LOCAL FULL-WAVE ENERGY IN NONUNIFORM PLASMAS

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Abstract

The subject of local wave energy in plasmas is treated via quasilinear theory from the dual perspectives of the action-angle formalism and gyrokinetic analysis. This work presents an extension to all orders in the gyroradius of the self-consistent wave-propagation / quasilinear-absorption problem using gyrokinetics. Questions of when and under what conditions local energy should be of definite sign are best answered using the action-angle formalism. An important result is that the "dielectric operators" of the linearized wave equation and of the local energy are not the same, a fact which is obscured when the eikonal or WKB assumption is invoked. Even though the two dielectrics are very different in character (one operates linearly on electric field for the plasma current, the other operates quadratically for the energy), it is demonstrated that they are nevertheless related by a simple mathematical statement.

This study was originally motivated by concern and lively discussion over the questions of local energy for rf-heating of plasmas, where in certain instances, full-wave effects such as refraction, strong absorption, and mode conversion are of primary importance. Fundamentally, the rf-absorption must equate with the energy moment of the quasilinear term to achieve a correct energy balance. This fact governs the derivation (as opposed to postulation) of the local absorption. The troublesome "kinetic flux" may then be chosen (it is not unique) to satisfy a wave-energy balance relation with the Poynting flux and local absorption. It is shown that at least one such choice reduces asymptotically to the Stix form away from nonuniformities, thereby demonstrating energy conservation to all orders in Larmor radius.

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MASTER

I. Introduction

Conceptually, it is easy to imagine a point particle in a plasma as a viable entity happily moving about with some well-prescribed kinetic energy. It is perhaps more difficult, but entirely correct, to imagine a *radio-frequency wave* as an equally viable kinetic entity, also transporting energy. But when plasma wave meets plasma particle, with exchange of energy, the picture suddenly becomes fuzzy. Historically, the question of local wave energy in dielectric media has been studied (and debated) for half a century.¹⁻³ Nevertheless, it is precisely this controversial question which is addressed in connection with rf-heating of plasmas. Exactly when and where does energy cease to be carried by a wave and start to be carried by the particles?

In a uniform weakly absorbing plasma, this question has been quite satisfactorily answered.⁴ It has also been answered for weakly nonuniform plasmas carrying a wave which satisfies an eikonal relation.^{5,6} Under certain restrictions, this question has been successfully answered for plasmas containing full waves of limited nonlocal character, i.e., truncation to second order in the gyroradius.⁷⁻¹⁰ The action-angle formalism removes these restrictions,^{11,12} but is not formulated explicitly in terms of the wave electric fields. Some gyrokinetic analyses have also removed the restrictions of finite order in the gyroradius;^{13,14} however, they did not deal explicitly with local wave energies in the rf-frequency range. Other gyrokinetic analysis in the appropriate frequency range has been made, but again truncated to finite order in the gyroradius.⁸ Recent work by McVey, Sund, and Sharer¹⁵ uses a postulated local absorption which is strikingly similar in form to gyrokinetics, and has been generalized to all orders in ρ_L by Vaclavik and Appert.¹⁶ Brambilla and Krüken¹⁷ have further generalized this latter formalism to include arbitrary magnetic geometry, and their work results in an important positivity-theorem for McVey *et al.*'s postulated absorption quantity.

Nearly all of the mathematics necessary to answer the question at the end of the first paragraph have already been developed in the above references. However, much of this work is singularly specific either to the linear wave propagation problem or to the quasilinear absorption problem; the two complementary aspects are rarely studied on their common ground, which is the energy balance relation. In the case of full waves to all orders in ρ_L , this complementary study has not yet been performed, and it is that study which is the major goal of this paper.

II. Local Energy from Wave Propagation Theory

Many studies of plasma wave propagation attempt to define local absorption in terms of the plasma dielectric. While this technique works well when there is only a single freely propagating wave, it can lead to ambiguities when more complicated situations arise.¹⁸ In fact, there is no way to resolve our question from within the confines of linear wave propagation theory alone, a fact which can be easily demonstrated. The fundamental ambiguity is closely related to the gauge

invariance of electromagnetics. Attempts to define energy in terms of the dielectric will invariably result in *nonsymmetric* forms, which allow local rf-absorption to be of indefinite sign even for plasmas in equilibrium.

Consider the propagation of a mostly electromagnetic wave, whereby the wave's energy is carried primarily by the Poynting flux. Set the scalar potential to zero, and then the vector potential, $A(r)$, differs from the electric field, $E(r)=i\omega A(r)$, by a factor of $(i\omega)$. The wave equation satisfied by the electric field is

$$\nabla \times \nabla \times E(r) - k_0^2 K\{E\} = 0 \quad (1a)$$

where $K\{E\}$ is the full-wave dielectric operator of the plasma, which may be defined in terms of the induced plasma current: $J_p(r)=\sigma\{E\}$, where $K=1-(i\omega\epsilon_0)^{-1}\sigma$. Produce an energy balance equation (Poynting's Theorem) by multiplying Eq. (1a) by $(-i\omega\mu_0)^{-1}E^*(r)$, and employing vector identities, to give

$$-\nabla \cdot (E^* \times H) = -i\omega\epsilon_0 E^* \cdot K\{E\} + i\omega\mu_0 |H|^2 \quad (1b)$$

where $\nabla \times E = -i\omega\mu_0 H$. The real part of this equation gives (twice) the energy conservation rule: $-\nabla \cdot P = W_a$, where $P = 1/2 \text{Re}[E^* \times H]$ is the Poynting flux and $W_a = 1/2\omega\epsilon_0 \text{Im}[E^* \cdot K\{E\}]$ is the local absorption.

Alternately, consider the propagation of an electrostatic wave. Now the vector potential is set to zero, and the wave equation for the scalar potential, Φ , is

$$\nabla \cdot K\{\nabla\Phi\} = 0 \quad (2a)$$

An energy balance relation is produced by multiplying equation (2a) through by $i\omega\epsilon_0\Phi^*$ and employing vector identities, giving

$$-\nabla \cdot (-i\omega\epsilon_0\Phi^* K\{\nabla\Phi\}) = -i\omega\epsilon_0 E^* \cdot K\{E\} \quad (2b)$$

where $E = -\nabla\Phi$. Again, the real part of this equation gives the energy balance equation: $-\nabla \cdot T = W_a$, where now $T = 1/2\omega\epsilon_0 \text{Im}[\Phi^* \cdot K\{\nabla\Phi\}]$ is the "kinetic flux" and $W_a = 1/2\omega\epsilon_0 \text{Im}[E^* \cdot K\{E\}]$ is still the local absorption. In a sufficiently dense plasma the kinetic flux approaches the value, $1/2 \text{Re}[\Phi^* J_p]$, which is a useful intuitive form.

The generalization of Eqs. (1) and (2) to situations where the plasma contains waves of indefinite electrostatic and electromagnetic character cannot be made in any unique definitive manner. This is a consequence of the arbitrary gauge. Since the energy is quadratic in the

potentials, there will inevitably be cross-terms between vector and scalar potentials, which depend critically on the particular choice of gauge. Furthermore, it is not clear which cross-terms contribute to absorption and which contribute to the energy fluxes, for any scalar quantity can always be represented as the divergence of some vector quantity (e.g., $T(r) = 1/4\pi \int d^3r' u(r') [r-r']/|r-r'|^3$ will satisfy $\nabla \cdot T(r) = u(r)$ for an arbitrary bounded function $u(r)$). In addition, it is not possible to specify uniquely the energy fluxes by the requirement that they reduce to some prescribed form asymptotically in position, since this condition effectively determines only the first term in an asymptotic series (e.g., if $T_x(x)$ satisfies the asymptotic condition for kinetic flux, then so does $T_x(x)[1+C(x)/(1+x^2)]$, where $C(x)$ is any bounded function). Similarly, asymptotic conditions in terms of plasma uniformity also do not specify the energies uniquely.

An equally important difficulty with the independent energy balances, Eqs. (1b) and (2b), is that they are not symmetric. While it has never been proven exactly when local absorption, W_a , must have a symmetric form, it is nevertheless reasonable to expect W_a to be of positive sign for a plasma in equilibrium. Symmetric forms allow conclusive proof of sign, hence it is widely believed that a plasma in equilibrium should produce a symmetric form. In both of the above energy balances W_a is given by $1/2\omega\epsilon_0 \text{Im}[E^* \cdot K \cdot E]$. Since the electric field on the right is evaluated nonlocally by the full-wave dielectric (e.g., K may contain integral or derivative operators), and the electric field on the left is evaluated locally (e.g., at the same position, r , as $W_a(r)$), this form is usually nonsymmetric.

One familiar example involves the second harmonic term in the expansion of K to second order in gyroradius, ρ_L , for a Maxwellian equilibrium.⁷⁻¹⁰ Here the second order contribution to W_a , in terms of the plasma parameters, is

$$-i\omega\epsilon_0 E^* \cdot K^{(2)} \cdot E = i\omega\epsilon_0 E_{\pm}^* \frac{\omega_p^2 \rho_L^2 Z_{\pm 2}}{4\omega |k_{\parallel}| v_{th}} \nabla_{\perp}^2 E_{\pm} \quad (3a)$$

where $E_{\pm}$ refers to the circularly polarized components of the electric field and $Z_{\pm 2}$ is the appropriate evaluation of the plasma dispersion function. Now if $E_{\pm}$ varies as in the WKB approximation, $E_{\pm} \sim e^{ik \cdot r}$, with real k , then the real part of Eq. (3a) has positive sign, since $\text{Im}[Z_{\pm 2}]$ does. But with a full-wave electric field, the real part of Eq. (3a) is of indefinite sign, which is clearly undesirable. Hence, in the references, Eq. (3a) is not used for the local absorption. Instead, a symmetric form is used, and any difference between the two is taken up within the kinetic flux in a natural manner. This symmetric form is

$$-i\omega\epsilon_0 \frac{\omega_p^2 \rho_L^2 Z_{\pm 2}}{4\omega |k_{\parallel}| v_{th}} |\nabla_{\perp} E_{\pm}|^2 \quad (3b)$$

where $\nabla_{\pm} \equiv \partial/\partial x \pm i\partial/\partial y$ (note: $|\nabla_{\pm}|^2 = \nabla_{\perp}^2$). The real part of expression (3b) is of definite sign. The unfortunate consequence of this intuitively more pleasing form, though, is that now both electric fields are evaluated nonlocally. Notationally, this can be indicated by writing the combination $|\nabla_{\pm} E_{\pm}|^2$ as $E_{\pm}^* \nabla^* \nabla_{\pm} E_{\pm}$, where " ∇^* " means that the operator operates to the left.

