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FREE-ELECTRON LASERS WITH VERY SLOW WIGGLER TAPER*

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Abstract — A highly accurate, explicit asymptotic solution of the electron energy and phase is found for a class of free-electron lasers with very long wavelength beams, very low electron energies, and very slow taper of the wiggler field relative to the wiggler period. Dimensionless variables are defined and normalized, and three small parameters which characterize the operation of the FEL are identified. Because of the explicit nature of the solution, our results may be directly used to calculate features such as the escape distance of the electron from the potential well and the effects of the various physical parameters. One important advantage of the very slow wiggler taper is the increased efficiency of the energy transfer from the electron beam to the signal field due to increased bucket width. Numerical calculations are performed to verify all results.

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I. INTRODUCTION

This paper is a continuation of work on the effects of wiggler taper rate and signal field gain rate in free-electron lasers reported in Li and Kevorkian [8]. In that paper, dimensionless, normalized equations governing the energy and phase of the electrons in the FEL were derived, and three small parameters which characterize the operation of the FEL were identified: ε , which measures the slow variation of the wiggler parameters; μ , the ratio of the signal field wavelength to the wiggler period; and κ , which measures the electron beam density and governs the rate at which the signal field amplitude grows along the length of the laser. In addition, two distinguished limits relating μ and ε and three distinguished limits relating κ and ε were identified. By assuming a relation between the magnitudes of the small parameters $\mu = O(\varepsilon^\alpha)$, these two distinguished limits were characterized by the values $\alpha = 1$ and $\alpha = 2/3$, referred to as "slow resonance" and "slower resonance" respectively. In actuality, these are simply the two leading values of a series of distinguished limits given by $\alpha = 1, 2/3, \dots, 2/n$. As higher order terms are necessary in the asymptotic equations for the $\alpha = 2/3$ case, only the analysis for the case $\alpha = 1$ is explicitly presented by Li and Kevorkian.

Here we focus on the distinguished limit characterized by the value $\alpha = 2/3$. This distinguished limit (or the limit for a smaller α) corresponds to an FEL with longer wavelength radiation beams and lower electron energies, which provides design advantages over the faster ($\mu = O(\varepsilon)$) variation. The main design advantage is that greater efficiency in the transfer of energy from the electron beam to the signal field is achieved since the ponderomotive potential well has a nearly full 2π width, allowing more electrons to be trapped in resonance for longer times.

The extension of the mathematical analysis from the $\alpha = 1$ case to the $\alpha = 2/3$ case is not trivial. The primary difficulty is that it is necessary to account for hundreds of higher order terms in order to calculate an asymptotic solution that remains valid

along the entire length of the laser. Direct numerical solutions of the differential equations also provide little advantage, as the explicit effects of the various parameters are lost. The analytical solution presented here relies heavily on the use of a symbolic manipulation program to account for the large proliferation of terms in the asymptotic expansions. Not only does such a program overcome the algebraic difficulties while preserving the parameter dependencies, but it also provides an easy evaluation of the final solutions obtained, since FORTRAN code can be directly generated by the symbolic manipulation program. In addition, an unexpected simplification occurs for the $\alpha = 2/3$ case. The $O(1)$ nonlinear oscillator equation which results has an explicit solution (unlike the situation for $\alpha = 1$), so that additional approximations are unnecessary, giving a highly accurate final asymptotic solution for the electron energy and phase. The analytical nature of this solution provides distinct advantages for further analysis. In particular, we derive an easily evaluated condition which predicts the distance at which the electron "escapes" from the ponderomotive potential well, allowing quick evaluation of energy transfer for a variety of parameter values.

In section 2, we review the derivation given by Li and Kevorkian [8] for the dimensionless, normalized equations governing the energy and phase of the electrons in the beam. All dimensionless parameters are defined and the final system of equations is stated for general α and is extended to the orders necessary for the analysis to be valid for the case $\alpha = 2/3$. Additional details can be found in Bosley [1]. The system of equations governing the energy and phase of the electrons exhibits what is referred to as sustained resonance (or phase-locking). Mathematical analysis for the solution of these types of systems can be found in Bosley and Kevorkian [2], Kevorkian [5], and Kevorkian and Li [6]. In section 3 we illustrate the nature of the distinguished limits characterized by the value of α , specifying $\alpha = 2/3$ to examine the slower resonance. The solution procedure described in [2], which

examines these very slow variations and involves canonical averaging techniques, can therefore be applied directly, resulting in final asymptotic solutions for the energy and phase of the electrons in the beam. As mentioned previously, we also derive an escape condition from these solutions illustrating their usefulness. In section 4, we examine a model FEL corresponding to the $\alpha = 2/3$ case, to verify numerically the results obtained. The accuracy of the asymptotic solutions as well as the escape condition are shown to be in excellent agreement with numerical results.

II. EQUATIONS OF MOTION FOR ELECTRON ENERGY AND PHASE

Since the nondimensionalizations and normalizations used here are identical to those in [8], we merely summarize the results for reference purposes, indicating where any extensions are necessary for the $\alpha = 2/3$ case. The reader is referred to [8] and [1] for details.

We consider an FEL with a helical magnetic wiggler of constant wavelength λ_w and variable field amplitude $B_w(Z)$, where the independent variable Z is the distance along the longitudinal axis. The analysis presented here can be easily modified to account for variable λ_w as well, but in [7] it was found that a constant λ_w has certain hardware and efficiency advantages. The signal field generated in the laser cavity is specified by constant wavelength λ_s , frequency $\omega_s = 2\pi c/\lambda_s$, amplitude $E_s(Z)$, and slowly varying phase shift ϕ_s . The evolution of the signal field parameters E_s and ϕ_s are not explicitly prescribed by the laser design but are determined by a solution of Maxwell's equations. This evolution is dependent on the parameter κ which measures the electron beam density and characterizes the gain of the signal field amplitude along the length of the laser.

The dimensionless energy γ and phase ϕ for a single electron in the beam are

$$\gamma = \frac{E}{mc^2} = \frac{1}{\sqrt{1 - v^2/c^2}} \quad (2.1)$$

$$\phi = 2\pi \left(\frac{1}{\lambda_w} + \frac{1}{\lambda_s} \right) Z - \omega_s T + \phi_s \quad (2.2)$$

where T is dimensional time. We introduce dimensionless length and time scales defined by

$$z = \frac{2\pi}{\lambda_w} Z \quad t = \frac{2\pi c}{\lambda_w} T \quad (2.3)$$

and the dimensionless field amplitudes

$$b_w = \frac{\lambda_w e B_w}{2\pi mc^2} \quad b_s = \frac{\lambda_w e B_s}{2\pi mc^2} \quad e_s = \frac{\lambda_w e E_s}{2\pi mc^2} \quad (2.4)$$

We also introduce a dimensionless small parameter ε , $0 < \varepsilon \ll 1$, which measures the rate of the variation of the wiggler field along the longitudinal axis,

$$\varepsilon = \frac{\lambda_w}{2\pi L} \quad (2.5)$$

where L is the dimensional length of the wiggler. We can therefore define the slow variable $z^* = \varepsilon z$. By this definition, $z^* \in [0, 1]$ along the wiggler, and the parameters which vary slowly along the wiggler (b_w, b_s, e_s , and ϕ_s) are best expressed as functions of this variable.

