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**Covers by Polars
of
Arrangements**

by

B. Curtis Eaves and Alan J. Hoffman

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COVERS BY POLARS OF ARRANGEMENTS

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Abstract: For a collection of hyperplanes passing through the origin in euclidean space let $\mathcal{S}$ be the induced subdivision. Let $\mathcal{T}$ be the collection of polars of the full cells of $\mathcal{S}$. If only the origin lies in all hyperplanes, $\mathcal{T}$ forms a $k \sim$ fold cover of euclidean space. If, in addition, the collection of hyperplanes is of size m and is regular, then k is $m - 1$ choose $n - 1$. Other similar results relate to spheres, hemispheres, and linear programming.

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1. Introduction

Let $\mathcal{A}$ be a collection of m distinct hyperplanes in R^n all passing through the origin, we refer to such $\mathcal{A}$ as an m -arrangement in R^n . Let $\mathcal{S}(R^n, \mathcal{A})$ be the subdivision of R^n induced by $\mathcal{A}$ and let $\mathcal{T}(R^n, \mathcal{A})$ be the collection of all faces of polars of the full cells of $\mathcal{S}(R^n, \mathcal{A})$. We refer to $\mathcal{T}(R^n, \mathcal{A})$ as the polar division of R^n induced by $\mathcal{A}$. Under the assumption that $\mathcal{A}$ is pointed, that is, $\cap \mathcal{A} = \{0\}$, we show that $\mathcal{T}(R^n, \mathcal{A})$ forms a k -fold cover of an open dense subset of R^n for some $k = 1, 2, \dots$. Under the assumption that $\mathcal{A}$ is pointed and regular we show that k equals $\binom{m-1}{n-1}$. If S is an s -sphere centered at the origin in R^n and in regular position with $\mathcal{A}$ we show that S meets

$$2 \sum_{i=1}^{s+1} \binom{m-1}{n-i}$$

full cells of $\mathcal{T}(R^n, \mathcal{A})$. If H is an s -hemisphere centered at the origin in R^n and in regular position with $\mathcal{A}$ we show that H meets

$$\binom{m-1}{n-1} + \sum_{i=1}^s \binom{m}{n-i}$$

full cells of $\mathcal{T}(R^n, \mathcal{A})$. Some results for linear programming are obtained by interpreting the full cells of $\mathcal{S}(R^n, \mathcal{A})$ as feasible regions of linear programs and interpreting the spheres S and hemispheres H as collections of objective vectors.

In Section 2 we lay out our tools as cells, hyperplanes, subdivisions $\mathcal{S}(R^n, \mathcal{A})$, and Buck's Theorem. In Section 3 we construct the polar division $\mathcal{T}(R^n, \mathcal{A})$ and show that it is a k -fold cover of an open dense subset of R^n . In Section 4 the number k is calculated under a regularity assumption. In Sections 5 and 6 results are extended to spheres, hemispheres, and linear programming. By reference to Lemma 1 and Lemma 2.1 we mean Lemma 1 of that section and of Section 2, respectively.

2. Cells and Subdivisions

For $n = 0, 1, 2, \dots$ let $R^n \equiv R \times R \times \dots \times R$ denote the n -tuples of reals. Given a set σ in R^n , a linear combination of points in σ is a point x in R^n of the form $\sum_{i=1}^k \theta_i x^i$ where k is a nonnegative integer, the x^i 's are vectors in σ and the θ_i 's are scalars in R . If, in addition, (1) $\sum_{i=1}^k \theta_i = 1$, (2) $\sum_{i=1}^k \theta_i = 0$, (3) $\sum_{i=1}^k \theta_i = 1$ and each $\theta_i \geq 0$, or (4) each $\theta_i \geq 0$, we call x (1) an affine, (2) a tangential, (3) a convex or (4) conic combination, respectively. The empty sum ($k = 0$) is regarded as the zero vector. The linear, affine, tangential, convex and conic hull of a set σ in R^n , denoted $\text{lin}(\sigma)$, $\text{aff}(\sigma)$, $\text{tng}(\sigma)$, $\text{conv}(\sigma)$, and $\text{cone}(\sigma)$, are the sets of all corresponding combinations of elements of σ . In particular, we have that $\text{lin}(\emptyset) = \text{tng}(\emptyset) = \text{cone}(\emptyset) = \{0\}$ and $\text{conv}(\emptyset) = \text{aff}(\emptyset) = \emptyset$. A set σ in R^n is called linear, affine, convex or conic if it is equal to its linear, affine, convex or conic hull, respectively. We frequently call a nonempty affine set a plane and, when we want a linear set to be the universe we call it a space. The dimension of a nonempty convex set σ is defined as the maximal number of linearly independent vectors in $\text{tng}(\sigma)$; the dimension of the empty set is taken as negative unity. With a prefix as in " ℓ -object" we indicate the dimension, namely ℓ , of the object. Through out the paper let N denote an n -space, that is, a universe which is a linear set of dimension $n = 0, 1, 2, \dots$; of course $N \subseteq R^d$ for some $d \geq n$. We use R^n when we have a choice, but occasionally the generality of N is required.

By a hyperplane in an n -plane we mean an $(n - 1)$ -plane in the n -plane.. A hyperplane separates the plane into two closed sets called halfspaces of the plane. A cell in a plane is defined to be a finite intersection of half spaces in the plane; a cell may be empty or not and bounded or not. By a pointed cell we mean a cell with an extreme point.

Let op be an operator which transforms subsets of R^n to subsets of R^n . If $\mathcal{S}$ is a collection sets of R^n by $op(\mathcal{S})$ we denote the collection $\{op(\sigma) : \sigma \in \mathcal{S}\}$. Given a cell σ , by $F(\sigma)$ we denote the facial completion of σ , that is, $F(\sigma)$ is the collection of all nonempty faces of σ including σ itself. For a collection of cells $\mathcal{S}$, by $F(\mathcal{S})$ we denote the collection of all nonempty faces of cells of $\mathcal{S}$.

Let $\mathcal{A} = \{h_1, \dots, h_m\}$ be an ordered collection of m distinct hyperplanes h_i in N all passing through the origin, that is, $0 \in \cap \mathcal{A}$. Such $\mathcal{A}$ is referred to as an m -arrangement in N . An arrangement is defined to be pointed, if only the origin lies in all of the hyperplanes, that is, $\cap \mathcal{A} = \{0\}$. Let $\mu \equiv \{1, \dots, m\}$ be the index set for the hyperplanes. For an arrangement $\mathcal{A}$ in N we define the redundancy to be the

size of the largest subset $B \subseteq A$ such that

$$N \cap (\cap A) = N \cap (\cap (A \setminus B))$$

where $A \setminus B$ denotes the hyperplanes in A but not B . The arrangement A is defined to be regular, if every subset of A of size n or less has zero redundancy.

Let A be an arrangement in the n -space N . Define $\mathcal{S}(N, A)$ to be the subdivision of N induced by A , that is, $\mathcal{S}(N, A)$ is the facial completion of all n -cells of form

$$\bigcap_{i \in \mu} h_i^+$$

where h_i^+ is either of the two halfspaces of N as separated by h_i . Let $\mathcal{S}^i(N, A)$ denote the collection of i -cells of $\mathcal{S}(N, A)$ for $i = 0, \dots, n$. We have $N = \cup \mathcal{S}(N, A)$, $\cup A = \cup \mathcal{S}^{n-1}(N, A)$, and $\mathcal{S}(N, A) = \cup_{i=0}^n \mathcal{S}^i(N, A)$. Cells of $\mathcal{S}(N, A)$ are pointed, if and only if A is pointed.

A matrix is defined to have full row rank if the rank equals the number of rows; similarly for columns. A matrix is defined to have full rank, if it has full row or column rank. A matrix is defined to be row regular, if each submatrix of rows has full rank; similarly for columns. A matrix is defined to be regular (that is, nonsingular), if it is both row and column regular.

Lemma 1: Let A be an m -arrangement in $N \subseteq R^d$ and let B be a regular $d \times d$ matrix. Then BA is an m -arrangement in BN and

$$B\mathcal{S}(NA) = \mathcal{S}(BN, BA)$$

Proof: As B is regular BN is an n -linear set. As $h_i \subseteq N$ we have $Bh_i \subseteq BN$ and as B is regular the Bh_i are distinct hyperplanes in BN . An n -cell of $\mathcal{S}(N, A)$ is of form

$$\cap_{i \in \mu} h_i^+$$

where h_i^+ is either of the two halfspaces of N as separated by h_i . We have

$$B(\cap_{i \in \mu} h_i^+) = \cap_{i \in \mu} Bh_i^+ = \cap_{i \in \mu} (Bh_i)^+$$

where $(Bh_i)^+$ is one of the two halfspaces of BN as separated by Bh_i . Further

$$BF(\cap_{i \in \mu} h_i^+) = F(\cap_{i \in \mu} (Bh_i)^+)$$

and the results follows. $\square$

For a space $N \subseteq R^d$ there is a regular (even orthogonal) matrix B where $BN = R^n \times \{0\}^{d-n}$. Hence, as $\mathcal{S}(N, \mathcal{A})$ is a linear transformation of $\mathcal{S}(R^n, BA)$ where BA an m -arrangement in R^n , we can draw certain conclusions about $\mathcal{S}(N, \mathcal{A})$ from those of $\mathcal{S}(R^n, B)$. We use $\mathcal{S}(R^n, \mathcal{A})$ when it is our choice, but occasionally $\mathcal{S}(N, \mathcal{A})$ is required with $N \neq R^n$.

