TRADEOFFS IN THE DESIGN OF MULTIVARIABLE FEEDBACK SYSTEMS*

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ABSTRACT

Some of the tradeoffs involved in the design of multivariable feedback systems are examined. The paper addresses the problem of obtaining good loop properties at both the plant input and output and the impact of non-minimum phase zeros.

Notation:

$\bar{\sigma}(\cdot)$ - maximum singular value
 $\underline{\sigma}(\cdot)$ - minimum singular value
 $\bar{\lambda}(\cdot)$ - eigenvalue
 $(\cdot)^T$ - transpose
 $(\cdot)^*$ - complex conjugate
 $(\cdot)^*$ - complex conjugate transpose
diag $[\cdot]$ - diagonal matrix

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I. INTRODUCTION

The intent of this paper is to help clarify the tradeoffs involved in the design of multivariable feedback systems. In one respect, multivariable systems are no different from single-loop systems. There is always uncertainty in the plant and uncertainty between design models and the actual plant. The feedback design problem centers around the tradeoffs involved in reducing the overall impact of model uncertainty on system performance.

These tradeoffs are well understood in the single-loop case, and classical control methods primarily focus on their analysis. Actual classical design involves trial and error manipulation of compensators using frequency-domain analysis tools to guide and evaluate each trial design. A similar design procedure could be used for multivariable systems, except that the classical analysis tools are inadequate for dealing with the fundamental feedback properties of coupled multivariable systems.

Some generalizations of classical analysis methods to the multivariable are found in [1]. Specifically, the algebraic tradeoff between large loop gain for good sensitivity and small loop gain for stability margins is discussed. The tradeoffs imposed on multivariable systems by the Bode gain/phase relations are also examined in [1]. These results are briefly reviewed in Section II. Sections III and IV discuss other multivariable phenomena in more detail. The intrinsically multivariable problem of obtaining loop properties (e.g., stability margins) at different points in a multivariable feedback configuration is considered in Section III. Section IV examines the effect of non-minimum phase zeros on multivariable systems. Some of these ideas are illustrated through an example of an electric generating plant in Section V. The conclusions are in Section VI.

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II. MULTIVARIABLE GENERALIZATIONS OF SINGLE LOOP PROPERTIES

Consider the feedback configuration shown in Figure 1. This represents a (nominally) stable closed-loop system consisting of a linear time-invariant plant $G(s)$ and controller $K(s)$. In this section, we will discuss multivariable generalizations of some classical properties of single-loop feedback systems.

One use of feedback is to reduce sensitivity to changes in the plant. It is well-known that one useful sensitivity function for a single-loop feedback system is the inverse of the return difference. To quantify the desensitizing effect of feedback in the multivariable case, we use the comparison sensitivity approach [2]. For the comparison, let G represent the nominal plant and $G(I+\Delta)^{-1}$ represent the perturbed plant. To obtain a direct comparison of sensitivity with and without feedback, we consider the feedback system with precompensation $(I+KG)$ shown in Figure 2. This precompensation is added so that the closed-loop transfer function for the nominal plant is identical with the nominal open-loop transfer function, G . The perturbed closed-loop transfer function is

$$\begin{aligned} G(I+\Delta)^{-1}(I+KG(I+\Delta))^{-1}(I+KG) &= G(I+(I+KG)^{-1}\Delta)^{-1} \\ &= G(I+\Delta_{CL})^{-1} \end{aligned} \quad (2.1)$$

where the effective closed-loop perturbations Δ_{CL} is

$$\Delta_{CL} = (I+KG)^{-1}\Delta \quad (2.2)$$

Thus for multivariable systems, the inverse return difference matrix, $(I+KG)^{-1}$, relates the sensitivity of the closed-loop system to that of the open-loop system. In terms of the spectral norm, if the condition

$$1 > \sigma[(I+KG)^{-1}] \quad (2.3)$$

or, equivalently,

$$\underline{\sigma}(I+KG) > 1 \quad (2.4)$$

holds at some frequency, then the use of feedback accomplishes desensitization at that frequency.