Since the FEL operates near the resonant state for the electrons, we can define a resonant energy γ_r by setting $d\phi/dT = 0$ to leading order to obtain

$$\gamma_r^2 = \frac{\lambda_w}{\lambda_s} \frac{1 + b_w^2}{2} \quad (2.6)$$

This gives a second small parameter μ which scales the electron energy γ and is defined as the ratio of the signal field wavelength to the wiggler period.

$$\mu = \frac{\lambda_s}{\lambda_w} \quad (2.7)$$

Since the relative orders of magnitude of μ and ε are not known at this time, we assume

$$\mu = O(\varepsilon^\alpha) \quad (2.8)$$

From equation (2.6), we can then say that the electron energy γ is of $O(\varepsilon^{-\alpha/2})$, and define the normalized energy $\bar{\gamma}$ of the electron by

$$\gamma = \varepsilon^{-\alpha/2} \bar{\gamma} \quad (2.9)$$

where all normalized variables are $O(1)$ and are indicated by an overbar. We also define a dimensionless wavenumber $\bar{k}_s$ of the signal field by

$$\frac{\lambda_w}{\lambda_s} = \varepsilon^{-\alpha} \bar{k}_s = \frac{1}{\mu} \quad (2.10)$$

and express the phase (2.2) in terms of dimensionless quantities

$$\phi = z + \varepsilon^{-\alpha} \bar{k}_s(z - t) + \phi_s(z^*) \quad (2.11)$$

From (2.6), it can be seen that b_w must be $O(1)$ in order to determine the resonant energy, so that $b_w(z^*) = \bar{b}_w(z^*)$. In addition, in [8] it was assumed that the magnitude of the signal field was also $O(1)$: $e_s(z^*) = b_s(z^*) = \bar{b}_s(z^*)$. This restriction is not necessary, but we continue to use it for the present analysis. Since all of the slowly varying parameters depend on z^* , it is convenient to use z instead of t as the independent variable. We therefore use the relation, $dz/dt = \beta_{||}$, to find the equations for $\bar{\gamma}$ and ϕ in terms of z . Omitting the details of the asymptotic expansions (see [8] and [1]), we find the system of equations for the normalized energy $\bar{\gamma}$ and phase ϕ

$$\begin{aligned} \frac{d\phi}{dz} = & 1 - \frac{\bar{k}_s(1 + \bar{b}_w^2)}{2\bar{\gamma}^2} + \varepsilon^\alpha \left[\frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}^2} \cos \phi - \frac{3\bar{k}_s(1 + \bar{b}_w^2)^2}{8\bar{\gamma}^4} \right] \\ & + \varepsilon \phi_s'(z^*) + \varepsilon^{2\alpha} \left[\frac{3\bar{b}_w \bar{b}_s(1 + \bar{b}_w^2)}{2\bar{\gamma}^4} \cos \phi - \frac{5\bar{k}_s(1 + \bar{b}_w^2)^3}{16\bar{\gamma}^6} - \frac{\bar{b}_s^2}{2\bar{k}_s \bar{\gamma}^2} \right] \\ & + O(\varepsilon^{3\alpha}) + O(\varepsilon^{(1+\alpha)}) + O(\varepsilon^2) \end{aligned} \quad (2.12a)$$

and

$$\begin{aligned} \frac{d\bar{\gamma}}{dz} = & -\varepsilon^\alpha \frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}} \sin \phi - \varepsilon^{2\alpha} \frac{\bar{b}_w \bar{b}_s(1 + \bar{b}_w^2)}{2\bar{\gamma}^3} \sin \phi \\ & - \varepsilon^{(1+\alpha)} \frac{\bar{b}_w' \bar{b}_s}{\bar{\gamma}} \cos \phi + \varepsilon^{3\alpha} \left[\frac{\bar{b}_w^2 \bar{b}_s^2}{2\bar{k}_s \bar{\gamma}^3} \sin 2\phi - \frac{3\bar{b}_w \bar{b}_s(1 + \bar{b}_w^2)^2}{8\bar{\gamma}^5} \sin \phi \right] \\ & + O(\varepsilon^{4\alpha}) + O(\varepsilon^{(1+2\alpha)}) + O(\varepsilon^{(2+\alpha)}) \end{aligned} \quad (2.12b)$$

where ' indicates d/dz^* .

The functional form of $\bar{b}_w(z^*)$ is given by the hardware design of the FEL. However, the parameters $\bar{b}_s(z^*)$ and $\phi_s(z^*)$ must be found from the wave equation for the signal field. These equations are also derived in [8] and are given by

$$\begin{aligned} \bar{b}_s'(z^*) = & \kappa \varepsilon^{-(1+3\alpha/2)} \frac{\bar{k}_s^2}{2} \bar{b}_w \left\langle \frac{\sin \phi}{\bar{\gamma}} \right\rangle + \kappa \varepsilon^{-3\alpha/2} \frac{\bar{k}_s^2}{2} \bar{b}_w' \left\langle \frac{\cos \phi}{\bar{\gamma}} \right\rangle \\ & + O(\kappa \varepsilon^{(1-3\alpha/2)}) + O(\kappa \varepsilon^{-\alpha/2}) + O(\varepsilon^{(1+\alpha)}) \end{aligned} \quad (2.13a)$$

and

$$\begin{aligned} \bar{b}_s(z^*) \phi_s'(z^*) = & \kappa \varepsilon^{-(1+3\alpha/2)} \frac{\bar{k}_s^2}{2} \bar{b}_w \left\langle \frac{\cos \phi}{\bar{\gamma}} \right\rangle \\ & - \kappa \varepsilon^{-3\alpha/2} \frac{\bar{k}_s^2}{2} \bar{b}_w' \left\langle \frac{\sin \phi}{\bar{\gamma}} \right\rangle - \kappa \varepsilon^{-(1+\alpha/2)} \frac{\bar{k}_s}{2} \bar{b}_s \left\langle \frac{1}{\bar{\gamma}} \right\rangle \\ & + O(\kappa \varepsilon^{(1-3\alpha/2)}) + O(\kappa \varepsilon^{-\alpha/2}) + O(\varepsilon^{(1+\alpha)}) \end{aligned} \quad (2.13b)$$

where $\langle \cdot \rangle$ indicates the average over all electrons in the beam, and where the dimensionless small parameter κ is given by

$$\kappa = \frac{\omega_p^2}{\omega_s^2} = \frac{\lambda_s^2 n_e e^2}{\pi m c^2} \quad (2.14)$$

The quantities ω_s and ω_p are the signal and plasma frequencies, respectively, and n_e is the electron beam density. Since equations (2.13) depend on the still undetermined $\bar{\gamma}$ and ϕ , giving a coupled system with (2.12), we postpone the solution for the wave evolution parameters $\bar{b}_s(z^*)$ and $\phi_s(z^*)$. In [8], Li and Kevorkian discuss three cases characterized by fast, slow, and nearly constant gain rates, indicated by the magnitude of κ . For $\alpha = 2/3$, only the case of nearly constant gain is physically consistent if we ignore the space-charge force. (We also have neglected the radiation-reaction force; see [9] for the physical conditions restricting these forces and [1] for details of the analysis here.) For the slow and fast gain rates, the space-charge force is of a significant order in the equations and must be systematically accounted for, slightly modifying the results obtained here.

III. HAMILTONIAN STANDARD FORM AND ASYMPTOTIC SOLUTION

The system of equations (2.12) is an example of a system in *Hamiltonian standard form*, and is derivable to $O(\epsilon^{2\alpha})$ from the Hamiltonian

$$\begin{aligned}
 H(\bar{\gamma}, \phi, z^*; \epsilon) = & \bar{\gamma} + \frac{\bar{k}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}} \\
 & + \epsilon^\alpha \left[-\frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}} \cos \phi + \frac{\bar{k}_s (1 + \bar{b}_w^2)^2}{8 \bar{\gamma}^3} \right] + \epsilon \bar{\gamma} \phi'_s(z^*) \\
 & + \epsilon^{2\alpha} \left[-\frac{\bar{b}_w \bar{b}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}^3} \cos \phi + \frac{\bar{k}_s (1 + \bar{b}_w^2)^3}{16 \bar{\gamma}^5} + \frac{\bar{b}_s^2}{2 \bar{k}_s \bar{\gamma}} \right]
 \end{aligned} \tag{3.1}$$

In this context, the terminology *standard form* indicates that to $O(1)$ H does not depend on ϕ , and that H is a 2π -periodic function of ϕ to higher order. See [1], [2] and [5] for further mathematical details.