Given an m -arrangement $\mathcal{A}$ in R^n and $m \times n$ matrix A is defined to be a matrix of normals of $\mathcal{A}$ is for each $i \in \mu$ the i th row of A , denoted A_i , is normal to h_i , that is,

$$h_i = \{x : A_i x = 0\}$$

Such A is unique up to nonzero scales of rows. If A is a matrix of normals for the arrangement $\mathcal{A}$, then A has full column rank, if and only if $\mathcal{A}$ is pointed, and A is row regular, if and only if $\mathcal{A}$ is regular. If $\mathcal{A}$ is pointed, then $m \geq n$, and if $\mathcal{A}$ is regular and $m \geq n$, then $\mathcal{A}$ is pointed. Let B be a regular $n \times n$ matrix, then A is a matrix of normals for $\mathcal{A}$ in R^n if and only if AB^{-1} is a matrix of normals for BA .

Several of our calculations are clarified though the use of the collection $\mathcal{J}_m$ of all $m \times m$ signed identity matrices, that is, $J \in \mathcal{J}_m$ if and only if $J = \text{diag}(\pm 1, \dots, \pm 1)$. Of course, $\#\mathcal{J}_m = 2^m$, where we use $\#$ to denote the cardinality of a set. Let A be a fixed matrix of normals for the m -arrangement $\mathcal{A}$ in R^n . Then σ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$, if and only if there is a signed identity J where $JA x > 0$ has a solution x and where $\sigma = \{x : JA x \geq 0\}$.

Lemma 2: Let B be an $n \times m$ matrix with nonzero columns and let J be an $n \times n$ signed identity matrix. If the cells

$$\{B\lambda : \lambda \geq 0\} \quad \{BJ\lambda : \lambda \geq 0\}$$

are pointed and one contains the other, then J is the identity. $\square$

Let $\mathcal{A}$ be an m -arrangement in R^n and τ be a cell in $\mathcal{S}(R^n, \mathcal{A})$. The normal index set of τ is defined to be the subset α of μ where $\tau \subseteq h_i$ for $i \in \alpha$. In this case there is a matrix of normals A of $\mathcal{A}$ where $Ax > 0$ has a solution, where $\tau = \{x : A_\alpha x = 0, A_\beta x \geq 0\}$ with $\beta = \mu \setminus \alpha$, and where $A_\alpha x = 0$ with $A_\beta x > 0$ has a solution. In this case τ is an ℓ -cell, if and only if $\text{rank}(A_\alpha) = n - \ell$. If $\alpha \subseteq \mu$, $\beta = \mu \setminus \alpha$, A a matrix of normals of $\mathcal{A}$, τ an ℓ -cell of $\mathcal{S}(R^n, \mathcal{A})$, and $\tau = \{x : A_\alpha x = 0, A_\beta x \geq 0\}$, then α is a subset of the normal index set of τ ; however, if $\mathcal{A}$ is regular and $1 \leq \ell \leq n$, then, α is the normal index set of τ .

$\mathcal{S}(R^n, \mathcal{A})$ is symmetric about the origin, for example: Let σ be a n -cell of $\mathcal{S}(R^n, \mathcal{A})$, τ a ℓ -face of σ , and α the normal index set of τ . Then $-\sigma$ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$, $-\tau$

is an ℓ -face of σ , and α is the normal index set of $-\tau$. The next lemma addresses a related but different matter.

Lemma 3: Let A be an arrangement in R^n , σ be an n -cell of $\mathcal{S}(R^n, A)$, A a matrix of normals of A with $\sigma = \{x : Ax \geq 0\}$, τ a nonempty ℓ -face of σ , α the normal index set of τ , and $\beta = \mu \setminus \alpha$. Then $\hat{\sigma} = \{x : A_\alpha x \geq 0, -A_\beta x \geq 0\}$ is an n -cell of $\mathcal{S}(R^n, A)$, $-\tau = \{x : A_\alpha x = 0, -A_\beta x \geq 0\}$ is an ℓ -face of $\hat{\sigma}$, and the normal index set of $-\tau$ is α .

Proof: Let x solve $Ax > 0$ with y solve $A_\alpha y = 0$ with $A_\beta y > 0$. Let $\hat{x} = -y + \theta(x - y)$ and $\hat{y} = -y$. Then $A_\alpha \hat{x} = -A_\alpha y + \theta A_\alpha x - \theta A_\alpha y = \theta A_\alpha x > 0$ for all positive θ . Further $-A_\beta \hat{x} = A_\beta y - \theta A_\beta x + \theta A_\beta y > 0$ for all small positive θ . Thus $\hat{\sigma}$ is an n -cell of $\mathcal{S}(R^n, A)$. As $A_\alpha \hat{y} = 0$ and $-A_\beta \hat{y} > 0$ we see that $-\tau$ is a ℓ -face of $\hat{\sigma}$ with α as the normal index set. $\square$

Let $\mathcal{S}$ be a finite collection of cells in the n -plane N . We define $\mathcal{S}$ to be a $k \sim$ fold cover of N if each point of $N \setminus E$ lies in exactly k n -cells of $\mathcal{S}$ where the exceptional set E is the carrier of the boundary points of the n -cells of $\mathcal{S}$, that is,

$$E \equiv \cup \{bd(\sigma) : \sigma \in \mathcal{S}, \dim(\sigma) = n\}$$

Observe that $N \setminus E$ is open and dense in N . Of course, $\mathcal{S}(R^n, A)$ is a $1 \sim$ fold cover of R^n ; actually a stronger statement can be made about $\mathcal{S}(R^n, A)$, but it is of no interest here. We shall show that if A is pointed the polar of $\mathcal{S}(R^n, A)$ forms a $k \sim$ fold cover of R^n . If A is a regular pointed arrangement we can calculate k by employing Buck's formula for the number of n -cells of $\mathcal{S}(R^n, A)$. We define m choose n by

$$\binom{m}{n} \equiv \begin{cases} \frac{m!}{n!(m-n)!} & \text{if } 0 \leq n \leq m \\ 0 & \text{otherwise} \end{cases}$$

Theorem 4 (Buck [1]): If A is a regular m -arrangement in R^n the number of n -cells in $\mathcal{S}(R^n, A)$ is given by

$$\#\mathcal{S}^n(R^n, A) = 2 \sum_{i=0}^{n-1} \binom{m-1}{i}$$

Proof: See Buck [1], Winder [5], or Eaves [2]. $\square$

For $x \in R^n$ let $x \neq 0$ denote that x has no zero components. Define $\text{sgn}(x) = (\text{sgn}(x_1), \dots, \text{sgn}(x_n))$ where $\text{sgn}(x_i) = -1, 0$, or 1 as $x_i < 0, x_i = 0$, or $x_i > 0$. A matrix version of Buck's result is given below.

Corollary 5: Let A be a row regular $m \times n$ matrix. Then

$$\#\{\operatorname{sgn}(Ax) \neq 0 : \forall x\} = \#\{J \in \mathcal{J}_m : \exists x, JAx > 0\} = 2 \sum_{i=0}^{n-1} \binom{m-1}{i}$$

Proof: Let $\mathcal{A}$ be the arrangement with A as the matrix of normals, and note such $\mathcal{A}$ is regular. Each possible $\operatorname{sgn}(Ax) \neq 0$ corresponds to a cell of $\mathbb{S}^n(R^n, \mathcal{A})$. Also $\operatorname{sgn}(Ax) \neq 0$, if and only if there is a signed identity with $JAx > 0$. $\square$

3. Polar Divisions

Given an m -arrangement $\mathcal{A}$ in R^n we show that the polars of n -cells of the subdivision $\mathcal{S}(R^n, \mathcal{A})$ form a $k \sim$ fold cover of R^n , however, let us get our tools in order for polars of cells, then move to consider polars of subdivisions.

