Magnetostatic Equilibria Bounded by a Nonanalytic X-Point

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Abstract

A solution of a magnetostatic equilibrium problem involving an x-point of a form given by Harold Grad [COO-3077-152, MF-92, Courant Inst., N.Y., (1978)] is applied to the case where the x-point and null lines emanating from it form the boundary of the plasma, separating it from a current-free vacuum region. Several conventional notions are violated. The solution has no Taylor expansion at the x-point. The inter-separatrix angle may not be 90 degrees and can range from 0 to 180 degrees. The current does not vanish in the vicinity of the x-point.

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MASTER

The purpose of this note is to point out the existence of exact nonanalytic solutions of a magnetostatic equilibrium problem that involve a plasma-bounding saddle point and separatrices. These solutions are of a form given by Grad [1] and violate familiar properties of conventional analytic solutions: 1) the solutions do not possess a Taylor expansion at the saddle point, 2) the angle at a "vacuum" saddle point, in which the plasma fills just one quadrant, does not have to be 90 degrees, and 3) the current need not vanish at the saddle point. These solutions are not singular. In the neighborhood of the saddle point, the flux is finite and differentiable.

The classical vacuum x-point has the flux in the vacuum region depending as $x^2 - y^2$, to leading order. The inter-separatrix or inter-null-line angle is, therefore, 90 degrees. The joining of this vacuum solution to an analytic plasma solution, occupying one quadrant, forces the plasma current to be zero in some finite neighborhood of the x-point. Therefore, in this model, the plasma must be artificially bounded away from the vicinity of the x-point, an assumption which has no physical justification.

Grad [1] has presented an exact solution for the problem of an x-point surrounded by a plasma. His solution differs from the usual ones [2] in that the flux is not analytic at the x-point. I have adapted his solution to the vacuum case, and, in doing so, I consider a wider range of the fundamental parameter of the model. Grad was aware of the applicability of his solution to the vacuum case, but he did not explore it in reference [1], where he was mainly concerned with another subject. The possibility of nonzero current at a bounding x-point was considered by Laval [3] some time ago. Some of the tentative results that he advanced are contradicted by the solution given here. The importance of this problem lies in its possible relevance to the H-mode phenomenon [4]. This state of improved confinement, characterized by steep profile gradients at the edge of the plasma, is usually associated with separatrix bounded plasmas.

I assume a straight two-dimensional geometry in which all quantities are independent of the cartesian z axis. In the absence of singularities this is a valid, near saddle point or large aspect ratio, limit of toroidal geometry. In this same limit, given certain assumptions, the Grad-Shafranov equilibrium equation for toroidally symmetric systems [5], to leading order, assumes the form

$$\nabla^2 \Psi = \lambda \Psi^\alpha, \quad (1)$$

Here Ψ is the poloidal flux and an arbitrary constant has been selected so that $\Psi = 0$ at the saddle point and along the separatrices. ∇^2 is the large aspect ratio limit of the Grad-Shafranov operator Δ^* . λ is a given constant, and Ψ^α is the leading term in a series expansion for the plasma profiles of the Grad-Shafranov equation. The assumptions which

lead to Eq. (1) in the large aspect ratio limit may not always be valid. In particular, a convergent expansion for the plasma profiles, of the implied form, may not always exist near the saddle point. However, the solutions given here are not approximations or the leading terms in an expansion. They are exact solutions of Eq. (1).

Figure 1 illustrates the geometry of the problem. Reflective symmetry about $y = 0$ is assumed. A plasma region Ω_P , in which the current is nonvanishing ($J = \nabla^2 \Psi \neq 0$), occupies one quadrant of an area delineated by four separatrices that emanate from a saddle point in the center of the figure. The remaining three quadrants form a vacuum region Ω_V in which the current vanishes ($\nabla^2 \Psi = 0$). The poloidal flux Ψ vanishes on the separatrices and at the saddle point. Equation (1) holds in the plasma region and the derivative of the poloidal flux normal to the separatrices is required to be continuous across those separatrices which form the boundary of the plasma.