The solution procedure for such systems involves a series of transformations, reducing the final system to quadrature. Since the FEL operates near the resonant state (corresponding to the vanishing of the $O(1)$ term in (2.12a)), we rescale the dependent and independent variables to reflect the behavior of the system in the resonance region. The resulting system of equations is still Hamiltonian but is no longer in standard form, so that canonical averaging techniques cannot be applied. We therefore use a canonical action-angle transformation to reformulate the system back into standard form, then apply another canonical near-identity transformation, averaging the system and eliminating the dependence of the Hamiltonian on the angle variable to the necessary order. This implies that the corresponding action variable is an adiabatic invariant of the system, constant to the specified order, and uncouples the system, so that the angle variable can be found by quadrature. Inversion of the transformations is straightforward, giving the final asymptotic solutions for the energy

and phase. The reader is referred to Bosley and Kevorkian [2] for a more detailed account of the mathematical theory and solution technique.

For the FEL problem at hand, we introduce the resonant energy for the electron $\bar{\gamma}_r(z^*)$ which sets the leading order frequency of the phase to zero, i.e.,

$$\bar{\gamma}_r(z^*) = \sqrt{\frac{\bar{k}_s(1 + \bar{b}_w^2)}{2}} \quad (3.2)$$

We then define new variables γ^* and ϕ^* and the new fast variable $\hat{z} = \varepsilon^{1/2} z$.

$$\bar{\gamma} = \bar{\gamma}_r(z^*) + \varepsilon^{1/2} \gamma^*(\hat{z}) \quad (3.3a)$$

$$\phi = \phi^*(\hat{z}) \quad (3.3b)$$

Expanding the differential equations (2.12) about $\bar{\gamma}_r$ we find the following equations for γ^* and ϕ^*

$$\begin{aligned} \frac{d\gamma^*}{d\hat{z}} = & -\frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r} \sin \phi^* - \varepsilon^{(1-\alpha)} \frac{d\bar{\gamma}_r}{dz^*} \\ & + \varepsilon^{1/2} \frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r^2} \gamma^* \sin \phi^* - \varepsilon^\alpha \left[\frac{\bar{b}_w \bar{b}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^3} \sin \phi^* + \frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r^3} \gamma^{*2} \sin \phi^* \right] \\ & + \varepsilon^{3/2} \left[\frac{3 \bar{b}_w \bar{b}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^4} \gamma^* \sin \phi^* + \frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r^4} \gamma^{*3} \sin \phi^* \right] \\ & - \varepsilon \frac{\bar{b}_w' \bar{b}_s}{\bar{\gamma}_r} \cos \phi^* + O(\varepsilon^{(1+1/2)}) + O(\varepsilon^{2\alpha}) \end{aligned} \quad (3.4a)$$

and

$$\begin{aligned} \frac{d\phi^*}{d\hat{z}} = & \frac{\bar{k}_s (1 + \bar{b}_w^2)}{\bar{\gamma}_r^3} \gamma^* \\ & + \varepsilon^{1/2} \left[\frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r^2} \cos \phi^* - \frac{3 \bar{k}_s (1 + \bar{b}_w^2)^2}{8 \bar{\gamma}_r^4} - \frac{3 \bar{k}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^4} \gamma^{*2} \right] \end{aligned} \quad (3.4b)$$

$$\begin{aligned}
& + \varepsilon^\alpha \left[\frac{3 \bar{k}_s (1 + \bar{b}_w^2)^2}{2 \bar{\gamma}_r^5} \gamma^* + \frac{2 \bar{k}_s (1 + \bar{b}_w^2)}{\bar{\gamma}_r^5} \gamma^{*3} - \frac{2 \bar{b}_w \bar{b}_s}{\bar{\gamma}_r^3} \gamma^* \cos \phi^* \right] \\
& + \varepsilon^{(1-\alpha/2)} \phi_s'(z^*) \\
& + \varepsilon^{3\alpha/2} \left[-\frac{5 \bar{k}_s (1 + \bar{b}_w^2)^3}{16 \bar{\gamma}_r^6} - \frac{15 \bar{k}_s (1 + \bar{b}_w^2)^2}{4 \bar{\gamma}_r^6} \gamma^{*2} - \frac{5 \bar{k}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^6} \gamma^{*4} \right. \\
& \quad \left. - \frac{\bar{b}_s^2}{2 \bar{k}_s \bar{\gamma}_r^2} + \frac{3 \bar{b}_w \bar{b}_s}{\bar{\gamma}_r^4} \gamma^{*2} \cos \phi^* + \frac{3 \bar{b}_w \bar{b}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^4} \cos \phi^* \right] \\
& + O(\varepsilon^{(1+\alpha/2)}) + O(\varepsilon^{2\alpha})
\end{aligned}$$

So far, the value of α is arbitrary; however, α must be dictated by the relative orders of magnitude of the leading terms in (3.4a). If the first two terms on the right hand side of (3.4a) are comparable we must have $\alpha = 1$. If however, the first term predominates, and the next two are comparable we must have $1 - \alpha = \alpha/2$ or $\alpha = 2/3$, etc. Thus, defining $\hat{\varepsilon} = \varepsilon^{\alpha/2}$, we have the following sequence of possible distinguished values for α

$$\begin{aligned}
\varepsilon^{(1-\alpha)} &= 1, \quad \text{for } \alpha = 1 \\
\varepsilon^{(1-\alpha)} &= \hat{\varepsilon}, \quad \text{for } \alpha = 2/3 \\
\varepsilon^{(1-\alpha)} &= \hat{\varepsilon}^2, \quad \text{for } \alpha = 1/2 \\
&\vdots \\
\varepsilon^{(1-\alpha)} &= \hat{\varepsilon}^{(n-2)}, \quad \text{for } \alpha = 2/n
\end{aligned} \tag{3.5}$$

Equivalently, we can specify the ratio of the slow variable z^* to the new fast variable $\hat{z} = \hat{\varepsilon} z$.

$$z^* = \varepsilon z = \hat{\varepsilon}^{(2/\alpha-1)} \hat{z} \Rightarrow \begin{cases} z^* = \hat{\varepsilon} \hat{z} & \text{for } \alpha = 1 \\ z^* = \hat{\varepsilon}^2 \hat{z} & \text{for } \alpha = 2/3 \\ z^* = \hat{\varepsilon}^3 \hat{z} & \text{for } \alpha = 1/2 \\ \vdots \\ z^* = \hat{\varepsilon}^{(n-1)} \hat{z} & \text{for } \alpha = 2/n \end{cases} \tag{3.6}$$