Given a set σ in the n -space N define the (positive) polar relative to N , denoted σ^* , by

$$\sigma^* \equiv \{y \in N : x \in \sigma \rightarrow x^T y \geq 0\}$$

where $\rightarrow$ means implies. Clearly, $\sigma \subseteq \tau$ implies $\sigma^* \supseteq \tau^*$. If $\mathcal{S}$ is a collection of sets define $\mathcal{S}^* = \{\sigma^* : \sigma \in \mathcal{S}\}$. We catalogue some required properties of the polar. The next lemma involves two spaces, namely N and BN where B is an orthogonal matrix, that is, $B^T B = I$.

Lemma 1: Given the space $N \subseteq R^d$, a $d \times d$ orthogonal matrix B , the space BN , and a set σ in N , let $*$ and $\oplus$ compute the polars relative to N and BN , then

$$B\sigma^* = (B\sigma)^\oplus$$

$$\begin{aligned} \text{Proof: } (B\sigma)^\oplus &= \{x \in BN : y \in B\sigma \rightarrow y^T x \geq 0\} \\ &= B\{x \in N : y \in \sigma \rightarrow (By)^T (Bx) \geq 0\} \\ &= B\{x \in N : y \in \sigma \rightarrow y^T x \geq 0\} = B\sigma^* \end{aligned}$$

□

The point of this last lemma is to show that we can understand polars in the space N by understanding polars in R^n , since there is an orthogonal matrix B with $BN = R^n \times \{0\}^{d-n}$. Now we return to the space R^n and polars relative to R^n .

Lemma 2: If σ is a linear set in R^n , then σ^* is the orthogonal complement $\sigma^\perp$ of σ in R^n , that is, $\sigma^* = \sigma^\perp \equiv \{y \in R^n : x \in \sigma \rightarrow y^T x = 0\}$. If σ is a linear subset of the closed convex set τ in R^n , then $\tau^* = \{y \in \sigma^\perp : x \in \tau \cap \sigma^\perp \rightarrow x^T y \geq 0\}$. □

Lemma 3: For a set σ in R^n and for an $n \times n$ regular matrix B we have $\sigma^* = B^T(B\sigma)^*$.

$$\begin{aligned} \text{Proof: } (B\sigma)^* &= \{y : z \in B\sigma \rightarrow z^T y \geq 0\} = \{y : x \in \sigma \rightarrow x^T B^T y \geq 0\} \\ &= \{(B^T)^{-1} z : x \in \sigma \rightarrow x^T z \geq 0\} = (B^T)^{-1} \{z : x \in \sigma \rightarrow x^T z \geq 0\} = (B^T)^{-1} \sigma^* \end{aligned}$$

□

We can conclude that $(\mathcal{S}^n(R^n, \mathcal{A}))^* = B^T(B\mathcal{S}^n(R^n, \mathcal{A}))^*$.

Lemma 4: If σ is a conic cell in R^n , then the polar of the polar of σ , denoted σ^{**} , is σ . □

Theorem 5: Let σ be a pointed conic n -cell in R^n and let $\sigma^* = \{y \in R^n : x \in \sigma \rightarrow x^T y \geq 0\}$ be the polar of σ . Then σ^* is a pointed conic n -cell. Furthermore, there is a one-to-one and onto map $\delta_\sigma : F(\sigma) \rightarrow F(\sigma^*)$ from the nonempty faces of σ to those of σ^* with the properties:

- (1) $\delta_\sigma(\tau) = \{y \in \sigma^* : x \in \tau \rightarrow x^T y = 0\}$
- (2) $\delta_\sigma^{-1}(\gamma) = \{x \in \sigma : y \in \gamma \rightarrow y^T x = 0\}$
- (3) $\dim(\tau) + \dim(\delta_\sigma(\tau)) = n$
- (4) $\tau \subseteq \rho$ if and only if $\delta_\sigma(\tau) \supseteq \delta_\sigma(\rho)$

where τ and ρ are any nonempty faces of σ , and γ is any nonempty face of σ^* . $\square$

For a pointed conic n -cell σ , we refer to δ_σ as in the theorem above as the polar face map of σ . For a pointed arrangement $\mathcal{A}$ in R^n we shall be interested in the δ_σ as σ ranges over the n -cells of $\mathcal{S}(R^n, \mathcal{A})$. Let A be a matrix of normals for the pointed m -arrangement $\mathcal{A}$ in R^n .

Lemma 6: Let $\sigma = \{x : Ax \geq 0\}$ be an n -cell of $\mathcal{S}(R^n, \mathcal{A})$. Then $\sigma^* = \{A^T y : y \geq 0\}$. Let δ_σ be the polar face map of σ . If τ is a nonempty face of σ , then

$$\delta_\sigma(\tau) = \{(A_\alpha)^T y : y \geq 0\}$$

where α is the normal index set of τ . If $\alpha \subseteq \mu$ and $\tau = \{(A_\alpha)^T y : y \geq 0\}$ is a face of σ^* , then $\delta_\sigma^{-1}(\tau) = \{x : A_\alpha x = 0, A_\beta x \geq 0\}$ where $\beta = \mu \setminus \alpha$.

Proof: For $\sigma = \{x : Ax \geq 0\}$ the form of σ^* follows from the matrix alternative theorem. If τ is a face of σ with normal index set α then $\delta_\sigma(\tau) = \{A^T y : y \geq 0, A_\alpha x = 0, A_\beta x \geq 0 \rightarrow x^T A^T y = 0\}$ where $\beta = \mu \setminus \alpha$. As there is an $\bar{x}$ with $A_\alpha \bar{x} = 0$ and $A_\beta \bar{x} > 0$ we have $y_\beta = 0$ and $\delta_\sigma(\tau) = \{(A_\alpha)^T z : z \geq 0\}$. If $\gamma = \{(A_\alpha)^T z : z \geq 0\}$ is a face of σ^* , then $\delta_\sigma^{-1}(\gamma) = \{x : Ax \geq 0, z \geq 0 \rightarrow z^T A_\alpha x = 0\} = \{x : A_\alpha x = 0, A_\beta x \geq 0\}$ where $\beta = \mu \setminus \alpha$. $\square$

Note that in the second conclusion of the lemma above the normal index set of $\delta_\sigma^{-1}(\gamma)$ may properly contain α .

Given the arrangement $\mathcal{A}$ in the n -linear set N define the polar division of N induced by $\mathcal{A}$, denoted $\mathcal{T}(N, \mathcal{A})$, to be the facial completion of the polars of the n -cells of $\mathcal{S}(N, \mathcal{A})$ in N , that is

$$\mathcal{T}(N, \mathcal{A}) \equiv F((\mathcal{S}^n(N, \mathcal{A}))^*)$$

See Figure 1 where $N = R^2$. Let $\mathcal{T}^i(N, \mathcal{A})$ denote the collection of i -cells of $\mathcal{T}(N, \mathcal{A})$ for $i = 0, 1, \dots, n$. As we shall see $\mathcal{T}(N, \mathcal{A})$ is very much like a subdivision of N except

that it is a $k \sim$ fold cover rather than a $1 \sim$ fold cover of N for some $k = 0, 1, 2, \dots$. See Figure 1 again where $k = 2$.

Lemma 7: Given the n -space $N \subseteq R^d$, an arrangement $\mathcal{A}$ in N , and a $d \times d$ orthogonal matrix B then

$$B\mathcal{T}(N, \mathcal{A}) = \mathcal{T}(BN, B\mathcal{A})$$

Proof: Let $\star$ and $\odot$ compute polars relative to N and BN . We have

$$\begin{aligned} \mathcal{T}(BN, B\mathcal{A}) &= \mathbf{F}(\{\sigma^\odot : \sigma \in \mathcal{S}^n(BN, B\mathcal{A})\}) \\ &= \mathbf{F}(\{\sigma^\odot : \sigma \in B\mathcal{S}^n(N, \mathcal{A})\}) \\ &= \mathbf{F}(\{(B\sigma)^\odot : \sigma \in \mathcal{S}^n(N, \mathcal{A})\}) \\ &= \mathbf{F}(\{B\sigma^\star : \sigma \in \mathcal{S}^n(N, \mathcal{A})\}) \\ &= B\mathbf{F}(\{\sigma^\star : \sigma \in \mathcal{S}^n(N, \mathcal{A})\}) \end{aligned}$$

using Lemmas 2.1 and 1. $\square$

Again the point of the lemma just above is that we can understand the polar division in any space N by understanding it in R^n .

