TRANSVERSE RESONANCES OF PERIODICALLY WIDENED

CYLINDRICAL TUBES WITH CIRCULAR
CROSS SECTION*

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ABSTRACT

The transverse resonances of the electromagnetic field in periodic cylindrical cavities connected by concentric side tubes are calculated for arbitrary azimuthal mode numbers by expansion of the Hertz potentials in subregions bounded by coordinate surfaces. The expansion coefficients are determined by matching of the tangential field components across the common surfaces of the subregions. Resonances are characterized by the existence of solutions of the homogeneous wave equation in the absence of source terms, and the resonant frequencies are given by the vanishing of the determinant of an infinite matrix. They can be calculated on a computer by truncating the matrix to finite size. Expressions for the stored energy, the loss-factor, and for R/Q are then obtained by integration over the fields at resonance. These results are of interest for the determination of the energy loss and the stability of the beam in high-energy particle accelerators and storage rings when cavities are formed by pairs of cross section variations of the vacuum chamber.

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MASTER

I. Introduction

A number of computer codes exist for the calculation of longitudinal resonances (axisymmetric modes) of rotationally symmetric cavities. Earlier programs based on over-relaxation methods have been superseded by the program SUPERFISH¹ which has better convergence properties and is faster due to the use of a triangular mesh. These programs solve essentially for standing waves in a closed cavity. Standing waves in periodically repeating cavities with rectangular cross section (in the r - z plane), connected by a concentric tube can be found with the program KN7C.² This program permits the calculation of a large number of resonances in a single run of a few minutes on a high-speed computer. The program is based on a field expansion and matching along a cylindrical cut extending the tubes through the cavity (see Fig. 1).

The lowest transverse (deflecting) modes in such a geometry have been calculated in a similar manner for RF particle-separators.^{3,4} In this report, we will extend the calculations to cover higher modes of arbitrary azimuthal variation. These modes are required for the calculation of the energy loss of displaced charged particle beams in high energy accelerators and storage rings, as well as for the calculation of the transverse stability of these beams in the presence of cross section variations of the vacuum chamber.

II. Hybrid Modes

The main difficulty in the calculation of azimuthally varying modes lies in the fact that the solutions of the wave equation no longer separate into the familiar TM and TE modes if the cross section of the chamber is not constant along the beam axis. The fields can still be

described by a superposition of these modes, but then another difficulty appears for modes with phase velocities at or near the light velocity: in that case two or more terms of the infinite series describing the transverse field components become infinite and cannot be evaluated by computer, although their sum remains finite.

This latter difficulty can be overcome by using an expansion in hybrid ("HM" and "HE") modes, which are defined in such a manner that the terms in the infinite sums remain finite for all phase-velocities. Such an expansion leads to exactly the same frequency condition as the expansion in TM/TE modes. However, since the expressions are simpler, we use the TM/TE expansion in this report, although a hybrid expansion is used for any term with phase velocity equal c .

Matching the field components across the common boundary of the subregions, we obtain an infinite set of homogeneous equations for the expansion coefficients. In order for non-vanishing solutions to exist, the determinant of the system matrix has to vanish. This occurs for an infinite number of "resonant frequencies" depending only on the geometry of the cavity with sidetubes and on the phase-shift over one period. For computer evaluation, the infinite matrix has to be truncated, and the resonant frequencies are found approximately by a zero finding routine. A last difficulty occurs due to the fact that this determinant has also an infinite number of poles, which unfortunately do not alternate with the zeros except for the lowest modes. However, since these poles occur when a single term in the (infinite) series for a single matrix element blows up, they can be found rather easily. The determinant then can be multiplied with a function which vanishes at every pole, and its sign changes only due to zeros.

Due to the hybrid character of the modes, the matrices are in general twice the size of the matrices used for axisymmetric modes, and four times as many elements have to be calculated. It is therefore important to use the series with the fastest convergence and to find means to shorten the computations.

III. Solutions of the Wave Equation

The components of the electric and magnetic field strengths in the absence of sources are given by the vector wave equation

$$\left(\nabla^2 + \frac{\partial^2}{\partial t^2} \right) \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = 0 \quad (3.1)$$

where the vector Laplacian is defined by the relation $\nabla^2 = \text{grad div} - \text{curl curl}$. The components of the vector Laplacian equal the scalar Laplacian only for rectangular coordinates. For cylindrical coordinates, the transverse components of the fields are coupled, but the longitudinal component is given by the solution of the scalar wave equation. The transverse field components can be obtained from the longitudinal ones with the help of Maxwell's curl relations. In general, however, this requires solution of a simple differential equation, which degenerates into pure differentiation only for single terms of the fields expressed by infinite sums. It is therefore preferable to introduce potentials, from which the field components can be obtained by direct differentiation. In the absence of sources, the Hertz vectors are simpler than the usual scalar and vector potentials, as one needs in general only two components to describe all six field components in a cylindrical subregion.⁵

The two components of the Hertz vectors also must be solutions of the scalar wave equation. In circular cylindrical coordinates, the wave equation for a single frequency component $\sim \exp(j\omega t)$ becomes

$$\left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} + k^2 \right] \begin{Bmatrix} \pi_z \\ \tilde{\pi}_z \end{Bmatrix} = 0 \quad (3.2)$$

where $k = \omega/c$. (3.3)

We now assume a product solution of the form

$$R(r) \cdot Z(z) \cdot \Theta(\theta) \quad (3.4)$$

For the azimuthal variation we find the equation

$$\frac{d^2 \Theta}{d\theta^2} + m^2 \Theta = 0 \quad (3.5)$$

where the separation constant must be the square of an integer in order to obtain solutions periodic with 2π . Without loss of generality we can choose $\cos m\theta$ for π_z and then $\tilde{\pi}_z$ must vary as $\sin m\theta$.