The $\alpha = 1$ case has already been examined in [8]; as indicated previously, here we choose $\alpha = 2/3$, requiring the retention of terms to one higher order. The system (3.4) can be derived from the Hamiltonian

$$\begin{aligned}
H^*(\gamma^*, \phi^*, z^*; \hat{\varepsilon}) = & \frac{\bar{k}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^3} \gamma^{*2} - \frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r} \cos \phi^* \\
& + \hat{\varepsilon} \left[\frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r^2} \gamma^* \cos \phi^* - \frac{3 \bar{k}_s (1 + \bar{b}_w^2)^2}{8 \bar{\gamma}_r^4} \gamma^* - \frac{\bar{k}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^4} \gamma^{*3} + \frac{d\bar{\gamma}_r}{dz^*} \phi^* \right] \\
& + \hat{\varepsilon}^2 \left[\frac{3 \bar{k}_s (1 + \bar{b}_w^2)^2}{4 \bar{\gamma}_r^5} \gamma^{*2} + \frac{\bar{k}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^5} \gamma^{*4} + \phi_s' \gamma^* \right. \\
& \quad \left. - \frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r^3} \gamma^{*2} \cos \phi^* - \frac{\bar{b}_w \bar{b}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^3} \cos \phi^* \right] \\
& + \hat{\varepsilon}^3 \left[-\frac{5 \bar{k}_s (1 + \bar{b}_w^2)^3}{16 \bar{\gamma}_r^6} \gamma^* - \frac{5 \bar{k}_s (1 + \bar{b}_w^2)^2}{4 \bar{\gamma}_r^6} \gamma^{*3} - \frac{\bar{k}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^6} \gamma^{*5} \right. \\
& \quad + \frac{\bar{b}_w \bar{b}_s}{\bar{\gamma}_r^4} \gamma^{*3} \cos \phi^* + \frac{3 \bar{b}_w \bar{b}_s (1 + \bar{b}_w^2)}{2 \bar{\gamma}_r^4} \gamma^* \cos \phi^* \\
& \quad \left. - \frac{\bar{b}_s^2}{2 \bar{k}_s \bar{\gamma}_r^2} \gamma^* + \frac{\bar{b}_w' \bar{b}_s}{\bar{\gamma}_r} \sin \phi^* \right] + O(\hat{\varepsilon}^4)
\end{aligned} \tag{3.7}$$

where $\bar{b}_w, \bar{b}_s, \bar{\gamma}_r, \phi_s$, and all their derivatives are functions of the slow variable $z^* = \hat{\varepsilon}^2 \hat{z}$. Note that the Hamiltonian is no longer in standard form since the $O(1)$ portion depends on the angle variable ϕ^* and the dependence on ϕ^* to higher order is no longer periodic.

We now observe an important distinction between the $\alpha = 1$ and $\alpha = 2/3$ cases. Since (3.4b) is linear in γ^* , it is possible to eliminate it to find a pendulum-like equation for ϕ^* .

$$\frac{d^2\phi^*}{d\bar{z}^2} + f(z^*) \sin \phi^* = \varepsilon^{(1-\alpha)} g(z^*) + O(\hat{\varepsilon}) \quad (3.8)$$

In the $\alpha = 1$ case, equation (3.8) has an $O(1)$ "tangential force" given by $g(z^*)$. If all dissipative and slow variational effects are ignored, the phase plane for this system appears as in Figure 1. However if $\alpha = 2/3$, then the tangential force of the pendulum equation becomes of $O(\hat{\varepsilon})$, and the $O(1)$ equation has a phase plane as in Figure 2. The significant difference between the phase planes is that for the $\alpha = 2/3$ case the full 2π width of the ϕ^* phases are possible for sustained resonance, giving a correspondingly higher percentage of electrons trapped in the potential well. This will result in an increased efficiency for the energy transfer to the signal field, since only those electrons trapped in resonance transfer an appreciable amount of energy. In both Figures 1 and 2, the area inside the heavy curve is the region of sustained resonance.

Figure 1. Integral Curves in the Phase Plane: $\alpha = 1$

Figure 2. Integral Curves in the Phase Plane: $\alpha = 2/3$

An additional simplification occurs in the solution of the problem in the $\alpha = 2/3$ case. If $\alpha = 1$, the $O(1)$ differential equation (3.8) does not have an explicit analytical solution in terms of known elementary or transcendental functions. However if $\alpha = 2/3$, then the tangential force of the pendulum equation becomes of $O(\hat{\epsilon})$, and the $O(1)$ equation does permit an explicit analytical solution in terms of elliptic functions. This means that no further approximations are necessary in the asymptotic solution procedure, giving a more precise final solution.

We use this known solution to define a canonical action-angle transformation (see H. Goldstein [4] for details) from the old variables (γ^*, ϕ^*) to new variables (p, q)

$$\phi^* = 2 \sin^{-1}(k \operatorname{sn}(4 K(k) q, k)) \quad (3.9a)$$

$$\gamma^* = \sqrt{2 b_w b_s} k \operatorname{cn}(4 K(k) q, k) \quad (3.9b)$$

where the modulus of the elliptic functions $k = k(p, z^*)$ is defined implicitly by

$$p = 16 \sqrt{\frac{\bar{b}_w \bar{b}_s}{2}} \{E(k) - (1 - k^2) K(k)\} \quad (3.10)$$

where K and E are the complete elliptic integrals of the first and second kinds respectively (see Byrd and Friedman [3] for definitions of all elliptic integrals and functions). The new Hamiltonian is given by the expansion

$$\begin{aligned} h(p, q, z^*; \hat{\varepsilon}) &= h_0(p, z^*) + \hat{\varepsilon} h_1(p, q, z^*) \\ &+ \hat{\varepsilon}^2 h_2(p, q, z^*) + \hat{\varepsilon}^3 h_3(p, q, z^*) + O(\hat{\varepsilon}^4) \end{aligned} \quad (3.11a)$$

where

$$h_0(p, z^*) = \lambda_{00}(z^*) (2k^2 - 1) \quad (3.11b)$$

$$\begin{aligned} h_1(p, q, z^*) &= \lambda_{10}(z^*) \sin^{-1}(k \operatorname{sn}(4K(k)q, k)) \\ &+ [k \lambda_{11}(z^*) + k^3 \lambda_{12}(z^*)] \operatorname{cn}(4K(k)q, k) \end{aligned} \quad (3.11c)$$

$$\begin{aligned} h_2(p, q, z^*) &= \lambda_{20}(z^*) E_{osc}(4K(k)q, k) + \lambda_{21}(z^*) k \operatorname{cn}(4K(k)q, k) \\ &+ [\lambda_{22}(z^*) k^2 + \lambda_{23}(z^*) k^4] \operatorname{cn}^2(4K(k)q, k) + \lambda_{24}(z^*) (2k^2 - 1) \end{aligned} \quad (3.11d)$$

$$\begin{aligned} h_3(p, q, z^*) &= \lambda_{30}(z^*) k \operatorname{sn}(4K(k)q, k) \operatorname{dn}(4K(k)q, k) \\ &+ [\lambda_{31}(z^*) k + \lambda_{32}(z^*) k^3] \operatorname{cn}(4K(k)q, k) \\ &+ [\lambda_{33}(z^*) k^3 + \lambda_{34}(z^*) k^5] \operatorname{cn}^3(4K(k)q, k) \end{aligned} \quad (3.11e)$$

where $E_{osc}(u, k) = E(\operatorname{am}(u, k), k) - E(k) u / K(k)$. Equation (3.10) gives an implicit definition of the modulus k as a function $k(p, z^*)$. The λ_{ij} are functions of z^* alone and are given by

$$\lambda_{00}(z^*) = \bar{b}_w(z^*) \bar{b}_s(z^*) \sqrt{\frac{2}{\bar{k}_s (1 + \bar{b}_w^2(z^*))}} \quad (3.12a)$$

$$\lambda_{10}(z^*) = \sqrt{\frac{\bar{k}_s}{2 (1 + \bar{b}_w^2(z^*))}} \bar{b}_w(z^*) \frac{d\bar{b}_w}{dz^*}(z^*) \quad (3.12b)$$