Lemma 8: Let $\mathcal{A}$ be an arrangement in R^n and B be an regular $n \times n$ matrix. Then $\mathcal{T}(R^n, \mathcal{A}) = B^T \mathcal{T}(R^n, B\mathcal{A})$

Proof: $\mathbf{F}((\mathcal{S}^n(R^n, \mathcal{A}))^\star) = \mathbf{F}(B^T(B\mathcal{S}^n(R^n, \mathcal{A}))^\star) = \mathbf{F}(B^T(\mathcal{S}^n(R^n, B\mathcal{A}))^\star) = B^T \mathbf{F}((\mathcal{S}^n(R^n, B\mathcal{A}))^\star)$ where $\mathbf{F}$ is the facial completion operator. $\square$

As a preview of the nature of the polar division we can give a transparent proof that $\mathcal{T}(R^n, \mathcal{A})$ is a subdivision of R^n when $\mathcal{A}$ is a pointed m -arrangement in R^n with $m = n$. Observe that for the n -arrangement $\mathcal{I} \equiv \{x : x_i = 0\} : i = 1, \dots, n$ in R^n we have $\mathcal{S}(R^n, \mathcal{I}) = \mathcal{T}(R^n, \mathcal{I})$ which is the orthant subdivision of R^n . Now we can see a (necessary and) sufficient condition for $\mathcal{T}(R^n, \mathcal{A})$ to be a subdivision.

Lemma 9: Let $\mathcal{A}$ be a pointed n -arrangement in R^n . Then $\mathcal{T}(R^n, \mathcal{A})$ is a subdivision of R^n .

Proof: Let A be the matrix of normals of $\mathcal{A}$. Then from Lemma 8

$$\begin{aligned} \mathcal{T}(R^n, \mathcal{A}) &= A^T \mathcal{T}(R^n, AA) \\ &= A^T \mathcal{T}(R^n, \mathcal{I}) = A^T \mathcal{S}(R^n, \mathcal{I}) \end{aligned}$$

$$R^2 \quad \mathcal{A} = \{h1, h2, h3\}$$

$$n = 2 \quad m = 3$$

Polar Division $\mathcal{I}(R^n, \mathcal{A})$, 2 ~ Fold Cover

Figure 1

Thus $\mathcal{T}(R^n, \mathcal{A})$ is A^T times the subdivision $\mathcal{S}(R^n, \mathcal{I})$, that is $\mathcal{T}(R^n, \mathcal{A})$ is a subdivision of R^n . $\square$

However, as we shall shortly see, $\mathcal{T}(R^n, \mathcal{A})$ is not a subdivision of R^n for an m -arrangement with $m \neq n$ or if $\mathcal{A}$ is not pointed. If $\cap \mathcal{A}$ is more than the origin, the polar division lies in the orthogonal complement of $\cap \mathcal{A}$ as the next lemma shows.

Lemma 10: Let $\mathcal{A}$ be an m -arrangement in R^n , let $L = \cap \mathcal{A}$ be the intersection of the hyperplanes of $\mathcal{A}$, and $L^\perp$ be the orthogonal complement of L , then

$$\mathcal{T}(R^n, \mathcal{A}) = \mathcal{T}(L^\perp, L^\perp \cap \mathcal{A})$$

where $L^\perp \cap \mathcal{A}$ is a pointed m -arrangement in $L^\perp$.

Proof: Let ℓ be the dimension of $L^\perp$. Let σ be an n -cell of $\mathcal{S}(R^n, \mathcal{A})$, then there is an ℓ cell τ in $\mathcal{S}(L^\perp, L^\perp \cap \mathcal{A})$ with $\sigma = \tau + L$ and we have $\sigma^* = \{x \in L^\perp : y \in \tau \rightarrow y^T x \geq 0\}$. Conversely, if τ is an ℓ -cell of $\mathcal{S}(L^\perp, L^\perp \cap \mathcal{A})$ then $\tau + L$ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ with $(\tau + L)^* = \{x \in L^\perp : y \in \tau \rightarrow y^T x \geq 0\}$. If $hi \cap L^\perp = hj \cap L^\perp$, then with $L \subseteq hi$ and $L \subseteq hj$ we have $(hi)^\perp + L = (hj)^\perp + L$ and $(hi)^\perp \subseteq L^\perp$ and $(hj)^\perp \subseteq L^\perp$. As each point in $(hi)^\perp + L = (hj)^\perp + L$ has a unique representation in $L^\perp + L$, we see that $(hi)^\perp = (hj)^\perp$ or $hi = hj$. Thus $\mathcal{A} \cap L^\perp$ is an m -arrangement. $\square$

From Lemma 8 we see that in the study of polar divisions that we may confine our study to polar divisions of pointed arrangements. In R^n for an arrangement $\mathcal{A}$ and set τ define the generator index set of τ to be the subset α of μ where some normal of hi is in τ for $i \in \alpha$, that is, $i \in \alpha$ if and only if $(hi)^\perp \cap \tau \neq \emptyset$. The next lemma indicates why we name the generator index set as we do.

Theorem 11: For a pointed m -arrangement $\mathcal{A}$ in R^n , let σ be an n -cell in $\mathcal{S}(R^n, \mathcal{A})$, γ a face of σ , and τ a cell of $\mathcal{T}(R^n, \mathcal{A})$ with $\delta_\sigma(\gamma) = \tau$. Then the normal index set α of γ and the generator index set of τ are equal. Let A be a matrix of normals of $\mathcal{A}$ with $(A_i)^T \in \tau$ for $i \in \alpha$. Then $\tau = \{(A_\alpha)^T y : y \geq 0\}$ and further $\delta_\sigma(\gamma) = \tau$ implies $\sigma \subseteq \{\bar{x} : A_\alpha \bar{x} \geq 0\}$.

Proof: First suppose $\delta_\sigma(\gamma) = \tau$. We can assume $\sigma = \{x : JA x \geq 0\}$ and $\gamma = \{x : (JA)_\alpha x = 0, (JA)_\beta x \geq 0\}$ where α is the normal index set of γ , $\beta = \mu \setminus \alpha$ and J is a signed identity. Let ξ be the generator set of τ . We know that $\delta_\sigma(\gamma) = \{((JA)_\alpha)^T \lambda : \lambda \geq 0\} = \tau$, see Lemma 6. Hence $\alpha \subseteq \xi$. As α is the normal index set of γ there is an $\bar{x}$ with $(JA)_\alpha \bar{x} = 0$ and $(JA)_\beta \bar{x} > 0$. Thus no column of $((JA)_\beta)^T$ lies in the hyperplane

$\{x : x^T \bar{x} = 0\}$, whereas τ lies in this hyperplane. Hence we see that $\beta \cap \xi = \emptyset$, and with $\alpha \subseteq \xi$ we get $\alpha = \xi$. Thus $\tau = \{((JA)_\alpha)^T \lambda : \lambda \geq 0\} = (JA_\xi)^T \lambda : \lambda \geq 0\} = \{(A_\xi)^T \lambda : \lambda \geq 0\}$ where the last equality is due to our selection of A . Thus $J_\alpha^\alpha = I$, see Lemma 2.2. $\square$

Corollary 12: Let τ be an n -cell of $\mathcal{T}(R^n, \mathcal{A})$, then there is exactly one n -cell σ of $\mathcal{S}(R^n, \mathcal{A})$ and face γ of σ , namely, $\gamma = \{0\}$, with $\delta_\sigma(\gamma) = \tau$.

Proof: As $\dim \gamma = n - \dim(\tau)$ we see that $\gamma = \{0\}$, see Theorem 5. If ρ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ with $\delta_\sigma(\{0\}) = \delta_\rho(\{0\})$ then $\sigma = \rho$ by the Theorem 5. Or alternatively, if $\sigma^* = \rho^*$ then $\sigma = \sigma^{**} = \rho^{**} = \rho$; see Lemma 4. $\square$

Lemma 13: If σ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$, τ is an $(n-1)$ -cell of $\mathcal{T}(R^n, \mathcal{A})$, and τ is contained in the boundary of σ^* , then τ is a facet of σ^* .