That $J = 0$ in the vacuum region is a consequence of the fact that the magnetic field lines in this region intersect physical limiters and the charged particles responsible for the current follow these field lines and are quickly lost to the limiters. Magnetic surfaces within the plasma are closed in the large and the particles are confined and continue to circulate. *This picture is much too simple. In reality, all sorts of effects may allow a finite current in the vacuum region. However, the intent of this note is only to give solutions that have not yet been considered as a starting point for a more complete analysis; as when the current is nonvanishing, nonanalytic solutions may turn out to be more physically relevant than analytic ones.*

In the vacuum region Ω_V a form for Ψ is assumed that obeys Laplace's equation.

$$\Psi = A r^p \cos p(\theta - \pi). \quad (2)$$

Then $\Psi = 0$ when $\theta/\pi = 1 + 1/2p, 1 - 1/2p, 1 - 3/2p, 1 + 3/2p \dots$

To obtain a four null line or true x-point, define

$$\theta_V = \pi \left(1 - \frac{3}{2p} \right) \quad (3)$$

to be the half-angle of the plasma-bounding separatrix required by the vacuum solution. Since $0 < \theta_V < \pi$, one must have $3/2 < p < \infty$.

In the plasma region Ω_P , following Grad [1], I assume $\Psi \geq 0$ and a solution of the separable form

$$\Psi = B r^\nu \nu(\theta), \quad (4)$$

where $0 \leq \nu(\theta) \leq 1$. Substitution of (4) into (1) and equating powers of r requires that $\alpha = 1 - 2/p$.

p	α
3/2	-1/3
2	0
5/2	1/5
3	1/3

A nonlinear second-order ordinary differential equation results for $\nu(\theta)$

$$\frac{d^2\nu}{d\theta^2} - p^2\nu = \epsilon\nu^\alpha, \quad (5)$$

where $\epsilon = \lambda/B^{2/p}$.

The “energy”

$$E = \frac{1}{2} \left(\frac{d\nu}{d\theta} \right)^2 - \frac{p^2\nu^2}{2} - \frac{\epsilon\nu^{1+\alpha}}{1+\alpha} \quad (6)$$

is readily verified to be a constant. $\nu(\theta)$ is assumed to have the form of Fig. 2. One has $\nu(0) = 1$ by definition. In the interval $0 \leq \theta \leq \theta_P$, ν is a monotone decreasing function. $\nu(\theta_P) = 0$ implicitly defines θ_P , the angle of the plasma-bounding separatrix required by the plasma solution.

Assuming that $d\nu/d\theta = 0$ at $\theta = 0$, the energy is

$$E = \frac{p^2}{2} - \frac{\epsilon}{1+\alpha} \quad (7)$$

and one finds

$$\frac{d\nu}{d\theta} = \pm p \left\{ (1 - \nu^2) \left[1 - \frac{\epsilon}{p(p-1)} \left(\frac{1 - \nu^{1+\alpha}}{1 - \nu^2} \right) \right] \right\}^{1/2}. \quad (8)$$

The term in [] in (8) may be verified to be finite, for $0 \leq \nu \leq 1$, provided that $\epsilon/p(p-1) < 1$. The minus sign is the appropriate choice for positive θ . Then

$$\theta_P = \int_0^1 \frac{d\nu}{|d\nu/d\theta|} \quad (9)$$

and one condition for a solution is $\theta_P = \theta_V$.

θ_P is easily verified to be a monotonically increasing function of ϵ . If $\epsilon = 0$, then $\theta_P = \pi/2p$. One has $\theta_V > \pi/2p$ if $p > 2$, and $\theta_V < \pi/2p$ if $p < 2$. As a result, if $p > 2$, one needs $\epsilon > 0$ to match the angles, while $p < 2$ requires that $\epsilon < 0$. $p = 2$ forces $\epsilon = 0$ and the classical 90-degree vacuum x-point is recovered.

Evidently, given p , one can always find an ϵ such that $\theta_F = \theta_V$. Having matched the angles, continuity of Ψ is trivial. Continuity of the normal derivative of Ψ for all r at $\theta = \theta_V = \theta_F$ requires that

$$A = B \left[1 + \frac{\epsilon}{p(p-1)} \right]^{1/2}. \quad (10)$$

One sees from (10) that giving B and ϵ (or λ) determines A , so that a solution is always possible. It now becomes apparent that λ can be scaled out and only enters via its sign. Hence, these are a one parameter family of solutions in the parameter p up to a normalization constant that can be fixed by assuming $B = 1$ while allowing ϵ to obtain the value required to match the half-angles. Figure 3 gives numerically computed values of ϵ as a function of p and Fig. 4 gives the ratio of A to B as a function of p .