$$\lambda_{11}(z^*) = \sqrt{2 \bar{b}_w(z^*) \bar{b}_s(z^*)} \frac{(4 \bar{b}_w(z^*) \bar{b}_s(z^*) - 3(1 + \bar{b}_w^2(z^*)))}{2 \bar{k}_s (1 + \bar{b}_w^2(z^*))} \quad (3.12c)$$

$$\lambda_{12}(z^*) = -\sqrt{2 \bar{b}_w(z^*) \bar{b}_s(z^*)} \frac{4 \bar{b}_w(z^*) \bar{b}_s(z^*)}{\bar{k}_s (1 + \bar{b}_w^2(z^*))} \quad (3.12d)$$

$$\lambda_{20}(z^*) = \left(\frac{2}{\bar{k}_s (1 + \bar{b}_w^2(z^*))} \right)^{1/4} (\bar{b}_w'(z^*) \bar{b}_s(z^*) + \bar{b}_w(z^*) \bar{b}_s'(z^*)) \quad (3.12e)$$

$$\lambda_{21}(z^*) = \sqrt{2 \bar{b}_w(z^*) \bar{b}_s(z^*)} \frac{d\phi_s}{dz^*}(z^*) \quad (3.12f)$$

$$\lambda_{22}(z^*) = \frac{-4 \sqrt{2 \bar{b}_w(z^*) \bar{b}_s(z^*)}}{(\bar{k}_s (1 + \bar{b}_w^2(z^*)))^{3/2}} (\bar{b}_w(z^*) \bar{b}_s(z^*) - (1 + \bar{b}_w^2(z^*))) \quad (3.12g)$$

$$\lambda_{23}(z^*) = \frac{8 \sqrt{2 \bar{b}_w^2(z^*) \bar{b}_s^2(z^*)}}{(\bar{k}_s (1 + \bar{b}_w^2(z^*)))^{3/2}} \quad (3.12h)$$

$$\lambda_{24}(z^*) = \sqrt{\frac{2}{\bar{k}_s (1 + \bar{b}_w^2(z^*))}} \frac{\bar{b}_w(z^*) \bar{b}_s(z^*)}{\bar{k}_s} \quad (3.12i)$$

$$\lambda_{30}(z^*) = 2 \sqrt{\frac{2}{\bar{k}_s (1 + \bar{b}_w^2(z^*))}} \bar{b}_s(z^*) \frac{d\bar{b}_w}{dz^*}(z^*) \quad (3.12j)$$

$$\lambda_{31}(z^*) = -\sqrt{2 \bar{b}_w(z^*) \bar{b}_s(z^*)} \frac{(2 \bar{b}_s^2(z^*) - 12 \bar{b}_w(z^*) \bar{b}_s(z^*) + 5(1 + \bar{b}_w^2(z^*)))}{2 \bar{k}_s^2 (1 + \bar{b}_w^2(z^*))} \quad (3.12k)$$

$$\lambda_{32}(z^*) = -\sqrt{2 \bar{b}_w(z^*) \bar{b}_s(z^*)} \frac{12 \bar{b}_w(z^*) \bar{b}_s(z^*)}{\bar{k}_s^2 (1 + \bar{b}_w^2(z^*))} \quad (3.12l)$$

$$\lambda_{33}(z^*) = 8 \sqrt{2 \bar{b}_w(z^*) \bar{b}_s(z^*)} \frac{\bar{b}_w(z^*) \bar{b}_s(z^*) (\bar{b}_w(z^*) \bar{b}_s(z^*) - (1 + \bar{b}_w^2(z^*)))}{\bar{k}_s^2 (1 + \bar{b}_w^2(z^*))^2} \quad (3.12m)$$

$$\lambda_{34}(z^*) = \frac{-16 \sqrt{2} (\bar{b}_w(z^*) \bar{b}_s(z^*))^{5/2}}{\bar{k}_s^2 (1 + \bar{b}_w^2(z^*))^2} \quad (3.12n)$$

The action-angle transformation (3.9) has reformulated the system back into standard form, so that canonical averaging techniques can now be performed (see [2] and [5] for details). The near-identity averaging transformation is carried out to $O(\hat{\varepsilon}^2)$, keeping track of averaged terms of $O(\hat{\varepsilon}^3)$, in order to have a uniformly valid solution to $O(\hat{\varepsilon})$ over the entire length of the laser. We transform (p, q) to (P, Q) using a generating function of the form

$$F(P, q, z^*) = Pq + \hat{\varepsilon} F_1(P, q, z^*) + \hat{\varepsilon}^2 F_2(P, q, z^*) \quad (3.13)$$

We then solve for F_1 and F_2 , as discussed in [2] to find the new averaged Hamiltonian (the solutions for F_1 and F_2 are not included here for brevity but can be found in [1]). The resulting Hamiltonian with the Q dependence removed to $O(\hat{\varepsilon}^2)$ is given by

$$\begin{aligned} H = & \underline{H}_0(P, z^*) + \hat{\varepsilon}^2 \underline{H}_2(P, z^*) \\ & + \hat{\varepsilon}^3 [\underline{H}_3(P, z^*) + \underline{H}_3(P, Q, z^*)] + O(\hat{\varepsilon}^4) \end{aligned} \quad (3.14a)$$

where

$$\underline{H}_0 = \lambda_{00} (2\bar{k}^2 - 1) \quad (3.14b)$$

$$\underline{H}_2 = \frac{(\lambda_{22} + \lambda_{23} \bar{k}^2) (E(\bar{k}) - (1 - \bar{k}^2) K(\bar{k}))}{K(\bar{k})} + \lambda_{24} (2\bar{k}^2 - 1) \quad (3.14c)$$

$$\begin{aligned} & - \frac{(\lambda_{11} + \lambda_{12} \bar{k}^2) (\lambda_{11} K(\bar{k}) + \lambda_{12} (4E(\bar{k}) - (4 - 5\bar{k}^2) K(\bar{k})))}{8\lambda_{00} K(\bar{k})} \\ & - \frac{\lambda_{10}^2}{8\lambda_{00} \bar{k} K(\bar{k})} \frac{\partial}{\partial \bar{k}} \left(K(\bar{k}) \int_0^1 [\sin^{-1}(\bar{k} \operatorname{sn}(4K(\bar{k})q, \bar{k}))]^2 dq \right) \end{aligned}$$

$$\underline{H}_3 = - \frac{\lambda_{21} [2\lambda_{12} E(\bar{k}) + (\lambda_{11} - \lambda_{12} (2 - 3\bar{k}^2)) K(\bar{k})]}{4\lambda_{00} K(\bar{k})} \quad (3.14d)$$

The variable $\bar{k}$ indicates dependence on the new averaged variable P defined by

$$P = 16 \sqrt{\frac{\bar{b}_w \bar{b}_s}{2}} \left\{ E(\bar{k}) - (1 - \bar{k}^2) K(\bar{k}) \right\} \quad (3.15)$$

The definite integral in (3.14c) cannot be expressed in terms of a finite number of elementary and elliptic functions. However, using the Fourier series expansions given in [3], we can evaluate this integral in terms of an infinite series

$$\int_0^1 \left[\sin^{-1}(k \operatorname{sn}(4K(k)q, k)) \right]^2 dq = \quad (3.16)$$

$$2 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \operatorname{sech}^2 \left[(2m+1) \frac{\pi K'}{2K} \right]$$

which is rapidly converging, differentiable, and easily evaluated. The canonical differential equations resulting from the Hamiltonian (3.14) for (P, Q) are

$$\frac{dQ}{d\hat{z}} = \frac{\partial H_0}{\partial P} + \hat{\varepsilon}^2 \frac{\partial H_2}{\partial P} + \hat{\varepsilon}^3 \left(\frac{\partial H_3}{\partial P} + \frac{\partial \hat{H}_3}{\partial P} \right) + O(\hat{\varepsilon}^4) \quad (3.17a)$$

and

$$\frac{dP}{d\hat{z}} = -\hat{\varepsilon}^3 \frac{\partial \hat{H}_3}{\partial Q} + O(\hat{\varepsilon}^4) \quad (3.17b)$$

