a complex frequency search, has been developed by White [19]. The final test of the WKB solution is that once the frequency has been determined, the resulting Stokes diagram should be such that a solution can be chosen which satisfies the physical boundary conditions.

For Eq. (2.4), the function $G(z, \omega)$ is given by

$$G = - \left[k_y^2 \rho_s^2 + \frac{z^2}{z^2 - ix_R^2} \left(\frac{\omega - \omega_{*e}}{\omega} \frac{z^2}{x_s^2} \right) \right] \quad (4.4)$$

It has two poles and four zeroes given respectively by

$$z_{\text{pole}}^2 = (ix_R^2) \quad (4.5)$$

and

$$z_{\text{zero}}^2 = \frac{1}{2} x_s^2 \left[k_y^2 \rho_s^2 + \frac{\omega - \omega_{*e}}{\omega} \right] \pm \left[\frac{1}{4} x_s^4 \left(k_y^2 \rho_s^2 + \frac{\omega - \omega_{*e}}{\omega} \right) - i k_y^2 \rho_s^2 x_R^2 x_s^2 \right]^{1/2}. \quad (4.6)$$

The poles and zeroes have obvious symmetries about the origin. Figure 1 shows a typical Stokes diagram for this function. The region A(B) can be shown to include the asymptotic positive (negative) real axis for $\text{Im } \omega > 0$. For either $\text{Im } \omega > 0$ or $\text{Im } \omega < 0$ the physical boundary condition demands the choice of a subdominant (decaying) solution in Region A, and the standard connection formula

$$\int_{z_1}^{z_2} [G(z, \omega)]^{1/2} dz = (n + \frac{1}{2}) \pi \quad (4.7)$$

produces a subdominant solution extending from z_1 toward the origin. Thus the solution will approximately vanish at the origin. A numerical integration of Eq. (4.1) would be expected to yield the same eigenvalue as Eq. (4.7), since from Eq. (4.2) the solutions behave as $\phi_{\pm} \sim \exp(\pm k_y \rho_s x)$ for small x . Any combination of $\phi_{\pm}$ will be completely dominated by ϕ_+ before reaching z_1 provided only that

$|z_1| \gg (k_y \rho_s)^{-1}$, which is well satisfied.

The insensitivity of the eigenvalue condition, which guarantees the boundary condition at $x = \infty$, to variations of ϕ at $x = 0$ indicates that even and odd modes will have practically the same eigenvalue. This result is in agreement with the large Λ limit of matched asymptotic treatment (Sec. 3) and also numerical calculations (Sec. 6). Unfortunately, the integration in Eq. (4.7) has to be done numerically so that we do not have a closed analytic form for the eigenvalue condition. Figure 2 presents the results of the WKB analysis for $m_e/m_i = 10^{-4}$, $L_s/L_n = 100$, $k_y^2 \rho_s^2 = .02$ and various values of v_{ei}/ω_* . Also presented for comparison are the results obtained from Eq. (3.30) of the matching treatment and those from a direct numerical integration of the differential equation by the shooting method (Sec. 6). Note that there is very good agreement between the various methods.

The above treatment thus permits us to conclude that even for $(m_e v_{ei} L_s / m_i \omega_* L_n) > 1$, where the matched asymptotic treatment breaks down, no unstable eigenmodes can be found for resistive drift waves in the slab geometry.

Finally, we wish to present some analytical WKB results in the limit $k_y |x_R| \rightarrow \infty$. In this limit G gets very large and the only way to satisfy the eigenvalue condition Eq. (4.7) is for the two turning points z_1 and z_2 to coalesce. From Eq. (4.6), it is readily seen that the condition for the coalescence

of the two zeroes of G is given by

$$k_y^2 \rho_s^2 + \frac{\omega - \omega_{*e}}{\omega} = \pm 2 k_y \rho_s \left[i \frac{x_R^2}{x_s^2} \right]^{1/2}$$