Feedback can also reduce the effect of a disturbance, d , on the output, y (Figure 1). The response at the output to a disturbance at the plant input is

$$y = G(I+KG)^{-1}d \quad (2.5)$$

Since the open-loop response to such a disturbance is simply Gd , satisfaction of condition (2.4) at some frequencies means that feedback provides some disturbance rejection at those frequencies.

In terms of sensitivity and disturbance rejection, good performance of a multivariable (as well as single-loop) feedback system translates into the requirement of a "large" return difference matrix. Since

$$|\underline{\sigma}(I+KG) - \underline{\sigma}(KG)| < 1 \quad (2.6)$$

this is equivalent to requiring "large" loop gain. A more detailed discussion of the relationship between performance and loop gain may be found in [1]. We will return to the desideratum of large loop gain after considering stability margins for multivariable feedback systems.

The representation of uncertainty as $(I+\Delta)^{-1}$ is convenient for studying sensitivity. Some uncertainties in physical systems, however, cannot be

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represented in this way. In particular, both gain and phase become highly uncertain at high frequencies. Modeling the plant with uncertainty as $G(I+\Delta)$ is quite useful in these situations.

Using this representation for the plant with uncertainty in the multivariable feedback system of Figure 1, it has been shown [3] that the closed-loop system remains stable for all (stable) perturbations $\Delta(s)$ satisfying

$$\bar{\sigma}(\Delta(j\omega)) < \underline{\sigma}[I + (K(j\omega)G(j\omega))^{-1}] \quad \forall \omega \geq 0 \quad (2.7)$$

This condition can be obtained as a consequence of the Small Gain Theorem [4] or the multivariable Nyquist criterion [5]. The size of $I + (KG)^{-1}$ measures the robustness of closed-loop stability with respect to multiplicative perturbations $(I+\Delta)$. This is directly analogous to the distance to the critical point on an inverse Nyquist diagram, $|1 + \frac{1}{KG}|$.

Adequate representations of physical systems inevitably require that the perturbation become large at high frequency, and so large $(KG)^{-1}$ (or, equivalently, small loop gain KG) is required at these frequencies to ensure stability.

This last statement leads to a trade-off in the design of feedback systems. At a particular frequency, it is possible to have large loop gain for desensitization and disturbance rejection or small loop gain for stability margins -- but not both. More precisely, consider the identity

$$(I+KG)^{-1} + (I+(KG)^{-1})^{-1} = I \quad (2.8)$$

A direct consequence of this identity is the string of inequalities

$$\frac{\underline{\sigma}(I+L)}{\underline{\sigma}(I+L)+1} \leq \underline{\sigma}(I+L^{-1}) \leq \frac{\underline{\sigma}(I+L)}{\underline{\sigma}(I+L)-1} \quad (2.9)$$

where either KG or $(KG)^{-1}$ may be substituted for L , and the right-hand inequality requires $\underline{\sigma}(I+L) > 1$. Clearly then,

$$\underline{\sigma}(I+L) \gg 1 \rightarrow \underline{\sigma}(I+L^{-1}) \approx 1$$

Thus, desensitization can be obtained only at frequencies where stability margins need not be large. Since large stability margins are always required at sufficiently high frequencies, no desensitization can be achieved there.

As an aside, note that the left-hand inequality in (2.9) may be rewritten as

$$\underline{\sigma}(I+L) \leq \frac{\underline{\sigma}(I+L^{-1})}{1-\underline{\sigma}(I+L^{-1})} \quad (2.10)$$

for $\underline{\sigma}(I+L^{-1}) < 1$. Hence

$$\underline{\sigma}(I+L^{-1}) \ll 1 \rightarrow \underline{\sigma}(I+L) \ll 1$$

Recall that L may be replaced by either KG or $(KG)^{-1}$. So roughly speaking, "poor" sensitivity at any frequency is equivalent to "poor" margins there.