The important conclusions which should be drawn from this section are that first, the local full-wave energies are not likely to result from linearized wave analysis alone. And second, that if the local rf-absorption is to be represented as a quadratic form in electric field with some sort of plasma dielectric, this dielectric will not be the same operator as found in the wave equation. Most importantly, if it is to have a symmetric form for equilibrium plasmas, it will need to operate nonlocally and individually on both electric fields in the quadratic. In other words the local absorption will require some energy dielectric which operates both right and left, i.e., as $(E \cdot) K (E)$. An example of such an energy dielectric, accurate to all orders in ρ_1 , is postulated in reference [16]. In the remainder of this paper I shall demonstrate that such a dielectric arises naturally from quasilinear theory, and that there is a precise mathematical relationship between the wave and energy dielectrics.

III. Local Energy in Action Space

The application of the *Action-Angle* formalism to quasilinear Vlasov theory is particularly elegant because of the natural way in which symmetric forms are produced.¹⁹ For many geometries the practical determination of the conserved actions and the transformation from Cartesian canonical variables is quite difficult. However, the universality of the action-angle's fundamental equations helps to emphasize the basic physical processes which affect the particles. Even without specifying the magnetic geometry, these equations shed considerable light on the local energy question.

Therefore, let us suppose that a system containing zero-order magnetic fields is set up in action-angles with Hamiltonian, $H(\Phi, J)$, where $\Phi(r, v)$ are the three angle variables, and $J(r, v)$ are the three actions. Then accordingly, the J are constants of particle motion, and the Φ coordinates of the particles vary linearly with time, and all physical quantities are periodic in them. It is useful to have in mind a specific example, such as the tokamak. In that particular case, we may crudely consider J_1 to be the action associated with magnetic moment, J_2 to be the action associated with parallel motion along the field lines, and J_3 to be the action associated with the flux surface on which a drift orbit is centered. Then Φ_1 is like the gyroangle (i.e., angle of $v_{\perp}$), Φ_2 is like the toroidal angle, and Φ_3 is like the poloidal angle. Notice that in this crude interpretation, Φ_2 , Φ_3 , and J_3 are "position-like" while Φ_1 , J_1 , and J_2 are "velocity-like." In particular, only one action is "position-like"; hence any physical quantity over just the action space contains but a single degree of position-space dependency.

The usual zero-order equations of particle motion are

$$\Phi^{(0)} = \frac{\partial H}{\partial \mathbf{J}} \equiv \Omega(\mathbf{J}) \quad , \quad \mathbf{J}^{(0)} = -\frac{\partial H}{\partial \Phi} = 0 \quad ,$$

where for our example, the angle frequencies are Ω_1 , a gyrofrequency, Ω_2 , a toroidal frequency, and Ω_3 , a bounce or passing frequency. The zero-order particle distribution is hypothetically randomized in angles, such that $F^{(0)} = F^{(0)}(\mathbf{J})$ depends only on the constants of motion. Then by a chain-rule evaluation of the Lorentz force,¹² the perturbations, $\Phi^{(1)}$ and $\mathbf{J}^{(1)}$, to the particle equations of motion in the linearized fields, can be found. (The actual expression of $\Phi^{(1)}$ and $\mathbf{J}^{(1)}$ in terms of the perturbed fields is quite tedious and requires a certain amount of agility with the phase-space transformation $(\mathbf{r}, \mathbf{v}) \leftrightarrow (\Phi, \mathbf{J})$. For this discussion, detailed expressions for $\Phi^{(1)}$ and $\mathbf{J}^{(1)}$ will not be needed.)

The Vlasov equation is then

$$\left[\frac{\partial}{\partial t} + (\Omega + \Phi^{(1)}) \cdot \frac{\partial}{\partial \Phi} + \mathbf{J}^{(1)} \cdot \frac{\partial}{\partial \mathbf{J}} \right] (F^{(0)}(\mathbf{J}, t) + F^{(1)}(\Phi, \mathbf{J}, t)) = 0 \quad . \quad (4a)$$

The first-order distribution is assumed to vary sinusoidally in time with angular frequency, ω , and to be decomposed into angle harmonics: $F^{(1)}(\Phi, \mathbf{J}, t) = \text{Re} \{ \sum_{\mathbf{m}} F_{\mathbf{m}}^{(1)}(\mathbf{J}) e^{i(\mathbf{m} \cdot \Phi - \omega t)} \}$. Similarly, the orbit perturbations, $\Phi^{(1)}$ and $\mathbf{J}^{(1)}$, are assumed sinusoidal in time, and are likewise decomposed in angle harmonics. The first-order terms of Eq. (4a) allow explicit solution for the perturbed distribution in terms of the orbit perturbations:

$$F_{\mathbf{m}}^{(1)}(\mathbf{J}) = \frac{\mathbf{J}_{\mathbf{m}}^{(1)} \cdot \frac{\partial F^{(0)}}{\partial \mathbf{J}}}{i(\omega - \mathbf{m} \cdot \Omega)} \quad . \quad (4b)$$

The second order, or quasilinear, terms of Eq. (4a) are,

$$\frac{\partial F^{(0)}}{\partial t} = -\frac{1}{2} \text{Re} \left\langle \Phi^{(1)*} \cdot \frac{\partial F^{(1)}}{\partial \Phi} + \mathbf{J}^{(1)*} \cdot \frac{\partial F^{(1)}}{\partial \mathbf{J}} \right\rangle \quad , \quad (4c)$$

where the brackets, $\langle \Phi(\Phi) \rangle$, refer to the angle average: $(2\pi)^{-3} \int d^3\Phi \Phi(\Phi)$.

One may then use the *Divergence Theorem*,¹²

$$\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{J}^{(1)} + \frac{\partial}{\partial \Phi} \cdot \Phi^{(1)} = 0$$

and employ Eq. (4b) for $F^{(1)}$ to get

$$\frac{\partial F^{(0)}}{\partial t} = -\frac{\partial}{\partial J} \cdot D(J) \cdot \frac{\partial F^{(0)}}{\partial J} , \quad (5a)$$

where

$$D(J) = \frac{1}{2} \operatorname{Im} \left\{ \sum_{\mathbf{m}} \frac{J_{\mathbf{m}}^{(1)*} J_{\mathbf{m}}^{(1)}}{\omega - \mathbf{m} \cdot \Omega} \right\} , \quad (5b)$$

which shows that $F^{(0)}$ evolves diffusively in action space. Define an *rf-driven current over action space* as

$$\Gamma_{rf}(J) = D(J) \cdot \frac{\partial F^{(0)}}{\partial J} , \quad (6a)$$

which allows the quasilinear equation to be expressed as the simple continuity equation:

$$\frac{\partial F^{(0)}}{\partial t} + \frac{\partial}{\partial J} \cdot \Gamma_{rf} = 0 . \quad (6b)$$

A natural energy balance relation can be arrived at by taking the energy moment of Eq. (6b). Multiplying by the Hamiltonian, $H(J)$, and separating out the energy flow, $H\Gamma_{rf}$, gives

$$\frac{\partial}{\partial t} (HF^{(0)}) + \frac{\partial}{\partial J} \cdot (H\Gamma_{rf}) = W_s(J) , \quad (7a)$$

which is also in the form of a continuity equation, except that there is now a source on the right-hand side; this defines the *local rf-absorption over action space*. It is

$$W_s(J) = \frac{\partial H}{\partial J} \cdot \Gamma_{rf} = \Omega \cdot D \cdot \frac{\partial F^{(0)}}{\partial J} = \frac{1}{2} \operatorname{Im} \left\{ \sum_{\mathbf{m}} \frac{\Omega \cdot J_{\mathbf{m}}^{(1)*} J_{\mathbf{m}}^{(1)}}{\omega - \mathbf{m} \cdot \Omega} \right\} \cdot \frac{\partial F^{(0)}}{\partial J} . \quad (7b)$$

A remarkable simplification occurs when the distribution is a function of only energy, $F^{(0)}(J) = F^{(0)}(H(J))$, such as for an equilibrium distribution. In this case $\partial F^{(0)} / \partial J = \Omega \, dF^{(0)} / dH$, giving

(8)

$$W_a(J) = \frac{1}{2} \operatorname{Im} \left\{ \sum_m \frac{|\Omega \cdot J_m^{(1)}|^2}{\omega - m \cdot \Omega} \right\} \frac{dF^{(0)}}{dH}$$

$$= \frac{\pi}{2} \left(\frac{dF^{(0)}}{dH} \right) \sum_m \delta(\omega - m \cdot \Omega) |\Omega \cdot J_m^{(1)}|^2,$$

where the Plemelj formula, and causality have been used to evaluate the resonant denominator. Absorption occurs only on the resonant action-surfaces defined by $\omega = m \cdot \Omega(J)$, and for a equilibrium distribution ($-dF^{(0)}/dH \geq 0$) is always positive in sign.

The above positivity-theorem must, of course, be qualified for our practical example, the tokamak. First, when there are minor-radius gradients in density and temperature, then $F^{(0)}(J)$ contains noncanonical dependence on the flux surface action, J_3 . In such a case, Eq. (7b) may yield negative absorption in action space, which signifies the presence of various drift-wave instabilities. Second, the positivity-theorem required *angle-averaging*, and recall that in our example two of the angles were "position-like," the toroidal angle, Φ_2 , and the poloidal angle, Φ_3 . Hence the positivity-theorem requires a certain amount of "position-averaging" even though it is local to action space. This "position-averaging" is, in fact, a common practice in quasilinear analysis, where it is called "bounce-averaging" or "flux-surface-averaging."²⁰ In the next section an investigation of local energy in the lab frame will be made. In general circumstances it will be possible to prove non-negativity of absorption for the canonical distribution only after the appropriate position "angle-averaging" is performed. It should be realized that local regions of negative absorption *before angle-averaging* (locally unstable regions) does not directly imply global instability. This fact is as true for high-power rf-heating as it is for global MHD stability.

Before proceeding on to the next section, it is useful to make a couple remarks on the results of this section. First let us note that an alternate intuitive definition of the local absorption in action space might be made by the formula, $W_a = 1/2 \operatorname{Re} \langle H'^{(1)*} F^{(1)} \rangle$ where $H'^{(1)} = \partial H / \partial J$, $J'^{(1)} = \Omega$. $J'^{(1)}$ is the first order time rate of change to the Hamiltonian. Happily, this definition produces Eq. (7b) precisely. Second, note that in the equation for local absorption, the perturbed fields, which are carried in the quantities $J_m^{(1)}$, appear in the symmetric form, $J_m^{(1)*} J_m^{(1)}$. It has been shown that the $J_m^{(1)}$ are, in general, nonlocally related to the electric fields even after they are expressed in action-space.¹² Hence, this symmetric quantity is a good example of a nonlocal dielectric which operates both right and left on electric field, symbolically as $\{E\} J_m^{(1)*} J_m^{(1)} \{E\}$, which was discussed in the last section.

