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Gyroelastic fluids *

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Gyroelastic Fluids

By

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DEDICATION

To my family:

ברוך אתה יי אלהינו מלך העולם.

אשר פקח עינינו במצותיו וצונו להדליק נר של דעה ותבונה:

ברוך אתה יי אלהינו מלך העולם.

שהחינו וקימנו והגיענו לזמן הזה:

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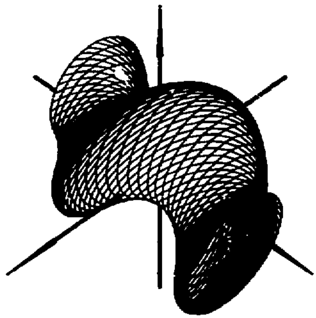
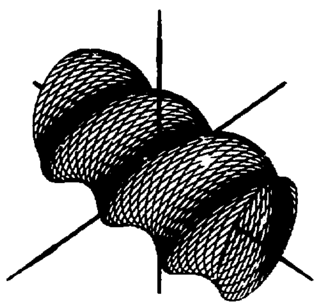


$m=1$ Helical Kink Perturbation



$m=2$ Helical Kink Perturbation

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ABSTRACT

A study is made of a scale model in three dimensions of a guiding center plasma within the purview of *gyroelastic* (also known as finite gyroradius-near theta pinch) magnetohydrodynamics. The (nonlinear) system sustains a particular symmetry called *isorrhopy* which permits the decoupling of fluid modes from drift modes. *Isorrhopic* equilibria are analyzed within the framework of geometrical optics, resulting in (local) dispersion relations and ray constants. A general scheme is developed to evolve an arbitrary linear perturbation of a screwpinch equilibrium as an invertible integral transform (over the complete set of generalized eigenfunctions defined naturally by the equilibrium.) Details of the structure of the function space and the associated spectra are elucidated. Features of the (global) dispersion relation owing to the presence of *gyroelastic* stabilization are revealed. An energy principle is developed to study the stability of the tubular screwpinch.

INTRODUCTION

The theoretical model is a metaphor used as a tool to represent some aspect of reality. We analyze the model as surrogate to nature. Its success is gauged in its simplicity and its pertinence. If the model is intractable little can be learned of it. If the model is inapplicable or irrelevant little can be learned from it.

To assure simplicity we symmetrize; to assure relevance we choose a symmetry which persists in accordance with physical law. The underlying symmetry dealt with in this work is permutation symmetry, a concept developed by Newcomb[21-22] who called it *isorrhopy*.

The presence of a symmetry can be expressed by identifying its associated infinitesmal symmetry operation. An operation which leaves the action integral of a Lagrangian system unchanged is a symmetry operation. Noether's theorem states that every such symmetry operation induces a conservation law for a certain physical quantity; the quantity can be given once the Lagrangian is known. For example, translation symmetry gives rise to linear momentum conservation. To find the conserved quantity one has only to find a variation of the action which leaves the action integral invariant, but does not vanish at the limits of integration. The essential feature of the exchange invariant or *isorrhopic* fluid system resides in the invariance of its action integral under symmetry operations called permutations. A permutation is a virtual displacement which leaves the solution at a fixed space-time point relevantly unaffected. The operation in effect exchanges the identity of neighboring fluid elements. Neighboring fluid point world lines are not uniquely identified in the *isorrhopic* fluid.

Chapter I is devoted to a discussion of some unique features of *isorrhopic gyroelastic* systems. *Gyroelastic* is the term I will use to denote the particular scaling regime to which attention is restricted in the present work. *Gyroelastic* scaling corresponds to what is often referred to in the literature as finite-gyro-radius scaling. However, here the key physical quantity is in fact not the size of the orbit of a charged particle in a magnetic field, but rather the angular momentum associated with its motion. It is this angular momentum (density) which gives rise to gyroscopic and quasielastic forces in addition to the usual pressure related forces of standard MHD.

A rudimentary understanding of the nature of the gyroscopic-quasielastic forces can be gleaned from a graphic though crude analogy. A spinning top will tend to wobble if disturbed from its equilibrium state. The wobble will tend to occur at a higher frequency the larger the angular momentum of the top (the faster it is spinning.) In analogy with a spring and mass, the angular momentum can be cast in the role of an elasticity in determining the (wobble) frequency. In the gyroelastic fluid, angular momentum is considered a continuum property. It is this angular momentum (density) which gives rise to the property of quasielasticity. In the case of the top, the influence of the presence of angular momentum is to stabilize the motion of the system. The top is prevented from falling over if its angular momentum is large enough. A similar effect exists in the gyroelastic fluid system.

Considerable literature exists on the subject of finite gyroradius effects in near theta-pinch or long-thin geometry.[3-5,11,16-17,21-22,26-34,36-39] Roberts and Taylor[28] were among the earliest researchers to recognize the stabilizing influence connected with the angular momentum of particle gyration in magnetized plasma. Rosenbluth and Simon[31] in a classic work developed the theory of low-β gyroscopic fluids with plasma flow. In the early 1970's there arose concurrently two approaches to the more general problem dealing with nonuniform magnetic field (long-thin geometry) and high-β: the Vlasov fluid model developed by Freidberg[3-5] and the theory of gyroscopic-quasielastic fluids developed by Newcomb[21-22]. Whereas the path used in each approach is substantively similar, the scenery along the way varies greatly. Certain results accessible to the Vlasov fluid model[27,36] derive also from the theory of *gyroelastic* systems. (An example is presented in Appendix II.) This study is an examination of the view at and beyond the periphery of the earlier work by Newcomb[22] as seen through the prism of that approach.