Proof: Select an n -cell $\hat{\sigma}$ of $\mathcal{S}(R^n, \mathcal{A})$ and 1-cells of γ and $\hat{\gamma}$ of σ and $\hat{\sigma}$ with $\delta_{\hat{\sigma}}(\hat{\gamma}) = \tau \subseteq \delta_\sigma(\gamma) \subseteq \sigma^*$. Select the matrix A of normals of $\mathcal{A}$ so that $\sigma = \{x : Ax \geq 0\}$, and select the signed identity J with $\hat{\sigma} = \{x : JAx \geq 0\}$. Let α and $\hat{\alpha}$ be the normal index sets of γ and $\hat{\gamma}$, in particular,

$$\begin{aligned} \gamma &= \{x : A_\alpha x = 0, A_\beta x \geq 0\} & \beta &= u \setminus \alpha \\ \hat{\gamma} &= \{x : (JA)_{\hat{\alpha}} x = 0, (JA)_{\hat{\beta}} x \geq 0\} & \hat{\beta} &= u \setminus \hat{\alpha} \end{aligned}$$

Then

$$\tau = \{((JA)_{\hat{\alpha}})^T y : y \geq 0\} \subseteq \{(A_\alpha)^T z : z \geq 0\}$$

Thus there is a nonnegative matrix B with

$$((JA)_{\hat{\alpha}})^T = (A_\alpha)^T B$$

As γ and $\hat{\gamma}$ are 1-cells the solutions $\bar{x}$ and $\hat{x}$ to $A_\alpha \bar{x} = 0$ with $A_\beta \bar{x} > 0$ with $(JA)_{\hat{\alpha}} \hat{x} = 0$, $(JA)_{\hat{\beta}} \hat{x} > 0$ are unique up to positive scale. In view of the matrix B we have $\hat{x} = \theta \bar{x}$ for some nonzero θ . Hence all columns of $(A_\beta)^T$ lie strictly to one side of the hyperplane $\{x : x^T \bar{x} = 0\} = \{x : x^T \hat{x} = 0\}$, and similarly for $((JA)_{\hat{\beta}})^T$. As τ is an $(n-1)$ -cell lying in the boundary of the n -cells $\delta_\sigma(\gamma)$ and $\delta_{\hat{\sigma}}(\hat{\gamma})$ it follows that $\alpha = \hat{\alpha}$. Hence

$$\{((JA)_\alpha)^T y : y \geq 0\} \subseteq \{(A_\alpha)^T y : y \geq 0\}$$

and $J_\alpha^\alpha = I$, see Lemmas 2.2 and 10. Thus $\delta_\sigma(\gamma) = \tau$ and τ is a facet of σ^* . $\square$

Lemma 14: Let $\mathcal{A}$ be a pointed m -arrangement in R^n . An $(n-1)$ -cell τ of $\mathcal{T}(R^n, \mathcal{A})$ is a facet of exactly two n -cells of $\mathcal{T}(R^n, \mathcal{A})$ and is the intersection of these two n -cells.

Proof: Let σ be an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ and γ a 1-cell of σ with $\delta_\sigma(\gamma) = \tau$. Select the matrix of normals of $\mathcal{A}$ so that $\sigma = \{x : Ax \geq 0\}$ and let α be the index of normals for γ . If $\hat{\sigma}$ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ and $\hat{\gamma}$ is a 1-face of $\hat{\sigma}$ with $\delta_{\hat{\sigma}}(\hat{\gamma}) = \tau$ then, from Theorem 11 we have $\hat{\sigma} = \{x : A_\alpha x \geq 0, JA_\beta x \geq 0\}$ for some signed identity J and $\beta = \mu \setminus \alpha$. Also α is the normal index set of $\hat{\gamma}$ as well as γ . Let $\bar{x}$ be the unique solution to $A_\alpha \bar{x} = 0$ and $A_\beta \bar{x} > 0$. As α is the normal index set of $\hat{\gamma}$, either $\bar{x}$ or $-\bar{x}$ is in $\hat{\sigma}$. If $\bar{x}$ is in $\hat{\sigma}$ then $J = I$, $\sigma = \hat{\sigma}$, and $\gamma = \hat{\gamma}$. Thus assume $-\bar{x}$ is in $\hat{\sigma}$, and then $J = -I$. From Lemma 2.2 we see that such $\hat{\sigma}$ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ and that $\hat{\gamma}$ is a 1-face of $\hat{\sigma}$. We have $A_\alpha \bar{x} = 0$, $A_\beta \bar{x} > 0$, and $JA_\alpha \bar{x} < 0$, and hence, that $\delta_\sigma(\gamma) \cap \delta_{\hat{\sigma}}(\hat{\gamma}) = \tau$. $\square$

Lemma 2.3 and the proof above also yields the following statement.

Lemma 15: Let σ be an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ and select the matrix of normals A with $\sigma = \{x : Ax \geq 0\}$. Let γ be a 1-face of σ with normal index set α . Then $-\gamma$ is a 1-face of the n -cell $\hat{\sigma} = \{x : A_\alpha x \geq 0, -A_\beta x \geq 0\}$ of $\mathcal{S}(R^n, \mathcal{A})$ where $\beta = \mu \setminus \alpha$ and $\delta_\sigma(\gamma) = \delta_{\hat{\sigma}}(-\gamma)$. $\square$

We show that $\mathcal{T}(R^n, \mathcal{A})$ is a $k \sim$ fold cover of R^n .

Theorem 16: For a pointed m -arrangement $\mathcal{A}$ in R^n , $\mathcal{T}(R^n, \mathcal{A})$ is a $k \sim$ fold cover of R^n .

Proof: Select any two points p and q in $R^n \setminus E$ where the exceptional set is $E = \cup \mathcal{T}^{n-1}(R^n, \mathcal{A})$. Then there is a neighborhood $N(p)$ of p which does not meet E . Thus the number of n -cells of $\mathcal{T}(R^n, \mathcal{A})$ covering points in $N(p)$ is invariant. Now there is a point $\hat{p}$ in $N(p)$ such that the line $[\hat{p}, q]$ from $\hat{p}$ to q does not meet any cells of $\mathcal{T}^{n-2}(R^n, \mathcal{A})$. Beginning with $\hat{p}$ and moving towards q we count the number of n -cells of $\mathcal{T}(R^n, \mathcal{A})$ which contain the point. If on the line $[\hat{p}, q]$ we are in n -cell σ and about the exit through facet τ , from Lemma 14, we see that there is a unique n -cell $\hat{\sigma}$ in $\mathcal{T}(R^n, \mathcal{A})$, distinct from σ , with facet τ and further, $\sigma \cap \hat{\sigma} = \tau$. Thus as we leave σ we enter $\hat{\sigma}$. Just as we leave σ we may also leave other n -cells of $\mathcal{T}(R^n, \mathcal{A})$. However, for each of these we leave through a facet and there is a distinct n -cell which shares this facet. Thus we see that all points on the line $[\hat{p}, q] \setminus E$ are covered by the same number, say k , of n -cells of $\mathcal{T}(R^n, \mathcal{A})$. Hence p and q lie in the same number, namely k , of n -cells of $\mathcal{T}(R^n, \mathcal{A})$. $\square$

We refer to k in the theorem above as the cover fold number of the polar division. As $\mathcal{A}$ is a pointed m -arrangement in R^n obviously $k \geq 1$ in the theorem above,

however, we suspect that the lower and upper bounds of k are given by $m - n + 1 \leq k \leq \binom{m-1}{n-1}$. In the next section, under the assumption that $\mathcal{A}$ is regular we prove that k is $\binom{m-1}{n-1}$.

Our final step in this section is to show that the carrier of the boundaries of the n -cells of the polar division $\mathcal{T}(R^n, \mathcal{A})$ is the carrier of an arrangement, that is

$$\cup \mathcal{T}^{n-1}(R^n, \mathcal{A}) = \cup \mathcal{B}$$

where $\mathcal{B}$ is an arrangement in R^n . If $\mathcal{S}$ is a collection of cells and τ is a set, by $\mathcal{S}|_\tau$ we denote the collection $\{\sigma \in \mathcal{S} : \sigma \subseteq \tau\}$.

Lemma 17: For any $(n-1)$ -cell τ of $\mathcal{T}(R^n, \mathcal{A})$, the collection $\mathcal{T}(R^n, \mathcal{A})|_{\text{lin}(\tau)}$ is a $k \sim$ fold cover of $\text{lin}(\tau)$.

Proof: Let σ be an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ and γ an 1 -cell of σ with $\delta_\sigma(\gamma) = \tau$. From Theorem 5 we know that γ is normal to $\text{lin}(\tau)$. Letting α be the normal index set of γ define the arrangement $\mathcal{B}$ in R^n by

$$\mathcal{B} \equiv \{h_i : i \in \alpha\}$$

The arrangement $\mathcal{B}$ is not pointed, indeed,

$$\cap \mathcal{B} = \text{lin}(\gamma) \equiv L$$

is a line, and

$$\mathcal{T}(R^n, \mathcal{B}) = \mathcal{T}(L^\perp, L^\perp \cap \mathcal{B})$$

where $L^\perp \cap \mathcal{B}$ is a pointed $\#\alpha$ -arrangement in the $(n-1)$ -linear set $L^\perp = \text{lin}(\tau)$. From Theorem 16, $\mathcal{T}(R^n, \mathcal{B}) = \mathcal{T}(L^\perp, L^\perp \cap \mathcal{B})$ is a $k \sim$ fold cover of $L^\perp$.