Since Q has been eliminated from the Hamiltonian to $O(\hat{\varepsilon}^2)$, P is an adiabatic invariant, constant to $O(\hat{\varepsilon}^2)$ along the entire length of the FEL, as seen from equation (3.17b). A simple quadrature of (3.17a) then gives Q .

$$Q(z^*) = Q(0) + \frac{1}{\hat{\varepsilon}^2} \int_0^{z^*} \frac{\partial H_0}{\partial P}(P, z^*) dz^* + \int_0^{z^*} \frac{\partial H_2}{\partial P}(P, z^*) dz^* \quad (3.18)$$

$$+ \hat{\varepsilon} \int_0^{z^*} \frac{\partial \hat{H}_3}{\partial P}(P, z^*) dz^* + O(\hat{\varepsilon}^2)$$

The near-identity transformation defined by the generating function (3.13) can be inverted to find P as a function of the original variables

$$P = \mathcal{A}(p, q, z^*; \hat{\varepsilon}) \equiv \text{constant} + O(\hat{\varepsilon}^3) \quad (3.19)$$

which for this problem gives

$$\begin{aligned} \mathcal{A}(p, q, z^*; \hat{\varepsilon}) = & 16 \rho(z^*) \{ E(k) - (1 - k^2) K(k) \} \\ & + \hat{\varepsilon} \frac{4 \rho(z^*) K(k)}{\lambda_{00}(z^*)} \{ \lambda_{10}(z^*) \sin^{-1}(k \operatorname{sn}(4 K(k) q, k)) \\ & + k (\lambda_{11}(z^*) + \lambda_{12}(z^*) k^2) \operatorname{cn}(4 K(k) q, k) \} \\ & + \hat{\varepsilon}^2 \left[\frac{\rho \lambda_{10}^2}{2 k \lambda_{00}^2} \frac{\partial}{\partial k} \left(K(k) \int_0^1 [\sin^{-1}(k \operatorname{sn}(4 K(k) q, k))]^2 dq \right) \right. \\ & + \frac{4 \rho K(k)}{\lambda_{00}} [\lambda_{20} E_{\operatorname{osc}} + \lambda_{21} k \operatorname{cn} + k^2 (\lambda_{22} + \lambda_{23} k^2) \operatorname{cn}^2] \\ & + \frac{\rho (E(k) - (1 - k^2) K(k))}{2 \lambda_{00}^2 k^2 (1 - k^2)} [\lambda_{10} \sin^{-1}(k \operatorname{sn}) + k (\lambda_{11} + \lambda_{12} k^2) \operatorname{cn}]^2 \\ & + \frac{\rho (\lambda_{11} + \lambda_{12} k^2)}{2 \lambda_{00}^2} [(\lambda_{11} + \lambda_{12} k^2) K(k) \\ & + 4 \lambda_{12} (E(k) - (1 - k^2) K(k))] \\ & \left. - \frac{4 \rho (\lambda_{22} + \lambda_{23} k^2)}{\lambda_{00}} [E(k) - (1 - k^2) K(k)] \right] + O(\hat{\varepsilon}^3) \end{aligned} \quad (3.20)$$

where $k = k(p, z^*)$ implicitly through (3.10), and where

$$\rho(z^*) = \sqrt{\frac{\bar{b}_w(z^*) \bar{b}_s(z^*)}{2}} \quad (3.21)$$

The expression (3.20) is primarily used to calculate the value of the adiabatic invariant from initial conditions for use in the final solution.

The solution for γ^* and ϕ^* is found to $O(\hat{\varepsilon})$ by expressing the action-angle transformation (3.9) asymptotically in terms of Q and $\bar{k}$

$$\begin{aligned}
\phi^* = & 2 \sin^{-1}(\bar{k} \operatorname{sn}(4 K(\bar{k}) Q, \bar{k})) \\
& - \hat{\epsilon} \left[\sqrt{\frac{\bar{b}_s \bar{b}_w}{\bar{k}_s (1 + \bar{b}_w^2)}} 4 \bar{k} \operatorname{cn} \sin^{-1}(\bar{k} \operatorname{sn}) \right. \\
& \left. + \frac{\bar{k}_s \bar{b}_w'}{2 \bar{b}_s} \frac{(\bar{k} \operatorname{cn}^2 - \bar{k}^2 \operatorname{cn} I_{SC2} + \operatorname{sn} \operatorname{dn} \sin^{-1}(\bar{k} \operatorname{sn}))}{\bar{k} (1 - \bar{k}^2)} \right]
\end{aligned} \tag{3.22a}$$

and

$$\begin{aligned}
\gamma^* = & \sqrt{2 \bar{b}_w \bar{b}_s} \bar{k} \operatorname{cn}(4 K(\bar{k}) Q, \bar{k}) \\
& + \hat{\epsilon} \left[\frac{\bar{k}_s \bar{b}_w'}{2 \bar{k} (1 - \bar{k}^2)} \sqrt{\frac{\bar{b}_w}{2 \bar{b}_s}} \left[\bar{k} \operatorname{sn} \operatorname{dn} (\operatorname{cn} - \bar{k} I_{SC2}) \right. \right. \\
& \left. \left. - \operatorname{cn} \sin^{-1}(\bar{k} \operatorname{sn}) (\bar{k}^2 \operatorname{cn}^2 + 1 - 2 \bar{k}^2) \right] \right. \\
& \left. + \frac{4 \bar{b}_s \bar{b}_w (4 \bar{k} \operatorname{sn} \operatorname{dn} \sin^{-1}(\bar{k} \operatorname{sn}) + 2 \bar{k}^2 - 1) + 3 (1 + \bar{b}_w^2)}{4 \sqrt{2 \bar{k}_s (1 + \bar{b}_w^2)}} \right]
\end{aligned} \tag{3.22b}$$

where we define

$$I_{SC2}(4K(\bar{k})Q, \bar{k}) = \int_0^{4K(\bar{k})Q} \operatorname{cn}^2(u, \bar{k}) \sin^{-1}(\bar{k} \operatorname{sn}(u, \bar{k})) du \tag{3.23}$$

The arguments of the elliptic functions are all $(4K(\bar{k})Q, \bar{k})$. The value for Q is found by the quadrature (3.18) and the value for $\bar{k}$ is found by inverting the algebraic relation

$$\frac{\mathcal{A}}{16 \rho(z^*)} = E(\bar{k}) - (1 - \bar{k}^2) K(\bar{k}) \tag{3.24}$$

Although the solutions (3.22) and the expression for the adiabatic invariant (3.20) contain a significant number of terms, they are not difficult to evaluate using the symbolic manipulator's automatic generation of FORTRAN code.

Since it is desirable for an electron to remain captured in sustained resonance, finding an easily evaluated condition for escape from resonance would be quite useful. Using the solution (3.22a) for ϕ^* , we asymptotically expand the quantity

$$\sin \frac{\phi^*}{2} \quad (3.25)$$

In $\hat{\epsilon}$ then find an envelope for the oscillations. When this envelope approaches ± 1 , then $\phi^* \rightarrow \pm \pi$ corresponding to the saddle points in the phase plane, where escape occurs. This calculation gives the envelope as

$$k_{CR}(z^*) = \bar{k}(z^*) - \hat{\epsilon} \frac{\bar{k}_s \bar{b}_w'(z^*)}{4 \bar{b}_s(z^*)} \frac{\sin^{-1}(\bar{k}(z^*))}{\bar{k}(z^*)} + O(\hat{\epsilon}^2) \quad (3.26)$$

Since $\bar{k}(z^*)$ is found from (3.24), this expression can be easily evaluated without any numerical integrations, and a very accurate distance of escape of the electron from the potential well can be found. Thus the length along the laser z_{CR}^* at which the electron escapes from resonance is given by the condition

$$k_{CR}(z_{CR}^*) = 1 \quad (3.27)$$

This condition is crucial, since escape from the potential well indicates the electron is no longer transferring energy to the signal field, and the relative simplicity of the expression (3.26) makes its potential for use in FEL design significant.