IV. Local Energy in the Lab Frame

Various concepts of local energy in configuration space have been in use for decades. They

are especially prevalent in ray-tracing computations, where the WKB approximation is in use. Full-wave local energy forms for finite-Larmor truncated dielectric operators have also been widely employed, although these have occasionally engendered considerable debate as to their correctness. In this section, I hope to come to some understanding of configuration-space local energy. One of the most urgent necessities, I feel, is to make some connection between the "local absorption" quantities and the particles. To do so, I shall try to follow the general pattern set forth in the previous section. *Most importantly, the local absorption quantity will derive from the energy moment of the quasilinear equation.*

The fundamental difference between the action-angle formalism and the traditional phase-space formalism is that the former is "particle-oriented" whereas the latter is "field-oriented." Only three quantities, J , are needed to label a particle orbit, and Maxwell's equations are expressed over only the three position variables, r . In action-angles, the electromagnetic fields appear as functions of six variables, $E(\Phi, J)$, instead of three. In traditional phase space, all six quantities, r and v , vary along a particle's orbit. If one is interested in *field quantities*, then obviously the traditional phase-space is favored. If one is interested in *particle quantities*, then conversely the action-angle phase-space is favored. Unfortunately, what is actually desired is an expression for *particle energies in terms of the perturbing fields*. Practically speaking, we need some formalism that is "in-between" action-angles and traditional (r, v) phase space.

Basic Gyrokinetics

Not surprisingly, such an "in-between" formalism already exists, and has been studied extensively. This alternate formalism is gyrokinetic analysis. Basically gyrokinetics is a "1/3" action-angle formalism. The traditional phase space, (r, v) , is mapped to a "gyro-centered" phase-space, (R, V) , according to the implicit relations

$$\begin{cases} r = R + \rho(R, V) \\ v = V \end{cases} \quad ; \quad \rho(R, V) = \frac{b(R)}{\Omega(R)} \times V \quad , \quad (9)$$

where R is the guiding center position and V is still the velocity. The gyroradius, $\rho(R, V)$, requires the unperturbed magnetic field, $q_m B_0(R) = \Omega(R) b(R)$. (Most gyrokinetic analyses then map V to E, μ , and ϕ , energy, magnetic moment, and gyroangle. For calculating the plasma current, it is more convenient to express V in terms of $V_{||}$, $V_{\perp}$, and ϕ .) Conceptually, gyrokinetics "action-anglicizes" the gyromotion, but not the guiding-center motion. In the end, the local absorption in gyro-centered position will be symmetric in gyromotion factors, but not symmetric in guiding-center factors. In the important limit of slab geometry with strictly perpendicular gradients, the guiding center motion is irrelevant, and gyrokinetics produces entirely symmetric forms, and they are equivalent to the results from action-angle formalism.

With transformation (9) the traditional phase-space derivatives are (exactly)

$$\begin{bmatrix} \frac{\partial}{\partial \mathbf{r}} \\ \frac{\partial}{\partial \mathbf{v}} \end{bmatrix} = \begin{bmatrix} [1 + \nabla \frac{\mathbf{b}}{\Omega} \times \mathbf{V}]^{-1} & 0 \\ \frac{\mathbf{b}}{\Omega} \times [1 + \nabla \frac{\mathbf{b}}{\Omega} \times \mathbf{V}]^{-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial \mathbf{R}} \\ \frac{\partial}{\partial \mathbf{V}} \end{bmatrix}, \quad (10)$$

where, in this section, " ∇ " will always refer to the $\partial/\partial \mathbf{R}$ derivatives. (Warning: diadic notation is used extensively!) The particle distribution functions will be written with a large "F" when they are over gyro-centered phase-space and with a small "f" when they are over traditional phase-space. They are, however, quantitatively equal when the $(\mathbf{r}, \mathbf{v})$ and $(\mathbf{R}, \mathbf{V})$ are related according to equations (9):

$$F^{(0)}(\mathbf{R}, \mathbf{V}, t) = f^{(0)}(\mathbf{r}(\mathbf{R}, \mathbf{V}), \mathbf{v}(\mathbf{R}, \mathbf{V}), t)$$

and

$$F^{(1)}(\mathbf{R}, \mathbf{V})e^{-i\omega t} = f^{(1)}(\mathbf{r}(\mathbf{R}, \mathbf{V}), \mathbf{v}(\mathbf{R}, \mathbf{V}))e^{-i\omega t}.$$

(Tangential remark. Collisional effects are not treated in this paper, hence the simplest separation involving a slowly evolving distribution and a fast oscillating distribution is employed. While accurate time-scale and small-parameter orderings are impressive, they are also unwieldy. This paper takes a more cavalier approach, mostly because the principle topics being discussed are still being studied at the zero-order level.)

The quasilinear Vlasov equation in the traditional phase space is

$$\left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \Omega(\mathbf{v} \times \mathbf{b}) \cdot \frac{\partial}{\partial \mathbf{v}} \right] f^{(0)} = -\frac{q}{m} \frac{\partial}{\partial \mathbf{v}} \cdot \frac{1}{2} \text{Re} \{ [\mathbf{E}^* + \mathbf{v} \times \mathbf{H}^*] f^{(1)} \}. \quad (11)$$

The transformation of the operator on the LHS to gyro-centered phase-space is

$$\begin{aligned} \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \Omega(\mathbf{v} \times \mathbf{b}) \cdot \frac{\partial}{\partial \mathbf{v}} &\rightarrow \frac{\partial}{\partial t} + \mathbf{V} \cdot \mathbf{b}\mathbf{b} \cdot \nabla + [\nabla \cdot \mathbf{V}\mathbf{V} \times \frac{\mathbf{b}}{\Omega}] \cdot \nabla + \Omega(\mathbf{V} \times \mathbf{b}) \cdot \frac{\partial}{\partial \mathbf{V}} \\ &\quad - \frac{\mathbf{b}}{\Omega} \cdot [\nabla \times \mathbf{V}\mathbf{V} \times \Omega\mathbf{b}] \cdot \frac{\partial}{\partial \mathbf{V}} - [1 - \mathbf{b}\mathbf{b}] : [\nabla \mathbf{V}\mathbf{V} \times \frac{\mathbf{b}}{\Omega}] \cdot \nabla \\ &\quad - \frac{\mathbf{b}}{\Omega} \cdot [\nabla \times \mathbf{V}\mathbf{V} \times \Omega\mathbf{b}] \cdot [\frac{\mathbf{b}}{\Omega} \times \nabla], \end{aligned} \quad (12)$$

where the guiding center drift approximation is employed, $\|\nabla \mathbf{b}/\Omega \times \mathbf{V}\| \ll 1$, by keeping only terms which are first order in this small quantity.

The tensor, $\nabla(\Omega\mathbf{b})/\Omega$, can be resolved in the $(\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{b})$ directions, with the elements

(13)

$$\frac{\nabla(\Omega \mathbf{b})}{\Omega} = \frac{\nabla \Omega}{\Omega} \mathbf{b} + \nabla \mathbf{b} = \begin{bmatrix} \nabla_{\perp} \mathbf{b} & \frac{\nabla_{\parallel} \Omega}{\Omega} \\ \mathbf{b} \cdot \nabla \mathbf{b} & \frac{\mathbf{b} \cdot \nabla \Omega}{\Omega} \end{bmatrix} = \begin{bmatrix} b_{xx} & b_{xy} & \frac{\cos \phi_{\perp}}{L_{\perp}} \\ b_{yx} & b_{yy} & \frac{\sin \phi_{\perp}}{L_{\perp}} \\ \frac{\cos \phi_{\parallel}}{R_c} & \frac{\sin \phi_{\parallel}}{R_c} & \frac{1}{L_{\parallel}} \end{bmatrix}$$

where use of the relation, $(\nabla \mathbf{b}) \cdot \mathbf{b} = 0$, has been made. The divergence condition, $\nabla \cdot (\Omega \mathbf{b}) = 0$, relates the parallel gradient, $1/L_{\parallel}$, to the other diagonal elements: $b_{xx} + b_{yy} + 1/L_{\parallel} = 0$. The zero-order Ampère's Law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_0$, relates the torsional elements, b_{xy} and b_{yx} , to the parallel current: $b_{xy} - b_{yx} = \mu_0 \mathbf{b} \cdot \mathbf{J}_0 / B_0 = 1/L_{\parallel}$, and relates the difference between curvature and $\nabla_{\perp} \mathbf{B}$ vectors to the perpendicular current: $\nabla_{\perp} \Omega / \Omega - \mathbf{b} \cdot \nabla \mathbf{b} = \mu_0 \mathbf{b} \times \mathbf{J}_0 / B_0$. Notice that if the zero-order current is null, then $\nabla(\Omega \mathbf{b})$ is a symmetric tensor.

With $\mathbf{V}$ expressed in terms of $V_{\parallel}$, $V_{\perp}$, and ϕ , assume gyrophase randomization of the zero-order distribution, such that $F^{(0)} = F^{(0)}(\mathbf{R}, V_{\parallel}, V_{\perp})$. The tensor elements, (13), then allow us to write explicitly the gyroaverage of the operators of Eq. (12). The well-known result is:

$$\left\langle \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \Omega(\mathbf{r} \times \mathbf{b}) \cdot \frac{\partial}{\partial \mathbf{v}} \right\rangle \rightarrow \frac{\partial}{\partial t} + \mathbf{V}_{gc} \cdot \nabla + \frac{V_{\perp}}{2L_{\parallel}} \left(V_{\parallel} \frac{\partial}{\partial V_{\perp}} - V_{\perp} \frac{\partial}{\partial V_{\parallel}} \right), \quad (14)$$

where now $\langle \dots \rangle$ indicates only the gyroangle average, $(2\pi)^{-1} \int d\phi$, and where the guiding-center velocity is:

$$\mathbf{V}_{gc} = V_{\parallel} \mathbf{b} + \frac{V_{\perp}^2}{2\Omega^2} \mathbf{b} \times \nabla_{\perp} \Omega + \frac{V_{\parallel}^2}{\Omega} \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}),$$

which includes the usual curvature and ∇B drifts; and where the parallel gradient, $1/L_{\parallel}$, acts on any anisotropy in the velocity.

(Another tangential remark. The Jacobian of the transformation $(\mathbf{r}, \mathbf{v}) \rightarrow (\mathbf{R}, \mathbf{V})$ is, to first order,

$$\frac{\partial(\mathbf{r}, \mathbf{v})}{\partial(\mathbf{R}, \mathbf{V})} = \det \left\{ 1 + \nabla \frac{\mathbf{b}}{\Omega} \times \mathbf{V} \right\} = 1 + \text{trace} \left\{ \nabla \frac{\mathbf{b}}{\Omega} \times \mathbf{V} \right\} = 1 + \nabla \cdot \rho$$

which gyroaverages to:

$$\left\langle \frac{\partial(\mathbf{r}, \mathbf{v})}{\partial(\mathbf{R}, \mathbf{V})} \right\rangle = 1 + \frac{V_{\perp}}{\Omega L_{\perp}}.$$

This contains the torsional scale length, $L_{\perp}$, which is proportional to the zero-order parallel plasma current. For plasma confinement by external fields this scale length should be quite large. In this paper I shall take the Jacobian to be unity, and simply remark that the entire torsional problem is

something which could benefit from further research, especially when fine details of the gyromotion could be important as for resonance heating.)