Substitution of x_R and x_s gives the eigenvalue condition

$$1 - \frac{\omega_{*e}}{\omega} = - k_y \rho_s \left[k_y \rho_s + (1 + i) \left(\frac{2 m_e v_{ei}}{m_i \omega_{*}} \right)^{1/2} \right] \quad (4.8)$$

Corrections of order $(k_y |x_R|)^{-1}$ then split the coalesced turning points apart giving the typical Stokes diagram shown in Fig. 1. The sign in front of square-root has been chosen in such a way that the Stokes diagram gives the proper boundary condition at $\text{Re} z \rightarrow \infty$. Note that the eigenvalue condition Eq. (4.8) is identical to Eq. (3.30) obtained by the large $k_y |x_R|$ limit. This suggests that the eigenvalue condition is unaltered by switching the inequality $|\tau_R|^2 = (m_e v_{ei} L_s / m_i \omega_{*} L_n) < 1$, as long as $k_y |x_R| > 1$ is satisfied.

V. PERTURBATION TREATMENT

In the previous two sections we have solved the eigenvalue problem associated with Eq. (2.4) by non-perturbative matched asymptotic and WKB treatments, respectively. We have found that the two treatments agree in the limit $k_y |x_R| \gg 1$. We now demonstrate that the $k_y |x_R| \ll 1$ limit of the matched asymptotic treatment can be recovered by a perturbation - theoretic analysis. Such an analysis for the collisional problem was first used by Hsu [20]. We shall then use the perturbation - theoretic treatment to analyze a more complicated eigenvalue equation, which arises when electron temperature fluctuations associated with the drift wave as well as finite electron temperature gradients are retained.

We now wish to solve Eq. (2.4) by a perturbation theory in which $\tau_R^2 = (m_e v_{ei} L_s / m_i \omega_* L_n) \ll 1$ is treated as the small parameter. We first rewrite Eq. (2.4) in the form Eq. (3.1) with the various quantities defined as before. We note now that for $\tau_R = 0$, the equation is a parabolic cylinder equation with the usual Hermite function solution

$$\phi_n(\tau) = 2^{-n/2} H_n \left(\frac{\tau}{\sqrt{2}} \right) \exp \left(- \frac{\tau^2}{4} \right) \quad (5.1)$$

and the eigenvalue given by

$$\gamma_n = n + \frac{1}{2} \quad ; \quad n = 0, 1, 2, \dots \quad (5.2)$$

The τ_R^2 terms give a perturbation of the eigenvalue which may be obtained by standard first order perturbation techniques yielding

$$\delta\gamma_n = \left[\int_{-\infty}^{\infty} d\tau \phi_n^2(\tau) \left(\frac{\lambda}{\tau^2 + \tau_R^2} \right) \right] / \int_{-\infty}^{\infty} d\tau \phi_n^2(\tau) . \quad (5.3)$$

For the lowest $n = 0$ (even- ϕ) mode, we get

$$\delta\gamma_0 \approx i \frac{\lambda}{\rho_s} \left(\frac{\pi}{2} \right)^{1/2} \left(1 - \frac{\omega_{*e}}{\omega} \right) \tau_R .$$

The corresponding dispersion relation is then given by

$$1 - \frac{\omega_{*e}}{\omega} = - (k_y^2 \rho_s^2 + i \frac{L_n}{L_s}) \left[1 + \left(\pi \frac{m_e}{m_i} \frac{v_{ei}}{\omega_{*}} \frac{L_s}{L_n} \right)^{1/2} \right] \quad (5.4)$$

Note that this equation is identical to Eq. (3.21) obtained in the matched asymptotic treatment in the limit $k_y |x_R| \ll 1$. Similar results are obtainable for the odd- ϕ modes.