The starting point for the construction of the model is the Vlasov equation in the adopted scaling. The Lagrangian for the general scale model is developed, then systematically specialized to the *isorrhopic* case. The specialization is in essence an initialization. A system once *isorrhopic* will evolve so as to remain isorrhopic. The class of *isorrhopic* MHD systems contains equilibria as a subclass. A closed set of equations governing the behavior of *isorrhopic gyroelastic* systems is presented. Transformation properties of these equations under changes of representation (transformations which continuously permute the identity of fluid elements) are reviewed. The theory is developed in fully nonlinear form.

In chapter II the *isorrhopic* configuration is examined in the geometrical optics limit. Ray constants and the local dispersion relation are revealed by a variational technique. A canonical theory of linear waves in the *isorrhopic gyroelastic* system is developed.

In chapter III the nonlinear system is linearized. The linearized Euler equation together with fixed boundary conditions then defines the eigensolution space for a chosen gyroelastic screwpinch equilibrium. The spectrum of eigenvalues is generally composed of discrete spectra and continua which touch at points of accumulation. The Euler equation may become singular over ranges of the parameters which map the solution space. These ranges then define the continua. Strictly speaking, the singular solutions are not functions, but rather generalized functions or distributions.

Distributions are linear functionals. Although not members of the Hilbert space of possible motions of the equilibrium they play an essential role in the construction of an invertible integral transform to evolve arbitrary perturbations that are possible motions.

Once the integral theorem is expressed, some of the details of the spectra are elaborated. At this point it is shown that fixed boundary equilibria which are normally Suydam unstable in the non-*gyroelastic* system seem to be stabilizable by *gyroelastic* effects. In particular, it is seen that no Kruskal-Shafranov instability behavior occurs when the eigenmode is convective ([helicity of perturbation equal to helicity of field lines) near the surface of the plasma.

In chapter IV the boundary condition is generalized to allow free boundary motions of the *gyroelastic* screwpinch. The resultant spectra are studied and compared with those of the fixed boundary model. In the limit that the vacuum region becomes thin an interesting discrepancy between the fixed and free boundary models arises.

In chapter V the boundary condition is used to develop an energy principle with which to streamline the study of the stability and stabilizability (through *gyroelasticity* effects) of a more general screwpinch configuration. Whereas in previous chapters the equilibria were columnar, now the more general tubular case is treated. Stability criteria are devised and growth rates are calculated for some typical systems.

The present work deals with the *isorrhopic* case. There exists of course also the *anisorrhopic* case. Though the subject is avoided throughout most of what follows it is tempting to mention briefly the relationship between the cases. As indicated, *isorrhopy* is an initialization. One chooses a set of initial conditions which obtain identically in time. The question of the stability of this configuration to fluid perturbations is addressed in this work. However, the stability of the configuration to perturbations which violate the *isorrhopy* of the system are not considered. Such perturbations come under the general heading of drift modes, which are beyond the scope of this analysis. Attention is restricted to the incompressible helical motions of the *isorrhopic gyroelastic* system.

NOTATION

We adopt a notation which is a slight modification of that used by Newcomb[22]. Vectors in the usual three dimensional space will be treated as an ordered pair of objects

$$A^{(3)} = \{\mathbf{a}, A\} = a_x \mathbf{e}_x + a_y \mathbf{e}_y + A\mathbf{e}_z \tag{0.1}$$

the first having two components, the second having one. This is simply a "hard-wired" distinction between perpendicular and parallel with respect to the direction of the magnetic field, to lowest order, say $\mathbf{e}_z$. The transposition of some common vector identities follows: let $A^{(3)}=(\mathbf{a},A)$ and $B^{(3)}=(\mathbf{b},B)$ be two three-vectors and let

$$\mathbf{a} \times \mathbf{e}_z = a_y \mathbf{e}_x - a_x \mathbf{e}_y = \mathbf{a}^* \tag{0.2}$$

then

$$\{\mathbf{a}, A\} \cdot \{\mathbf{b}, B\} = \mathbf{a}\cdot\mathbf{b} + AB \tag{0.3.1}$$

$$\{\mathbf{a}, A\} \times \{\mathbf{b}, B\} = \{\mathbf{a}^* B - \mathbf{b}^* A \ , \ \mathbf{a}\cdot\mathbf{b}^*\} \tag{0.3.2}$$

$$\nabla^{(3)} \cdot \{\mathbf{a}, A\} = \nabla\cdot\mathbf{a} + \partial_z \tag{0.3.3}$$

$$\nabla^{(3)} \times \{\mathbf{a}, A\} = \{\nabla^* A - \partial_z \mathbf{a}^* , \nabla\cdot\mathbf{a}^*\} \tag{0.3.4}$$

$$(\mathbf{AB}\cdot\mathbf{C}-\mathbf{C}\cdot\mathbf{AB})^{(3)} = \{(\mathbf{a}\cdot\mathbf{b}^*)\mathbf{c}^* + C(B\mathbf{a}-A\mathbf{b})\ ,\ -(B\mathbf{a}-A\mathbf{b})\cdot\mathbf{c}\} \tag{0.3.5}$$

$$\nabla\cdot\mathbf{ab} = \mathbf{b}\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla\mathbf{b} \tag{0.3.6}$$

