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SHEAR STRESS ENHANCEMENT OF VOID COMPACTION

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September 1977

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Shear Stress Enhancement of Void Compaction

by

J.H. Curran* and M.M. Carroll**

Abstract

Motivated by the success of the spherical model in predicting the volumetric compaction behavior of both porous rocks and metals to a uniform pressure, we consider the applicability of the spherical model to nonhydrostatic loading conditions. Specifically, the spherical model is used to examine the influence of the presence of a shear stress on the volumetric compression of a porous solid. We first obtain the linear, elastic solution for a hollow sphere subject to homogeneous tractions on the outer boundary. Then, assuming that the matrix material is governed by the Drucker-Prager yield criterion, we use the elastic solution to derive an analytic expression for the onset of yield in the hollow sphere. The expression for the initial yield surface shows that the presence of a shear stress hastens the onset of yield in the sphere in comparison to a purely hydrostatic loading condition. This result agrees well with experimental data which shows that, for porous solids, permanent crush-up begins at a lower mean stress under a nonhydrostatic loading than when the applied loading is a uniform pressure. At this point, due to difficulty in obtaining an analytic solution, we turn to

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a numerical scheme (finite element method) to extend the analysis of the hollow sphere problem into the elasto-plastic range. The spherical model results clearly exhibit the experimental finding that the presence of a shear stress tends to enhance the volumetric compaction of porous solids in comparison to a purely hydrostatic loading condition. For both a porous rock and metal sample, agreement between the spherical model and experimental results is excellent.

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1. Introduction

Over the last decade there has been considerable research on the effects of voids on the deformation characteristics of porous geologic materials. Naturally occurring rocks have porosities from a few tenths of 1% to 50% and even more in some special circumstances. Much of the recent interest in the behaviour of porous rocks has been stimulated by the necessity of understanding and predicting explosively induced ground motion (1). A knowledge of the mechanical properties of porous rocks is also required in the calculation of ground surface subsidence due to reservoir depletion (2) and for the effective use of the hydraulic fracturing technique used to enhance energy production from geothermal energy reservoirs (3). Most recently, research has centered on a response property of porous rocks, known as volume dilatancy, due to its application in earthquake prediction (4).

In addition to the considerable research on porous rocks, much attention has been focused on the deformation characteristics of porous metals. Initial research was in the phenomenology of shock wave propagation (5). Among the several reasons for this interest are the opportunities to investigate the equations of state of solids at high pressures and temperatures and the ability of porous metals to act as efficient shock wave attenuators. More recently, the industrial use of porous metals in the production of parts of unusual shape or controlled porosity has given additional importance to these studies (6).

The aforementioned applications span a large range of strain rates and stress amplitudes. In this paper we are concerned with a static response property of porous materials, known as shear-enhanced compaction. In addition the pore phase will be considered to be empty. An understanding of the static behavior is a prerequisite to comprehending the dynamic response of porous solids. Material models representing the static or quasi-static behavior of porous materials will clearly be applicable at low strain rates as long as volumetric creep is small. They will also have application in dynamic situations where the rate-dependent effects are negligible over the range of strain rates between static and dynamic conditions. With the purpose of illustrating the influence of shear stress on void compaction, we briefly review the mechanical response of a porous solid determined from a static test. In addition to being less expensive to perform than dynamic tests, static tests have the advantage that the stress path is more easily controlled.

A schematic representation of the pressure-volume relationship typical of a porous geologic material* is shown in Fig. 1. In the low pressure range the material response is initially soft but stiffens with increasing pressure. This nonlinear behavior is attributed to the presence of microcracks in the rock sample (7). At some intermediate stress level, a softening occurs, which is associated with the onset of the major part of pore closure, and is due to internal yielding, fracturing, and grain motion. At the highest pressures the void space

* The P-V response in Fig. 1 is appropriate for rocks in which the major part of the void volume consists of pores (i.e. rocks with void volume/solid volume $\geq .05$).