We now derive the modification of Eq. (2.4), which includes effects associated with finite electron thermal conductivity along the lines of force and ambient electron temperature gradients. We shall then solve that equation by a perturbation treatment similar to the one given above. The ion dynamics

and the electron perpendicular (to B_0) motion is the same as discussed in Sec. 2. The electron parallel equation of motion is modified to [21]

$$\begin{aligned} \omega v_{||e} = & -k_{||} e \phi / m_e - i v_{ei} (v_{||e} - v_{||i}) \\ & + k_{||} (T_e n_e / N_e + 1.71 t_e) / m_e . \end{aligned} \quad (5.5)$$

Here, t_e is the perturbed electron temperature. Furthermore, we have the electron equation of energy given by

$$\frac{3}{2} N_e (\omega t_e - \omega_{*T} e \phi) - N_e T_e k_{||} v_{||e} = k_{||} q_{e||} , \quad (5.6)$$

where

$$q_{e||} = 0.71 N_e T_e v_{||e} - i k_{||} N_e T_e t_e / m_e v_{ek} , \quad (5.7)$$

$\omega_{*T} = - (ck_y T_e / e B_0) (d \ln T_e / dx)$ and $v_{ek} \approx v_{ei} / 1.6$ for singly charged ions. The final (radial) wave equation then can be shown to be

$$\begin{aligned} [1 + g_2(x)] \rho_s^2 \left(\frac{d^2}{dx^2} - k_y^2 \right) \phi(x) = & \left\{ 1 - \frac{\omega_{*e}}{\omega} - 1.71 \frac{\omega_{*T}}{\omega} g_1(x) \right. \\ & \left. - [1 + 1.95 g_1(x)] \frac{x^2}{x_s^2} \right\} \phi(x) , \end{aligned} \quad (5.8)$$

where

$$g_1(x) = -ix_k^2 / (x^2 - ix_k^2) \quad , \quad (5.9)$$

$$g_2(x) = 1.95g_1(x) - ix_R^2/x^2 \quad , \quad (5.10)$$

$$\text{and } x_k^2 = 1.5\omega v_{ek}/k_{||}^2 v_{Te}^2 \quad .$$

We note that we can recover the resistive wave equation, Eq. (2.4), by setting $x_k^2 = 0$ in Eq. (5.8). Again, by changing the variable x and rearranging terms, we can rewrite Eq. (5.8) in the following form

$$\left[\frac{d}{d\tau} + \gamma_k - \frac{\tau^2}{4} - F(\tau) \right] \psi(\tau) = 0 \quad (5.11)$$

$$\text{Here, } \tau = x \exp(i\pi/4)/\lambda \quad , \quad \lambda^2 = \rho_s x_s/2 \quad ,$$

$$\gamma_k = i \left(\frac{\lambda}{\rho_s} \right)^2 \left(k_Y^2 \rho_s^2 + 1 - \frac{\omega^* e}{\omega} \right) \quad , \quad (5.12)$$

$$\begin{aligned} F(\tau) = i \left(\frac{\lambda}{\rho_s} \right)^2 & \left[\left(1 - \frac{\omega^* e}{\omega} \right) \left(1.95 \tau_k^2 + \tau_R^2 \right) \tau^2 + \tau_R^2 \tau_k^2 \right. \\ & \left. + 1.71 \frac{\omega^* \tau}{\omega} \tau_k^2 \tau^2 + i \left(\frac{x_R}{x_s} \right)^2 \tau^2 \left(\tau^2 + \tau_k^2 \right) \right] / g_3(\tau) \quad , \quad (5.13) \end{aligned}$$

$$g_3(\tau) = \tau^4 + (2.95\tau_K^2 + \tau_R^2)\tau^2 + \tau_R^2\tau_K^2,$$

$$\tau_R^2 = x_R^2/\lambda^2, \text{ and } \tau_K^2 = x_K^2/\lambda^2. \quad (5.14)$$

Due to the complexities in $F(\tau)$, we have, so far, been unable to solve Eq. (5.11) using the matching procedures developed for the resistive case. Hence, we shall confine our discussions on Eq. (5.11) to a perturbation analysis, in which both τ_R and τ_K are treated as small parameters. We thus write

$$\gamma_{kn} = n + 1/2 + \delta\gamma_n; \quad n = 0, 1, 2, \quad (5.15)$$

$$\delta\gamma_n = [\int_{-\infty}^{\infty} \phi_n^2 F(\tau) d\tau] / \int_{-\infty}^{\infty} \phi_n^2 d\tau. \quad (5.16)$$