We have seen that the trade-off between desensitization and stability margins applies to multivariable feedback systems as well as to single-loop systems. Typically, desensitization and disturbance rejection are desired at low frequencies, and large stability margins are required at high frequencies. A critical factor in the design process is the necessity to make the transition from large loop gain to small loop gain. Throughout this transition region, poor sensitivity and poor stability margins must be avoided.

The sensitivity and stability margins of a single-loop feedback system near crossover are known to be highly dependent on the phase of the loop transfer function. Also, Bode has shown that for a class of functions which includes proper, rational functions with no poles or zeros in the right-half plane, either the gain or phase is sufficient to reconstruct the entire function [6]. Thus the necessity to avoid poor sensitivity and poor stability margins near crossover of single-loop feedback systems (i.e., carefully control the phase of KG) imposes limitations on $|KG|$ (and, in turn, on $|1+KG|$ and $|1+\frac{1}{KG}|$) away from crossover. The Bode gain/phase relation provides a fundamental limitation on the achievable performance of single-loop feedback systems.

The multivariable generalization of the gain/phase relation is complicated by the fact that the singular values of a rational transfer function matrix are not analytic. The eigenvalues of such a matrix, however, do satisfy a gain/phase relation and can be related to the singular values by the inequalities [1].

$$\underline{\sigma}(I+L) \leq |\lambda_i(I+L)| \leq \overline{\sigma}(I+L) \quad \forall i \quad (2.11)$$

The resulting restrictions on the singular values are just as severe as in the single-loop case. The conclusion is that multivariable systems have limitations on their achievable performance analogous to those imposed on single-loop systems by the Bode gain/phase relations.

Summarizing this section, classical sensitivity and stability margins for single-loop systems have natural multivariable generalizations. The trade-off between sensitivity and stability margins at any frequency is reflected in the algebraic constraint (2.8). The Bode gain/phase relation also has a natural multivariable generalization and introduces tradeoffs across frequencies. These multivariable generalizations, of course, come with additional complicating factors compared to the single-loop case. However, the fundamental classical tradeoffs involved in the design of feedback systems naturally extend to the multivariable case.

III. DIRECTIONALITY IN MULTIVARIABLE FEEDBACK SYSTEMS

The preceding section discussed frequency-domain tradeoffs which occur in the design of single-loop and multivariable feedback systems. In this section, we examine some tradeoffs that are unique to multivariable systems.

For multivariable systems, unlike scalar systems, signals have a "spatial" as well as frequency distribution. Some sensors are noisier than others. Certain actuators might saturate at lower signal levels than others. Disturbances enter particular loops, but not others, etc. One is lead to consider, for example, different "bandwidths" in various loops.

The stability condition (2.7) is tight if all that is known about the perturbation Δ is its maximum singular value. However, if more is known about Δ — large in one direction, small in another — then condition (2.7) can be sharpened through the use of frequency-dependent weighting matrices [7]. That is, for a perturbed plant $G(s)(I+\Delta(s))$, the closed-loop system remains stable for all (stable) perturbations $\Delta(s)$ satisfying

$$\overline{\sigma}[Q_I(j\omega)\Delta(j\omega)R_I^{-1}(j\omega)] < \underline{\sigma}[R_I(j\omega)(I+(K(j\omega)G(j\omega))^{-1})Q_I^{-1}(j\omega)] \quad \forall \omega \geq 0 \quad (3.1)$$

The known characteristics of the allowable perturbations are reflected in the (nonsingular) weighting matrices R_I and Q_I .