Fig. 1. Schematic diagram of the pressure-volume response of a porous rock. Dashed lines show the behavior if the initial loading and final unloading stages were elastic.

Fig. 2. Schematic diagram of the mean stress-volume response of a porous rock in a triaxial test. Dashed line shows hydrostat from Fig. 1 for comparison.

is almost completely eliminated and the response approaches that of the solid phase. For unloading, the material at first appears to be quite stiff, but eventually a distinct inelastic softening associated with pore recovery begins, and significant portions of the original porosity may return. Further load cycling produces hysteresis loops, and the material cannot, in general, be returned to its initial state because permanent damage has been created. The pressure-volume response for a porous metal is similar but less complex than that of a porous rock. On loading, the initial response is essentially linearly elastic and for unloading, the response parallels the initial elastic behavior and is accompanied by almost no pore space recovery.

A nonhydrostatic test produces yet different behaviour in the pressure (mean stress)-volume space. Initially, both porous rocks and metals exhibit the phenomenon of shear-enhanced compaction (Fig. 2) which refers to the fact that the presence of a shear stress generally tends to enhance the volume compaction of porous materials in comparison to purely hydrostatic conditions. For porous rocks, however, when some of the porosity has been removed dilatation may begin and the material may expand under increasing load. This latter property, known as volume dilatancy, is a result of the opening up of microcracks in the body (8). The situation is further complicated in the case of some porous rocks by the effect of recoverable dilatancy (9) in which the dilatant sample undergoes a certain amount of permanent compaction even though the stress is being reduced.

A good deal of attention has been given to the mathematical description of the low-pressure elastic behavior of porous solids. Mackenzie (10) used a "self-consistent" method to derive expressions for the elastic constants of a homogeneous, isotropic, porous, linearly elastic solid. Walsh (7,11) has been able to model the nonlinear behavior in the low pressure range by including the effect of flat cracks.

The first significant contribution to the description of the inelastic properties of porous materials was the P - α model proposed by Herrmann (12). The main assumption of this model is that the pressure-volume-energy (P - V - E) relation for the matrix material is the same in both the porous and nonporous conditions. By introducing the porosity α , defined as the ratio of the specific volume of the matrix material to the specific volume of the porous material, and assuming a relation between the pressure P and the porosity α , Herrmann was able to obtain a P - V - E equation for the porous material. Carroll and Holt (13) made an important step in the development of pore-collapse relations based on the analysis of the collapse of a hollow sphere. The spherical model calculation (13) assumed ideal plasticity but subsequent generalization included strain-hardening (14) (Fig. 3). This approach has distinct advantages over previous P - α models which relied on purely phenomenological relationships between α and P . The spherical model, on the other hand, is based on the specific micromechanical model of noninteracting spherical pores. The only parameters in the model are the shear modulus, yield

Fig. 3. Comparison of the spherical model results (...) and experimental data (—) for a 78% dense porous aluminum subject to a hydrostatic pressure. From Butcher et al. (14).

Fig. 4. Spherical model prediction (—) for the pressure-volume response of 38% porous Mt. Nevada tuff in comparison with experimental data. From Bhatt et al. (15) .

strength and work hardening law for the matrix material, and the initial porosity, all of which have a definite, unambiguous, physical interpretation. Assuming that the matrix material followed the Mohr-Coulomb yield criterion, Bhatt et al. (15) were able to successfully predict the volumetric response of porous rocks subject to a uniform pressure (Fig. 4). In addition to the initial porosity and the shear modulus, they found that the P - α relation depends on the matrix yield parameters ϕ (angle of internal friction) and c (cohesion). Schatz (16) has incorporated two physically motivated changes into the spherical model. The first is to allow for curvature in the Mohr-Coulomb surface to account for the fact that the material strength does not increase without bound with infinite pressures. The second modification is to assume that the matrix bulk modulus, which is used to reduce results from P - α to P - V space*, is no longer taken to be ideally elastic, but instead includes the effect of flat cracks. Agreement is improved with these changes.