Similarly we find for the longitudinal variation

$$\frac{d^2 Z}{dz^2} + \beta^2 Z = 0 \quad (3.6)$$

with the traveling-wave solutions

$$Z(z) = e^{\pm j\beta z} \quad (3.7)$$

For a periodic structure, the solutions also have to fulfill the Floquet condition

$$Z(z + 2\pi R) = e^{-j\beta_0 z} \cdot Z(z) \quad (3.8)$$

where $2\pi R$ is the period length, and β_0 is the phase-shift per period

divided by the period length. Hence only discrete values of β are permitted which are given by

$$\beta_n = \beta_0 + \frac{n}{R} \quad (3.9)$$

We subdivide the whole structure into a cylindrical tube region (I) and an annular cavity region (II) (see Fig. 1). We can use the traveling wave solution in region (I), but in region (II) we need standing-wave solutions. Now calling the axial propagation constant α we find

$$Z(z) = \begin{cases} \sin \alpha z \\ \cos \alpha z \end{cases} \quad (3.10)$$

The boundary conditions for perfectly conducting walls require that the tangential electric and the normal magnetic field components vanish there. For side walls at $z = \pm g$ we can fulfill all these requirements by taking

$$Z(z) = \begin{cases} \cos \alpha_s (z + g) & \text{for } z \\ \sin \alpha_s (z + g) & \text{for } \tilde{z} \end{cases} \quad (3.11)$$

$$\text{with } \alpha_s = \pi/2g. \quad (3.12)$$

Finally, the radial variation is given by

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \left(\chi^2 + \frac{m^2}{r^2} \right) R = 0 \quad (3.13)$$

$$\text{where } \chi^2 = \beta^2 - k^2. \quad (3.14)$$

The solutions of this equation are modified Bessel functions of order m . Since region (I) contains the origin, only Bessel functions of the first kind are permitted there, and we get

$$R(r) = I_m(\chi r) \quad (3.15)$$

In region (II), we call the transverse propagation constant Γ , which is given by

$$\Gamma^2 = \alpha^2 - k^2 \quad (3.16)$$

In this region, both kinds of modified Bessel functions can be used, but the boundary conditions at a perfectly conducting outer cavity wall ($r=d$) require solutions of the form

$$\begin{aligned} R_m(\Gamma r) &= K_m(\Gamma d) I_m(\Gamma r) - I_m(\Gamma d) K_m(\Gamma r) \quad \text{for } \pi_z \\ S_m(\Gamma r) &= K'_m(\Gamma d) I_m(\Gamma r) - I'_m(\Gamma d) K_m(\Gamma r) \quad \text{for } \tilde{\pi}_z \end{aligned} \quad (3.17)$$

$$\text{where } R'_m(\Gamma d) = S'_m(\Gamma d) = 0. \quad (3.18)$$

The general solutions of the wave equation are obtained by summing the product solutions over all possible propagation constants. Each term may be multiplied by an arbitrary "expansion coefficient" which will be determined by the matching conditions. For convenience in the expressions for the fields, we divide by the squares of the propagation constants and find for region (I) (tube region)

$$\begin{aligned} \pi_z &= - \sum_{n=-\infty}^{\infty} \frac{A_n}{\chi_n^2} \frac{I_m(\chi_n r)}{I_m(\chi_n b)} e^{-j\beta_n z} \cos m\theta \\ \tilde{\pi}_z &= - \sum_{n=-\infty}^{\infty} \frac{B_n}{\chi_n^2} \frac{I_m(\chi_n r)}{I_m(\chi_n b)} e^{-j\beta_n z} \sin m\theta \end{aligned} \quad (3.19)$$

where β_n and χ_n are given by Eqs. (3.9) and (3.14). χ_n may become imaginary for small values of n , and the modified Bessel functions then may be replaced by regular Bessel functions of real argument.

In region (II) (cavity region) we find similarly

$$\begin{aligned}\pi_z^{\text{II}} &= - \sum_{s=0}^{\infty} \frac{\bar{C}_s}{\Gamma_s^2} \frac{R_m(\Gamma_s r)}{R_m(\Gamma_s b)} \cos \alpha_s(z+g) \cos m\theta \\ \tilde{\pi}_z^{\text{II}} &= j \sum_{s=1}^{\infty} \frac{\bar{D}_s}{\Gamma_s^2} \frac{S_m(\Gamma_s r)}{S_m(\Gamma_s b)} \sin \alpha_s(z+g) \sin m\theta\end{aligned}\quad (3.20)$$

where α_s and Γ_s are given by Eqs. (3.12) and (3.16), and the functions R_m and S_m by Eqs. (3.17). Again the modified Bessel functions may be replaced by regular ones. The factor $-j$ has been taken out of $\tilde{\pi}_z$ in order to make the expressions for the transverse fields purely imaginary.