The stability margin condition (3.1) yields margins at a particular point in the feedback configuration of Figure 1. These margins are at the inputs to the plant. The weighting matrices in (3.1) are useful in handling the issue of directionality at this point. For single-loop systems, margins at the input are identically equivalent to margins at the output. Input and output margins are not equivalent, however, in the multivariable case. Output margins are obtained by considering a perturbed plant $(I + \Delta_o(s))G(s)$. The stability condition (using weighting matrices R_o and Q_o) may be written as

$$\sigma[Q_o(j\omega)\Delta_o(j\omega)R_o^{-1}(j\omega)] < \sigma[R_o(j\omega)(I + (G(j\omega)K(j\omega))^{-1})Q_o^{-1}(j\omega)] \quad \forall \omega > 0 \quad (3.2)$$

This is the same condition as (3.1), with G and K interchanged.

From a design viewpoint, good margins are typically desired at both the inputs and the outputs of the plant. One may express this objective slightly differently by first fixing the input margins and then examining the limits imposed on the output margins. More precisely, we pose the following formal optimization problem at a single frequency:

$$\begin{aligned} \max_K \quad & \sigma[R_o(I + (GK)^{-1})Q_o^{-1}] \\ \text{s.t.} \quad & \sigma_i[R_i(I + (KG)^{-1})Q_i^{-1}] = 1 \quad \forall i \end{aligned} \quad (P)$$

and now we require both G and K to be square.

This constraint of all singular values equal to 1 is equivalent to constraining the matrix to be unitary. Hence

$$I + (KG)^{-1} = R_i^{-1} U Q_i \quad (3.3)$$

for some unitary U . The function to be maximized can then be written as

$$\begin{aligned} \sigma[R_o(I + (GK)^{-1})Q_o^{-1}] &= \sigma[R_o G(I + (KG)^{-1})G^{-1}Q_o^{-1}] \\ &= \sigma[(R_o G R_i^{-1})U(Q_o G Q_i^{-1})^{-1}] \end{aligned} \quad (3.4)$$

This means the original problem (P) is equivalent to

$$\begin{aligned} \max_U \quad & \sigma[(R_o G R_i^{-1})U(Q_o G Q_i^{-1})^{-1}] \\ \text{s.t.} \quad & U U^* = I \end{aligned} \quad (P^1)$$

To solve (P^1) , let $R_o G R_i^{-1} = U_1 \Sigma_1 V_1^*$ and $Q_o G Q_i^{-1} = U_2 \Sigma_2 V_2^*$ be singular value decompositions where

$$\begin{aligned} \Sigma_1 &= \text{diag}[\alpha_1, \alpha_2, \dots, \alpha_n] \quad , \quad \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \\ \Sigma_2 &= \text{diag}[\beta_1, \beta_2, \dots, \beta_n] \quad , \quad \beta_1 \geq \beta_2 \geq \dots \geq \beta_n \end{aligned}$$

Making these substitutions into (P^1) ,

$$\sigma[(R_o G R_i^{-1})U(Q_o G Q_i^{-1})^{-1}] = \sigma[\Sigma_1 \hat{U} \Sigma_2^{-1}] \quad (3.5)$$

where $\hat{U} = V_1^* U V_2$ is a unitary matrix.

Proposition

$$\begin{aligned} \max_{\hat{U}} \quad & \sigma[\Sigma_1 \hat{U} \Sigma_2^{-1}] = \min_i \left(\frac{\alpha_i}{\beta_i} \right) \\ \text{s.t.} \quad & \hat{U} \hat{U}^* = I \end{aligned}$$

proof: Appendix

The formal problem (P) can be interpreted in terms of this proposition. The best possible output margins equal the minimum of the ordered ratio of the singular values of $R_0 G R_1^{-1}$ to the singular values of $Q_0 G Q_1^{-1}$. When the uncertainty at the input and output is unstructured, i.e., R_1, R_0, Q_1, Q_0 are all scalar multiples of the identity, the maximum achievable margins at the outputs equal 1 — the same as the margins at the inputs. This unstructured case causes the multivariable problem to be analogous to the single-loop case. When the uncertainties do have some directionality or structure (as reflected in non-identity R 's and Q 's), the feedback problem is inherently multivariable. This directionality may impose additional limitations on the achievable performance.