Motivated by the success of the spherical model in predicting the compaction behavior of porous solids under a uniform pressure, we use the model to examine the volumetric behavior of porous solids under an applied stress field which is nonhydrostatic. In particular, we study the phenomenon of shear-enhanced compaction and show that the spherical model results are in good agreement with the experimental data for a 38% porous tuff and a 78% dense porous aluminum sample. In section 2, we obtain the elastic solution for a hollow sphere subject to homogeneous

* This reduction is discussed in reference (15).

tractions on the outer boundary. Next, using the elastic solution and assuming that yielding of the matrix material is governed by the Drucker-Prager criterion, we derive an expression for the onset of yield in the hollow sphere. In section 4, due to considerable difficulty in obtaining an analytical solution, we turn to the finite element method and study the response of the hollow sphere in both elasto-plastic and fully plastic phases. We conclude, in the final section, with a discussion of the relation between the results of sections 3 and 4 and the phenomenon of shear-enhanced compaction and comment upon using the spherical model to obtain constitutive equations for the void compaction of porous solids.

2. Elastic Solution for a Hollow Sphere Subject to Homogeneous Traction on the Outer Boundary

In this section we determine the linear, elastic response of a hollow sphere of homogeneous, isotropic material subject to the boundary conditions

$$t_i = 0 \quad (2.1)$$

on the inner boundary (i.e. $r = a$) and

$$t_i = \overset{\circ}{T}_{ij} n_j \quad (2.2)$$

on the outer boundary, where t_i is the applied stress vector, $\overset{\circ}{T}_{ij}$ are constants, n_j is the outward unit normal to the outer surface of the sphere and repeated indicies indicate the usual summation convention.

We refer the initial and final positions of a typical particle, denoted by X_i and x_i respectively, to the same fixed set of rectangular cartesian axes and designate the displacement of the particle by:

$$u_i = u_i(x_1, x_2, x_3) \quad (2.3)$$

In addition to depending on the spatial coordinates x_i , the displacements u_i will also be a functions of $\overset{\circ}{T}_{ij}$. The conditions of isotropy of the matrix material and symmetry of the spherical geometry require that

$$u_i(Q_{mn} x_n, Q_{mp} Q_{nq} \overset{\circ}{T}_{pq}) = Q_{ij} u_j(x_m, \overset{\circ}{T}_{mn}) \quad (2.4)$$

and linearity implies that

$$u_i(x_m, \bar{c} \overset{\circ}{T}_{mn}) = \bar{c} u_i(x_m, \overset{\circ}{T}_{mn}) \quad (2.5)$$

where Q_{ij} is a proper orthogonal tensor and $\bar{c}$ is a scalar. In order that (2.4) and (2.5) hold for all arbitrary Q_{ij} and $\bar{c}$, the displacements u_i must have the form

$$u_i = \left[A(r) \overset{\circ}{T}_{kk} + B(r) \overset{\circ}{T}_{km} x_k x_m \right] x_i + C(r) \overset{\circ}{T}_{ik} x_k \quad (2.6)$$

where A, B and C are functions of the radial coordinate r. It is noteworthy that, the conditions of isotropy, symmetry and linearity determine the deformed shape of the surface $r = r_0$ to within the three constants $A(r_0)$, $B(r_0)$ and $C(r_0)$.

With the use of (2.6) the constitutive equation for the stress tensor σ_{ij} becomes:

$$\begin{aligned} \sigma_{ij} = & \lambda \delta_{ij} \left[(3A + A'r + C) \overset{\circ}{T}_{kk} \right. \\ & + (5B + B'r + \frac{C''}{r}) \overset{\circ}{T}_{km} x_k x_m \left. \right] \\ & + \mu \left[2(A\delta_{ij} + A' \frac{x_i x_j}{r}) \overset{\circ}{T}_{kk} \right. \\ & + 2(B\delta_{ij} + \frac{B'}{r} x_i x_j) \overset{\circ}{T}_{km} x_k x_m \\ & + (2B + \frac{C'}{r}) (\overset{\circ}{T}_{jk} x_k x_i + \overset{\circ}{T}_{ik} x_k x_j) + 2C \overset{\circ}{T}_{ij} \left. \right] \end{aligned} \quad (2.7)$$