We also use the solutions (3.22) for γ^* and ϕ^* to find the wave evolution governed by the equations (2.13). We note that the first two terms in the right hand side of (2.13a) can be rewritten in terms of $d\bar{\gamma}/dz$ from (2.12b). This gives the equation

$$\begin{aligned} \bar{b}_s'(z^*) = & -\kappa \hat{\epsilon}^{-8} \frac{\bar{k}_s^2}{2 \bar{b}_s} \frac{d\langle \bar{\gamma} \rangle}{dz} \\ & - \kappa \hat{\epsilon}^{-4} \frac{\bar{k}_s^2}{4} \bar{b}_w (1 + \bar{b}_w^2) \left\langle \frac{\sin \phi}{\bar{\gamma}^3} \right\rangle \\ & + O(\kappa \hat{\epsilon}^{-2}) + O(\hat{\epsilon}^5) \end{aligned} \quad (3.28)$$

where we use the value $\alpha = 2/3$. We note that (2.12b) can again be used to express the second term in (3.28) in terms of $d\bar{\gamma}/dz$, then express all derivatives in terms of d/dz^* .

$$\begin{aligned}
2 \bar{b}_s \bar{b}_s'(z^*) &= -\kappa \hat{\epsilon}^{-5} \bar{k}_s^2 \frac{d\langle \bar{\gamma} \rangle}{dz^*} \\
&+ \kappa \hat{\epsilon}^{-3} \frac{\bar{k}_s^2}{2} (1 + \bar{b}_w^2) \left\langle \bar{\gamma}^{-2} \frac{d\bar{\gamma}}{dz^*} \right\rangle \\
&+ O(\kappa \hat{\epsilon}^{-2}) + O(\hat{\epsilon}^5)
\end{aligned} \tag{3.29}$$

We now integrate (3.29) using the approximation $\bar{\gamma} = \bar{\gamma}_r$ in the second term. This gives the result

$$\begin{aligned}
\bar{b}_s^2(z^*) &= \bar{b}_s^2(0) + \kappa \hat{\epsilon}^{-5} \bar{k}_s^2 \langle \bar{\gamma}(0) - \bar{\gamma}(z^*) \rangle \\
&- \kappa \hat{\epsilon}^{-3} \bar{k}_s \langle \bar{\gamma}_r(0) - \bar{\gamma}_r(z^*) \rangle \\
&+ O(\kappa \hat{\epsilon}^{-2}) + O(\hat{\epsilon}^5)
\end{aligned} \tag{3.30}$$

Since the signal field evolution is slow, we know κ must be a small parameter. Also, the space-charge force restriction found in [1] gives

$$\kappa = O(\hat{\epsilon}^6) \tag{3.31}$$

The three signal field gain rates identified in [8] were characterized by the orders:

$$\begin{aligned}
\text{(i) fast gain} & \quad \kappa = O(\hat{\epsilon}^5) \\
\text{(ii) slow gain} & \quad \kappa = O(\hat{\epsilon}^6) \\
\text{(iii) nearly constant gain} & \quad \kappa = O(\hat{\epsilon}^7)
\end{aligned} \tag{3.32}$$

For the restriction on the space-charge force (3.31), only the nearly constant gain rate can be considered valid. We therefore write

$$\kappa = \bar{\omega} \hat{\epsilon}^7 \tag{3.33}$$

From the Hamiltonian (3.7), we must determine $\bar{b}_s$ to $O(\hat{\epsilon}^3)$ to find the averaged terms which contribute to the $O(\hat{\epsilon})$ phase of the final solution. This gives the final solution for the signal field amplitude

$$\begin{aligned}\bar{b}_s(z^*) = & \bar{b}_s(0) + \hat{\epsilon}^2 \frac{\bar{\omega}}{2} \bar{k}_s^2 (\bar{\gamma}_r(0) - \bar{\gamma}_r(z^*)) \\ & + \hat{\epsilon}^3 \frac{\bar{\omega}}{2} \bar{k}_s^2 \langle \gamma^*(0) - \gamma^*(z^*) \rangle + O(\hat{\epsilon}^4)\end{aligned}\tag{3.34}$$

The resulting equation for the phase shift $\phi_s(z^*)$ from (2.13b) is

$$\phi_s'(z^*) = \hat{\epsilon} \frac{\bar{\omega}}{2} \frac{\bar{k}_s^2 \bar{b}_w}{\bar{b}_s} \left\langle \frac{\cos \phi}{\bar{\gamma}} \right\rangle + O(\hat{\epsilon}^2)\tag{3.35}$$

This term will contribute to the $O(\hat{\epsilon}^3)$ Hamiltonian (3.7). However, since we are only keeping track of averaged terms of $O(\hat{\epsilon}^3)$ and this term has zero average at that order, it can be set to zero without influencing the solution to $O(\hat{\epsilon}^2)$. (For the $\alpha = 1$ case, the larger value of κ representing the slow signal field gain rate can also be considered, as the restriction on the space-charge force is not as strong.) If we wish to relax the condition on the space-charge force, we can examine the distinguished limits characterized by the two larger values, $\kappa = O(\hat{\epsilon}^6)$ and $\kappa = O(\hat{\epsilon}^5)$, corresponding to the slow and fast gain rates of the signal field. In these cases, however, we would need to systematically account for the effects of the space-charge force in the equations of motion and thus in the final solution. Additionally, we have assumed the wiggler field to have a particular functional form. However, the analysis presented here is not altered significantly by including the higher order effects of the non-uniformity of this field. These correction terms could easily be included in the equations of motion, correcting the final asymptotic solutions.

IV. A Model FEL

In order to carry out numerical experiments and to specify a concrete example for the class of FELs considered in this chapter, we adopt the following parameters:

$$\begin{aligned}\lambda_w &= 2.5 \text{ cm} & L &= 15.0 \text{ m} \\ B_w &= (4.28 - 3.21 Z/L) \text{ kG} & \lambda_s &= 100 \mu\text{m}\end{aligned}\tag{4.1}$$

These parameters are similar to the "standard" wiggler design proposed in [7] for discussing a linearly tapered magnetic wiggler amplitude. The primary modification here is the change to a signal wavelength of $100 \mu\text{m}$ for a longer wavelength, lower energy FEL.