For three-vectors $\mathbf{A}^{(3)}$ and $\mathbf{B}^{(3)}$ we write the tensor product $\mathbf{A}^{(3)}\mathbf{B}^{(3)}$ as

$$\{\mathbf{a},A\}\{\mathbf{b},B\} = \{\{\ \mathbf{ab},\mathbf{a}B\ ;\ A\mathbf{b},AB\ \}\} \tag{0.4}$$

The unit two-tensor can be written as

$$\underline{\mathbf{I}} = \mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y \tag{0.5}$$

and the following identities with regard to it apply:

$$\mathbf{a}\cdot\underline{\mathbf{I}} = \underline{\mathbf{I}}\cdot\mathbf{a} = \mathbf{a} \tag{0.6}$$

$$\mathbf{a}\ \mathbf{a} + \mathbf{a}^*\mathbf{a}^* = \mathbf{a}\cdot\mathbf{a}\ \underline{\mathbf{I}} \tag{0.7}$$

The antisymmetric part of a two-tensor $\mathbf{ab}$ is

$$\alpha(\mathbf{ab}) = a_x b_y - a_y b_x = \mathbf{a}\cdot\mathbf{b}^* \tag{0.8}$$

from which derives the useful relation

$$\mathbf{ab}\cdot\mathbf{c} - \mathbf{c}\cdot\mathbf{ab} = \mathbf{c}^*(\mathbf{a}\cdot\mathbf{b}^*) \tag{0.9}$$

If two two-vector fields differ by the gradient of a scalar field κ

$$\mathbf{a} - \mathbf{b} = \nabla\kappa \tag{0.10}$$

then $\mathbf{a}$ and $\mathbf{b}$ will be said to be congruent

$$\mathbf{a} \sim \mathbf{b} \tag{0.11}$$

Likewise, if two two-tensor fields satisfy

$$\nabla\cdot(\mathbf{ab}-\mathbf{cd}) = \nabla\kappa \tag{0.12}$$

then they are said also to be congruent to one another, and their difference congruent to zero

$$\mathbf{ab} \sim \mathbf{ed} \tag{0.13}$$

There derives from these last considerations that

$$\kappa \underline{l} \sim 0 \tag{0.14.1}$$

$$\mathbf{a}\,\mathbf{a} + \mathbf{a}^*\mathbf{a}^* \sim 0 \tag{0.14.2}$$

$$\nabla^*\nabla^*\kappa \sim 0 \tag{0.14.3}$$

$$\nabla\,\nabla\,\kappa = 0 \tag{0.14.4}$$

The operator denoting total or convective time differentiation along a fluid two-trajectory is written D, so that

$$D\mathbf{x} = \mathbf{v}(\mathbf{x}, z; t) \tag{0.15}$$

or, for an arbitrary function $F(\mathbf{x}, t)$

$$DF = \partial_t F + \mathbf{v}\cdot\nabla F \tag{0.16}$$

The operators D and ∇ do not commute and it can be easily shown that

$$[\nabla, D] = \nabla D - D\nabla = \nabla\mathbf{v}\cdot\nabla \tag{0.17.1}$$

and also that

$$[\nabla^*, D] = \nabla^* D - D\nabla^* = \nabla^*\mathbf{v}\cdot\nabla \tag{0.17.2}$$

An addition, the following symbols are used throughout the text:

$$\mathbf{B}^{(3)} = \{B\tau\ ,\ B\} = magnetic\ field \tag{0.18.1}$$

$$\mathbf{E}^{(3)} = \{\mathbf{E}\ ,\ \mathcal{E}\} = electric\ field \tag{0.18.2}$$

$$\mathbf{J}^{(3)} = \{\mathbf{J}\ ,\ J\} = \textit{electric current density} \tag{0.18.3}$$

$$\mathbf{V}^{(3)} = \{\mathbf{v}\ ,\ V\} = \textit{fluid velocity} \tag{0.18.4}$$

$$\eta = \textit{electric charge density} \tag{0.18.5}$$

CHAPTER 1

Description of the model

The Gyroelastic Fluid

The guiding center plasma forms the basis for our model of a gyroelastic fluid. Since I intend to examine collisionless plasma behavior I have taken the Vlasov equation to apply and neglected all transport processes and particle correlations. The gyroelastic ordering is entirely characterized in the following expression of the Vlasov equation:

$$
\begin{aligned}
&\varepsilon^0 \ \frac{B}{m}\, \boldsymbol{p}^{*}\cdot\nabla_{\boldsymbol{p}} f \\
&+ \varepsilon^1 \ \Big(\Big(\frac{\boldsymbol{p}}{m}\cdot\nabla + e\Big(-\frac{q}{m}\, B\boldsymbol{\tau}^{*}\cdot\nabla_{\boldsymbol{p}} + \frac{B}{m}\,\boldsymbol{p}\cdot\boldsymbol{\tau}^{*}\partial_q + \boldsymbol{E}\cdot\nabla_{\boldsymbol{p}}\Big)\Big)f \\
&+ \varepsilon^2 \ \Big(\partial_t + \frac{q}{m}\,\partial_z + e\mathcal{E}\partial_q\Big) f = 0
\end{aligned}
\tag{1.1}
$$