For the lowest even- ϕ ($n = 0$) mode, one obtains

$$\begin{aligned} \delta\gamma_0 \approx & \left(\frac{\pi}{2}\right)^{1/2} \frac{i\lambda^2}{\rho_s^2(\tau_1 + \tau_2)} \left[\left(1 - \frac{\omega_{*e}}{\omega}\right) (1.95\tau_K^2 + \tau_R^2) \right. \\ & \left. + 1.71 \frac{\omega_{*T}}{\omega} \tau_K^2 + \frac{\tau_R^2\tau_K^2}{\tau_1\tau_2} + i \frac{x_R^2}{x_s^2} \left(\frac{2}{\pi}\right)^{1/2} (\tau_1 + \tau_2) \right], \end{aligned} \quad (5.17)$$

where

$$\tau_{1,2}^2 = \left\{ 2.95\tau_K^2 + \tau_R^2 \pm \left[\left(2.95\tau_K^2 + \tau_R^2 \right)^2 - 4\tau_R^2\tau_K^2 \right]^{1/2} \right\} / 2 . \quad (5.18)$$

The corresponding dispersion relation that is given

$$1 - \frac{\omega_{*e}}{\omega} \approx - \left(k_Y^2 \rho_s^2 + i \frac{L_n}{L_s} \right) - \left(\frac{\pi}{2} \right)^{1/2} \left[\left(k_Y^2 \rho_s^2 + i \frac{L_n}{L_s} \right) \left(1.95\tau_K^2 + \tau_R^2 \right) - 1.71\eta_e\tau_K^2 + \frac{\tau_R^2\tau_K^2}{\tau_1\tau_2} \right] / (\tau_1 + \tau_2) . \quad (5.19)$$

Here, $\eta_e = \omega_{*T}/\omega_{*e} = d\ln T_e(x)/d\ln N_e(x)$.

Noting that $\tau_1^2, \tau_2^2, \tau_R^2, \tau_K^2 \propto v_{ei}^2$, Eq. (5.19) indicates that dissipative effects further enhance the shear damping rates by factors proportional to $v_{ei}^{1/2}$. It is also interesting to note that electron temperature gradient (the η_e term) only modifies the real frequencies and has no effects on stability of the eigenmode.

As to the $n = 1$ eigenmodes; i.e., the lowest odd- ϕ mode, we have,

$$\delta\gamma_1 \approx i \left(\frac{\lambda}{\rho_s} \right)^2 \left[\left(1 - \frac{\omega_{*e}}{\omega} \right) \left(1.95\tau_K^2 + \tau_R^2 \right) + i \left(\frac{x_R}{x_s} \right)^2 + 1.71 \frac{\omega_{*T}}{\omega} \tau_K^2 \right] , \quad (5.20)$$

and the corresponding dispersion relation is

$$1 - \frac{\omega_{*e}}{\omega} \approx - \left(k_y^2 \rho_s^2 + i 3 \frac{L_n}{L_s} \right) \left(1 + 1.95 \tau_k^2 \right) + 1.71 \eta_e \tau_k^2 \\ - \left(k_y^2 \rho_s^2 + i \frac{5}{2} \frac{L_n}{L_s} \right) \tau_R^2 . \quad (5.21)$$

Equation (5.21) indicates that dissipative effects again only enhance the shear damping rates by factors proportional to v_{ei} and that finite η_e only modifies the real frequencies.

Thus, using a perturbation analysis, we have demonstrated that finite electron thermal conductivity along the lines of force and ambient temperature gradients do not alter the stability properties of dissipative drift waves in a slab-geometry sheared magnetic field. Specifically, we find that electron thermal conductivity further damps the waves in a fashion similar to the resistive effects. Meanwhile, electron temperature gradients are found to only modify the real frequencies and hence, have no effects on the stability.