The gyro-transformation of the RHS of Eq. (11) is:

$$\begin{aligned} & -\frac{q}{m} \frac{\partial}{\partial \mathbf{v}} \cdot \frac{1}{2} \operatorname{Re} \{ [\mathbf{E}^* + \mathbf{v} \times \mathbf{H}^*] f^{(1)} \} \\ & \rightarrow -\frac{q}{m} \frac{\partial}{\partial \mathbf{V}} \cdot \frac{1}{2} \operatorname{Re} \{ [\mathbf{E}^*(\mathbf{r}(\mathbf{R}, \mathbf{V})) + \mathbf{V} \times \mathbf{H}^*(\mathbf{r}(\mathbf{R}, \mathbf{V}))] F^{(1)}(\mathbf{R}, \mathbf{V}) \} \\ & \quad - \frac{q}{m} \left(\frac{\mathbf{b}}{\Omega} \times \nabla \right) \cdot \frac{1}{2} \operatorname{Re} \{ [\mathbf{E}^*(\mathbf{r}(\mathbf{R}, \mathbf{V})) + \mathbf{V} \times \mathbf{H}^*(\mathbf{r}(\mathbf{R}, \mathbf{V}))] F^{(1)}(\mathbf{R}, \mathbf{V}) \} \quad , \end{aligned} \quad (15)$$

where the higher order term in $(\mathbf{b}/\Omega \times \nabla) : \nabla (\mathbf{E}^* F^{(1)})$ has been neglected, and guiding-center position and velocity dependencies are explicitly shown. The second line of (15) is the usual velocity diffusion, and the third line encompasses the rf-induced transport of guiding centers and drifts associated with the ponderomotive force.²¹

Proceed with an energy balance relation in the same manner as for the action-angle formalism, Eqs. (5) - (7). Hence, let us define an rf-driven current on velocity space as:

$$\Gamma_{\text{rf}} \equiv \frac{q}{2m} \operatorname{Re} \{ [\mathbf{E}^*(\mathbf{r}(\mathbf{R}, \mathbf{V})) + \mathbf{V} \times \mathbf{H}^*(\mathbf{r}(\mathbf{R}, \mathbf{V}))] F^{(1)}(\mathbf{R}, \mathbf{V}) \}$$

which contributes to the time evolution of the zero-order distribution function in the following continuity equation:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}_{\text{gc}} \cdot \nabla + \frac{\mathbf{v}_{\perp}}{2L_{\parallel}} \left(v_{\parallel} \frac{\partial}{\partial v_{\perp}} - v_{\perp} \frac{\partial}{\partial v_{\parallel}} \right) \right] F^{(0)} + \left\langle \left[\frac{\partial}{\partial \mathbf{V}} + \frac{\mathbf{b}}{\Omega} \times \nabla \right] \cdot \Gamma_{\text{rf}} \right\rangle = 0 \quad .$$

Next multiply by the Hamiltonian, $H = 1/2 m V^2$, and isolate the energy flux, $H \Gamma_{\text{rf}}$, with whatever is left becoming the source,

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \mathbf{v}_{\text{gc}} \cdot \nabla + \frac{\mathbf{v}_{\perp}}{2L_{\parallel}} \left(v_{\parallel} \frac{\partial}{\partial v_{\perp}} - v_{\perp} \frac{\partial}{\partial v_{\parallel}} \right) \right] (H F^{(0)}) \\ & \quad + \left\langle \left[\frac{\partial}{\partial \mathbf{V}} + \frac{\mathbf{b}}{\Omega} \times \nabla \right] \cdot (H \Gamma_{\text{rf}}) \right\rangle = W_{\text{a}}(\mathbf{R}, v_{\perp}, v_{\parallel}) \quad . \end{aligned}$$

This defines the local absorption in gyro-centered phase-space as

$$W_{\text{a}}(\mathbf{R}, v_{\perp}, v_{\parallel}) \equiv \left\langle \frac{\partial H}{\partial \mathbf{V}} \cdot \Gamma_{\text{rf}} \right\rangle = \frac{q}{2} \operatorname{Re} \{ \mathbf{E}^*(\mathbf{r}(\mathbf{R}, \mathbf{V})) \cdot \mathbf{V} F^{(1)}(\mathbf{R}, \mathbf{V}) \} \quad (16)$$

which, just as with action-angles, corresponds to $1/2 \text{Re}\langle H^{(1)*} F^{(1)} \rangle$. Please compare and note the differences between this and McVey *et al.*'s postulated absorption. To make further progress, in particular to be able to determine when the above form will be symmetric, it is necessary to further evaluate the perturbed distribution, $F^{(1)}$, in terms of $F^{(0)}$ and the electric field.

Evaluation of the Local Absorption

Returning briefly to the lab phase-space, $(\mathbf{r}, \mathbf{v})$, the perturbed distribution satisfies the complex linearized Vlasov equation:

$$\left[-i\omega + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \Omega(\mathbf{r}) (\mathbf{v} \times \mathbf{b}(\mathbf{r})) \cdot \frac{\partial}{\partial \mathbf{v}} \right] f^{(1)} = -\frac{q}{m} \left[\frac{\partial}{\partial \mathbf{v}} f^{(0)} \right] \cdot \left[1 + \frac{1}{i\omega} \mathbf{v} \times \frac{\partial}{\partial \mathbf{r}} \times 1 \right] \cdot \mathbf{E}(\mathbf{r})$$

which has solution

$$f^{(1)}(\mathbf{r}, \mathbf{v}) = -\frac{q}{m} \int_0^\infty d\tau e^{i\omega\tau} \left[\frac{\partial}{\partial \mathbf{v}'} f^{(0)}(\mathbf{r}', \mathbf{v}') \right] \cdot \left[1 + \frac{1}{i\omega} \mathbf{v}' \times \frac{\partial}{\partial \mathbf{r}'} \times 1 \right] \cdot \mathbf{E}(\mathbf{r}') \quad (17)$$

where $\mathbf{r}'(\mathbf{r}, \mathbf{v}, \tau)$ and $\mathbf{v}'(\mathbf{r}, \mathbf{v}, \tau)$ are the unperturbed orbits (characteristics) satisfying:

$$-\frac{\partial}{\partial \tau} \mathbf{r}' = \mathbf{v}', \quad -\frac{\partial}{\partial \tau} \mathbf{v}' = \Omega(\mathbf{r}') \mathbf{v}' \times \mathbf{b}(\mathbf{r}'); \quad \mathbf{r}'|_{\tau=0} = \mathbf{r}, \quad \mathbf{v}'|_{\tau=0} = \mathbf{v} \quad (18)$$

The nonlocal evaluation, $\mathbf{E}(\mathbf{r}')$, in Eq. (17) can be generically treated by giving $\mathbf{E}(\mathbf{r})$ some predetermined spatial variation of a chosen set of basis functions. If the basis set is complete for all possible variations in electric field, then no loss of generality occurs. The Fourier basis set is satisfactory for our purposes. Hence, define

$$\mathbf{E}(\mathbf{r}) = \int d^3 \mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}} \mathbf{E}(\mathbf{k}) \quad \Leftrightarrow \quad \mathbf{E}(\mathbf{k}) = (2\pi)^{-3} \int d^3 \mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \mathbf{E}(\mathbf{r})$$

Linearity then guarantees

$$f^{(1)}(\mathbf{r}, \mathbf{v}) = -\frac{q}{m} \int d^3 \mathbf{k} \int_0^\infty d\tau e^{i(\omega\tau + \mathbf{k} \cdot \mathbf{r})} \left[\frac{\partial}{\partial \mathbf{v}'} f^{(0)}(\mathbf{r}', \mathbf{v}') \right] \cdot \left[1 + \frac{1}{i\omega} \mathbf{v}' \times \mathbf{k} \times 1 \right] \cdot \mathbf{E}(\mathbf{k}) \quad (19)$$

Next transform Eq. (19) to gyro-centered phase space, $(\mathbf{R}, \mathbf{V})$. Introducing the guiding-center-drift orbits, $\mathbf{R}'(\mathbf{R}, \mathbf{V}, \tau)$ and $\mathbf{V}'(\mathbf{R}, \mathbf{V}, \tau)$, it is possible to avoid the more complicated orbit Eqs. (18). With the gyroradius, $\rho' = b'/\Omega' \times \mathbf{V}'$, where notationally, $\mathbf{b}' = \mathbf{b}(\mathbf{R}')$ and $\Omega' = \Omega(\mathbf{R}')$, and ignoring the higher order term in $(b'/\Omega' \times \nabla \rho) \cdot \nabla F^{(0)}$, this is

$$\begin{aligned}
F^{(1)}(\mathbf{R}, \mathbf{V}) = & \\
& -\frac{q}{m} \int d^3 \mathbf{k} \int_0^\infty d\tau e^{i(\omega\tau + \mathbf{k} \cdot (\mathbf{R}' + \rho'))} \left[\frac{\partial}{\partial \mathbf{V}} F^{(0)}(\mathbf{R}', \mathbf{V}') \right] \cdot \left[1 + \frac{1}{i\omega} \mathbf{V}' \times \mathbf{k} \times \mathbf{1} \right] \cdot \mathbf{E}(\mathbf{k}) \\
& -\frac{q}{m} \int d^3 \mathbf{k} \int_0^\infty d\tau e^{i(\omega\tau + \mathbf{k} \cdot (\mathbf{R}' + \rho'))} \left[\frac{\mathbf{b}'}{\Omega} \times \nabla F^{(0)}(\mathbf{R}', \mathbf{V}') \right] \cdot \left[1 + \frac{1}{i\omega} \mathbf{V}' \times \mathbf{k} \times \mathbf{1} \right] \cdot \mathbf{E}(\mathbf{k}) .
\end{aligned}$$

One of the beneficial aspects of the gyro-transformation is that the nonlocal spatial evaluation of $F^{(0)}$ occurs only along the guiding-center trajectory, $\mathbf{R}'$, which should vary weakly in space and contains none of the implicit cyclotron variations involving the perpendicular gradients and the gyromotion. Instead, these diamagnetic effects are explicitly shown as the last line of Eq. (20), which resulted from straight forward transformation of derivatives according to Eq. (10).