Based on the nondimensionalizations defined in section 2 we find

$$\varepsilon = \frac{\lambda_w}{2\pi L} = 2.65 \times 10^{-4}\tag{4.2}$$

and

$$\mu = \frac{\lambda_s}{\lambda_w} = 4.0 \times 10^{-3}\tag{4.3}$$

This gives the dimensionless signal wavenumber for the value $\alpha = 2/3$

$$\bar{k}_s = \frac{\varepsilon^\alpha}{\mu} = 1.03\tag{4.4}$$

and using (2.4) we find the normalized wiggler field

$$\bar{b}_w = \frac{\lambda_w e B_w}{2\pi m c^2} = 1.0 - 0.75 z^*\tag{4.5}$$

The resonant energy is defined by (2.6), which for $z^* = 0$ gives an incoming electron energy

$$\gamma_r(0) = \sqrt{\frac{\lambda_w}{\lambda_s} \frac{1 + \bar{b}_w^2(0)}{2}} = 15.8\tag{4.6}$$

Thus, the accelerator must generate electrons with energies of $E = 8.08$ MeV. The normalized resonant energy is given by

$$\bar{\gamma}_r(z^*) = \sqrt{\frac{\bar{k}_s (1 + \bar{b}_w^2)}{2}} = \sqrt{1.03 - 0.77 z^* + 0.29 z^{*2}} \quad (4.7)$$

A constraint is placed on the optical power density in order to obtain a desired change in γ ; this is given in [7] as

$$k_w L a_s \geq \frac{1}{2} \eta_b \frac{2 - \eta_b}{1 - \eta_b} \quad (4.8)$$

where $a_s = e A_s / m c^2$ is the dimensionless vector potential and η_b is the bucket efficiency, defined by

$$\eta_b = \frac{\gamma(0) - \gamma(L)}{\gamma(0)} \quad (4.9)$$

The leading order solution for γ is given by $\varepsilon^{-\alpha/2} \bar{\gamma}_r$ so that the bucket efficiency has an approximate value

$$\eta_b = 1 - \sqrt{\frac{1 + \bar{b}_w^2(1)}{1 + \bar{b}_w^2(0)}} = 0.27 \quad (4.10)$$

Using this numerical result and noting that $k_w L = \varepsilon^{-1}$, we rewrite (4.8) to get a constraint on the signal field amplitude

$$\bar{b}_s \geq 0.32 \varepsilon^{(1-\alpha)} \bar{k}_s \quad (4.11)$$

which is easily satisfied for any $\bar{b}_s$ of $O(1)$ for the case $\alpha = 2/3$; we therefore choose

$$\bar{b}_s(0) = 1.0 \quad (4.12)$$

for convenience.

The signal field evolution is governed by the parameter κ which is dependent on the electron beam density. We can find the beam current necessary for an FEL

with the above parameters using the additional formulas specified in [7] for the minimum circulating peak power, P_m

$$P_m = 2 k_w^2 L^2 a_s^2 \left(\frac{m^2 c^5}{e^2} \right) \quad (4.13)$$

and for the required peak beam current, I_e

$$Q I_e = \left(\frac{e P_m}{m c^2 \gamma_i \eta_e} \right) \quad (4.14)$$

where Q is the cavity quality factor and $\eta_e \approx .2$ is the electronic efficiency. For this model we find

$$Q I_e = 2.4 \times 10^6 \text{ Amps} \quad (4.15)$$

Using the leading order expression for the current

$$I_e = \pi r_e^2 n_e e c \quad (4.16)$$

where r_e is the electron beam radius, we find the necessary electron beam density

$$n_e = \frac{I_e}{\pi r_e^2 e c} = 1.6 \times 10^8 \text{ cm}^{-3} \quad (4.17)$$

where we have used typical values for $Q = 10^8$ and $r_e = 1 \text{ mm}$. The wave evolution parameter κ is then given by

$$\kappa = \frac{\omega_p^2}{\omega_s^2} = \frac{\lambda_s^2 n_e e^2}{\pi m c^2} = 1.5 \times 10^{-9} \quad (4.18)$$

which gives the nearly constant gain rate,

$$\kappa = \bar{\omega} \hat{\varepsilon}^7 \quad (4.19)$$

for the value $\bar{\omega} = .32$. This is sufficient to ignore the space-charge force and gives the signal field amplitude from equation (3.34) for the parameter values specified above.

Using these parameter values, we proceed to verify our analytical solutions. In Figures 3(a) and 3(b), we plot the numerical solution of equations (2.12) of the original system for ϕ and $\bar{\gamma}$, and the asymptotic solution given by (3.22) for the initial

conditions $\bar{\gamma} = \bar{\gamma}_r(0)$ and $\phi = \pi/3$. For this case escape does not occur and the difference between the two solutions is imperceptible.

(a)

(b)

Figure 3. Asymptotic Solution for FEL Electron

If we choose initial conditions farther out in the potential well, for example, $\bar{\gamma} = \bar{\gamma}_r(0)$ and $\phi = 2\pi/3$, we find that escape does occur.

(a)

(b)

Figure 4. Asymptotic Solution with Escape

The asymptotic solutions (Figures 4(a) and 4(b)) exhibit slightly more error than for the previous initial conditions, but are still excellent to the point of escape. Since our asymptotic solutions fail at the point of escape, we use the escape condition (3.27) to predict this distance. In Figure 5, we plot the function $k_{CR}(z^*)$ given by (3.26), noting the distance z_{CR}^* where $k_{CR} = 1$. We also scale and plot the numerical solution for ϕ showing when escape actually does occur, and see that the escape distance predicted by (3.27) is quite accurate.

Figure 5. Escape Condition for FEL Electron

The function $k_{CR}(z_{CR}^*)$ can also be used to calculate the critical interval, the range of initial phases ϕ_0 for which the electron remains in the potential well over the entire length of the FEL. In Figure 6, we plot z_{CR}^* as a function of the initial phase $-\pi < \phi_0 < \pi$.

Figure 6. Critical Interval of Initial Phases

Notice that a certain range of initial phases nearest the center of the potential well will not escape before the end of the wiggler. All of these electrons will transfer $(\gamma_r(0) - \gamma_r(L))mc^2$ MeV to the signal field to leading order. For this model, the critical interval given by the range

$$-1.62 < \phi_0 < 1.71 \quad (4.20)$$

results in sustained resonance for the entire wiggler length. This interval is 53 percent of the total 2π width for initial phases, compared with a critical interval of $-0.4 < \phi_0 < 1.8$, or 35 percent for the $\alpha = 1$ test case considered in [8] – a substantial improvement.

Because of the explicit nature of the asymptotic solution it is possible to use the final analytical results to improve certain features of FEL design. For example, in [7] it is stated that *in principle* one could computationally solve the equations of motion for a large number of particles to choose the optimum functional form for $B_w(Z)$

and $\lambda_w(Z)$. Using the the escape condition (3.27), this optimization is easily realizable since no numerical integrations are required. Maximization of an energy transfer integral (measuring the total energy lost from the electron beam in its passage through the wiggler)

$$E_T = \int_0^{2\pi} [\bar{\gamma}(0) - \bar{\gamma}(L; \phi_0)] \sigma(\phi_0) d\phi_0 \quad (4.21)$$

with respect to $B_w(Z)$ and $\lambda_w(Z)$, where σ is an initial phase distribution function and ϕ_0 is the initial phase, is just one potential use in FEL design. The bucket efficiency specified in (4.10) gives a maximum value for this integral of 27 percent of the entire initial beam energy, if all electrons remain trapped for the full length of the laser. Using the approximation $\bar{\gamma} \approx \bar{\gamma}_r$ and a uniform initial phase distribution, the model presented here gives an energy transfer of fully 20 percent. (Of course, the $O(\hat{\epsilon})$ terms and the initial energy distribution will affect this result.) In addition, the solution (3.22) can give the form of the final energy distribution of the electron beam, a useful piece of information should one desire to recirculate the electron beam back through the wiggler. These are just two possibilities for potential use in FEL design.

These results can also be modified to account for other higher order effects, such as non-uniformity in the wiggler field and the influence of space-charge forces, achieving qualitatively similar results with the same ease of evaluation and usefulness as the results obtained here.

In this research, we have applied the mathematical techniques developed in the previous mathematical research [1], [2], and [5] to find a highly accurate asymptotic solution for the equations of motion of an electron in a lower energy, longer wavelength FEL. Coupled with the enhancement of wiggler tapering, these results show the significant design advantages for this class of lasers.

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