In the Appendix, it is shown that using $\hat{U}$ equal to the identity in the proposition achieves the maximum. With this choice of $\hat{U}$, we can solve for K from (3.3) as

$$K = (R_1^{-1} V_1 V_2^* Q_1 - I)^{-1} G^{-1} \quad (3.6)$$

assuming that $\hat{U} = I$ yields $R_1^{-1} V_1 V_2^* Q_1 - I$ invertible. As far as actually achieving this limit at each frequency, note that V_1 and V_2^* need not be rational functions of frequency. Hence, in general, we cannot achieve this limit with a causal compensator $K(s)$. The question of the limitations imposed on the output margins by both fixed input margins and a causal compensator is still open.

In concluding this section, we point out that the formal problem considered here is not especially important in and of itself. One could pose several other such problems, e.g. maximizing $\alpha(I+KG)$ subject to constraints on $I+GK$, etc. The reason for posing this problem is that it illustrates how directionality can introduce uniquely multivariable phenomena.

IV. NON-MINIMUM PHASE BEHAVIOR IN MULTIVARIABLE SYSTEMS

Given a desired rational, strictly proper loop transfer function $H(s)$ that yields a stable closed-loop system, there exists rational, strictly proper compensation $K(s)$ for the minimum phase plant $G(s)$ such that the desired loop transfer function is fit arbitrarily closely and the resulting closed-loop system is stable. Indeed, it suffices to introduce the feedback compensation

$$K(s) = G^{-1}(s)H(s) \left(\frac{\alpha}{s+\alpha}\right)^m \quad (4.1)$$

where m is chosen to make $K(s)$ strictly proper and α is chosen sufficiently large. This means that if $H(s)$ is designed to achieve the sensitivity and stability margin tradeoff discussed in Section II, then any minimum phase plant can be compensated to realize this loop transfer function as closely as desired. The same result applies to multivariable systems as well as single-loop systems by interpreting (4.1) in a matrix sense. In the multivariable case, however, such compensation yields the desired loop properties only at the plant inputs; the output loop properties can be quite different.

This is not true for non-minimum phase systems. A transfer function with right-half-plane zeros may be factored as

$$G(s) = M(s)P(s) \quad (4.2)$$

where $M(s)$ is minimum phase and $P(s)$ is an all-pass ($|P(j\omega)| = 1, \forall \omega$). The all-pass cannot be cancelled as in (4.1). It contributes a negative phase angle and thereby aggravates the tradeoff problem. Roughly speaking, a non-minimum phase zero λ requires small loop gain throughout some frequency interval "near" λ [8]. That is, the advantages of feedback are achievable at frequencies below and above the non-minimum phase zero's magnitude but not around it.

Multivariable non-minimum phase systems are subject to analogous restrictions. A plant transfer matrix, with right-half-plane zeros may be factored in two ways as

$$G(s) = M_1(s)P_1(s) = P_2(s)M_2(s) \quad (4.3)$$

where the P's are all-pass ($P^*(j\omega)P(j\omega) = I \forall \omega$) and the M's are minimum phase. These factorizations can be expressed in state-space terms. Let $G(s)$ be the transfer matrix of the quadruple (A, B, C, D) . Then for our purposes, $G(s)$ has a zero at λ means

$$\begin{bmatrix} \lambda I - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0 \quad (4.4)$$

has a solution with $w^*w = 1$. For a single real zero at λ , one factorization (unique up to a unitary transformation) $M_1(s)P_1(s)$ is given by

$$P_1(s) = I - \left(\frac{2\lambda}{s+\lambda}\right)ww^T \quad (4.5)$$

and $M_1(s)$ is the transfer matrix of the quadruple $(A, \hat{B}, C, D)$ where

$$\hat{B} = B - 2\lambda xw^T \quad (4.6)$$

For the case of a single pair of complex zeros at λ and λ^* ,

$$P_1(s) = I - \left(\frac{4\alpha\gamma}{s^2 + 2\alpha s + \lambda\lambda^*}\right) [s \operatorname{Re}(ww^*) - \frac{\alpha\beta}{\lambda} \bar{w}w^* + \operatorname{Re}(\lambda ww^* - \alpha\beta \bar{w}w^*)] \quad (4.7)$$