To find the local absorption, the quantity $E^*(\mathbf{r}(\mathbf{R}, \mathbf{V}))$ is also required. Using the Fourier basis this is $E^*(\mathbf{r}(\mathbf{R}, \mathbf{V})) = \int d^3 \mathbf{k} e^{-i\mathbf{k} \cdot (\mathbf{R} + \rho(\mathbf{R}, \mathbf{V}))} E^*(\mathbf{k})$. Then using equation (20) for $F^{(1)}$ in equation (16) for the local absorption, find:

$$\begin{aligned}
W_s(\mathbf{R}, \mathbf{V}_\perp, \mathbf{V}_\parallel) = & \\
& \frac{1}{2} \text{Im} \left(\omega \epsilon_0 \iint d^3 \mathbf{k}_1 d^3 \mathbf{k}_2 e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{R}} E^*(\mathbf{k}_2) \cdot \mathbf{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}_2) \cdot \mathbf{E}(\mathbf{k}_1) \right) ,
\end{aligned} \tag{21a}$$

where

$$\begin{aligned}
\mathbf{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}_2) = & -i \frac{q^2}{m \omega \epsilon_0} \int_0^\infty d\tau e^{i(\omega\tau + \mathbf{k}_1 \cdot (\mathbf{R}' - \mathbf{R}))} e^{i(\mathbf{k}_1 \cdot \rho' - \mathbf{k}_2 \cdot \rho)} \\
& \mathbf{V} \left[1 - \frac{\mathbf{k}_1 \cdot \mathbf{V}}{\omega} + \frac{\mathbf{V} \cdot \mathbf{k}_1}{\omega} \right] \cdot \left[\left(\frac{\partial}{\partial \mathbf{V}} + \frac{\mathbf{b}'}{\Omega} \times \nabla \right) F^{(0)}(\mathbf{R}', \mathbf{V}') \right]
\end{aligned} \tag{21b}$$

is the local absorption kernel, which acts on the electric fields both right and left. Equations (21) are the primary goal of this paper. Let us pause to examine their forms.

The most important aspect of the local absorption kernel is the quadratically operating gyromotion, as shown by the gyrophase factor, $e^{i(\mathbf{k}_1 \cdot \rho' - \mathbf{k}_2 \cdot \rho)}$, which contains both $\mathbf{k}_1$ and $\mathbf{k}_2$ "operators." Note that the guiding-center motion is *not* quadratically operating, as evidenced by the factor, $e^{i\mathbf{k}_1 \cdot (\mathbf{R}' - \mathbf{R})}$, which contains only $\mathbf{k}_1$. Also notice that the anisotropy effects also do not operate quadratically. Hence, the assertion that gyrokinetics is only a "1/3" - symmetrized theory.

The gyroangle average in Eq. (21a) may be accomplished via the usual Bessel function / harmonic expansion of the gyrophase term $e^{i(\mathbf{k}_1 \cdot \rho' - \mathbf{k}_2 \cdot \rho)}$. However, the process must now distinguish between $\mathbf{k}_1$ and $\mathbf{k}_2$. These calculations are given in Appendix A. Further integration over the remaining two velocity variables will yield a *local absorption in guiding-center position space* which is also shown in Appendix A for a Maxwellian distribution. At that point it is

possible to demonstrate positive definiteness under the constraints of slab geometry with strictly perpendicular gradients. To demonstrate positive definiteness in tokamak geometry it is necessary to perform an additional position-space "bounce-averaging." This is done in Appendix B.

Relation between the Energy Kernel and the Dielectric Kernel

Suppose we were to apply Eq. (21) to a uniform plasma, to calculate the phase-space integrated absorption. Since $W \rightarrow W(V, k_1, k_2)$ would be independent of R , the volume integral may be performed first, giving $\int d^3R e^{i(k_1 - k_2) \cdot R} \rightarrow (2\pi)^{-3} \delta(k_1 - k_2)$. Hence we would have

$$\int d^3R \int d^3V W_a(R, V) = \frac{1}{2} \text{Im} \left\{ \omega \epsilon_0 \frac{1}{(2\pi)^3} \int d^3k \int d^3V E^*(k) \cdot W(V, k_1 = k_2 = k) \cdot E(k) \right\}.$$

From Eq. (21) notice that if $K(k)$ is the uniform plasma dielectric, then $K(k) = 1 + \sum_a \int d^3V W_a(V, k_1 = k_2 = k)$. Actually, it is possible to formulate a more general local property for nonuniform plasmas.

In the traditional lab frame, let us seek the dielectric kernel in the form $K(r, k)$, that is,

$$K\{E\} = \int d^3k e^{ik \cdot r} K(r, k) \cdot E(k).$$

The kernel is related to the perturbed distribution via the plasma current, $J_p(r) = -i\omega \epsilon_0 [K - 1] \{E\} = \sum_a q_a \int d^3v v f^{(1)}(r, v)$. And using $f^{(1)}(r(R, V), v(R, V)) = F^{(1)}(R, V)$ and transformation equation (9), relate $f^{(1)}$ and $F^{(1)}$ via a Taylor series

$$f^{(1)}(r, v) = e^{-\rho_1 \cdot \nabla_2} F^{(1)}(R_2, V_2) \left| \begin{array}{l} R_1 = R(r, v) \\ R_2 = r \\ V_1 = V_2 = v \end{array} \right.$$

using the notation, $\rho_1 = p(R_1, V_1)$, and $R(r, v)$ is the inverse relation (9). Then using Eq. (20) for $F^{(1)}(R, V)$, employing several operator identities (such as: $e^{d/dx} [uv] = [e^{d/dx} u] [e^{d/dx} v]$), and using notation similar to above, the plasma current is

$$J_p(\mathbf{r}) = -\sum_s \frac{q^2}{m} \int d^3\mathbf{k} \int d^3\mathbf{v} e^{-\rho_1 \nabla_2} e^{i\mathbf{k} \cdot (\rho_2 - \rho_1)} \int_0^\infty d\tau e^{i(\omega\tau + \mathbf{k} \cdot (\mathbf{R}'_2 - \mathbf{R}_2))} e^{i\mathbf{k} \cdot (\rho'_2 - \rho_2)} \\ V_2 \left[\left(\frac{\partial}{\partial \mathbf{V}_2} + \frac{\mathbf{b}'_2}{\Omega_2} \times \mathbf{V}_2 \right) F^{(1)}(\mathbf{R}'_2, \mathbf{V}_2) \right] \cdot \left[1 + \frac{1}{\omega} \mathbf{V}_2 \times \mathbf{k} \times 1 \right] \cdot \mathbf{E}(\mathbf{k}) \Bigg|_{\substack{\mathbf{R}_1 = \mathbf{R}(\mathbf{r}, \mathbf{v}) \\ \mathbf{R}_2 = \mathbf{r} \\ \mathbf{V}_1 = \mathbf{V}_2 = \mathbf{v}}}$$

from which it follows that

$$\mathbb{K}(\mathbf{r}, \mathbf{k}) = 1 - i \sum_s \frac{q^2}{m} \int d^3\mathbf{v} e^{-\rho_1 \nabla_2} e^{i\mathbf{k} \cdot (\rho_2 - \rho_1)} \int_0^\infty d\tau e^{i(\omega\tau + \mathbf{k} \cdot (\mathbf{R}'_2 - \mathbf{R}_2))} e^{i\mathbf{k} \cdot (\rho'_2 - \rho_2)} \\ V_2 \left[\left(\frac{\partial}{\partial \mathbf{V}_2} + \frac{\mathbf{b}'_2}{\Omega_2} \times \mathbf{V}_2 \right) F^{(1)}(\mathbf{R}'_2, \mathbf{V}_2) \right] \cdot \left[1 + \frac{1}{\omega} \mathbf{V}_2 \times \mathbf{k} \times 1 \right] \Bigg|_{\substack{\mathbf{R}_1 = \mathbf{R}(\mathbf{r}, \mathbf{v}) \\ \mathbf{R}_2 = \mathbf{r} \\ \mathbf{V}_1 = \mathbf{V}_2 = \mathbf{v}}}$$

Comparison to Eq. (21) then shows

$$\mathbb{K}(\mathbf{r}, \mathbf{k}) = 1 + \sum_s \int d^3\mathbf{v} e^{-\rho_1 \nabla_2} e^{i\mathbf{k} \cdot (\rho_2 - \rho_1)} \mathbf{W}(\mathbf{R}_2, \mathbf{V}_2, \mathbf{k}_1, \mathbf{k}_2) \Bigg|_{\substack{\mathbf{R}_1 = \mathbf{R}(\mathbf{r}, \mathbf{v}) \\ \mathbf{R}_2 = \mathbf{r} \\ \mathbf{V}_1 = \mathbf{V}_2 = \mathbf{v} \\ \mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}}} \quad (22)$$

which is a relation between the dielectric kernel in the laboratory frame and the absorption kernel in the gyro-centered phase-space. In practice, since the guiding center drift orbits are used, it is appropriate to ignore the difference between ρ_1 evaluated at the guiding center position and ρ_2 evaluated at the particle location, i.e., $\rho_1 = \rho_2 = \rho$. Furthermore, let us continue to keep only first order terms in $\nabla \mathbf{B}$ and/or $\nabla F^{(0)}$. Finally, note that the operator $\partial/\partial \mathbf{k}_2$ on $\mathbf{W}$ produces a factor of $-i\rho$. Hence, relation (22) simplifies to

$$\mathbb{K}(\mathbf{r}, \mathbf{k}) = 1 + \left(1 - i \frac{\partial}{\partial \mathbf{k}_2} \cdot \nabla_2 \right) \sum_s \int d^3\mathbf{v} \mathbf{W}(\mathbf{R}_2, \mathbf{V}_2, \mathbf{k}_1, \mathbf{k}_2) \Bigg|_{\substack{\mathbf{R}_2 = \mathbf{r} \\ \mathbf{V}_2 = \mathbf{v} \\ \mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}}} \quad (23)$$

This relation is exact in so far as the guiding-center-drift particle motions are concerned. In other words, it is exact to all orders in $k\rho$ of the electric field, and contains all first order terms in ρ/L , where L is the variation scale lengths of $\mathbf{B}$ and/or $F^{(0)}$. However, since there are ρ/L_B errors in the gyroradius of the guiding-center-drift approximation, the above becomes inaccurate at the range

$k\rho \approx L_B/\rho$. Under most circumstances this still allows a sizable $k\rho > 1$ capability. (The same restriction applies to the action-angle formalism, which employs a guiding-center Lagrangian.) One very new aspect of Eq. (23) is that the $\partial/\partial \mathbf{k} \cdot \nabla$ operator combination provides a generalization of the "odd-order derivative" terms²² to all orders in $\mathbf{k}_\perp$. (See Appendix A for details.)

The Kinetic Flux

Relation (23) can also be used to clarify the role of the so-called "kinetic flux" and the exact energy balance equation. With the Poynting Flux defined as $1/2 \text{Re}[\mathbf{E} \times \mathbf{H}]$, then recalling equation (1b), the divergence of the Poynting flux is $-\partial/\partial t \cdot \mathbf{P}(\mathbf{r}) = \text{Im}[\omega \epsilon_0 \mathbf{E}^* \cdot \mathbf{K}(\cdot \mathbf{E})]$; and the local absorption is

$$W_a(\mathbf{R}) = \frac{1}{2} \omega \epsilon_0 \text{Im} \left\{ \int d^3 k_1 \int d^3 k_2 e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{R}} \sum_j \int d^3 V \mathbf{E}^*(\mathbf{k}_2) \cdot [\cdot \cdot] \cdot \mathbf{E}(\mathbf{k}_1) \right\} .$$

$\uparrow$
 $[\mathbf{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}_2)]$

(In this section, pay closer attention to the second line of all two-line equations!)