where λ and μ are the Lamé constants for the matrix material, δ_{ij} is the Kronecker delta and a prime indicates differentiation with respect to r. Substituting for σ_{ij} in the equilibrium equation and requiring that the resulting expression hold for arbitrary $\overset{\circ}{T}_{ij}$, we obtain the following system of three simultaneous, second order, linear, ordinary differential equations in the functions A(r), B(r) and C(r)

$$\begin{aligned} (\lambda+2\mu) A'' + 4(\lambda+2\mu) \frac{A'}{r} + 2\mu B + (\lambda+\mu) \frac{C'}{r} &= 0, \\ (\lambda+2\mu) B'' + (6\lambda+14\mu) \frac{B'}{r} + (\lambda+\mu) \frac{C''}{r^2} - (\lambda+\mu) \frac{C'}{r^3} &= 0, \\ 2(\lambda+\mu) B'r + (10\lambda+14\mu) B + \mu C'' + 2(\lambda+3\mu) \frac{C'}{r} &= 0. \end{aligned} \quad (2.8)$$

The solution of (2.8) is

$$A(r) = \frac{-D_2}{5r^5} + \frac{D_3 \lambda r^2}{(2\lambda+7\mu)} + \frac{D_5}{r^3} + D_6 ,$$

$$B(r) = \frac{D_1}{r^5} + \frac{D_2}{r^7} + D_3 , \quad (2.9)$$

$$C(r) = \frac{2\mu D_1}{3(\lambda+\mu)r^3} - \frac{2D_2}{5r^5} - \frac{(5\lambda+7\mu)}{(2\lambda+7\mu)} D_3 r^2 + D_4 ,$$

where $D_1, \dots, D_6$ are constants. These constants in (2.9) are determined by employing the boundary conditions (2.1) and (2.2). Using the boundary conditions on the inner and outer surfaces of the hollow sphere we obtain a system of six simultaneous linear equations in $D_1, \dots, D_6$, the solution of which is given by

$$D_i = \frac{\Delta_i}{\Delta} \quad (i = 1, \dots, 4) , \quad (2.10)$$

where

$$\Delta = \frac{32\mu^2}{15(2\lambda+7\mu)(\lambda+\mu)a^{10}b^{10}} \left[(b^{10}+a^{10})(171\lambda^2+392\mu\lambda+196\mu^2) \right. \\ \left. + 1008 a^5 b^5 (\lambda^2+2\mu\lambda+\mu^2) - 25(a^7 b^3 + a^3 b^7)(27\lambda^2+56\mu\lambda+28\mu^2) \right]$$

$$\Delta_1 = \frac{16\mu (19\lambda+14\mu) (b^7-a^7)}{(2\lambda+7\mu) a^7 b^7} ,$$

$$\Delta_2 = -\frac{16\mu (19\lambda+14\mu) (b^5-a^5)}{(2\lambda+7\mu) a^5 b^5} ,$$

$$\Delta_3 = \frac{128\mu (b^2-a^2)}{a^7 b^7} , \quad (2.11)$$

$$\Delta_4 = \frac{16\mu^2}{15(2\lambda+7\mu)(\lambda+\mu)} \left[\frac{1}{a^{10}} (171\lambda^2 + 392\mu\lambda + 196\mu^2) \right. \\ \left. + \frac{504}{a^5b^5} (\lambda^2+2\lambda\mu+\mu^2) - \frac{25}{a^3b^7} (27\lambda^2+56\mu\lambda+28\mu^2) \right] ,$$

$$D_5 = \frac{\lambda D_1}{6(\lambda+\mu)} + \frac{2D_2}{5} \frac{(b^5-a^5)}{a^2b^2 (b^3-a^3)} + \frac{\lambda D_3}{4(2\lambda+7\mu)} \frac{a^3b^3 (b^2-a^2)}{(b^3-a^3)} ,$$

$$D_6 = \frac{1}{3\lambda+2\mu} \left[\frac{8\mu D_2}{5a^2b^2} \frac{(b^2-a^2)}{(b^3-a^3)} + \frac{\lambda\mu D_3}{(2\lambda+7\mu)} \frac{(b^5-a^5)}{(b^3-a^3)} - \lambda D_4 \right] .$$