IV. Field Matching

The electric and magnetic field strengths can be obtained from the Hertz vectors by the relations

$$\begin{aligned}\vec{E} &= \text{curl curl } \pi - \text{curl } \frac{1}{c} \frac{\partial \tilde{\pi}}{\partial t} \\ \vec{H} &= \text{curl curl } \tilde{\pi} + \text{curl } \frac{1}{c} \frac{\partial \pi}{\partial t}\end{aligned}\quad (4.1)$$

In cylindrical subregions only the components π_z and $\tilde{\pi}_z$ are required. In circular cylindrical coordinates, for a time variation $\exp(jkct)$, we obtain the field components

$$\begin{aligned}E_r &= \frac{\partial^2 \pi_z}{\partial r \partial z} - \frac{jk}{r} \frac{\partial \tilde{\pi}_z}{\partial \theta} & Z_0 H_r &= \frac{\partial^2 \tilde{\pi}_z}{\partial r \partial z} + \frac{jk}{r} \frac{\partial \pi_z}{\partial \theta} \\ E_\theta &= \frac{1}{r} \frac{\partial^2 \pi_z}{\partial \theta \partial z} + jk \frac{\partial \tilde{\pi}_z}{\partial r} & Z_0 H_\theta &= \frac{1}{r} \frac{\partial^2 \tilde{\pi}_z}{\partial \theta \partial z} - jk \frac{\partial \pi_z}{\partial r} \\ E_z &= \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \pi_z & Z_0 H_z &= \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \tilde{\pi}_z\end{aligned}\quad (4.2)$$

We see that the "electric" vector component π_z corresponds to TM modes ($H_z = 0$), and the "magnetic" one $\tilde{\pi}_z$ to TE modes ($E_z = 0$). Except for the axisymmetric modes with $m = 0$, for which TM and TE modes separate, in general a superposition of both types of modes is required to fulfill the boundary conditions. TM and TE modes exist for $m \neq 0$ only in axially uniform structures such as wave-guides and closed ("pill-box") cavities. For the case of periodically widened circular cylindrical tubes investigated here, we need in general modes in which both E_z and H_z are finite. We can describe them by a superposition of TM and TE modes, or by any other pair of independent solutions of the wave-equation. One particular pair, called HM and HE modes, has been used in Ref. 4 in order to avoid convergence difficulties when the phase-velocity of the waves is near or at the light-velocity. However, one can show that these modes can also be described as a superposition of a hybrid mode and a TEM mode in order to avoid the convergence problems (see Section VI). In any case, the condition for resonance becomes the same in both descriptions.

The Hertz vector components are given by infinite sums over product solutions derived in the last section. The product solutions have already been chosen to fulfill the boundary conditions on the cavity walls as well as the periodicity requirement. We must further impose the boundary conditions on the tube wall and equate the field components at the interface of the two subregions. For perfectly conducting tube walls, the tangential electric and the normal magnetic field components must vanish, but only two of the three conditions are independent. At the interface of the two subregions, we equate the four tangential

components of the fields, which is sufficient for equality of all field components. We thus obtain the matching conditions

$$r=b: \quad E_{z,0}^I = \begin{cases} E_{z,0}^{II} & |z| < g \\ 0 & g < |z| < \pi R \end{cases} \quad (4.3)$$

$$H_{z,0}^I = H_{z,0}^{II} \quad |z| < g$$

Substituting the expressions for the field components given in Eq. (4.2), we obtain the four equations

$$\sum_n A_n e^{-j\beta_n z} = \begin{cases} \sum_s \bar{C}_s \cos \alpha_s(z+g) & |z| < g \\ 0 & g < |z| < \pi R \end{cases} \quad (4.4)$$

$$\sum_n (P_n A_n + I_n B_n) e^{-j\beta_n z} = \begin{cases} -j \sum_s (Q_s \bar{C}_s + S_s \bar{D}_s) \sin \alpha_s(z+g) & |z| < g \\ 0 & g < |z| < \pi R \end{cases}$$

$$\sum_n B_n e^{-j\beta_n z} = -j \sum_s \bar{D}_s \sin \alpha_s(z+g) \quad |z| < g \quad (4.5)$$

$$\sum_n (I_n A_n + P_n B_n) e^{-j\beta_n z} = \sum_s (\bar{R}_s \bar{C}_s + Q_s \bar{D}_s) \cos \alpha_s(z+g) \quad |z| < g$$

where $\bar{D}_0 = 0$. We have used the abbreviations

$$P_n = \frac{\pi \beta_n}{2 \chi_n b}, \quad I_n = \frac{k}{\chi_n} \frac{I_n'}{I_n} \bigg|_{\chi_n b} \quad (4.6)$$

and

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$$Q_s = \frac{m\alpha}{\Gamma_s^2 b}, \quad \bar{R}_s = \frac{k}{\Gamma_s} \frac{R'_m}{R_m} \bigg|_{\Gamma_s b}, \quad S_s = \frac{k}{\Gamma_s} \frac{S'_m}{S_m} \bigg|_{\Gamma_s b} \quad (4.7)$$