VI. NUMERICAL RESULTS AND DISCUSSIONS

We have now demonstrated by several analytical treatments that dissipative drift waves in a plasma slab with sheared magnetic fields do not form unstable eigenmodes in any domain of collisionality. Instability is recovered only if the conventional shear damping effects are nullified either by localizing the density gradient or by including strong

toroidal coupling effects. Since the various analytic treatments have restricted domains of validity in parameter space, we have verified and extended many of the results by a direct numerical integration of the differential equation, Eq. (2.4), using the so-called "shooting method". The numerical integration of the equation utilizes a Numerov finite difference scheme. Making a guess about the eigenvalue, the WKB solution for x is utilized for initializing the integration scheme. The equation is integrated to the origin where the derivative $d\phi/dx$ (if we are looking for even modes) or ϕ (for odd modes) is ascertained. The deviation of either $d\phi/dx$ or ϕ (as the case may be) from zero is essentially used in a Mueller's root-finding package for predicting a new eigenvalue. This is then used as a new guess and the iteration is continued until convergence is obtained. In the present problem, in view of the fact that the effective potential at large x is an anti well, and that for damped modes the solution finds to diverge in x , it turns out to be more convenient to integrate along the -45° line in the complex plane. The transformation is chosen such that the outward propagating wave along the real axis goes into the subdominant solution along the -45° line.

Figure 2 presents a plot of the numerically obtained rate γ/ω_* vs. the collision frequency, ν_{ei}/ω_* for $k_{y0}^2/k_s^2 = 0.2$, $m_e/m_i = 10^{-5}$, $L_s/L_n = 125$. Also plotted in Fig. 2 are

the results of various analytic treatments. We see that the agreement is very good. We note, as mentioned previously, that the perturbation theory gives reasonable results when $|\Lambda| = k_y^2 |x_p|^2$ is small and the WKB theory is reasonable when $|\Lambda| \gg 1$. The physical reason for this is clear from Eq.(3) and (4). In these figures we have plotted the real and imaginary parts of the eigenfunction along τ (viz the -45° degree line in the complex x plane, along which the solutions decay) for various values of $|\Lambda|$. It is noted that for $|\Lambda|$ small, the eigenfunctions are very close to the unperturbed form viz Hermite functions. On the other hand for $|\Lambda|$ large, the eigenfunctions are strongly distorted and have no resemblance to the unperturbed Hermite functions. They, however, tend to get localized in small regions of τ -axis and hence make the WKB approximation good. It is also interesting to note that in this limit, the eigenfunction almost vanishes near the origin; thus the boundary condition at the origin does not play a very important role and the odd- and even- parity solutions satisfy the same eigenvalue condition [of Eq. (3.30) and (4.7)].

Very recently we have been able to find a quadratic form which demonstrates that Eq. (2.4) has no unstable eigenmodes. The procedure followed below was inspired by a trick used by Antonson [22] for the corresponding collisionless problem. The existence of this quadratic form then puts the above analyses on a firm footing. Let us transform the independent variable

x to s defined by

$$x^2 = -i x_R^2 s^2 \quad (6.1)$$

The outgoing solution at large x viz $\exp(-ix^2/x_s)$ goes to a decaying solution $\exp(-i(m_e/m_i)(v_{ei}/\omega_{*e})(L_s/L_n)\}s^2)$ along the s direction. Equation (2.4) now takes the form

$$\frac{d^2\phi}{ds^2} + \left\{ i k^2 x_R^2 + \frac{s^2}{s^2 + 1} \left[\frac{i(\omega - \omega_{*e})}{\omega} \frac{x_R^2}{\rho_s^2} - \left(\frac{m_e}{m_i} \frac{v_{ei}}{\omega_{*e}} \frac{L_s}{L_n} \right)^2 s^2 \right] \right\} \phi = 0 \quad (6.2)$$