Now suppose we (ad hoc) require a vector flux, $\mathbf{T}(\mathbf{R})$, such that

$$-\nabla \cdot \mathbf{T}(\mathbf{R}) = W_a(\mathbf{R}) + \left[\frac{\partial}{\partial \mathbf{r}} \cdot \mathbf{P}(\mathbf{r}) \right]_{\mathbf{r}=\mathbf{R}} . \quad (24)$$

(Since the Poynting flux is a field quantity defined over the lab position, and the absorption is a particle quantity which is defined over the gyro-centered position, we are in effect "comparing apples to oranges." Which means that the kinetic flux should come out to be ... !)

Humor aside, use relation (23) for $\mathbf{K}$ to get

$$-\nabla \cdot \mathbf{T}(\mathbf{R}) = \frac{1}{2} \omega \epsilon_0 \text{Im} \left\{ \int d^3 k_1 \int d^3 k_2 e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{R}} \sum_j \int d^3 V \mathbf{E}^*(\mathbf{k}_2) \cdot [\cdot \cdot] \cdot \mathbf{E}(\mathbf{k}_1) \right\} .$$

$\uparrow$
 $\left[\mathbf{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}_2) - \left(1 - i \frac{\partial}{\partial \mathbf{k}} \cdot \nabla \right) \mathbf{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_P, \mathbf{k}) \right]_{\mathbf{k}=\mathbf{k}_1}$

For very large $\mathbf{R}$, i.e., very far from regions of nonuniformity, the oscillating factor, $e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{R}}$, will effectively limit the width of the spectral convolution between $\mathbf{k}_1$ and $\mathbf{k}_2$ which contributes to $-\nabla \cdot \mathbf{T}$. Substitution of the following expansion of $\mathbf{W}$ in $\mathbf{k}$ -space can then be made arbitrarily accurate

$$\mathbb{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}_2) \equiv \mathbb{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}) \Big|_{\mathbf{k}=\mathbf{k}_1} + (\mathbf{k}_2 - \mathbf{k}_1) \cdot \frac{\partial}{\partial \mathbf{k}} \mathbb{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}) \Big|_{\mathbf{k}=\mathbf{k}_1}$$

With this restriction, the Kinetic flux satisfies

$$-\nabla \cdot \mathbf{T}(\mathbf{R}) = \frac{1}{2} \omega \epsilon_0 \operatorname{Re} \left\{ \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{R}} \sum_{\mathbf{s}} \int d^3 \mathbf{V} \mathbf{E}^*(\mathbf{k}_2) \cdot \left[\frac{\partial}{\partial \mathbf{k}} \mathbb{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}) \Big|_{\mathbf{k}=\mathbf{k}_1} \right] \right\}$$

for which an exact solution is

$$\mathbf{T}(\mathbf{R}) = -\frac{1}{2} \omega \epsilon_0 \operatorname{Re} \left\{ \int d^3 \mathbf{k}_1 \int d^3 \mathbf{k}_2 e^{i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{R}} \sum_{\mathbf{s}} \int d^3 \mathbf{V} \frac{\partial}{\partial \mathbf{k}} \mathbf{E}^*(\mathbf{k}_2) \cdot \mathbb{W}(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}) \cdot \mathbf{E}(\mathbf{k}_1) \Big|_{\mathbf{k}=\mathbf{k}_1} \right\} \quad (25)$$

This is similar in appearance to the Stix kinetic flux⁵ in a weakly nonuniform plasma, in that it contains a $\partial/\partial \mathbf{k}$ operator. In Appendix A it is shown that the above corresponds exactly to the Stix form for a plasma in equilibrium, hence demonstrating exact energy balance to all orders in $k_{\perp} \rho$ for a Maxwellian plasma. Note however, that Eq. (25) for $\mathbf{T}$ is merely the zero-order term in a series involving spectrum width. In regions of strong damping and mode-conversion when a wide spectral sampling is required, it is possible to find solutions to the exact Eq. (24) which connect asymptotically to (25); they are, however, as previously discussed, not unique.

V. Summary

The question of local absorption in plasmas containing full-wave perturbed fields has been studied from several different viewpoints, including the action-angle formalism and gyrokinetics. A direct link between the local absorption and the particles is made via quasilinear analysis of the Vlasov equation. In all instances, the local absorption can be written symbolically as $\frac{1}{2} \operatorname{Re} \langle \mathbf{H}'^{(1)*} \mathbf{F}^{(1)} \rangle$, where $\mathbf{H}'^{(1)}$ is the linearized time rate of change in the Hamiltonian due to the perturbing fields, and $\langle \dots \rangle$ refers to "angle-averaging" over any hypothetically randomized or cyclical phase-space coordinates.

It was shown that this general expression for local absorption is strongly influenced by the choice of phase-space within which it is evaluated. It is known that a plasma in equilibrium should experience net absorption of energy. To demonstrate this fact, though, it is necessary to employ a phase-space in which coordinates label the particle trajectories, e.g., action-angles. Local

absorption is positive in action space. But since "angle-averaging" in action-angles can involve "position-averaging" in configuration space (e.g., toroidal and poloidal angles in a torus), it is not possible to relate directly this local absorption to configuration space.

It was shown that gyrokinetics utilizes a phase-space in which the configuration-space variables label a particle's guiding-center coordinates. When the guiding-center motion is irrelevant, as in slab geometry, local absorption is demonstrably positive; see Appendix A. However, when the guiding-center motion is not ignorable, as in a tokamak, factors such as the parallel field gradients lead to absorption of indefinite sign. Net positive absorption is then recouped by averaging over the cyclical guiding-center motions, as shown in Appendix B.

The local absorption is, in general, the quadratic product of the electric field with a "dielectric operator," which operates nonlocally and individually on each of the fields within the quadratic. A simplified calculation of this operator is given in Appendix A for a Maxwellian plasma in the gyrokinetics phase space. By Taylor expansion in the gyroradius, it is possible to relate mathematically the energy dielectric to the linearly operating wave dielectric of the more traditional laboratory coordinates. This process produces the familiar "odd-order derivative" terms, which are generalized here to all orders in $k_{\perp\rho}$.

An energy balance equation involving Poynting flux, kinetic flux, and local absorption was also discussed. In terms of physical considerations, the electric field and the quasilinear absorption must be unique and well-defined. Hence, this paper has concentrated on the wave and energy operators associated with each of these quantities. One may then *define* the kinetic flux, to within the curl of an arbitrary vector, as the solution of an energy balance equation. It was shown that this leads to a quantity which reduces to the Stix form asymptotically away from nonuniformities.

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Appendix A. Example Kernel

This Appendix presents an example calculation of the kernel, $W(R, V, k_1, k_2)$, for a Maxwellian distribution. Similar calculations can be found in the gyrokinetics reference of L.

Chen¹⁴ and also in Vaclavik and Appert.¹⁶ This presentation differs from these primarily by inclusion of parallel gradient effects following reference [23] and in the discussion of the question of sign for W_a .

The starting point is Eq. (21b), with substitution of an isotropic Maxwellian for $F^{(0)}$,

$$F^{(0)}(\mathbf{R}, \mathbf{V}) = \frac{N}{(V_{th} \sqrt{\pi})^3} \exp(-V^2 / V_{th}^2)$$

with density, N , and thermal velocity V_{th} . I would like to concentrate on the quadratic operation of W , i.e., the separate identities of k_1 and k_2 . Therefore, to simplify the algebra, spatial derivatives on N and V_{th} , which are the diamagnetic terms, will be ignored. However, parallel gradients of the magnetic field will be kept since these may have a zero-order effect on the absorptive part of W .²⁴ The tensor and vector field elements are indexed as the three directions $e_{\perp 1}$, $e_{\perp 2}$, and b . Obviously, the resolution of the electric field, E , into these three components must be performed prior to the decomposition onto the Fourier basis. A legitimate covariant system, with proper metric tensor, is not used here; clearly there are terms in the field curvature which are being ignored. Except for the time varying gyrofrequency, $\Omega'(\mathbf{R}, \mathbf{V}, \tau) = \Omega(\mathbf{R})[1 - V_{\parallel}^2/L_{\parallel}]$, which contains the parallel gradient, the particle orbits will appear as standard cyclotron motion along straight field lines. The Fourier basis indices, k_1 and k_2 , can be resolved onto the vectors, $e_{\perp 1}$, $e_{\perp 2}$, and b , at the local position, $\mathbf{R}$, but will not be corrected for curvature along the guiding center trajectory. Clearly then, there is also another large class of terms in field curvature which are being ignored.

Denoting the angle with respect to $e_{\perp 1}$ of $k_{\perp}$ as β , and of $V_{\perp}$ as ϕ , the absorption kernel is

$$W(\mathbf{R}, \mathbf{V}, k_1, k_2) =$$

$$i \frac{2\omega_p^2}{\omega V_{th}^3 \pi^{3/2}} e^{-\frac{V^2}{V_{th}^2}} \int_0^\infty d\tau e^{i(\omega - k_{\parallel 1} V_{\parallel})\tau} e^{i\left(\frac{k_{\perp 2} V_{\perp}}{\Omega} \sin(\phi - \beta_2) - \frac{k_{\perp 1} V_{\perp}}{\Omega} \sin(\theta(\tau) + \phi - \beta_1)\right)} \quad \uparrow$$

$$C^V(\beta_2) \cdot \begin{bmatrix} \frac{V_{\perp}}{\mathcal{J}_2} e^{-i(\phi - \beta_2)} \\ \frac{V_{\perp}}{\mathcal{J}_2} e^{i(\phi - \beta_2)} \\ V_{\parallel} \end{bmatrix} \begin{bmatrix} \frac{V_{\perp}}{\mathcal{J}_2} e^{i(\theta(\tau) + \phi - \beta_1)} & \frac{V_{\perp}}{\mathcal{J}_2} e^{-i(\theta(\tau) + \phi - \beta_1)} & V_{\parallel} \end{bmatrix} \cdot C(\beta_1)$$

where the gyrophase is

$$\theta(\tau) = \int_0^\tau d\tau' \Omega' = \Omega\tau \left(1 - \frac{V_{\parallel} \tau}{2L_{\parallel}}\right)$$

and the unitary polarization matrices are

$$C(\beta) = \begin{bmatrix} \frac{1}{\sqrt{2}} e^{i\beta} & \frac{-i}{\sqrt{2}} e^{i\beta} & 0 \\ \frac{1}{\sqrt{2}} e^{-i\beta} & \frac{i}{\sqrt{2}} e^{-i\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow C^{-1}(\beta) = C^V(\beta) = \begin{bmatrix} \frac{1}{\sqrt{2}} e^{-i\beta} & \frac{1}{\sqrt{2}} e^{i\beta} & 0 \\ \frac{i}{\sqrt{2}} e^{-i\beta} & \frac{-i}{\sqrt{2}} e^{i\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Expanding in Bessel functions and cyclotron harmonics, and performing the gyroangle average then yields (A1)

$$\langle W(\mathbf{R}, \mathbf{V}, \mathbf{k}_1, \mathbf{k}_2) \rangle = i \frac{\omega_p^2}{\omega V_{th}^2} \frac{e^{-\frac{V^2}{V_{th}^2}}}{\pi^{3/2}} \sum_{n=-\infty}^{+\infty} e^{in(\beta_1 - \beta_2)} \int_0^\infty d\tau e^{i[(\omega - k_{11} V_{||})\tau - n\theta(\tau)]} \uparrow$$

$$C^V(\beta_2) \cdot \begin{bmatrix} V_{\perp} J_{n+1}(\xi_2) \\ V_{\perp} J_{n-1}(\xi_2) \\ \sqrt{2} V_{||} J_n(\xi_2) \end{bmatrix} \left[\begin{array}{ccc} V_{\perp} J_{n+1}(\xi_1) & V_{\perp} J_{n-1}(\xi_1) & \sqrt{2} V_{||} J_n(\xi_1) \end{array} \right] \cdot C(\beta_1)$$

where $\xi_1 = k_{\perp 1} V_{\perp} / \Omega$, and $\xi_2 = k_{\perp 2} V_{\perp} / \Omega$.