Our next step is to show that $\mathcal{T}^{n-1}(R^n, \mathcal{B})$ is equal to $\mathcal{T}^{n-1}(R^n, \mathcal{A})|_{L^\perp}$. First observe that all cells of $\mathcal{T}^{n-1}(R^n, \mathcal{A})|_{L^\perp}$ can be expressed as $\delta_\sigma(\gamma)$ or $\delta_{-\sigma}(-\gamma)$ where σ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ which has γ as a 1 -face and indeed, from Lemma 14 we see that they come in equal pairs, that is, $\delta_\sigma(\gamma) = \delta_{-\sigma}(-\gamma)$. There is a one-to-one correspondence between n -cells of σ of $\mathcal{S}(R^n, \mathcal{A})$ with γ as an 1 -cell and n -cells $\tilde{\sigma}$ of $\mathcal{S}(R^n, \mathcal{B})$, this correspondence is given by $\sigma \subseteq \tilde{\sigma}$. Furthermore, $\delta_\sigma(\gamma) = \delta_{\tilde{\sigma}}(L)$, indeed, they both equal $\{(A_\sigma)^T y : y \geq 0\}$ where the matrix of normals A of $\mathcal{A}$ is chosen so that $\sigma = \{x : Ax \geq 0\}$, see Lemma 6. $\square$

Lemma 18: For the pointed arrangement $\mathcal{A}$ we have $\cup \mathcal{B} = \cup \mathcal{T}^{n-1}(R^n, \mathcal{A})$ where $\mathcal{B}$ is the arrangement $\{\gamma^\perp : \gamma \in \mathcal{S}^1(R^n, \mathcal{A})\}$.

Proof: Just apply Lemma 17 repeatedly and note that if $\tau = \delta_o(\gamma)$ is an $(n-1)$ -cell of $\mathcal{T}(R^n, \mathcal{A})$ then γ is an 1-cell of $\mathcal{S}(R^n, \mathcal{A})$ and $\tau \subseteq \gamma^\perp$. $\square$

Thus we see that the number of $(n-1)$ -hyperplanes needed to cover $\cup \mathcal{T}^{n-1}(R^n, \mathcal{A})$ is one half of $\#\mathcal{S}^1(R^n, \mathcal{A})$. Under the assumption that $\mathcal{A}$ is regular this number is easy to compute as is done in the next section. At present the authors have little to say about the following type of question. Suppose $\mathcal{T}(R^n, \mathcal{A})$ is a k -fold cover of R^n and that an ℓ -arrangement is needed to cover $\mathcal{T}^{n-1}(R^n, \mathcal{A})$, do k and ℓ determine $\mathcal{T}(R^n, \mathcal{A})$?

4. Regular Polar-Divisions

Given the pointed m -arrangement $\mathcal{A}$ in R^n we have already proved that $\mathcal{T}(R^n, \mathcal{A})$ is a $k \sim$ fold cover of R^n . We now assume, in addition, that $\mathcal{A}$ is regular where upon k becomes fixed, namely, as we shall show k is $m - 1$ choose $n - 1$. Other related results are developed.

Let ν be the collection of all subsets of μ of size $n - 1$. As $\mathcal{A}$ is regular there is a two-to-one map from the 1-cells of $\mathcal{S}(R^n, \mathcal{A})$ to elements of ν . Namely if γ is an 1-cell of $\mathcal{S}(R^n, \mathcal{A})$ and α is the normal index set of γ then $\alpha \in \nu$ and $\gamma \cup (-\gamma) = \{x : A_\alpha x = 0\}$.

Lemma 1: For a pointed regular m -arrangement $\mathcal{A}$ in R^n the set $\cup \mathcal{T}^{m-1}(R^n, \mathcal{A})$ is the carrier of an $\binom{m}{n-1}$ -arrangement. Namely,

$$\begin{aligned} \cup \mathcal{T}^{m-1}(R^n, \mathcal{A}) &= \cup \{\gamma^\perp : \gamma \in \mathcal{S}^1(R^n, \mathcal{A})\} \\ &= \cup \{(A_\alpha)^T y : y\} : \alpha \in \nu \end{aligned}$$

Proof: For $\gamma \in \mathcal{S}^1(R^n, \mathcal{A})$ we have $\gamma^\perp = (-\gamma)^\perp = \{(A_\alpha)^T y : y\}$ where α is the normal index set of γ . Now we use Lemma 3.18. Also $\#\nu = \binom{m}{n-1}$ and each $\{(A_\alpha)^T y : y\}$ is a hyperplane through the origin. $\square$

Lemma 2: Let A be a matrix of normals for the regular pointed arrangement $\mathcal{A}$ in R^n and let b be a $1 \times n$ vector. Then the matrix

$$\begin{pmatrix} A \\ b \end{pmatrix}$$

is row regular, if and only if b^T is in $R^n \setminus E$ where the exceptional set is $E = \cup \mathcal{T}^{m-1}(R^n, \mathcal{A})$.

Proof: We know that $\cup \mathcal{T}^{m-1}(R^n, \mathcal{A}) = \cup_{\alpha \in \nu} (A_\alpha)^T y : y$ where ν is the collection of all subsets of μ of size $n - 1$. As each A_α has full row rank, b is not in $\{(A_\alpha)^T y : y\}$ if and only if

$$\begin{pmatrix} A_\alpha \\ b \end{pmatrix}$$

has full row rank. Hence b is not in

$$\cup_{\alpha \in \nu} \{(A_\alpha)^T y : y\}$$

if and only if

$$\begin{pmatrix} A \\ b \end{pmatrix}$$

is row regular. $\square$

Theorem 3: Let $\mathcal{A}$ be a regular pointed m -arrangement in R^n . Then the polar division is a $k \sim$ fold cover of R^n where

$$k = \binom{m-1}{n-1}$$

Proof: Let A be a normal matrix for $\mathcal{A}$ and select b^T in $R^n \setminus E$ where $E = \cup T^{n-1}(R^n, \mathcal{A})$. The matrix

$$\begin{pmatrix} A \\ b \end{pmatrix}$$

is row regular, see Lemma 2. Let us define two propositions P and Q concerning signed identities J in J_m .

$P : JA x > 0$ has a solution x .

$Q : A^T J y = b^T$ has a solution $y \geq 0$.

Proposition P asserts that $\sigma = \{x : JA x \geq 0\}$ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$. Proposition Q asserts that σ^* contains the vector b^T , see Lemma 3.6. Thus to compute the number k of n -cells σ in $\mathcal{S}(R^n, \mathcal{A})$ with $b^T \in \sigma^*$ we may commute the number $\#\{J : P \wedge Q\}$ of $J \in J_m$ where both P and Q are satisfied. Q is equivalent to Q_1 .

$Q_1 : A^T J y = b^T \theta$ has a solution (y, θ) with $y \geq 0$ and $\theta > 0$.

As $\begin{pmatrix} A \\ b \end{pmatrix}$ is regular Q_1 is equivalent to Q_2 .

$Q_2 : A^T J y = b^T \theta$ has a solution (y, θ) with $0 \neq (y, \theta) \geq 0$.

Using the matrix alternative theorem an equivalent statement of Q_2 is Q_3 .

$Q_3 : JA x > 0$ with $-b x > 0$ has no solution x .

As the negation of Q_3 implies P , that is, $\neg Q_3 \rightarrow P$, we see that for each J exactly one of $\neg P$, $\neg Q$, and $P \wedge Q$ is satisfied.

Thus we can compute by $\#\{J : P \wedge Q\} = 2^m - \#\{J : \neg P\} - \#\{J : \neg Q\}$.

Calculate $\#\{J : \neg P\}$: From Corollary 2.5

$$\#\{J : \neg P\} = 2^m - 2 \sum_{i=0}^{n-1} \binom{m-1}{i}$$

Calculate $\#\{J : Q\}$. From Corollary 2.5 for an $(m+1)$ -arrangement $B = A \cup \{x : bx = 0\}$ the number of $J \in \mathcal{J}_{m+1}$ where

$$J \begin{pmatrix} A \\ -b \end{pmatrix} x > 0$$

has a solution x is

$$2 \sum_{i=0}^{n-1} \binom{m}{i}$$

But if there is a solution x for J then $-x$ is a solution for $-J$. Thus the number of $J \in \mathcal{J}_m$ where $JAx > 0$ and $-bx > 0$ is one-half the previous number, that is,

$$\#\{J : Q\} = \sum_{i=0}^{n-1} \binom{m}{i}$$

Calculate $\#\{J : P \wedge Q\}$:

$$\begin{aligned} \#\{J : P \wedge Q\} &= 2^m - \#\{J : P\} - \#\{J : Q\} \\ &= 2 \sum_{i=0}^{n-1} \binom{m-1}{i} - \sum_{i=0}^{n-1} \binom{m}{i} \\ &= \binom{m-1}{n-1} + \sum_{i=0}^{n-1} \binom{m}{i} - \sum_{i=0}^{n-1} \binom{m}{i} \\ &= \binom{m-1}{n-1} \end{aligned}$$

as

$$\binom{m}{i} = \binom{m-1}{i} + \binom{m-1}{i-1}$$

for $i \geq 1$. $\square$

If B is a regular $(m+1)$ -arrangement obtained by adjoining a hyperplane to the regular pointed m -arrangement A the cover fold number of the polar division increases by

$$\binom{m-1}{n-2} = \binom{m}{n-1} - \binom{m-1}{n-1}$$

Even if A is only pointed and not regular we know that the $(n-1)$ -cells of $\mathcal{T}(R^n, A)$ are facets of exactly two n -cells of $\mathcal{T}(R^n, A)$. But now assuming A is pointed and regular we want to answer such questions for lower dimensional cells of $\mathcal{T}(R^n, A)$. Namely, an ℓ -cell γ of $\mathcal{T}(R^n, A)$, is a face of how many n -cells of $\mathcal{T}(R^n, A)$?