where

$$f = f(\{\boldsymbol{x},z\};\{\boldsymbol{p},q\};t) \tag{1.2}$$

is the one particle distribution function,

$$\boldsymbol{p}^{(3)} = \{\boldsymbol{p},q\} \tag{1.3}$$

is the particle three-momentum and ε is a formal expansion (*smallness*) parameter. The Vlasov equation as written in (1.1) dictates the ordering of quantities as

$$\nabla \sim \varepsilon \sim \frac{\rho_0}{L_{\perp s}} = \frac{gyroradius}{perpendicular\ scale\ length} \tag{1.4.1}$$

$$\partial_t \sim \varepsilon^2 \sim \frac{1}{\Omega\tau_s} = \frac{gyroperiod}{plasma\ flow\ time\ scale} \tag{1.4.2}$$

$$\partial_z \sim \varepsilon^2 \sim \frac{\rho_0}{L_s} = \frac{gyroradius}{parallel\ scale\ length} \tag{1.4.3}$$

$$\{ \mathbf{E} \sim \varepsilon\ ,\ \mathcal{E} \sim \varepsilon^2 \} = electric\ field \tag{1.4.4}$$

$$\{ B\tau \sim \varepsilon\ ,\ B \sim 1 \} = magnetic\ field \tag{1.4.5}$$

The set of equations governing the behavior of the gyroelastic fluid is completed with Maxwell's equations which in the adopted notation are written as

$$\partial_t (B\tau) + \nabla^* \mathcal{E} - \partial_z \mathbf{E}^* = 0 \tag{1.5.1}$$

$$\partial_t B + \nabla\cdot\mathbf{E}^* = 0 \tag{1.5.2}$$

$$\chi_0 \nabla\cdot\mathbf{E} + \varepsilon^2 \chi_0 \partial_z \mathcal{E} = \varepsilon^2 \eta \tag{1.5.3}$$

$$\nabla^* B - \varepsilon^2 \partial_z (B\tau^*) = \mu_0 \mathbf{J} + \varepsilon^2 \mu_0 \chi_0 \partial_t \mathbf{E} \tag{1.5.4}$$

$$\nabla\cdot(B\tau^{*}) = \mu_0 J + \varepsilon^2 \mu_0 \chi_0 \partial_t \mathcal{E} \tag{1.5.5}$$

$$\nabla\cdot(B\tau) + \partial_z B = 0 \tag{1.5.6}$$

Factors of ε are included in (1.5.1)–(1.5.6) to indicate the relative ordering of terms.

Essentially all the relevant physics necessary to derive an equation of motion for the gyroelastic system is contained in equations (1.1) and (1.5.1)–(1.5.6). There remains to derive from this system, in standard form, the law of conservation of momentum

$$\partial_t \boldsymbol{s}^{(3)} + \nabla^{(3)}\cdot \boldsymbol{T}^{(3)} = 0 \tag{1.6.1}$$

where $\boldsymbol{s}^{(3)}$ is the three-momentum-density

$$\boldsymbol{s}^{(3)} = | \; \boldsymbol{s} \; , \; S \; | \tag{1.6.2}$$

and $\boldsymbol{T}^{(3)}$ is the three-stress tensor

$$\underline{\boldsymbol{T}}^{(3)} = \|\; \underline{\boldsymbol{t}}, \boldsymbol{T} \; ; \; \boldsymbol{u}, U \;\| \tag{1.6.3}$$

It will prove expedient to separate f into two constituents respectively even and odd in $\boldsymbol{p}$:

$$2\, f^{+}(\boldsymbol{p}) = (f(\boldsymbol{p}) + f(-\boldsymbol{p})) \tag{1.7}$$

and

$$2\varepsilon f^{-}(\boldsymbol{p}) = (f(\boldsymbol{p}) - f(-\boldsymbol{p})) \tag{1.8}$$

The presence of the factor ε in (1.8) will be discussed in more detail below. Now write (1.1), the Vlasov equation, in the form

$$\mathcal{V}(\boldsymbol{p})\; f(\boldsymbol{p}) = 0 \tag{1.9}$$

where $\mathcal{V}(\boldsymbol{p})$ is the Vlasov operator as indicated. Since $f(\boldsymbol{p})$ and $f(-\boldsymbol{p})$ both solve the Vlasov equation, two equivalent versions of (1.9) can be displayed as

$$\mathcal{V}(\pm\boldsymbol{p})\; f(\pm\boldsymbol{p}) = 0 \tag{1.10}$$

Substitution of (1.7) and (1.8) in (1.10) gives

$$e\,\frac{B}{m}\,\boldsymbol{p}\cdot\nabla_{\boldsymbol{p}}^{*}f(\pm\boldsymbol{p}) = \varepsilon^{2}(\partial_{t}+\frac{q}{m}\,\partial_{z}+\,e\mathcal{E}\partial_{q})f(\pm\boldsymbol{p})$$
$$\pm\varepsilon\,\,(\frac{\boldsymbol{p}}{m}\cdot\nabla + e(-\frac{q}{m}\,B\tau^{*}\cdot\nabla_{\boldsymbol{p}}+\frac{\boldsymbol{p}}{m}\cdot B\tau^{*}\partial_{q}+\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}}))f(\pm\boldsymbol{p}) \tag{1.11}$$