In the important limit of planar geometry with uniformity along a straight field line in the z-direction, the substitution of the above into Eq. (21a) produces a symmetric form which demonstrates positive absorption: (A2)

$$\int dZ W_s(\mathbf{R}, V_{\perp}, V_{||}) = \sum_s \frac{\omega_p^2}{V_{th}^2} \frac{e^{-\frac{V^2}{V_{th}^2}}}{\pi^{3/2}} \int d\mathbf{k}_z \sum_n \frac{e_s}{8\pi} \left| \dots \right|^2 \uparrow$$

$$\int d^2 \mathbf{k}_{\perp} e^{i\mathbf{k} \cdot \mathbf{R}_{\perp}} e^{in\beta} \left[\begin{array}{ccc} V_{\perp} J_{n+1}(\xi) & V_{\perp} J_{n-1}(\xi) & \sqrt{2} V_{||} J_n(\xi) \end{array} \right] \cdot C(\beta) \cdot \mathbf{E}(\mathbf{k}) \Big|_{k_z = \frac{\omega - n\Omega}{V_{||}}}$$

(although note that it is still necessary to "position average" along the ignorable coordinate, Z.) The above corresponds roughly to Eq. (24) of reference [16], except that it is simpler by one Bessel function summation. This is a result of their use of laboratory position, $\mathbf{r}$, as the independent variable for local absorption, in contrast to our use of guiding-center position, $\mathbf{R}$. Except for this difference I believe that their postulated absorption, which uses time-averaging, should be equivalent to Eq. (A2), which is derived from the quasilinear equation using gyrophase averaging. A truncation of Eq. (A2) in powers of the gyroradius may be performed as in

references [15] and [16]. Such a truncation does not, however, preserve the property of definite sign for $k_{\perp}\rho \approx 1$; for example, see reference [18].

If the static field is curved, or if there are parallel gradients, then the simplifications leading to Eq. (A2) are no longer possible and the local absorption is no longer positive. It is known from action-angle analysis, though, that the absorption averaged over the two position variables encompassing a drift surface must be positive. The time dependence of Ω' is especially troublesome, since the imaginary part of the phase integral in τ is no longer of definite sign.²⁵ In reference [23] it is shown that this more difficult phase integral may be performed semi-analytically by doing the parallel velocity summation prior to the time integration. It is also possible to perform the perpendicular velocity summation analytically. The result, which can be used directly in Eq. (23) for the wave dielectric, is

$$\sum_s \int d^3v \, W(\mathbf{R}, \mathbf{v}, \mathbf{k}_1, \mathbf{k}_2) = \quad (A3)$$

$$\sum_s \frac{\omega_p^2}{\omega |k_{\parallel 1}| v_{th}} \sum_n e^{-\lambda} e^{in(\beta_1 - \beta_2)} C^V(\beta_2) \cdot \begin{bmatrix} W_{xx}^n & W_{xy}^n & W_{xz}^n \\ W_{yx}^n & W_{yy}^n & W_{yz}^n \\ W_{zx}^n & W_{zy}^n & W_{zz}^n \end{bmatrix} \cdot C(\beta_1)$$

$$\begin{aligned} W_{xx}^n &= \left[(1 - \lambda) I_{n+1}(\lambda) + \lambda I'_{n+1}(\lambda) \right] Z(\zeta_n, \alpha_n) \\ W_{xy}^n &= \left[\frac{1}{2} \lambda_2 I_{n-1}(\lambda) - \lambda I_n(\lambda) + \frac{1}{2} \lambda_1 I_{n+1}(\lambda) \right] Z(\zeta_n, \alpha_n) \\ W_{yx}^n &= \left[\frac{1}{2} \lambda_1 I_{n-1}(\lambda) - \lambda I_n(\lambda) + \frac{1}{2} \lambda_2 I_{n+1}(\lambda) \right] Z(\zeta_n, \alpha_n) \\ W_{yy}^n &= \left[(1 - \lambda) I_{n-1}(\lambda) + \lambda I'_{n-1}(\lambda) \right] Z(\zeta_n, \alpha_n) \\ W_{xz}^n &= \frac{1}{2} \frac{k_{\parallel 1}}{|k_{\parallel 1}|} \left[\lambda_1^{1/2} I_{n+1}(\lambda) - \lambda_2^{1/2} I_n(\lambda) \right] \left(\frac{\partial}{\partial \zeta} Z(\zeta_n, \alpha_n) + \frac{1}{2} \alpha_n \frac{\partial^2}{\partial \zeta^2} Z(\zeta_n, \alpha_n) \right) \\ W_{zx}^n &= \frac{1}{2} \frac{k_{\parallel 1}}{|k_{\parallel 1}|} \left[\lambda_2^{1/2} I_{n+1}(\lambda) - \lambda_1^{1/2} I_n(\lambda) \right] \left(\frac{\partial}{\partial \zeta} Z(\zeta_n, \alpha_n) + \frac{1}{2} \alpha_n \frac{\partial^2}{\partial \zeta^2} Z(\zeta_n, \alpha_n) \right) \\ W_{yz}^n &= \frac{1}{2} \frac{k_{\parallel 1}}{|k_{\parallel 1}|} \left[\lambda_1^{1/2} I_n(\lambda) - \lambda_2^{1/2} I_{n-1}(\lambda) \right] \left(\frac{\partial}{\partial \zeta} Z(\zeta_n, \alpha_n) + \frac{1}{2} \alpha_n \frac{\partial^2}{\partial \zeta^2} Z(\zeta_n, \alpha_n) \right) \\ W_{zy}^n &= \frac{1}{2} \frac{k_{\parallel 1}}{|k_{\parallel 1}|} \left[\lambda_2^{1/2} I_n(\lambda) - \lambda_1^{1/2} I_{n-1}(\lambda) \right] \left(\frac{\partial}{\partial \zeta} Z(\zeta_n, \alpha_n) + \frac{1}{2} \alpha_n \frac{\partial^2}{\partial \zeta^2} Z(\zeta_n, \alpha_n) \right) \\ W_{zz}^n &= -I_n(\lambda) \left(\zeta_n \frac{\partial}{\partial \zeta} Z(\zeta_n, \alpha_n) + \alpha_n \frac{\partial}{\partial \alpha} Z(\zeta_n, \alpha_n) \right) \end{aligned}$$

where

$$\lambda_1 = \frac{k_{\perp 1}^2 V_{th}^2}{2\Omega^2}, \quad \lambda_2 = \frac{k_{\perp 2}^2 V_{th}^2}{2\Omega^2}, \quad \bar{\lambda} = \frac{1}{2}(\lambda_1 + \lambda_2), \quad \tilde{\lambda} = \sqrt{\lambda_1 \lambda_2},$$

$$\zeta_n = \frac{\omega - n\Omega}{|k_{\parallel 1}| V_{th}}, \quad \alpha_n = \frac{n\Omega}{k_{\parallel 1} |k_{\parallel 1}| V_{th} L_{\parallel}}$$

and $Z(\zeta, \alpha)$ is a generalized plasma dispersion function defined in reference [23], which contains the parallel gradient parameter, α , and whose imaginary part is no longer strictly positive,

$$Z(\zeta, \alpha) = \frac{i}{\sqrt{\pi}} \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dx e^{-u^2 + i[(\zeta - u)x + \frac{\alpha}{2}ux^2]} = i \int_{-\infty}^{+\infty} dx e^{-\frac{x^2}{4}(1 - \frac{\alpha}{2})^2 + i\zeta x}.$$

For the purpose of calculating the asymptotic kinetic flux via Eq. (25), notice that the $\partial/k_{\perp}$ operator followed by the substitution $k_1, k_2 \rightarrow k$ on the symmetric part of (A3) yields the same result as performing the substitution followed by $1/2 \partial/\partial k$. Therefore, Eq. (25) will produce the Stix forms for the asymptotic kinetic flux of a Maxwellian plasma.

Appendix B. Curved Geometries.

A recent paper by Brambilla and Krüicken¹⁷ rigorously demonstrates a positivity-theorem for an equilibrium plasma including the more complicated rf-phase variations of a magnetic field possessing parallel gradients. Unfortunately, since that reference utilizes the formalism of McVey, Sund, and Scharer¹⁵ and Vaclavik and Appert,¹⁶ it does not derive explicitly from known quasilinear theory, and hence is not directly applicable to this paper. However, by performing a "bounce-average" as in reference [21], it is possible to produce an equivalent result for quasilinear theory. The mathematics follows reference [17] quite closely.