Lemma 4: Let

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

be a row regular $(m \times \ell) \times n$ matrix. Then $\#\{\text{sgn}(Ax) \neq 0 : Bx = 0\}$ is

$$2 \sum_{i=0}^{n-\ell-1} \binom{m-1}{i}$$

Proof: If $\ell \geq n$, $Bx = 0$ implies $x = 0$ and the result is trivial. Assume $\ell < n$. By rearranging columns of $\begin{pmatrix} A \\ B \end{pmatrix}$ we may assume it has the form

$$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}$$

where B_2 is square and regular. Post multiplying by the regular matrix

$$\begin{pmatrix} I & 0 \\ -B_2^{-1}B_1 & B_2^{-1} \end{pmatrix}$$

to get the row regular matrix

$$\begin{pmatrix} A_1 - A_2 B_2^{-1} B_1 & A_2 B_2^{-1} \\ 0 & I \end{pmatrix}$$

Using a change of variable our target number becomes

$$\#\{\text{sgn}(A_1 - A_2 B_2^{-1} B_1)y \neq 0 : y\}$$

where $A_1 - A_2 B_2^{-1} B_1$ is $m \times (n - \ell)$ and row regular. From Corollary 2.5 the target number is obtained.

Lemma 5: Each ℓ -cell with $\ell < n$ of $\mathcal{T}(R^n, A)$ is a face of

$$2 \sum_{i=0}^{n-\ell-1} \binom{m-\ell-1}{i}$$

n -cells of $\mathcal{T}(R^n, A)$.

Proof: Let τ be a ℓ -cell of $\mathcal{T}(R^n, A)$. Suppose τ is a face of σ^* for some n -cell σ in $\mathcal{S}(R^n, A)$. Then for some face γ of σ we have $\delta_\sigma(\gamma) = \tau$. From Theorem 3.11 we know that if α is the generator index set of τ then α is the normal index set for γ and $\sigma \subseteq \{x : A_\alpha x \geq 0\}$ where the normal matrix is chosen so that $(A_i)^T \in \tau$ for $i \in \alpha$. As A is regular and $\ell < n$ we have $\#\alpha = \ell$. Let $\beta = \mu \setminus \alpha$. Thus we want to count the number of signed identities $J \in \mathcal{J}_{m-\ell}$ so that

$$A_\alpha x > 0 \quad JA_\beta x > 0$$

$$A_\alpha y = 0 \quad JA_\beta y > 0$$

has a solution (x, y) . However, we know that $A_\alpha x > 0$ has a solution as τ is a cell of $\mathcal{T}(R^n, A)$, hence if we can solve $A_\alpha y = 0$ with $JA_\beta y > 0$, then we can solve $A_\alpha x > 0$ with $JA_\beta x > 0$. Thus we want the number of J where we can solve

$$A_\alpha y = 0 \quad JA_\beta y > 0$$

By Lemma 4 this is the asserted number. $\square$

For $\ell = n - 1$ we know from Lemma 14 as well as the lemma just stated that an $(n - 1)$ -cell is a face of exactly two n -cells.

5. Spheres and Hemispheres

In R^n let $\mathcal{A}$ be a pointed m -arrangement, S an s -sphere and H an s -hemisphere centered at the origin. We are interested in the number of n -cells of the polar division $\mathcal{I}(R^n, \mathcal{A})$ which meet S and which meet H . Under regularity conditions these numbers are computed.

By an s -sphere S in R^n where $s = 0, \dots, n-1$, we mean a set of form $S = \{x \in R^n : \|x\| = 1\}$ where τ is an $(s+1)$ -linear set and $\|\cdot\|$ is the euclidean norm. By an s -hemisphere H we mean a set of form $\{x \in S : v^T x \geq 0\}$ where S is an s -sphere and v is a point in S . We refer to v as the direction of the hemisphere H . Our spheres and hemispheres are always centered at the origin.

Two linear sets σ and τ in R^n are defined to be $\perp$ -transverse, if either

$$\sigma + \tau^\perp = R^n \quad \text{or} \quad \sigma \cap \tau^\perp = \{0\}$$

This is a symmetric relation, see Eaves [2] and Eaves, Hoffman, and Hu [3]. Two cells σ and τ are defined to be $\perp$ -transverse, if their tangential hulls are, that is, if $\text{tng}(\sigma)$ and $\text{tng}(\tau)$ are $\perp$ -transverse. Two cells σ and τ are defined to be facially $\perp$ -transverse if every face of σ is $\perp$ -transverse to every face of τ . Most cells are facially $\perp$ -transparent in the sense that the set of such pairs are open and dense in the set of all pairs of cells. A collection of cells $\mathcal{S}$ and a cell τ are defined to be facially $\perp$ -transverse if every cell σ of $\mathcal{S}$ and τ are facially $\perp$ -transverse. We regard the arrangement $\mathcal{A}$ and an s -sphere or s -hemisphere T to be in regular position, if $\mathcal{S}(R^n, \mathcal{A})$ and $\text{cone}(T)$ are facially $\perp$ -transverse.

Lemma 1: In R^n let $\mathcal{A}$ be a regular m -arrangement, S an s -sphere, and A a matrix of normals of $\mathcal{A}$. Then $\mathcal{S}(R^n, \mathcal{A})$ and $\text{lin}(S)$ are facially $\perp$ -transverse, if and only if there is an $(s+1) \times n$ matrix B such that

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

is row regular and $\text{lin}(S) = \text{lin}(B^T)$

Proof: See Lemma 5.13 of Eaves [2]. $\square$

Theorem 2: Given a regular pointed arrangement $\mathcal{A}$ and an s -sphere S assume $\mathcal{S}(R^n, \mathcal{A})$ and $\text{lin}(S)$ are facially $\perp$ -transverse. Then the number of n -cells of the polar division $\mathcal{I}(R^n, \mathcal{A})$ meeting S is

$$2 \sum_{i=1}^{s+1} \binom{m-1}{n-i}$$

Proof: Let A be a matrix of normals of A and select B according to Lemma 1. Let us define two propositions P and Q concerning signed identities J in $\mathcal{J}_m$.

P : $JAx > 0$ has a solution x .

Q : $A^T Jy = B^T z$ has a solution (y, z) with $y \geq 0$ and $z \neq 0$.

Proposition P asserts that $\sigma = \{x : JAx \geq 0\}$ is an n -cell of $\mathcal{S}(R^n, A)$ and proposition Q asserts that σ^* and S meet, see Lemma 3.6. Our target number is $\#\{J : P \wedge Q\}$. In view of the row regularity an equivalent statement of Q is Q_1 .

Q_1 : $A^T Jy = B^T z$ has a solution (y, z) with $0 \neq y \geq 0$.

Using the matrix alternative theorem an equivalent statement of Q_1 is Q_2 .

Q_2 : $JAx > 0$ with $Bx = 0$ has no solution x .

As $\neg Q_2 \rightarrow P$ we see that for each J exactly one of the propositions $\neg P$, $\neg Q$, and $Q \wedge P$ is true. Thus we can compute by $\#\{J : P \wedge Q\} = 2^m - \#\{J : \neg P\} - \#\{J : \neg Q\}$.