For the following calculation factors of ε will be deleted for clarity. The sum and difference of the two equations represented by (1.11) give the useful relationships

$$(\partial_{t}+\frac{q}{m}\,\partial_{z})f^{\pm}+\frac{\boldsymbol{p}}{m}\cdot\nabla f^{\mp} = e(-\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}}+\frac{q}{m}\,B\tau^{*}\cdot\nabla_{\boldsymbol{p}}-\frac{\boldsymbol{p}}{m}\cdot B\tau^{*}\partial_{q})f^{\mp}$$
$$+\,e(\,\frac{B}{m}\,\boldsymbol{p}\cdot\nabla_{\boldsymbol{p}}^{*}-\mathcal{E}\partial_{q})f^{\pm} \tag{1.12}$$

Retaining the relativistic form of the particle mass

$$m^{2} = m_{0}^{2} + (q/c)^{2} + (\boldsymbol{p}/c)^{2} \tag{1.13}$$

define $\boldsymbol{s}_{\mu}^{(3)}$ to be the total material three-momentum density

$$\boldsymbol{s}_{\mu}^{(3)} = \}\boldsymbol{s}_{\mu},S_{\mu}\{ = \sum_{s}\int d\boldsymbol{p}\;dq\;\;\}\boldsymbol{p},q\{\;f_{s} \tag{1.14}$$

where s denotes the species of particle. Now, to develop an expression of the conservation of momentum for the gyroelastic system, begin with the definition:

$$\boldsymbol{F}^{(3)} = \}\boldsymbol{f},F\{ \equiv \sum_{s}\int d\boldsymbol{p}\;dq\;\,(\partial_{t}+\}\frac{\boldsymbol{p}}{m},\frac{q}{m}\{\cdot\}\nabla,\partial_{z}\{)\;\,(\}\boldsymbol{p},q\{\;f_{s}(\boldsymbol{p},q;t)) \tag{1.15}$$

It will turn out that the parallel component of this equation is reducible to a more or less obvious relation whereas the transverse component embodies the essential dynamical features unique to the gyroelastic system. In the interest of completeness the parallel component is kept in the following analysis.

Separating f_{s} into odd and even parts as prescribed above, equation (1.15) becomes

$$\boldsymbol{f} = \sum_s \int d\boldsymbol{p}\, dq\, \boldsymbol{p}((\partial_t + \frac{q}{m}\partial_z) f_s^- + \frac{\boldsymbol{p}}{m}\cdot\nabla f_s^+)$$

$$F = \sum_s \int d\boldsymbol{p}\, dq\, q((\partial_t + \frac{q}{m}\partial_z) f_s^+ + \frac{\boldsymbol{p}}{m}\cdot\nabla f_s^-)$$

(1.16)

The sum and difference formulae (1.11) can now be used to reexpress the right-hand member of (1.16), leaving the equation of motion in the form

$$\sum_s \int d\boldsymbol{p}\, dq\, \boldsymbol{p}((\partial_t + \frac{q}{m}\partial_z) f_s^- + \frac{\boldsymbol{p}}{m}\cdot\nabla f_s^+)$$

$$= \sum_s \int d\boldsymbol{p}\, dq\, \boldsymbol{p}((\frac{B}{m}\boldsymbol{p}\cdot\nabla_{\boldsymbol{p}}^* - \mathcal{E}\partial_q) f_s^- + (-\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}} + \frac{q}{m}B\tau^*\cdot\nabla_{\boldsymbol{p}} - \frac{\boldsymbol{p}}{m}\cdot B\tau^*\partial_q) f_s^+)$$

$$\sum_s \int d\boldsymbol{p}\, dq\, q((\partial_t + \frac{q}{m}\partial_z) f_s^+ + \frac{\boldsymbol{p}}{m}\cdot\nabla f_s^-)$$

$$= \sum_s \int d\boldsymbol{p}\, dq\, q((\frac{B}{m}\boldsymbol{p}\cdot\nabla_{\boldsymbol{p}}^* - \mathcal{E}\partial_q) f_s^+ + (-\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}} + \frac{q}{m}B\tau^*\cdot\nabla_{\boldsymbol{p}} - \frac{\boldsymbol{p}}{m}\cdot B\tau^*\partial_q) f_s^-)$$

(1.17)

Gor a function G which vanishes at the limits of integration, an integration by parts gives the result

$$\int dx\, dy\, \boldsymbol{a}\boldsymbol{b}\cdot\nabla G = -\int dx\, dy\, (\boldsymbol{a}\nabla\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{a})G$$

(1.18)

(also valid with ∇ replaced by ∇^*.) Using this result and noting that the relativistic mass m depends on the particle momentum through (1.13) a short calculation recasts the right-hand member of (1.17) as

$$\boldsymbol{f} = \boldsymbol{E}\eta + B\boldsymbol{J}^* - B\tau^* J$$

$$F = \mathcal{E}\eta + B\tau^*\cdot\boldsymbol{J}$$

(1.19.1)

where η is the electric charge density, J is the parallel electric current density and $\boldsymbol{J}$ is the electric two-current density. These sources of the electromagnetic fields are related to the distribution function as follows:

$$\eta = \sum_s e_s \int d\boldsymbol{p}\, dq\, f_s^+$$

(1.19.2)

$$
\mathbf{J} = \sum_s e_s \int d\mathbf{p}\, dq\, \frac{\mathbf{p}}{m} f_s
$$

(1.19.3)

$$
J = \sum_s e_s \int d\mathbf{p}\, dq\, \frac{q}{m} f_s^{+}
$$

(1.19.4)