Note that if $\epsilon > 0$ or $p > 2$, then the solution of (1) cannot describe a plasma region of finite extent since, on average, Ψ and $\nabla^2 \Psi$ must have opposite signs. In this case, the solution must be regarded as a near x-point approximation and the current must reverse direction far from the x-point. Such a current reversal occurs naturally in high β toroidal plasmas. Positive ϵ solutions have $J \approx r^{p-2} \rightarrow 0$ at the x-point; however, this approach to 0 is arbitrarily steep as p nears 2. In contrast to the 90-degree vacuum x-point, these solutions have nonvanishing current in any neighborhood of the x-point. The angle of the plasma-bounding separatrix is greater than 90 degrees for these solutions.

One surprise is that the angle of the plasma-bounding separatrix is greater than 180 degrees if $p > 3$. In other words, the plasma is actually indented or concave at the x-point. Intuitively, this seems nonphysical and, based on numerical evidence, Fig. 4, $A/B \rightarrow 0$ as $p \rightarrow 3$, implying that $\epsilon/p(p-1) \rightarrow 1$ or $\epsilon \rightarrow 6$. Since $\theta_V = \pi/2$ in this limit, this requires that

$$\theta_F = \frac{\pi}{2} = \frac{1}{3} \int_0^1 \frac{d\nu}{\sqrt{\nu^{4/3} - \nu^2}}. \quad (11)$$

a formula which is easily proved with the substitution $\cos \mu = \nu^{1/3}$. Thus, p is effectively restricted to $3/2 < p < 3$ and $0 < \theta_F < \pi/2$, a result that agrees with intuition.

The $\epsilon < 0$ or $p < 2$ solutions can be globally valid for a finite plasma (see comments by Grad [1]). These have an arbitrarily mild current singularity at the x-point and along the separatrix. The angle of the plasma bounding separatrix is less than 90 degrees for these solutions.

The poloidal magnetic field goes as r^{p-1} near the x-point. The safety factor or winding number ratio behaves as

$$q \approx \int \frac{dr}{r \Psi} \approx r^{2-p}. \quad (12)$$

and is finite for the $p < 2$ singular current case. $dJ/d\Psi$ behaves as $1/r^{2/2+p}$ and is much larger near the x-point ($\approx 1/d^2$, where d is distance) than it is near the separatrix, but away from the x-point ($\approx 1/d^{2/p}$).

In conclusion, I have presented a solution of the magnetostatic equilibrium problem, in which an x-point and the separatrices emanating from it form the boundary of the plasma and separate it from a current-free vacuum region. This solution is a straightforward modification of that given by Grad¹. The solution violates several notions which were previously thought to be valid, at least by the author, in the neighborhood of a plasma-bounding x-point. The angle of the plasma-bounding separatrix does not have to be 90 degrees and can range from 0 to 180 degrees. The current does not necessarily vanish in the vicinity of the x-point and the flux is not necessarily Taylor expandable at the x-point. The solution given here is not the only solution of Eq. (1) and the question of whether a finite nonzero current can exist at a vacuum x-point is left open. The latter question may be academic given that the present model can approach arbitrarily close to the constant current state. However, even more pathological solutions might exist where the x-point becomes a line null. Whether these facts are relevant to actual plasmas remains to be seen, but more complete models should allow for these possibilities.

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Figure Captions

- Fig. 1 A plasma region Ω_P , in which the current is nonvanishing occupies one quadrant of an area delineated by four separatrices that emanate from a saddle point in the center of the figure. The remaining three quadrants Ω_V form a vacuum region in which the current vanishes.
- Fig. 2 ν is a positive, monotone decreasing function for positive θ less than θ_P , the half-angle of the separatrix. $\nu(\theta_P) = 0$ implicitly defines θ_P . $\nu(0) = 1$ and $d\nu(0)/d\theta = 0$.
- Fig. 3 $\epsilon(p)$, is determined by matching the half-angles of the separatrix in the plasma and vacuum solutions and is a monotonically increasing function of p , that tends to 6 as $p \rightarrow 3$.
- Fig. 4 A/B , the ratio of vacuum to plasma flux strengths, is uniquely determined by the solution for a given p . A/B tends to zero as p approaches 3.

Fig. 1

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Fig. 2

Fig. 3

Fig. 4