By using the orthogonality of $\exp(-j\beta_n z)$ in the interval $(-\pi R, +\pi R)$, and the orthogonality of $\cos \alpha(z+g)$ and $\sin \alpha(z+g)$ in the interval $(-g, +g)$, we can solve these relations to obtain

$$A_n = \mu \sum_s \bar{N}_{ns} \bar{C}_s \quad (4.8)$$

$$P_n A_n + I_n B_n = \mu \sum_s \bar{M}_{ns} (Q_s \bar{C}_s + S_s \bar{D}_s)$$

$$\bar{D}_s = 2 \sum_n \bar{M}_{sn}^\dagger B_n \quad (4.9)$$

$$(1 + \delta_{so}) \bar{R}_s \bar{C}_s + Q_s \bar{D}_s = 2 \sum_n \bar{N}_{sn}^\dagger (I_n A_n + P_n B_n)$$

where

$$\begin{aligned} \begin{Bmatrix} \bar{N}_{ns} \\ \bar{M}_{ns} \end{Bmatrix} &= \frac{1}{2g} \int_{-g}^{+g} e^{j\beta_n z} \begin{Bmatrix} \cos \alpha_s(z+g) \\ -j \sin \alpha_s(z+g) \end{Bmatrix} dz \\ &= \frac{1}{g} \frac{\begin{Bmatrix} \beta_n \\ \alpha_s \end{Bmatrix}}{\beta_n^2 - \alpha_s^2} \begin{Bmatrix} \sin \beta_n g & s \text{ even} \\ j \cos \beta_n g & s \text{ odd} \end{Bmatrix} \end{aligned} \quad (4.10)$$

where $\mu = g/\pi R$ and $\bar{M}^\dagger, \bar{N}^\dagger$ are the Hermitian conjugate matrices of $\bar{M}, \bar{N}$.

In order to have purely real equations, we define

$$C_s, D_s = \begin{cases} \bar{C}_s, \bar{D}_s & s \text{ even} \\ j\bar{C}_s, j\bar{D}_s & s \text{ odd} \end{cases} \quad (4.11)$$

and

$$N_{ns}, M_{ns} = \begin{cases} \bar{N}_{ns}, \bar{M}_{ns} & s \text{ even} \\ -j\bar{N}_{ns}, -j\bar{M}_{ns} & s \text{ odd} \end{cases} \quad (4.12)$$

Finally, we define

$$R_s = (1 + \delta_{s0}) \bar{R}_s \quad (4.13)$$

and then obtain the four infinite sets of equations

$$A_n = \mu \sum_s N_{ns} C_s \quad (4.14)$$

$$P_n A_n + I_n B_n = \mu \sum_s M_{ns} (Q_s C_s + S_s D_s)$$

$$D_s = 2 \sum_n M_{sn}^T B_n \quad (4.15)$$

$$R_s C_s + Q_s D_s = 2 \sum_n N_{sn}^T (I_n A_n + P_n B_n)$$

(where M^T, N^T are the transpose matrices of M, N) which determine the four infinite sets of expansion coefficients A_n, B_n, C_s and D_s . For the axisymmetric case, only the first and last of these equations are valid (the others are multiplied by $\sin m\theta$ on both sides), and furthermore $B_n = D_s = 0$ for pure TM modes. In that case we get

$$A_n = \mu \sum_s N_{ns} C_s \quad (4.16)$$

$$R_s C_s = 2 \sum_n N_{sn}^T I_n A_n$$

which can be immediately solved for either A_n or C_s . For the general

case we cannot solve for a single coefficient, but for a pair as shown below.

V. Matrix Formalism

The equations for the expansion coefficients for $m \geq 1$ can be written in matrix form as

$$\begin{pmatrix} 1 & 0 \\ P & I \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \mu \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & S \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \quad (5.1)$$

$$\begin{pmatrix} R & Q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = 2 \begin{pmatrix} N^T & 0 \\ 0 & M^T \end{pmatrix} \begin{pmatrix} I & P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

where each element is an infinite matrix itself. Since the submatrices I, P, Q, R, S are all diagonal matrices, we can solve Eqs. (5.1) for the (infinite) vectors (A,B) or (C,D) by the rules for 2×2 matrices to get

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mu \begin{pmatrix} 1 & 0 \\ -PI^{-1} & I^{-1} \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & S \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} \quad (5.2)$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = 2 \begin{pmatrix} R^{-1} & -QR^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^T & 0 \\ 0 & M^T \end{pmatrix} \begin{pmatrix} I & P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

We now have the choice to eliminate either (A,B) or (C,D). For geometries with a long period length compared to the cavity radius it is preferable to use the first option, for short periods the second one. This permits truncation of the matrices at a smaller size, while the slower converging series are shifted to the matrix elements. As the calculation of the determinant is proportional to the third power of its dimension, while the number of elements grows only as the second

power, this will minimize the required computer time. Following the first option, we obtain a matrix equation which can be written

$$\left\{ \begin{pmatrix} R & Q \\ 0 & I \end{pmatrix} - 2\mu \begin{pmatrix} N^T & 0 \\ 0 & M^T \end{pmatrix} \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -PI^{-1} & I^{-1} \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q & S \end{pmatrix} \right\} \begin{pmatrix} C \\ D \end{pmatrix} = 0 \quad (5.3)$$