Maxwell's equations are now used to free the equation of motion of explicit reference to sources, in favor of fields. Evaluate $\mathbf{E}\eta$ using (1.5.3) to find

$$
\mathbf{E}\eta = \chi_0(-\nabla\cdot(\mathbf{E}^{\bullet}\mathbf{E}^{\bullet}) + \nabla\frac{\mathbf{E}^2}{2} + \mathbf{E}^{\bullet}\nabla\cdot\mathbf{E}^{\bullet} + \mathbf{E}\partial_z\mathcal{E}\ )
$$

(1.20)

where (0.4.6) and (0.8) have been used. Ampere's law (1.5.4) is applied next to express $B\mathbf{J}^{\bullet}$ as

$$
B\mathbf{J}^{\bullet} = \frac{1}{\mu_0}(-\nabla\frac{B^2}{2} + B\partial_z(B\tau)\ ) - \chi_0(\partial_t(\mathbf{E}^{\bullet}B) + \mathbf{E}^{\bullet}\nabla\cdot\mathbf{E}^{\bullet})
$$

(1.21)

Adding these expressions gives

$$
\mathbf{E}\eta + B\mathbf{J}^{\bullet} = \chi_0(-\nabla\cdot(\mathbf{E}^{\bullet}\mathbf{E}^{\bullet}) + \nabla\frac{\mathbf{E}^2}{2} - \partial_t(\mathbf{E}^{\bullet}B) + \mathbf{E}\partial_z\mathcal{E}\ )
$$
$$
\frac{1}{\mu_0}(-\nabla\frac{B^2}{2} + B\partial_z(B\tau)\ )
$$

(1.22)

Similarly, $B\tau^{\bullet}\cdot\mathbf{J}$ and $\mathcal{E}\eta$ are evaluated resulting in

$$
\mathcal{E}\eta + B\tau^{\bullet}\cdot\mathbf{J} = \chi_0(-B\tau^{\bullet}\cdot\partial_t\mathbf{E} + \mathcal{E}\nabla\cdot\mathbf{E} + \partial_z\frac{\mathcal{E}^2}{2}) + \frac{1}{\mu_0}(B\tau\cdot\nabla B - \partial_z\frac{(B\tau)^2}{2})
$$

(1.23)

Finally (1.5.5) is used to represent the parallel current density with the result

$$
-B\tau^{\bullet}J = -\frac{1}{\mu_0}(\ \nabla\cdot(B^2\tau^{\bullet}\tau^{\bullet}) - \nabla\frac{(B\tau)^2}{2} - B\tau\partial_z B\ ) + \chi_0 B\tau^{\bullet}\partial_t\mathcal{E}
$$

(1.24)

where (0.4.6),(0.10) and (1.5.6) have been applied.

Reintroducing the scaling factor ε into these latter expressions the equation of motion can be written in terms of field quantities as

$$\boldsymbol{f} = \sum_s \int d\boldsymbol{p}\, dq\, \boldsymbol{p}(\varepsilon^2(\partial_t + \frac{q}{m}\partial_z)f_s^- + \frac{\boldsymbol{p}}{m}\cdot\nabla f_s^+)$$
$$= (-\nabla\cdot\frac{B^2}{2\mu_0}\boldsymbol{I}) + \varepsilon^2(-\nabla\cdot(\frac{B^2}{2\mu_0}(\tau^*\tau^* - \tau\,\tau) + \frac{\chi_0}{2}(\boldsymbol{E}^*\boldsymbol{E}^* - \boldsymbol{E}\,\boldsymbol{E}))$$
$$+ \partial_z(\frac{1}{\mu_0}B^2\tau) - \partial_t(\chi_0\boldsymbol{E}^*B)) + O(\varepsilon^4)$$

$$F = \sum_s \int d\boldsymbol{p}\, dq\, q((\partial_t + \frac{q}{m}\partial_z)f_s^+ + \frac{\boldsymbol{p}}{m}\cdot\nabla f_s^-)$$
$$= (\nabla\cdot\frac{1}{\mu_0}(B^2\tau) + \partial_z\frac{1}{\mu_0}B^2) + O(\varepsilon^2) \tag{1.25}$$

In these expressions the relative ordering of the terms involving field quantities is evident on the basis of (1.4.1)–(1.4.5). However, the reason for the ε expansion of the distribution function consistent with the adopted ordering scheme has not yet been made apparent. Here a plausibility argument will suffice, to be justified *a posteriori*.