3. Initial Yield

In the introduction we recalled that, with the use of an appropriate yield condition for the matrix material, the spherical model successfully predicted the volumetric response of both a porous rock and metal sample to a uniform pressure. A yield criterion which reduces to the Mohr-Coulomb or Mises criterion under certain conditions is the Drucker-Prager yield criterion (17). The Drucker-Prager yield function is

$$f = \beta J_1 + J_2^{1/2} = k \quad (3.1)$$

where β and k are positive constants at each point of the material; J_1 is the first invariant of the stress tensor

$$J_1 = \sigma_{kk} \quad (3.2)$$

and J_2 is the second invariant of the stress deviation

$$J_2 = \frac{1}{2} S_{ij} S_{ij} \quad (3.3)$$

The stress deviation is defined by

$$S_{ij} = \sigma_{ij} - \left(\frac{J_1}{3}\right) \delta_{ij} \quad (3.4)$$

The yield surface $f = k$, in principal stress space is a right circular cylinder for the Mises criterion ($\beta=0$). For $\beta>0$ the surface is a right circular cone with its axis equally inclined to the coordinate axes and with its apex in the tension octant. By choosing

$$\begin{aligned} \beta &= \frac{2 \sin \phi}{\sqrt{3} (3 + \sin \phi)} , \\ k &= \frac{5 c \cos \phi}{\sqrt{3} (3 + \sin \phi)} , \end{aligned} \quad (3.5)$$

where c is cohesion and ϕ the angle of internal friction, (3.1) becomes identical to the Mohr-Coulomb criterion used by Bhatt et al. (15) to predict the volumetric response of porous rocks on loading.

Similar to the discussion in section 2 for the displacements u_i , it can be shown that the conditions of isotropy, symmetry and linearity determine the form of the expression for the onset of yield in the sphere. For the stress tensor, these conditions require that:

$$\sigma_{ij} (Q_{mn} x_n, Q_{mp} Q_{nq} \overset{\circ}{T}_{pq}) = Q_{ir} Q_{js} \sigma_{rs} (x_m, \overset{\circ}{T}_{mn}) \quad , \quad (3.6)$$

$$\sigma_{ij} (x_m, \bar{c} \overset{\circ}{T}_{mn}) = \bar{c} \sigma_{ij} (x_m, \overset{\circ}{T}_{mn})$$

where Q_{ij} is a proper orthogonal tensor and $\bar{c}$, is a constant. In order that 3.6 holds for all arbitrary Q_{ij} and $\bar{c}$, σ_{ij} must have the form

$$\begin{aligned} \sigma_{ij} = & A_1(r) T_{kk} \delta_{ij} + A_2(r) \overset{\circ}{T}_{km} x_k x_m \delta_{ij} \\ & + A_3(r) \overset{\circ}{T}_{kk} x_i x_j + A_4(r) \overset{\circ}{T}_{km} x_k x_m x_i x_j \\ & + A_5(r) (\overset{\circ}{T}_{jk} x_k x_i + \overset{\circ}{T}_{ik} x_k x_j) + A_6(r) \overset{\circ}{T}_{ij} \quad , \end{aligned} \quad (3.7)$$

where $A_1, \dots, A_6$ are functions of the radial coordinate r . With the use of (3.7) and (3.2) to (3.4), and noting that (3.1) will take the critical value on the inner boundary (i.e. $r = a$), we get