$$\hat{B} = B - 2\alpha\gamma [xw^* - \frac{\alpha\beta}{\lambda} \bar{x}w^*] \quad (4.8)$$

where $\alpha = \operatorname{Re}(\lambda)$, $\beta = w^T w$, and $\gamma = (1 - \frac{\alpha\beta}{\lambda\lambda^*})^{-1}$. The proofs of (4.5) - (4.8) follow from the generalized eigenvalue problem (4.4) with algebraic manipulation. Multiple non-minimum phase zeros can be handled by repeated application of (4.5) - (4.8). We note that similar formulas may be obtained for the factorization $P_2(s)M_2(s)$ by using the left generalized eigenvector.

$$\begin{bmatrix} \hat{x}^T & v^T \end{bmatrix} \begin{bmatrix} \lambda I - A & -B \\ -C & -D \end{bmatrix} = 0 \quad (4.9)$$

The non-minimum phase zero λ adds extra phase lag in the input direction w and the output direction v . This is the same effect and has the same restrictions for feedback design as in the single-loop case. However, the transfer matrix from inputs orthogonal to w to outputs orthogonal to v does not have any additional phase lag caused by λ . Thus, the non-minimum zero λ does not impose any limitations on the loop gain in these orthogonal directions.

Because a non-minimum phase zero λ does not impose limitations in directions orthogonal to w and v , it could be the case that a feedback design has large loop gain in these directions*. Such a design would necessarily have a large spread in the singular values of KG at frequencies near λ . This large spread is the same situation that occurs in the presence of structured uncertainty, i.e., weighting matrices R_1 and Q_1 not equal to a scalar multiple of the identity. As discussed in Section III, the tradeoff between loop properties at the plant input vs. the output enters the design problem when there is a spread in the singular values of KG . This additional tradeoff is uniquely multivariable.

* Other considerations may restrict this loop gain. For example, the transfer matrix associated with this subspace may itself be non-minimum phase, although not having a zero at λ .

Thus, non-minimum phase zeros affect multivariable feedback design in two ways. First, they contribute additional phase lag in certain input and output directions. This limits the achievable performance in these directions and is directly analogous to non-minimum phase zeros in single-loop systems. The resulting unequal gains in different directions can lead to a conflict between input and output loop properties. This second effect has no single-loop analog.

V. AN ELECTRIC GENERATING PLANT EXAMPLE

The impact of a non-minimum phase zero on multivariable feedback design will be illustrated in this section by comparing two feedback designs for an electric generating plant. The plant is Philadelphia Electric Company's Cromby Number 2 unit, a 250 MW coal-fired boiler-turbine-generator system. For design purposes, we use a linearized version of the nonlinear 14th order model of this unit described in [9], [10]. The model is numerically linearized at steady-state conditions and a generation of 220 MW. The six control variables and six output variables used in the design are listed in Table 1. In order to regulate the output variables, six integrators are appended to the original model. This model has a non-minimum phase zero at 0.35 rad/sec. The non-minimum phase input vector w has non-zero components in the air flow and both burner tilts.

Table 1. Cromby Number 2 Control and Output Variables Used For Controller Design

Controls:	Length of the coal feeder stroke	Outputs:	Generation
	Governor valve		Drum level
	Feedwater valve		Secondary superheater pressure
	Air flow		Secondary superheater temperature
	Superheat burner tilt		Reheat temperature
	Reheat burner tilt		Coal flow

The design approach used here is a two-step LQG procedure. The first step is the design of an acceptable LQ state-feedback regulator. The second step consists of employing the asymptotic filter design procedure described in [11]. This filter design procedure asymptotically yields full-state loop properties at the inputs for a minimum phase plant. We will now illustrate what can happen with a non-minimum phase zero.