Multiplying (6.2) by ϕ^* , integrating all over s, using $\omega = \Omega + i\gamma$ and separating the real part we find

$$- \int \left| \frac{\partial \phi}{\partial s} \right|^2 ds + \int \left[-\gamma k^2 \rho_s^2 \left(\frac{m_e}{m_i} \frac{v_{ei}}{\omega_{*e}} \frac{L_s^2}{L_n^2} \right) - \frac{s^2}{s^2 + 1} \left(\gamma \frac{m_e}{m_i} \frac{v_{ei}}{\omega_{*e}} \frac{L_s^2}{L_n^2} + \frac{m_e^2}{m_i^2} \frac{v_{ei}^2}{\omega_{*e}^2} \frac{L_s^2}{L_n^2} s^2 \right) \right] |\phi|^2 ds = 0. \quad (6.3)$$

For $\gamma > 0$, all the above terms are individually negative definite and so their sum cannot go to zero. The conclusion then is that there can be no growing modes for Eq. (2.4).

We now present a possible physical mechanism explaining the results. An examination of the local dispersion relation for dissipative drift waves indicates that $\omega \approx \omega_*$ only for values of $k_{||}$ such that $(k^2 v_{eT}^2 / v_{ei} \omega_{*e}) > (k_{ys}^2 / \rho_s^2)$. This is the range $k_{||}$ in which the electron motion along the lines of force gives predominantly the usual Boltzmann response $n/n_0 \approx e\phi/T$, which is essential for drift waves.

For smaller $k_{\parallel}$, electron parallel response is far from Boltzmann like and the normal mode frequency shifts [22] to $\omega_i \ll \omega_*$.

In a sheared magnetic field geometry, $k_{\parallel}$ takes all values from zero up. Thus approximately, within the resistive layer (where $k_y^2 v_{eT}^2 / \omega_e^2 \ll 1$), the mode frequency wants to be $\omega \ll \omega_*$ and outside, it wants to be $\omega \gg \omega_*$. Concentrating on modes with $\omega \approx \omega_*$ then produces solutions which localize themselves away from the mode-rational surface and the resistive layer (this is emphasized in the plots in Fig. 4, which are valid for large $k_y |x_R|$). Thus, considerable changes in γ and β take place in the transition from the inside to the outside of the resistive layer. This then has severe consequences on shear damping (which is governed by the usual parabolic cylinder equation but with effectively a new boundary condition on the inside) and enhances it. This enhancement always overpowers the subdominant growth terms, so we have only damped eigenmodes. However, if one gets rid of the shear damping by some physical effects which localize the mode energy, then the instability can be recovered since the above mechanism does not operate.

The above discussion brings out the sensitivity of the eigenmode analysis to processes occurring within a small resistive layer near the mode-rational surface. It is tempting to speculate that physical processes which might mix the electron fluid over this region (e.g., small magnetic islands, convective cells etc.) will provide the electrons with an effective $k_{\parallel}$

in the inner region which is not too small; this might, in turn, drastically alter the conclusions of the eigenmode analysis. An investigation of these effects is in progress.

Finally, a few words about experiments. Experiments on drift waves in the collisional regime have been done in stellarators [24] and multipoles [25]. We have recently shown [26] that the Kharkov stellarator experiments [24] can be explained in terms of a density gradient localized mode. Similar conclusions about the FM-1 experiment cannot be firm because no direct measurement of strong density gradients was made (although indirect evidence [27] exists from collisionless instability experiments). Finally, we must always leave open the possibility that the observed fluctuations were associated with convectively amplified wave-packets.