$$\begin{aligned}
 f|_{r=a} = & \beta(P \overset{\circ}{T}_{kk} + Q \overset{\circ}{T}_{km} n_k n_m) \\
 & + (R \overset{\circ}{T}_{kk} \overset{\circ}{T}_{mm} + S \overset{\circ}{T}_{kk} \overset{\circ}{T}_{mn} n_m n_n + U \overset{\circ}{T}_{mn} \overset{\circ}{T}_{kp} n_m n_n n_k n_p \\
 & + V \overset{\circ}{T}_{jm} \overset{\circ}{T}_{jk} n_m n_k + W \overset{\circ}{T}_{ij} \overset{\circ}{T}_{ij})^{\frac{1}{2}} \\
 = & k,
 \end{aligned} \tag{3.8}$$

where $P, \dots, W$ are constants. It is interesting to note that (3.8), which expresses the initial yield surface in terms of the applied surface tractions and the normal vector n_i , is determined to within the constants $P, \dots, W$ using only the condition of linearity, isotropy and symmetry of the hollow sphere.

In order to obtain expressions for the constants $P, \dots, W$, we need the results from section 2. Using (2.7), (2.9), (2.10) and (3.2) - (3.4) expressions for J_1 and J_2 can be easily obtained. Then, substituting the resulting expressions for J_1 and J_2 in (3.1) and taking $r = a$, we obtain:

$$\begin{aligned}
 P &= (3\lambda + 2\mu) (3A + A'a + C), \\
 Q &= (3\lambda + 2\mu) (5B + B'a + \frac{C'}{a}) a^2, \\
 R &= \frac{2}{3} \mu^2 (2A'^2 a^2 - 2ACa - C^2), \\
 S &= \frac{2}{3} \mu^2 a^2 (8A'Ba - 4BC + 4A'B'a^2 - 2B'Ca + 4A'C' - \frac{2CC'}{a} + \frac{6A'C}{a}), \\
 U &= \frac{2}{3} \mu^2 a^4 (2B^2 + 8BB'a + \frac{2BC'}{a} + 2B'^2 a + 4B'C' + \frac{C'^2}{2a^2} + \frac{6B'C}{a}),
 \end{aligned} \tag{3.9}$$

$$V = \frac{2}{3} \mu^2 a^2 (6B^2 a^2 + 6BC'a + \frac{3}{2} C'^2 + 12BC + \frac{6CC'}{a}) ,$$

$$W = 2 \mu^2 C^2 .$$

Since almost all static testing of porous solids is conducted using a "triaxial" test configuration, we equate $\overset{0}{T}_{ij}$ with the stress tensor in a standard "triaxial" test in order to illustrate the above theory. Thus, in matrix form, we choose

$$\overset{0}{T}_{ij} = \begin{bmatrix} -(\omega+p) & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \quad (3.10)$$

where ω is the axial stress applied along the x_1 -axis and p is the confining pressure. Using (3.9) and (3.10), (3.8) becomes:

$$\begin{aligned} f|_{r=a} &= -\beta \left[P(\omega+3p) + Q(\omega n_1^2 + p) \right] \\ &+ \left\{ R(\omega+3p)^2 + S(\omega+3p)(\omega n_1^2 + p) + U(\omega n_1^2 + p)^2 \right. \\ &+ \left. V \left[(\omega^2 + 2p\omega) n_1^2 + p^2 \right] + W(\omega^2 + 2p\omega + 3p^2) \right\}^{\frac{1}{2}} \\ &= k \end{aligned} \quad (3.11)$$

The initial yield surface (3.11), for the Drucker-Prager condition is represented in Fig. 5 by plotting $\sqrt{J_2}/k$ vs. J_1/k at yield for a number of initial porosities. The influence of the elastic constants of the matrix material on the onset of yield is shown in Fig. 6 for a 78% dense

Fig. 5. Initial yield surface (3.11) assuming matrix material governed by Drucker-Prager yield criterion. Values of the elastic constants are $\lambda = 1206$ kb. and $\mu = 517$ kb.. Numbers above curves indicate initial ratio of solid volume to solid and void volume.