Figure 3 and 4 show the singular values of the return difference matrix with the loops broken at the plant inputs under state-feedback for two different LQ designs. Since these are LQ designs, the minimum singular value is guaranteed to be no less than one. The design in Figure 3 does not roll off the non-minimum phase direction before 0.35 rad/sec; the design in Figure 4 does provide this roll off. The second design has rolled off both burner tilts and the air flow by about 0.1 rad/sec.

The results of using the asymptotic filter designs with these two state feedback gains are shown in Figures 5 and 6. These figures show the singular values of the return difference matrix with the loops broken at the inputs under output-feedback. The impact of the non-minimum phase zero is evident. Figure 5 shows a minimum singular value less than one from 0.1 rad/sec to almost 10 rad/sec. This feedback design amplifies disturbances (in some direction) and increases sensitivity in this frequency range.

From Figure 6, the design which rolled off the non-minimum phase direction does not have this problem. The asymptotic filter design procedure was able to achieve the full-state loop properties in this case.

These results are presented for illustrative purposes only. The controllers are not intended to be considered as a practical design. They are intended only to provide information on the impact of non-minimum phase zeros.

VI. CONCLUSIONS

This paper has discussed design tradeoffs which impact the achievable performance of multivariable feedback systems. Multivariable generalizations of the well-understood single-loop sensitivity vs. stability margins tradeoff and Bode gain/phase relations were reviewed in Section II. Using these "singular value" characterizations of loop properties, the intrinsically multivariable problem of obtaining good loop properties at both the plant inputs and outputs was examined. This tradeoff was illustrated by considering the question of finding the best output margins for given input margins. In Section IV, the influence of non-minimum phase zeros was discussed. These zeros add extra phase lag, just as in the single-loop case, and also may lead to a conflict between desired loop properties at different points in the feedback configuration. In closing, we point out that the approach taken here allows a great deal of single-loop intuition to be carried over to the multivariable case. What is less satisfying, is that the results presented are fragmented; essentially one-at-a-time results. Much work remains to be done to unify these results in a coherent theory.

REFERENCES

1. Doyle, J.C. and Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," to appear IEEE Trans. Automat. Contr., February, 1981
2. Cruz, J.B. and Perkins, W.R., "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design," IEEE Trans. Automat. Contr., vol. AC-9, pp 216-223, 1964.
3. Doyle, J.C., "Robustness of Multiloop Linear Feedback Systems," Proc. 1978 CDC, San Diego, Calif., January, 1978.
4. Desoer, C.A. and Vidyasagar, M., Feedback Systems: Input-Output Properties, Academic Press, N.Y., (1975).
5. Rosenbrock, H.H., "The Stability of Multivariable Systems," IEEE Trans. Automat. Contr., vol. AC-17, pp. 105-107, 1972.
6. Bode, H.W., Network Analysis and Feedback Amplifier Design, D. Van Nostrand, Princeton, 1945.
7. Safonov, M.G., "Tight Bounds on the Response of Multivariable Systems with Component Uncertainty," 16th Allerton Conference, October, 1978.
8. Horowitz, I.M., Synthesis of Feedback Systems, Academic Press, N.Y., (1963).
9. McDonald, J.P., Kwatny, H.G., and Spare, J.H., "A Nonlinear Model for Reheat Boiler-Turbine-Generator Systems, Part I - General Description and Evaluation," Proc. 12th JACC, pp. 219-226, 1971.
10. Kwatny, H.G., McDonald, J.P., and Spare, J.H., "A Nonlinear Model for Reheat Boiler-Turbine-Generator Systems, Part II - Development," Proc. 12th JACC, pp. 227-236, 1971.
11. Doyle, J.C. and Stein, G., "Robustness with Observers," IEEE Trans. Automat. Contr., vol. AC-24, pp. 607-611, 1979

Figure 1
Multivariable
Feedback System

Figure 3 Singular values of the return difference matrix under state-feedback for design 1.