Calculate $\#\{J : \neg P\}$: From Corollary 2.5

$$\#\{J : \neg P\} = 2^m - 2 \sum_{i=0}^{n-1} \binom{m-1}{i}$$

Calculate $\#\{J : \neg Q\}$: Using Lemma 4.4 we obtain

$$\#\{J : \neg Q\} = 2 \sum_{i=0}^{n-(s+1)-1} \binom{m-1}{i}$$

Calculate $\#\{J : P \wedge Q\}$:

$$\begin{aligned} \#\{J : P \wedge Q\} &= 2^m - \#\{J : \neg P\} - \#\{J : \neg Q\} \\ &= 2 \sum_{i=0}^{n-1} \binom{m-1}{i} - 2 \sum_{i=0}^{n-s-2} \binom{m-1}{i} \\ &= 2 \sum_{i=n-s-1}^{n-1} \binom{m-1}{i} = 2 \sum_{i=1}^{s+1} \binom{m-1}{n-i} \quad \square \end{aligned}$$

Lemma 3: In R^n let A be a regular m -arrangement, H be an s -hemisphere, and A a matrix of normals of A . Then $\mathcal{S}(R^n, A)$ and $\text{cone}(H)$ are facially $\perp$ -transverse, if and only if there is an $(s \times n)$ -matrix B and a $(1 \times n)$ -vector C such that the $(m+s+1) \times n$ matrix

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

is row regular and

$$\text{cone}(H) = \text{lin}(B^T) + \text{cone}(C^T)$$

Proof: See Lemma 5.14 of Eaves [2]. $\square$

Lemma 4: Let

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

be a row regular $(m + \ell + 1) \times n$ matrix. Then $\#\{\text{sgn}(Ax) \neq 0 : Bx = 0, Cx > 0\}$ is

$$\sum_{i=1}^{n-\ell-1} \binom{m}{i}$$

Proof: From Lemma 4.4 $\#\{\text{sgn} \begin{pmatrix} Ax \\ Cx \end{pmatrix} \neq 0 : Bx = 0\}$ is

$$2 \sum_{i=0}^{n-\ell-1} \binom{m}{i}$$

But the sgn's come in pairs, namely

$$\text{sgn} \begin{pmatrix} Ax \\ Cx \end{pmatrix} = -\text{sgn} \begin{pmatrix} A(-x) \\ C(-x) \end{pmatrix}$$

Hence we want only half the number as we require $Cx > 0$. $\square$

Theorem 5: Given a regular pointed arrangement $\mathcal{A}$ and an s -hemisphere H assume $\text{cone}(H)$ and $\mathcal{S}(R^n, \mathcal{A})$ are facially $\perp$ -transverse. Then the number of n -cells of the polar division $\mathcal{T}(R^n, \mathcal{A})$ meeting H is

$$\binom{m-1}{n-1} + \sum_{i=1}^i \binom{m}{n-i}$$

Proof: Let A be a matrix of normals of $\mathcal{A}$ and select B and C according to Lemma 3. Let us define two propositions P and Q concerning signed identities J in J_m .

P : $JA x > 0$ has a solution x .

Q : $A^T J y = Bz + Cw$ has a solution (y, z, w) with $(y, w) \geq 0$ and $(z, w) \neq 0$.

Proposition P asserts that $\sigma = \{x : JA x \geq 0\}$ is an n -cell of $\mathcal{S}(R^n, \mathcal{A})$ and proposition Q asserts that σ^* meets H . Our target number is $\#\{J : P \wedge Q\}$.

In view of the row regularities an equivalent statement of Q is Q_1

Q_1 : $A^T J y = B^T z + C^T w$ has a solution (y, z, w) with $0 \neq (y, w) > 0$.

Using the matrix alternative theorem an equivalent statement of Q_1 is Q_2 .

Q_2 : $JA x > 0$ with $Bx = 0$ and $-Cx > 0$ has no solution x .

As $\neg Q_2 \rightarrow P$ we see that for each J exactly one of the propositions $\neg P$, $\neg Q$, and $Q \wedge P$ is true. Thus we can compute by $\#\{J : P \wedge Q\} = 2^m - \#\{J : \neg P\} - \#\{J : \neg Q\}$

Calculate $\#\{J : P\}$: From Corollary 2.5

$$\#\{J : P\} = 2^m - 2 \sum_{i=0}^{n-1} \binom{m-1}{i}$$

Calculate $\#\{J : Q\}$: From Lemma 4

$$\#\{J : Q_2\} = \sum_{i=0}^{n-s-1} \binom{m}{i}$$

Calculate $\#\{J : P \wedge Q\}$:

$$\begin{aligned} \#\{J : P \wedge Q\} &= 2^m - \#\{J : P\} - \#\{J : Q\} \\ &= 2 \sum_{i=0}^{n-1} \binom{m-1}{i} - \sum_{i=0}^{n-s-1} \binom{m}{i} \\ &= \binom{m-1}{n-1} + \sum_{i=0}^{n-1} \binom{m}{i} - \sum_{i=0}^{n-s-1} \binom{m}{i} \\ &= \binom{m-1}{n-1} + \sum_{i=n-s}^{n-1} \binom{m}{i} \\ &= \binom{m-1}{n-1} + \sum_{i=1}^s \binom{m}{n-i} \quad \square \end{aligned}$$

Example of Theorem 5: If $s = 0$, then H is merely a point and we see that Theorems 5 and 4.3 both apply and yield $m-1$ choose $n-1$. $\square$

Example of Theorem 5: In Figure 2 insert a 1-hemisphere and observe that it almost always meets

$$5 = \binom{3-1}{2-1} + \sum_{i=1}^1 \binom{3}{2-i} = \binom{m-1}{n-1} + \sum_{i=1}^s \binom{m}{n-i}$$

polar cells where $m = 3$, $n = 2$, and $s = 1$. $\square$

The number of n -cells σ^* meeting the s -sphere and the s -hemisphere of Theorems 2 and 5 are closely related as the following corollary shows; this observation extends corresponding results of Eaves [2].

Corollary 6: In R^n let $\mathcal{A}$ be an m -arrangement, S an s -sphere, H an s -hemisphere in S , and T the $(s-1)$ -sphere in H . Assume $\mathcal{A}$ is regular and that $\mathcal{S}(R^n, \mathcal{A})$ is facially $\perp$ -transverse to $\text{cone}(H)$. Then the number of n -cells σ of $\mathcal{S}(R^n, \mathcal{A})$ meeting H equals one-half the number meeting S plus one-half the number meeting T that is,

$$\binom{m-1}{n-1} + \sum_{i=1}^s \binom{m}{n-i} = \sum_{i=1}^{s+1} \binom{m-1}{n-i} + \sum_{i=1}^s \binom{m-1}{n-i}$$

Proof: As

$$\binom{m}{n-i} = \binom{m-1}{n-i} + \binom{m-1}{n-i-1}$$

for $i = 2, \dots, s+1$ the result follows. $\square$

6. Linear Programming

Our results are now interpreted in the context of linear programming. Consider a linear program where the cell σ is the feasible region and c is the objective vector. Define $X(\sigma, c)$ be the set of optimal solutions for this linear program, that is,

$$X(\sigma, c) \equiv \arg \max_{x \in \sigma} c^T x$$

If C is a set of vectors define $X(\sigma, C)$ by

$$X(\sigma, C) \equiv \bigcup_{c \in C} X(\sigma, c)$$

Let $\mathcal{A}$ be a pointed arrangement in R^n . For any vector c in $R^n \setminus E$ where $E = \bigcup \mathcal{T}^{n-1}(R^n, \mathcal{A})$, the number of n -cells in $\mathcal{S}(R^n, \mathcal{A})$ where $X(\sigma, c)$ is nonempty is invariant. That is, for fixed $\mathcal{A}$ there is some fixed k where

$$\#\{\sigma \in \mathcal{S}^n(R^n, \mathcal{A}) : X(\sigma, c) \neq \emptyset\} = k$$

for any c in $R^n \setminus \bigcup \mathcal{T}^{n-1}(R^n, \mathcal{A})$. To see this we observe that $X(\sigma, c) \neq \emptyset$ if and only if $c \in \sigma^*$, and we apply Theorem 3.16. If $\mathcal{A}$ is a regular pointed arrangement, then k is $(m-1)$ choose $(n-1)$.

Next in R^n , let S be an s -sphere, H an s -hemisphere, and $\mathcal{A}$ a regular pointed m -arrangement. If $\mathcal{S}(R^n, \mathcal{A})$ is facially $\perp$ -transverse to $\text{lin}(S)$ then

$$\#\{\sigma \in \mathcal{S}^n(R^n, \mathcal{A}) : X(\sigma, S) \neq \emptyset\} = 2 \sum_{i=1}^{s+1} \binom{m-1}{n-i}$$

If $\mathcal{S}(R^n, \mathcal{A})$ is facially $\perp$ -transverse to $\text{cone}(H)$ then

$$\#\{\sigma \in \mathcal{S}^n(R^n, \mathcal{A}) : X(\sigma, H) \neq \emptyset\} = \binom{m-1}{n-1} + \sum_{i=1}^s \binom{m}{m-i}$$

See theorems 5.2 and 5.5. Our intention is to bring these results to bear on parametric linear programming. Much less general results of this type have been used in the study of the average speed of the simplex method, see, for example, Haimovitch [4].

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