(—) $K = 8000 \text{ kb.}, \mu = 2000 \text{ kb.}$

(---) $K = 500,000 \text{ kb.}, \mu = 2000 \text{ kb.}$

(-.-) $K = 8000 \text{ kb.}, \mu = 16,000 \text{ kb.}$

where K and μ are the bulk modulus and shear modulus, respectively.

Fig. 6. Influence of elastic constants of matrix material on initial yield surface (3.11)

Fig. 7. Initial yield surface (3.11) assuming matrix material governed by Mises yield criterion. Values of the elastic constants are $\lambda = 5447$ kb. and $\mu = 2806$ kb.. Numbers above curves indicate initial ratio of solid volume to solid and void volume.

hollow sphere obeying the Mises yield criterion. Fig. 7 shows the relation between J_2/Y and J_1/Y , at yield, for a hollow sphere of matrix material with yield strength Y governed by the Mises condition and various initial porosities.

4. Elasto-Plastic Analysis

Once yielding has occurred the elastic solution obtained in section 2 is no longer applicable. As the load intensity is increased the region or regions of plastic yielding will spread and the body will consist of both elastic and plastic zones until the fully plastic load is reached. Thus, to continue the analysis of the hollow sphere we must solve an elasto-plastic problem in which the shape and position of the elasto-plastic interface is unknown a priori. Since finding an analytical solution to this problem is extremely difficult for all but the simplest cases, we turn to a numerical scheme.

We analyze the hollow sphere problem in the elasto-plastic range using the NONSAP (18,19) finite element program. NONSAP, which stands for Nonlinear Structural Analysis Program, includes an elasto-plastic material model based on the Drucker-Prager yield condition.

In addition to the yield function (3.1), a flow rule for the calculation of the plastic strain increments is required. In NONSAP, the associated flow rule is employed; the plastic strain increments are taken normal to the yield surface. Although the normality rule is applicable for metals, (i.e. Mises yield condition), this is not the case for porous rocks*. It is well known that the associative flow rule for the Drucker-Prager yield criterion (3.1) predicts too much dilatation. In view of this, a new elasto-plastic material model using

* A number of papers on the applicability of the normality rule to geological materials are cited by Schatz (16).

the Drucker-Prager yield condition and a nonassociative flow rule* was programmed and implemented into the NONSAP program. Specifically, we assumed that the flow rule was the same as the associative flow rule except that the term involving β was assumed to be zero.

The finite element program **, employing the mesh shown in Fig. 8, was used to examine the response of a hollow sphere subject to a number of loading conditions. In the first case, the matrix material was assumed to follow the Drucker-Prager criterion, while in the second case, the Mises condition was assumed to hold. In Fig. 9, the spherical model and experimental results for a 38% porous tuff are shown in a plot of mean pressure versus volume change/initial volume. The experimental and spherical model results for a nominally 78% dense porous aluminum are shown in Fig. 10.

* Nayak and Zienkiewicz (20) discuss some of the numerical problems arising from the use of a non-associative flow rule.

** In the finite element analysis of the hollow sphere problem both the elastic and plastic strains are assumed to be small. (i.e. geometrical nonlinearities are neglected).

Fig. 8. Finite element mesh of hollow sphere, consisting of 24, 8 node, isoparametric, axisymmetric elements.

Fig. 9. Comparison of spherical model prediction (—) and experimental results (---) for 38% porous Mt. Helen Nevada tuff. Lower curves indicate response for uniaxial compression test at 0.5 kb. confining pressure. Upper curves show calculated and measured hydrostat for comparison. Values of the elastic constants and yield parameters are $\lambda = 1206$ kb., $\mu = 5.7$ kb., $c = 1.55$ kb., $\phi = 39.5^\circ$. Experimental data from Heard et al. (22).

Fig. 10. Comparison of spherical model prediction (—) and experimental results (---) for a 78% dense porous aluminum. Values of the elastic constants are $\lambda = 5447$ kb. and $\mu = 2806$ kb. Initial yielding of the matrix material in simple tension occurs at $Y = 5$ kb. and linear strain hardening, with strain hardening modulus $E_t = 116$ kb., is assumed. Experimental data is from Johnson et al. (21).