We further multiply the first term with the inverse of the last matrix of the second term, which consists of diagonal submatrices, and change the sign of the first column to get

$$\left\{ \begin{pmatrix} G_1 & G_{12} \\ G_{12} & G_2 \end{pmatrix} - 2\mu \begin{pmatrix} N^T & 0 \\ 0 & M^T \end{pmatrix} \begin{pmatrix} H_1 & H_{12} \\ H_{12} & H_2 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \right\} \begin{pmatrix} C' \\ D' \end{pmatrix} = 0 \quad (5.4)$$

$$\begin{aligned} \text{where } G_1 &= Q^2 S^{-1} - R & H_1 &= P^2 I^{-1} - I \\ G_{12} &= QS^{-1} & H_{12} &= PI^{-1} \\ G_2 &= S^{-1} & H_2 &= I^{-1} \end{aligned} \quad (5.5)$$

$$\text{and } C' = -C, \quad D' = QC + SD. \quad (5.6)$$

The matrix equations are now symmetric, since $G_1^T = G_1$ and $(N^T H_{12} M)^T = M^T H_{12} N$. Furthermore, the G 's and H 's have been brought into a form where they do not diverge for $\Gamma \rightarrow 0$ or $\chi \rightarrow 0$, although each of the two terms of G_1 and H_1 diverges by itself. The condition that a solution exist for the homogeneous system of Eq. (5.4) is

$$\det \left[G - 2\mu \begin{pmatrix} N^T H_1 N & N^T H_{12} M \\ M^T H_{12} N & M^T H_2 M \end{pmatrix} \right] = 0 \quad (5.7)$$

This equation has an infinite number of solutions which determine the resonant frequencies. They are found on a computer by truncating the matrices to a finite size, stepping up in frequency until the sign of the determinant changes, and finding the zeros to a predetermined accuracy. Only the diagonal submatrices of G and H depend on frequency, while the full matrices M and N are constant.

A difficulty in the search for the zeros is the fact that the determinant has also an infinite number of poles, which unfortunately do not alternate with the zeros except for the lowest modes. They occur whenever one of the terms in the denominator of the infinite sums vanishes. This happens at zeros of the Bessel function J_m or their derivatives J'_m , as well as at the zeros of the Bessel function combinations R_m and S'_m . The location of the poles thus can be determined rather easily, and the determinant is multiplied with a function which passes through zero linearly at every pole.

VI. Synchronous Case

For the case $k = \beta_0$ the $n=0$ term in the transverse components of the fields will blow up unless $B_0 = -A_0$. Then letting χ_0 go to zero we obtain

$$\begin{aligned} E_{z0} &= A_0 \left(\frac{r}{b}\right)^m e^{-j\beta_0 z} \cos m\theta \\ Z_0 H_{z0} &= -A_0 \left(\frac{r}{b}\right)^m e^{-j\beta_0 z} \sin m\theta \\ E_{\theta 0} &= -\frac{jA_0}{2\beta_0 r} \left(\frac{r}{b}\right)^m \left[m - \frac{\beta_0^2 r^2}{(m+1)} \right] e^{-j\beta_0 z} \sin m\theta \\ Z_0 H_{\theta 0} &= -\frac{jA_0}{2\beta_0 r} \left(\frac{r}{b}\right)^m \left[m - \frac{\beta_0^2 r^2}{(m+1)} \right] e^{-j\beta_0 z} \cos m\theta \end{aligned} \quad (6.1)$$

We need to add another independent solution to this. Let

$$\begin{aligned} \Pi_{zo}^I &= -\frac{B_o}{2\beta_o^2} \left(\frac{r}{b}\right)^m e^{-j\beta_o z} \cos m\theta \\ \tilde{\Pi}_{zo}^I &= 0 \end{aligned} \quad (6.2)$$

This gives a TEM mode with

$$\begin{aligned} E_{zo} &= 0 \\ H_{zo} &= 0 \\ E_{\theta o} &= -\frac{jB_o}{2\beta_o r} \left(\frac{r}{b}\right)^m e^{-j\beta_o z} \sin m\theta \\ z_o H_{\theta o} &= \frac{jB_o}{2\beta_o r} \left(\frac{r}{b}\right)^m e^{-j\beta_o z} \cos m\theta \end{aligned} \quad (6.3)$$

If we define

$$P_o = \frac{\left[m - \frac{\beta_o^2 b^2}{(m+1)} \right]}{2\beta_o b} \quad \text{and} \quad I_o = \frac{m}{2\beta_o b},$$

then matching the fields as before gives

$$\begin{aligned} A_n &= \mu \sum_s N_{ns} C_s \\ P_n A_n + I_n B_n &= \mu \sum_s M_{ns} (Q_s C_s + S_s D_s) \\ D_s &= 2 \left(\sum_{n \neq 0} M_{sn}^T B_n - M_{so}^T A_o \right) \\ R_s C_s + Q_s D_s &= 2 \left(\sum_{n \neq 0} N_{sn}^T (I_n A_n + P_n B_n) - N_{so}^T (P_o A_o - I_o B_o) \right) \end{aligned} \quad (6.4)$$

The first of matrix Eq. (5.1) remains the same while the second becomes

$$\begin{pmatrix} R & Q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2} \begin{pmatrix} N^T & 0 \\ 0 & M^T \end{pmatrix} \begin{pmatrix} TI - UP & TP + UI \\ -U & T \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \quad (6.5)$$

where

$$T_{nm} = \delta_{nm} (1 - \delta_{no})$$

$$U_{nm} = \delta_{no} \delta_{mo}$$

Therefore

$$\begin{aligned} H_1 &= T(P^2 I^{-1} - I) + 2UP \\ H_2 &= TI^{-1} \end{aligned} \quad (6.6)$$

$$H_{21} = H_{12} = TPI^{-1} + U$$

replace H of Eq. (5.5).