It is implicit in the ordering that guiding centers drift slowly with respect to the velocity of gyration of a particle. The electric drift two-velocity given by

$$\boldsymbol{v}_e = \frac{\boldsymbol{E}^*}{B} \tag{1.26}$$

is clearly $O(\varepsilon)$ on the basis of (1.4.4) and (1.4.5). The part of f antisymmetric in $\boldsymbol{p}$ must then have its lowest order nonvanishing contribution at $O(\varepsilon)$. Take the formal representation of the distribution function as an asymptotic series of the form

$$f^+ = f_0^+ + \varepsilon^2 f_2^+ + \cdots \tag{1.27.1}$$

and

$$\varepsilon\, f^- = \varepsilon\, f_1^- + \varepsilon^3 f_3^- + \cdots \tag{1.27.2}$$

The fact that the successive terms in these series are smaller by a factor ε^2 results from the choice of ordering in the Vlasov equation. This can be seen more clearly by the following: Adjust (1.12) as

$$\frac{B}{m}\boldsymbol{p}\cdot\nabla_{\boldsymbol{p}}^* f^+ = \varepsilon^2(\partial_t + \frac{q}{m}\partial_z + e\mathcal{E}\partial_q)f^+$$
$$+ \varepsilon^2(\frac{\boldsymbol{p}}{m}\cdot\nabla + e(-\frac{q}{m}B\tau^*\cdot\nabla_{\boldsymbol{p}} + \frac{\boldsymbol{p}}{m}\cdot B\tau^*\partial_q + \boldsymbol{E}\cdot\nabla_{\boldsymbol{p}}))f^- \tag{1.28.1}$$

and

$$e\,\frac{B}{m}\,\boldsymbol{p}\cdot\nabla^{*}_{\boldsymbol{p}}f^{-} = \varepsilon^{2}(\partial_{t}+\frac{q}{m}\,\partial_{z}+e\mathcal{E}\partial_{q})f^{-}$$
$$+\,(\frac{\boldsymbol{p}}{m}\cdot\nabla + e(-\frac{q}{m}\,B\tau^{*}\cdot\nabla_{\boldsymbol{p}}+\frac{\boldsymbol{p}}{m}\cdot B\tau^{*}\partial_{q}+\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}}))f^{+} \tag{1.28.2}$$

The operator $\boldsymbol{p}\cdot\nabla^{*}_{\boldsymbol{p}}$ can be written also as

$$\boldsymbol{p}\cdot\nabla^{*}_{\boldsymbol{p}} = \boldsymbol{p}\cdot(\boldsymbol{e}_{p}\partial_{p} + \boldsymbol{e}_{\theta}\frac{1}{p}\partial_{\theta})^{*} = p\boldsymbol{e}_{p}\cdot(\boldsymbol{e}_{p}\frac{1}{p}\partial_{\theta} - \boldsymbol{e}_{\theta}\partial_{p}) = \partial_{\theta} \tag{1.29}$$

where $\boldsymbol{e}_{p}$ and $\boldsymbol{e}_{\theta}$ are unit vectors in the two-momentum space. Using this abbreviation and substituting the representations of f given by (1.27.1) and (1.27.2) into the Vlasov equation in the form (1.28.1) gives to lowest order the result that

$$\partial_{\theta}f^{+}_{0} = 0 \tag{1.30}$$

The lowest order contribution to f is evidently gyrophase independent. Define this gyrophase independent component of f as

$$f^{+}_{0} = g(\boldsymbol{x},z;\sigma,q;t) \tag{1.31}$$

where $\sigma = \boldsymbol{p}^{2}/2$.

The ordering of terms in the equation of motion (1.25) is now complete. The moments appearing explicitly there are ordered as

$$\boldsymbol{f} = \sum_{s}\int d\boldsymbol{p}\;dq(\frac{1}{m}\nabla\cdot\boldsymbol{pp}g_{s} + \varepsilon^{2}(\frac{1}{m}\nabla\cdot\boldsymbol{pp}f^{+}_{2s} + (\partial_{t}+\frac{q}{m}\,\partial_{z})\boldsymbol{p}f^{-}_{1s}))$$

$$F = \sum_{s}\int d\boldsymbol{p}\;dq((\frac{1}{m}\nabla\cdot q\boldsymbol{p}f^{-}_{1s} + (\partial_{t}+\frac{q}{m}\,\partial_{z})qg_{s}) + \varepsilon^{2}(\partial_{t}+\frac{q}{m}\,\partial_{z})qf^{+}_{2s}) \tag{1.32}$$

The standard form (1.6.1) of the equation of motion is apparent in (1.25) and (1.32). The following identifications can thus be made: Including the factors of ε to specify the relative ordering, the components of the equation of motion are

$$\partial_{t}\boldsymbol{s} + \nabla\cdot\underline{\boldsymbol{t}} + \partial_{z}\boldsymbol{T} = 0 \tag{1.33.1}$$

and

$$\partial_t S + \nabla\cdot\boldsymbol{s} + \partial_z U = 0 \tag{1.33.2}$$

where

$$\boldsymbol{s} = \sum_s \int d\boldsymbol{p}\, dq\, (\boldsymbol{p} f_{1s}^- + \chi_0 \boldsymbol{E}^* B) + O(\varepsilon^2) \tag{1.34.1}$$

$$S = \sum_s \int d\boldsymbol{p}\, dq\, (q g_s + \varepsilon^2\, q f_{2s}^+) + \varepsilon^2\, \chi_0 \boldsymbol{E}\cdot B\tau^* + O(\varepsilon^4) = S_\mu + O(\varepsilon^2) \tag{1.34.2}$$