5. Discussion and Conclusions

The results of sections 3 and 4 demonstrate that the spherical model is consistent with the experimentally observed phenomenon of shear-enhanced compaction in porous solids.

For porous rocks, Fig. 5 shows that the mean stress (i.e. $\frac{J_1}{3}$) at yield is highly sensitive to the presence of a shear stress. For example, the spherical model predicts that a 38% porous tuff will yield at a mean pressure (in nondimensional form $\frac{J_1}{3Y}$) equal to 2.4 under purely hydrostatic conditions; in contrast, yielding will occur at a mean stress of approximately 0.18 under a uniaxial compressive loading. The effect of the presence of a shear stress on the volume compaction for Mt. Helen Nevada tuff is shown in Fig. 9, where both the spherical model and experimental results are plotted. Agreement is good and the spherical model clearly shows the influence of shear stress on volume compaction.

The initial yield surface for a porous aluminum with various initial porosities is plotted in Fig. 7. As with a porous rock we find that the mean pressure at yield is a function of the applied shear stress. A comparison of Fig. 6 and Fig. 7 shows that the initial yield surface is relatively insensitive to changes in the elastic constants but does depend strongly on the initial porosity. The influence of a shear stress on the volume compaction for a 78% dense porous aluminum is shown in Fig. 10. The computed and experimental results are in very good agreement. Again the spherical model predicts that the presence

of a shear enhances the volume compression in comparison to a purely hydrostatic loading condition.

An interesting feature of the spherical model is that it provides a physical mechanism for the enhancement of void compaction in porous solids due to the presence of a shear stress. Under a uniform pressure loading yielding will spread out more or less uniformly around the boundary of the pore. However, under a nonhydrostatic loading, initiation of yield will take place at an isolated location on the pore boundary and the mean stress at the onset of yield will be lower in comparison to a hydrostatic loading. As the load intensity is increased, the volume of the matrix material which has yielded will continue to be greater for the nonhydrostatic vs. uniform pressure loading at the same mean stress value. In this way yielding of the matrix material surrounding the pore is promoted by the presence of a shear stress and this is manifested by an increase in the volume compaction of the porous solid in comparison to the uniform pressure loading.

Although the results are very encouraging so far, much more comparison with experimental data is needed before it can be established that the spherical model is capable of representing certain response properties of porous solids. Many porous materials, in particular those with a high ratio of void to solid phase, undergo considerable deformation even under moderate levels of applied stress. In order to analyze the crush-up behavior of these materials, it will be necessary to develop an elasto-plastic finite element program which incorporates both geometrical and material nonlinearities. If it is established that the spherical

model is capable of describing certain aspects of the mechanical behavior of porous solids, then it will be possible to, in effect, perform experiments using the computer*. The ability to examine the influence of variables such as the initial porosity and the yield parameters of the matrix material on the mechanical properties should prove very helpful in developing general constitutive equations for porous solids.

Certainly, the spherical model cannot describe all the mechanical properties of porous solids. Studies using a scanning electron microscope have shown that the phenomenon of volume dilatancy in porous rocks is related to the opening up of microcracks (8). A mechanistic model capable of describing the complex response of materials like porous rocks, which exhibit both ductile and brittle behavior, will have to be based on a void population consisting of both cracks and pores. However, the spherical model may provide an indirect means of predicting volume dilatancy in certain rocks. For rocks with a porosity greater than about five percent, compaction and dilatancy occur simultaneously and since both depend nonlinearly on the stress their separation is difficult (4). By using the spherical model to calculate the volume decrease due to void compaction, the volume increase resulting from the opening up of microcracks can be determined by subtracting the volume compaction from the overall volume change.

* A major advantage of the spherical model is that it is amenable to numerical analysis. Although the model is inherently three-dimensional, loading conditions with axial symmetry (which in the case for all experiments conducted on porous solids) can be analyzed using two-dimensional axisymmetric elements, which greatly increases computational efficiency.

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