VII. Stored Energy and Loss Factor

The calculation of the energy loss of charged particles can be done using the loss factors for each resonance,⁶ defined by

$$k_L = \frac{VV^*}{4U} \quad (7.1)$$

where V is the equivalent voltage acting on the particles, and U is the stored energy. The loss factor is related to the "geometrical factor" R/Q which is simply $4k_L/\omega$.

Usually the stored energy is obtained from the electromagnetic field strengths by the integral

$$U = \frac{1}{4} \int (\epsilon E \cdot E^* + \mu H \cdot H^*) dV \quad (7.2)$$

At resonance, the stored energy in the electric and magnetic field are equal, and we can reduce the calculation to

$$U = \frac{\epsilon}{2} \int E \cdot E^* dV \quad (7.3)$$

Using 3.19, 3.20 and 4.2 we get for Region I

$$U^I = \frac{\pi^2 \epsilon b^2}{2} \sum_n \left[|A_n|^2 \left(\frac{B_n^2}{\chi_n^2} K_{2n} + K_{1n} \right) + |B_n|^2 \frac{k^2}{\chi_n^2} K_{2n} + \frac{4A_n B_n m s k}{\chi_n^4 b^2} \right] \quad (7.4)$$

For Region II

$$U^{II} = \frac{\pi \epsilon g b^2}{4} \sum_s \left[(1 + \epsilon_{so}) |C_s|^2 \left(\frac{\alpha_s^2}{\Gamma_s^2} L_{2s} + L_{1s} \right) + (1 - \epsilon_{so}) |D_s|^2 \frac{k^2}{\Gamma_s^2} L_{3s} - \frac{4C_s D_s m s k}{\Gamma_s^4 b^2} \right] \quad (7.5)$$

where

$$K_{1n} = \frac{2}{I_m^2(x)} \int_0^1 I_m^2(xr) r dr \Big|_{x=x_n b} = \left[\left(1 + \frac{m^2}{x^2} \right) - \frac{I_m'^2(x)}{I_m^2(x)} \right] \Big|_{x=x_n b}$$

$$K_{2n} = \frac{2}{I_m^2(x)} \int_0^1 \left[I_m'^2(xr) + \frac{m^2}{x^2 r^2} I_m^2(xr) \right] r dr \Big|_{x=x_n b} \\ = \left[\frac{2I_m'(x)}{xI_m(x)} - K_{1n} \right] \Big|_{x=x_n b} \quad (7.6)$$

$$L_{1s} = \frac{2}{R_m^2(x)} \int_1^{d/b} R_m^2(xr) r dr \Big|_{x=\Gamma_s b} \\ = \left[- \left(1 + \frac{m^2}{x^2} \right) + \frac{R_m'^2(x)}{R_m^2(x)} - \frac{1}{x^2 R_m^2(x)} \right] \Big|_{x=\Gamma_s b}$$

$$\begin{aligned}
 L_{2s} &= \frac{2}{R_m^2(x)} \int_1^{d/b} \left[R_m'^2(xr) + \frac{m^2}{x^2 r^2} R_m^2(xr) \right] r dr \Big|_{x=\Gamma_s b} \\
 &= \left[-\frac{2R_m'(x)}{xR_m^2(x)} - L_{1s} \right] \Big|_{x=\Gamma_s b}
 \end{aligned} \quad (7.7)$$

$$\begin{aligned}
 L_{3s} &= \frac{2}{S_m^2(x)} \int_1^{d/b} \left[S_m'^2(xr) + \frac{m^2}{x^2 r^2} S_m^2(xr) \right] r dr \Big|_{x=\Gamma_s b} \\
 &= \left[-\frac{2S_m'(x)}{xS_m^2(x)} + \left(1 + \frac{m^2}{x^2}\right) - \frac{S_m'^2(x)}{S_m^2(x)} - \frac{1}{x^2 S_m^2(x)} \left(1 + \frac{m^2}{x^2 (d/b)^2}\right) \right] \Big|_{x=\Gamma_s b}
 \end{aligned}$$

The derivation of integrals K_{2n} , L_{2s} and L_{3s} are given in Appendix A.