In this Appendix, then, I shall seek an absorption quantity which is positive definite for toroidal geometry. The absorption quantity must be "local" to a phase-space comprised of three particle constants-of-motion to provide the necessary symmetry. It is not necessary to utilize the three actions of action-angle formalism, though. Instead, it is possible to use the following more tangible quantities: $E = (V_{\perp}^2 + V_{\parallel}^2)/2$, the total energy, $\mu = V_{\perp}^2/2\Omega$, the magnetic moment, and ψ , the poloidal flux surface around which the particle orbit is centered. The absorption, $W_{\pm\pm}(\psi, E, \mu)$, will be found by averaging along the drift orbit, taking into account the amount of time spent at each position. (The " $\pm$ " subscript refers to the degeneracy in the sign of $V_{\parallel}$ as calculated from the energy.) As in reference [21], the following Jacobian identity is employed to provide the proper time weight:

$$\frac{1}{\Omega} V_{\perp} dV_{\perp} dV_{\parallel} \Leftrightarrow \frac{1}{|V_{\parallel}|} d\mu dE$$

Denote the guiding-center orbit by $\mathbf{R}'_{\psi, E, \mu, \pm}(t)$, where the initial point at $t=0$, $\mathbf{R}'_{\psi, E, \mu, \pm}(0)$, may be chosen arbitrarily along the orbit and has no effect on the final result. The bounce-averaged absorption, $W_{\pm}(\psi, E, \mu) dE d\mu$, (units of power/volume) is:

$$W_{\pm}(\psi, E, \mu) dE d\mu =$$

$$\frac{\oint_{\psi \Omega'} \frac{dl}{\Omega'} W_{\pm}(\mathbf{R}'_{\psi, E, \mu, \pm}(t), V_{\perp}(\psi, E, \mu, t), V_{\parallel \pm}(\psi, E, \mu, t)) V_{\perp} dV_{\perp} dV_{\parallel}}{\oint_{\psi \Omega'} \frac{dl}{\Omega'}}$$

where Ω' refers to $\Omega(\mathbf{R}'_{\psi, E, \mu, \pm}(t))$. Since, in general, the particle orbits do not reconnect, the cyclical line integrals may be performed as the ratio of two finite integrals taken in the limit of infinite length. One way to do this is to increment the line integral by the time variable of the orbit, that is, take $dl=|V_{\parallel}|dt$. Then employ the small positive causality parameter, γ , which is the imaginary part of the rf frequency, $\omega=\omega_{\text{real}}+i\gamma$. The limits of integration will be from $t=-\infty$ to $t=0$; however, a "causal" factor of $e^{2\gamma t}$ will be introduced to the integrands of both numerator and denominator to make them finite. The limit of infinite length is then taken by letting $\gamma \rightarrow 0^+$; hence, both linear and quasilinear causality are treated by the same limit. Thus:

$$W_{\pm}(\psi, E, \mu) = \lim_{\gamma \rightarrow 0^+} \frac{\int_{-\infty}^0 dt e^{2\gamma t} W_{\pm}(\mathbf{R}'_{\pm}, V_{\perp}, V_{\parallel \pm})}{\int_{-\infty}^0 dt \frac{|V_{\parallel}|}{\Omega'} e^{2\gamma t}} \quad (B1)$$

where, from here on, the prime (') denotes dependence on the orbit time and constants-of-motion. Substituting from Eqs. (21) and (A1) this is:

$$W_{\pm}(\psi, E, \mu) = \lim_{\gamma \rightarrow 0^+} \left[\int_{-\infty}^0 dt \frac{|V_{\parallel}|}{\Omega'} e^{2\gamma t} \right]^{-1} \frac{\omega_p^2}{V_b^5 \pi^{3/2}} \frac{e^{-\frac{V^2}{V_b^2}}}{\sum_{n=-\infty}^{+\infty} \frac{\epsilon_n}{2}} \text{Re} \left\{ \int_{-\infty}^0 dt \int_0^{\infty} d\tau \dots \right\} \quad (B2)$$

$$\uparrow$$

$$\iint d^3 k_1 d^3 k_2 e^{2\gamma t} e^{i\omega \tau} e^{-i n \theta'(\tau)} e^{i \mathbf{k}_1 \cdot \mathbf{R}'_{\pm}(t-\tau) - i \mathbf{k}_2 \cdot \mathbf{R}'_{\pm}(t)} e^{i n (\beta'_1(t-\tau) - \beta'_2(t))}$$

$$\mathbf{E}^*(\mathbf{k}_2) \cdot \mathbf{C}^V(\beta'_2(t)) \cdot J_{n\pm}(\xi'_2(t)) J_{n\pm}(\xi'_1(t-\tau)) \cdot \mathbf{C}(\beta'_1(t-\tau)) \cdot \mathbf{E}(\mathbf{k}_1),$$

where for sake of brevity the vectors containing the Bessel functions has been abbreviated as:

$$\mathbf{J}_{n\pm}(\xi) = \begin{bmatrix} V_{\perp} J_{n\pm}(\xi) \\ V_{\perp} J_{n\pm}(\xi) \\ \sqrt{2} V_{\parallel} J_n(\xi) \end{bmatrix}.$$

Notice that because of the curved field lines the two perpendicular wavenumber quantities, β and ξ , must be resolved at each guiding-center position. All diamagnetic, anisotropy, and first order parallel gradient terms are ignored in (B2). I am emphasizing the particle-rf phase which comprises the second line of (B2). The cyclotron phase is:

$$\theta'(\tau) = \int_0^{\tau} d\tau' \Omega'(t - \tau') = \int_{t-\tau}^0 d\tau'' \Omega'(\tau'') - \int_t^0 d\tau'' \Omega'(\tau'') .$$

Also notice that the rf-phase and causal factors can be rewritten as

$$e^{2\gamma t} e^{i\omega\tau} = e^{2\gamma t} e^{i\omega_{\text{res}} t} e^{-\gamma\tau} = e^{i\omega^* t} e^{-i\omega(t-\tau)} .$$

Evidently, the integrand of the time variables t and τ , i.e., lines 2 and 3 of Eq. (B2), is symmetric with respect to the quantities t and $t-\tau$. In other words replacing t with $t-\tau$ will not alter the real part of the last two lines of Eq. (B2). Unfortunately, the time limits-of-integration do not have the same symmetry. This can be remedied, though. Redefine the time variables as: $t_1 \equiv t-\tau$ and $t_2 \equiv t$. Then the time integral operators are

$$\int_{-\infty}^0 dt \int_0^{\infty} d\tau \dots = \int_{-\infty}^0 dt_2 \int_{-\infty}^{t_2} dt_1 \dots = \int_{-\infty}^0 dt_1 \int_{t_1}^0 dt_2 \dots .$$

The two alternate forms merely result from switching the order of integration. However, since the integrand is symmetric in t_1 and t_2 , the following identity is also true:

$$\text{Re} \left\{ \int_{-\infty}^0 dt_2 \int_{-\infty}^{t_2} dt_1 \dots \right\} = \text{Re} \left\{ \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 \dots \right\} .$$

By combining two of these several equivalent forms, it follows that

$$\operatorname{Re}\left\{\int_{-\infty}^0 dt \int_0^{\infty} d\tau \dots\right\} = \frac{1}{2} \operatorname{Re}\left\{\int_{-\infty}^0 dt_1 \int_{-\infty}^0 dt_2 \dots\right\}$$

which now has the appropriate symmetry. Finally, take advantage of the symmetry to rewrite Eq. (B2) in a form which clearly illustrates the positive definiteness of the bounce-averaged absorption. This result is:

(B3)

$$W_{a\pm}(\psi, E, \mu) = \lim_{\gamma \rightarrow 0} \left[\int_{-\infty}^0 dt \frac{|V_{\parallel}|}{\Omega'} e^{2\gamma t} \right]^{-1} \frac{\omega_p^2}{V_{th}^5 \pi^{3/2}} \sum_{n=-\infty}^{+\infty} \frac{\epsilon_n}{4} \left| \dots \right|^2$$

$$\int_{-\infty}^0 dt \int d^3k e^{i \int_0^t dt' (\omega - n\Omega'(t') - \mathbf{k} \cdot \mathbf{V}_{ex}(t'))} e^{i \mathbf{k} \cdot \mathbf{R}_s(0)} e^{in\beta'(t)} J_{n\pm}(\xi'(t)) \cdot \mathbf{C}(\beta'(t)) \cdot \mathbf{E}(\mathbf{k})$$

Equation (B3) is gyrokinetic-quasilinear theory's equivalent to Eq. (6) of reference [17], and is the curved-field-line analog to Eq. (A2). It proves that even though the imaginary part of the phase integral can be negative in the presence of parallel gradients, the net sign of absorption must always be positive when averaged over the drift surface. This fact might have been anticipated by noting that the net area under the imaginary part of the generalized phase integral, $Z(\zeta, \alpha)$, is always positive.²³

References.

- ¹ J.D. Jackson, *Classical Electrodynamics*, 2e (Wiley, New York, 1975), chapter 6.
- ² J.A. Stratton, *Electromagnetic Theory*, (McGraw-Hill, New York, 1941), chapter 2.
- ³ L.D. Landau and E.M. Lifshitz, *Electrodynamics of Continuous Media*, (Pergamon Press, Oxford, 1960), section 61.
- ⁴ C.F. Kennel and F. Engelmann, *Physics of Fluids* **9**, 2377 (1966).
- ⁵ T.H. Stix, *Theory of Plasma Waves*, (McGraw-Hill, New York, 1962), chapter 3.
- ⁶ Liu Chen and Shih-Tung Tsai, *Plasma Physics* **25**, 349 (1983).
- ⁷ P.L. Colestock and R.J. Kazhuba, *Nuclear Fusion* **23**, 763 (1983).
- ⁸ S.C. Chiu, *Plasma Physics and Controlled Nuclear Fusion* **27**, 1525 (1985).
- ⁹ D.G. Swanson, *Physics of Fluids* **24**, 2035 (1981).
- ¹⁰ Marco Brambilla and Maurizio Ottaviani, *Plasma Physics and Controlled Nuclear Fusion* **27**, 1 (1985).
- ¹¹ Allan N. Kaufman, *Physics of Fluids* **15**, 1063 (1972).
- ¹² S.M. Mahajan and C.Y. Chen, *Physics of Fluids* **28**, 3538 (1985);

- John R. Cary, H. Vernon Wong, and S.M. Mahajan, *Physics of Fluids* **31**, 706 (1988).
- ¹³ W. W. Lee, *Physics of Fluids* **26**, 556 (1983).
- ¹⁴ Liu Chen, F.-J. Ye, M.-J. Xu, S.-T. Tsai, Y.C. Lee, and T.M. Antonsen, Jr., *Plasma Physics* **24**, 743 (1982).
- ¹⁵ B.D. McVey, R.S. Sund, and J.E.Scharer, *Physical Review Letters* **55**, 507 (1985).
- ¹⁶ J. Vaclavik and K. Appert, *Plasma Physics and Controlled Nuclear Fusion* **29**, 257 (1986).
- ¹⁷ M. Brambilla and T. Krücken, 15th European Conference on Controlled Fusion and Plasma Physics, Dubrovnik, 968 (1988).
- ¹⁸ E.F. Jaeger, D.B. Batchelor, and H. Weitzner **28**, 53 (1988).
- ¹⁹ Allan N. Kaufman, *Physics of Fluids* **14**, 387 (1971).
- ²⁰ T.H. Stix, *Nuclear Fusion* **15**, 737 (1975);
I.B. Bernstein and D.C.Baxter, *Physics of Fluids* **24**, 108 (1981).
- ²¹ Liu Chen, J. Vaclavik, and G.W. Hammett, *Nuclear Fusion* **28**, 389 (1988).
- ²² D.G. Swanson, *Physics of Fluids* **28**, 2645 (1985).
- ²³ D. Smithe, P. Colestock, T. Kammash, and R. Kashuba,
Physical Review Letters **60**, 801 (1988).
- ²⁴ D.J. Gambier and A. Samain, *Nuclear Fusion* **25**, 283 (1985).
- ²⁵ D. W. Faulconer, *Plasma Physics and Controlled Fusion* **29**, 433 (1987).