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REFERENCES

- [1] Kadomtsev, B. B., Plasma Turbulence, Academic Press, (1965), 88.
- [2] Rutherford, P. H. and Frieman, E. A., Phys. Fluids 10, (1967) 1007.
- [3] Kadomtsev, B. B. and Pogutse, O. P., Reviews of Plasma Phys., Consultants Bureau, 5, (1970) 249.
- [4] Pearlstein, L. D. and Berk, H. L. Phys. Rev. Letters, 23, (1969) 220.
- [5] Coppi, B., Phys. Letters 14, (1965) 172.
- [6] Rukhadze, A. A. and Silin, V. P., Sov. Phys. Usp. 11 (1969), 659 [Usp. Fiz. Nauk 96, (1968) 877.
- [7] Krall, N. A. and Rosenbluth, M. N., Phys. Fluids 8, (1965) 1488.
- [8] Moiseev, S. S. and Sagdeev, R. Z., Sov. Phys. Tech. Phys. 9, (1964) 196 [Ah. Tekh. Fiz. 34, (1964) 248].
- [9] Taylor, J. B. Proc. of IAEA Conf. on Plasma Physics and Controlled Fusion Research, Berchetsgarden, 2 (1967) 323.
- [10] Ross, D. W. and Mahajan, S. M., Phys. Rev. Letters 40, (1978) 324.
- [11] Tsang, K. T., Catto, P. J., Whitson, J. C. and Smith, J., Phys. Rev. Letters 40, (1978) 327.
- [12] Chen, Liu, Guzdar, P. N., White, R. B., Kaw, P. K. and Oberman, C., Princeton University Plasma Physics Lab. Report No. PPPL-1446, (1978).
- [13] Rosenbluth, M. N. and Catto, P. J., Nucl. Fusion 15, (1975) 573; Gladd, N. T. and Horton, W., Phys. Fluids 16, (1973) 879.

- [14] Coppi, B., Laval, G. Pellat, R. and Rosenbluth, M. N.,
Nucl. Fusion 6, (1966) 261.
- [15] Kent, A. and Taylor, J. B., Phys. Fluids 12, (1969) 209.
- [16] Dobrott, D., et al., Phys Rev. Letters 30, (1977) 943.
- [17] Abramowitz, M. and Stegon, I. A., Handbook of Mathematical
Functions, NBS Appl. Math. Series 55 (1964) 257.
- [18] Heading, J., An Introduction to Phase Integral Methods
(John Wiley and Sons, NY) 1962.
- [19] White, R. B., Princeton University Plasma Physics Lab.
Report PPPL-1482 (1978).
- [20] Hsu, J., PhD. Thesis, Princeton University (1977), Unpublished.
- [21] Tsai, S. T., Perkins, F. W. and Stix, T. H., Phys. Fluids
13, (1970) 2108.
- [22] Antonsen, T. Jr., Proceedings of the Annual Controlled
Fusion Conference, Gatlinburg, Tennessee, April, 1978.
- [23] Mahajan, S. M., Guzdar, P. N., Kaw, P. K. and Oberman, C.
Princeton University Plasma Phys. Lab. Rep., PPPL-1351 (1977).
- [24] Vojtsenya, V. S., Voloshko, A. Yu., Zalkind, V. M.,
Solodovchenko, S. I., Tarasenko, V. P., Stan, A. F.,
Nucl. Fusion 17, (1977) 651.
- [25] Okabayashi, M. and Arunasalam, V., Nucl. Fusion 17, (1977) 497.
- [26] Guzdar, P. N., Chen, Liu, Kaw, P. K. and Oberman, C.,
Phys. Rev. Letters 40, (1978) 1566.
- [27] Sauthoff, N., Ph.D. Thesis, Princeton University (1975)
Unpublished.

FIGURE CAPTIONS

- Fig. 1. Anti-Stokes plot for the function $G(x, \omega)$ defined in Eq. (44).
- Fig. 2. Comparison of $-\gamma/\omega_*$ vs v_{ei}/ω_* curve obtained by numerical shooting method and analytic (WKB and matched asymptotic) theories.
- Fig. 3. Real and Imaginary parts of the eigenfunction vs. distance along the -45° line (i.e., along a line in the complex x plane where the solution decays at large x) for $|\Lambda| \ll 1$.
- Fig. 4. Same as in Fig. 3 but for $|\Lambda| \gg 1$.

Fig. 1. 782128

Fig. 2. 782130

Fig. 3. 782129

Fig. 4. 782131