If $\beta_0 = k$ the $n=0$ harmonic of the fields must be replaced by the form of the fields derived in Section VI. This will change the $n=0$ contribution to the energy in Region I to

$$U_o^I = \frac{\pi^2 R_0 b^2}{2} \left[\frac{|A_o|^2}{(m+1)} + \frac{|A_o|^2 \beta_o^2 b^2}{2(m+1)^2(m+2)} + \frac{|A_o + B_o|^2 m}{2\beta_o^2 b^2} \right] \quad (7.8)$$

The average accelerating voltage per cell on a vertically displaced particle is given by

$$V = \lim_{N \rightarrow \infty} \frac{1}{N} \int_{-N\pi R}^{N\pi R} E_z^I(r=y, \theta=0) e^{j\omega t} \Big|_{t=z/\beta c} dz \quad (7.9)$$

where β is the particle velocity divided by c and y is the vertical displacement.

$$V = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_n A_n \frac{I_m(x_n y)}{I_m(x_n b)} \int_{-N\pi R}^{N\pi R} e^{-j(\beta_n - k/\beta)z} dz \quad (7.10)$$

$$V = \lim_{N \rightarrow \infty} \frac{1}{(2N+1)} \sum_{n=-N}^N e^{-j2\pi R n (\beta_0 - k/\beta)} \sum_n A_n \frac{I_m(x_n y)}{I_m(x_n b)} \int_{-\pi R}^{\pi R} e^{-j(\beta_n - k/\beta)z} dz$$

The factor $1/(2N+1)$ will cause V to vanish unless $\beta_0 - k/\beta = n'$, with n' an integer. If $\beta_0 - k/\beta$ is an integer, then $\beta_n - k/\beta$ will also be an integer. Unless this second integer is zero, the integral vanishes.

$$V = \begin{cases} 2\pi R A_n \frac{I_m(x_n y)}{I_m(x_n b)} & \beta_n - k/\beta = 0 \\ 0 & \beta_n - k/\beta \neq 0 \end{cases} \quad (7.11)$$

Only the space harmonic synchronous with the particle contributes an average accelerating voltage. In particular let $\beta = 1$, $\beta_0 = k$ then

$$V = 2\pi R A_0 \left(\frac{y}{b}\right)^m \quad (7.12)$$

VIII. Conclusions

The resonant frequencies and fields of arbitrary azimuthal variation have been calculated for a circular cylindrical tube with periodically widened sections. In general, the fields have a hybrid character and are described by a superposition of TM and TE modes. The fields are expanded in infinite series in the "tube" and "cavity" subregions. Since the expansion functions are orthogonal, the expansion coefficients can be determined from two infinite, homogeneous sets of linear equations. Resonances thus can be found by determining those frequencies where the determinant of a double-infinite matrix vanishes.

A computer program has been written which calculates the resonant frequencies and the expansion coefficients by truncating the matrices to finite size. The input to the program requires only the geometric dimensions of the structure, the phase-shift per cell, the derived dimensions of the matrices, and the frequency range and step size. The program determines the poles of the determinant and multiplies the matrix with a function which vanishes there in order to facilitate the finding of the zeros. One option permits automatic determination of an optimum step size. But since the zeros of the determinant may come arbitrary close, a second run over a limited frequency region with reduced step size is sometimes necessary. Another option permits one to find only the lowest zero of the determinant as a function of the phase shift per cell which thus yields the ω - β diagram for a particular mode. Finally, the phase-shift per cell may be zero to yield standing waves, or determined such that the waves have a prescribed phase velocity. The program has been written in such a form that the phase velocity may be equal to the light velocity. However, since all matrices have to be recalculated for every frequency when the phase-shift per cell is not constant, the option with prescribed phase velocity is considerably slower.

The expansion coefficients are calculated at each resonance, and in the near future we shall add the required matrix-multiplications to calculate the stored energy, voltage gain and loss-factor (the required formulas are given in Section VII).

The program has been tested and compared with previous results for the lowest dipole mode as well as for the azimuthally uniform case ($m = 0$). However, the matrix dimensions have to be chosen carefully to guarantee

correct results, and may become excessively large for practical use for particular geometries. However, satisfactory results have been obtained with only 10×10 matrices for the test cases run so far.

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APPENDIX A

Derivation of Integrals K_{2n} , L_{2s} , L_{3s}

Let the n 'th term of $\vec{E}$ in Region I be denoted by $\vec{E}_n^I$, then

$$\vec{E}_n^I = \vec{\nabla}_t \frac{\partial \Pi_{zn}^I}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \Pi_{zn}^I \hat{z} - jk \vec{\nabla}_t \tilde{\Pi}_{zn}^I \times \hat{z} \quad (A.1)$$

where Π_{zn}^I , $\tilde{\Pi}_{zn}^I$ are the n 'th terms of Π_z^I , $\tilde{\Pi}_z^I$ respectively, $\vec{\nabla}_t$ is the transverse gradient operator $\left(\vec{\nabla}_t = \vec{\nabla} - \frac{\partial}{\partial z} \hat{z} \right)$ and $\hat{z}$ the unit vector in the longitudinal direction. Denote the first term in $\vec{E}_n^I$ by $\vec{E}_{1n}^I$ and the third by $\vec{E}_{3n}^I$. (The second is just $\vec{E}_{2n}^I$.) We can see that $\vec{E}_{1n}^I$ and $\vec{E}_{3n}^I$ are orthogonal to $\vec{E}_{2n}^I$. Therefore

$$\begin{aligned} U_n^I &= \frac{\epsilon}{2} \int \left[\vec{E}_{1n}^I \cdot \vec{E}_{1n}^{*I} + \vec{E}_{2n}^I \cdot \vec{E}_{2n}^{*I} + \vec{E}_{3n}^I \cdot \vec{E}_{3n}^{*I} + 2\vec{E}_{1n}^I \cdot \vec{E}_{3n}^{*I} \right] dV \\ &= U_{1n}^I + U_{2n}^I + U_{3n}^I + U_{13n}^I \end{aligned} \quad (A.2)$$