Figure 5 Singular values of the return difference matrix under output-feedback for design 1.

Figure 2
Precompensation For
Comparative Sensitivity

Figure 4 Singular values of the return difference matrix under state-feedback for design 2.

Figure 6 Singular values of the return difference matrix under output-feedback for design 2.

APPENDIX

Proposition: Let $\Sigma_1 = \text{diag}[\alpha_1, \alpha_2, \dots, \alpha_n]$, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ and $\Sigma_2 = \text{diag}[\beta_1, \beta_2, \dots, \beta_n]$, $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$. Then

$$\max_{\hat{U}} \sigma[\Sigma_1 \hat{U} \Sigma_2^{-1}] = \min_i \left(\frac{\alpha_i}{\beta_i} \right)$$

$$\text{s.t. } \hat{U} \hat{U}^* = I$$

Proof: For any unitary $\hat{U}$,

$$\begin{aligned} \sigma[\Sigma_2 \hat{U}^* \Sigma_1^{-1}] &= |\Sigma_2 \hat{U}^* \Sigma_1^{-1}| \\ &\geq \frac{\beta_i}{\alpha_i} \cdot \underbrace{|\text{diag}[1, \dots, 1, 0, \dots, 0]|}_{i \text{ times}} \underbrace{|\hat{U}^* \text{diag}[0, \dots, 0, 1, \dots, 1]|}_{n-i+1 \text{ times}} \end{aligned} \quad (\text{A.1})$$

where $\frac{\beta_i}{\alpha_i} = \max_j (\beta_j / \alpha_j)$. By compatibly partitioning $\hat{U}$, this may be written as

$$\begin{aligned} \sigma[\Sigma_2 \hat{U}^* \Sigma_1^{-1}] &\geq \frac{\beta_i}{\alpha_i} \cdot \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{U}_1^* & \hat{U}_4^* & \hat{U}_7^* \\ \hat{U}_2^* & \hat{U}_5^* & \hat{U}_8^* \\ \hat{U}_3^* & \hat{U}_6^* & \hat{U}_9^* \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| \\ &= \frac{\beta_i}{\alpha_i} \cdot \left| \begin{pmatrix} 0 & \hat{U}_4^* & \hat{U}_7^* \\ 0 & \hat{U}_5^* & \hat{U}_8^* \\ 0 & 0 & 0 \end{pmatrix} \right| \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \text{But } \begin{pmatrix} 0 & 0 & 0 \\ \hat{U}_4 & \hat{U}_5 & 0 \\ \hat{U}_7 & \hat{U}_8 & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{U}_4^* & \hat{U}_7^* \\ 0 & \hat{U}_5^* & \hat{U}_8^* \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - \hat{U}_6 \hat{U}_6^* & -\hat{U}_6 \hat{U}_9^* \\ 0 & -\hat{U}_9 \hat{U}_6^* & 1 - \hat{U}_9 \hat{U}_9^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \hat{U}_6 \\ \hat{U}_9 \end{pmatrix} \begin{pmatrix} 0 & \hat{U}_6^* & \hat{U}_9^* \end{pmatrix} \end{aligned} \quad (\text{A.3})$$

Hence the norm of the matrix on the rhs of (A.2) is one, and so

$$\sigma[\Sigma_2 \hat{U}^* \Sigma_1^{-1}] \geq \frac{\beta_i}{\alpha_i} \quad (\text{A.4})$$

Equation (A.4) is true for all unitary $\hat{U}^*$, with equality being obtained for $\hat{U}^* = I$. So

$$\max_{\hat{U}} \sigma[\Sigma_1 \hat{U} \Sigma_2^{-1}] = \left(\min_{\hat{U}} \sigma[\Sigma_2 \hat{U}^* \Sigma_1^{-1}] \right)^{-1} = \min_i \left(\frac{\alpha_i}{\beta_i} \right)$$

Q.E.D.

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