The first term is

$$U_{1n}^I = \frac{\epsilon}{2} \int_{\mathcal{A}} dz \int \vec{\nabla}_t \frac{\partial \Pi_{zn}^I}{\partial z} \cdot \vec{\nabla}_t \frac{\partial \Pi_{zn}^{*I}}{\partial z} dA$$

where $\mathcal{A}$ is the cross section area of Region I.

$$U_{1n}^I = \frac{\epsilon}{2} \int_{-\pi R}^{\pi R} dz \int_0^{2\pi} d\theta \int_0^b r dr \vec{\nabla}_t \frac{\partial \Pi_{zn}^I}{\partial z} \cdot \vec{\nabla}_t \frac{\partial \Pi_{zn}^{*I}}{\partial z} \quad (A.3)$$

$$U_{1n}^I = \pi^2 R \epsilon \frac{|A_n|^2 \beta_n^2}{\chi_{nm}^2(\chi_n b)} \int_0^b r dr \left[I_m'^2(\chi_n r) + \frac{m^2}{\chi_{nr}^2} I_m^2(\chi_n r) \right]$$

From Green's identity we also get

$$U_{1n}^I = \frac{\epsilon}{2} \int_{\mathcal{L}} dz \left[\oint_{\mathcal{L}} \frac{\partial \Pi_{zn}^I}{\partial z} \cdot \frac{\partial}{\partial n} \left(\frac{\partial \Pi_{zn}^{*I}}{\partial z} \right) dz - \int_{\mathcal{A}} \frac{\partial \Pi_{zn}^I}{\partial z} \cdot \nabla^2 \frac{\partial \Pi_{zn}^{*I}}{\partial z} dA \right] \quad (A.4)$$

where $\mathcal{L}$ is the path that bounds $\mathcal{A}$, $\hat{n}$ is the unit normal vector pointing outward.

$$U_{1n}^I = \pi^2 R \epsilon \int_{-\pi R}^{\pi R} \left[\int_0^{2\pi} r d\theta \frac{\partial \Pi_{zn}^I}{\partial z} \cdot \frac{\partial}{\partial r} \left(\frac{\partial \Pi_{zn}^{*I}}{\partial z} \right) \right]_{r=b} \\ + \int_0^{2\pi} d\theta \int_0^b r dr \frac{\partial \Pi_{zn}^I}{\partial z} \cdot \left(\frac{\partial^2}{\partial z^2} + k^2 \right) \Pi_{zn}^{*I} \quad (A.5)$$

where the wave equation was used in the second term. Using Eq. (3.19) for Π_{zn}^I gives

$$U_{1n}^I = \frac{\pi^2 R b^2 \epsilon}{2} \frac{|A_n|^2 \beta_n^2}{\chi_n^2} \left[\frac{2I_m'(x_n b)}{\chi_n b I_m(x_n b)} - K_{1n} \right] \quad (A.6)$$

where⁷

$$K_{1n} = \frac{2}{I_m^2(x)} \int_0^1 I_m^2(xr) r dr \bigg|_{x=\chi_n b} = \left[\left(1 + \frac{m^2}{x^2} \right) - \frac{I_m'^2(x)}{I_m^2(x)} \right] \bigg|_{x=\chi_n b}$$

Equations (A.3) and (A.6) show that

$$\frac{2}{I_m^2(x)} \int_0^1 \left[I_m^2(xr) + \frac{m^2}{x^2 r^2} I_m^2(xr) \right] r dr \bigg|_{x=\chi_n b} = \left[\frac{2I_m'(x)}{x I_m(x)} - K_{1n} \right] \bigg|_{x=\chi_n b} \\ = K_{2n} \quad (A.7)$$

which is the result given in Section VII. L_{2s} is derived in an analogous manner using Π_{zs}^{II} from Eq. (3.20) in place of its Region I counterpart.

To find L_{3s} we use

$$U_{3s}^{II} = \frac{\epsilon k^2}{2} \int dz \int_V \vec{\nabla}_t \tilde{\Pi}_{zs}^{II} \times \hat{z} \cdot \vec{\nabla}_t \tilde{\Pi}_{zs}^{*II} \times \hat{z} \, dV$$

$$U_{3s}^{II} = \frac{\epsilon k^2}{2} \int dz \int_V \vec{\nabla}_t \tilde{\Pi}_{zs}^{II} \cdot \vec{\nabla}_t \tilde{\Pi}_{zs}^{*II} \, dV$$

Using Green's identity

$$U_{3s}^{II} = \frac{\epsilon k^2}{2} \int dz \left[\int_S \tilde{\Pi}_{zs}^{II} \frac{\partial}{\partial n} \tilde{\Pi}_{zs}^{*II} \, dl - \int_V \tilde{\Pi}_{zs}^{II} \cdot \nabla_t^2 \tilde{\Pi}_{zs}^{*II} \, dV \right]$$

These two relations for U_{3s}^{II} can be used to give L_{3s} in the same manner as in the derivation of K_{2n} above.

Fig. 1. Figure showing subdivision of axisymmetric periodic structure into a cylindrical tube Region I and an annular cavity Region II.