$$\underline{\boldsymbol{t}} = \sum_s \int d\boldsymbol{p}\, dq\, \frac{1}{m}\boldsymbol{p}\boldsymbol{p}(g_s + \varepsilon^2\, f_{2s}^+)$$
$$\varepsilon^2 \left(\frac{B^2}{2\mu_0}(\tau^*\tau^* - \tau\,\tau) + \frac{\chi_0}{2}(\boldsymbol{E}^*\boldsymbol{E}^* - \boldsymbol{E}\,\boldsymbol{E})\right) + O(\varepsilon^4) \tag{1.34.3}$$

$$\boldsymbol{u} = \boldsymbol{T} = \sum_s \int d\boldsymbol{p}\, dq\, \frac{1}{m}(\boldsymbol{p} q f_{1s}^-) - \frac{1}{\mu_0} B^2 \tau + O(\varepsilon^2) \tag{1.34.4}$$

$$U = \sum_s \int d\boldsymbol{p}\, dq\, \frac{1}{m}(q^2 g_s + \varepsilon^2\, q^2 f_{2s}^+) - \frac{B^2}{2\mu_0} + \varepsilon^2 \left(\frac{(B\tau)^2}{2\mu_0} + \frac{\chi_0 \boldsymbol{E}^2}{2}\right) + O(\varepsilon^4)$$
$$= \sum_s \int d\boldsymbol{p}\, dq\, \frac{1}{m}(q^2 g_s) - \frac{B^2}{2\mu_0} + O(\varepsilon^2) \tag{1.34.5}$$

The next task toward developing a fluid Lagrangian for the gyroelastic system is to evaluate the gyrophase dependent components of f in (1.27.1). and (1.27.2). This is done order by order, expressing successive corrections to f in terms of (gyrophase dependent) operators on $g(\sigma,q;\boldsymbol{x},z;t)$. First, to find f_1^-, use (1.29) to express (1.28.2) at lowest order:

$$\partial_\theta f_1^- = \boldsymbol{p}\cdot\boldsymbol{h}^* \tag{1.35}$$

Here the gyrophase-independent function $\boldsymbol{h}$ is defined as

$$\boldsymbol{h} = (-\frac{1}{eB}\nabla^* - m\boldsymbol{v}_e\partial_\sigma + \tau(q\partial_\sigma - \partial_q))g = \boldsymbol{L}g \tag{1.36}$$

Since g and $\boldsymbol{h}$ are both independent of θ, noting that $\partial_\theta \boldsymbol{p} = -\boldsymbol{p}^*$ and $\partial_\theta \boldsymbol{p}^* = -\boldsymbol{p}^{**} = \boldsymbol{p}$, find

$$\boldsymbol{p}\cdot\boldsymbol{h}^* = -\boldsymbol{p}^*\cdot\boldsymbol{h} = \partial_\theta \boldsymbol{p}\cdot\boldsymbol{h} = \partial_\theta(\boldsymbol{p}\cdot\boldsymbol{h}) \tag{1.37}$$

This equation can be integrated simply to give f_1^-

$$f_1^- = \boldsymbol{p}\cdot\boldsymbol{h} \tag{1.38}$$

The constant of integration is zero since f_1^- must be odd in $\boldsymbol{p}$.

A somewhat more tedious though straightforward calculation yields f_2^+ at next lowest order. The analogue to (1.35) is

$$\boldsymbol{p}\cdot\nabla^*_{\boldsymbol{p}} f_2^+ = -\frac{1}{2}(\boldsymbol{p}^*\boldsymbol{p} + \boldsymbol{p}\,\boldsymbol{p}^*):\boldsymbol{L}\boldsymbol{h} = -\frac{1}{2}(\boldsymbol{p}^*\boldsymbol{p} + \boldsymbol{p}\,\boldsymbol{p}^*):\boldsymbol{L}\boldsymbol{L}g \tag{1.39}$$

Noting again the comment proceeding (1.37) there results from a revision of (1.39) that the θ-derivative of f_2^+ is given by

$$\partial_\theta f_2^+ = \partial_\theta(\frac{1}{4}(\boldsymbol{p}\,\boldsymbol{p} - \boldsymbol{p}^*\boldsymbol{p}^*):\boldsymbol{L}\boldsymbol{L}g) \tag{1.40}$$

Again, an integration can be performed on sight resulting in

$$f_2^+ = \frac{1}{4}(\boldsymbol{p}\,\boldsymbol{p} - \boldsymbol{p}^*\boldsymbol{p}^*):\underline{\boldsymbol{k}} + \ell \tag{1.41}$$

where $\underline{\boldsymbol{k}} = \boldsymbol{L}\boldsymbol{L}g$. The constant of integration is $\ell = \ell(\sigma,q)$ a function independent of θ, not necessarily zero.

The equation of motion is now expressed as a relation among electromagnetic field quantities and various moments of the lowest order gyrophase-independent distribution function. Further simplification is afforded by performing the angular integrations indicated in (1.34.1)-(1.34.5).

Define the operation of *gyrophase-averaging* as follows:

$$(F)_{av} = \text{gyrophase average of } F = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, F \tag{1.42}$$

then it is obvious that

$$\int d\boldsymbol{p}\, dq\, F = 2\pi \int d\sigma\, dq\, (F)_{av} \tag{1.43}$$

A few identities useful in carrying out the gyrophase integration are

$$(\boldsymbol{p})_{av} = (\boldsymbol{p}^*)_{av} = 0 \tag{1.44.1}$$

$$(\boldsymbol{p}\,\boldsymbol{p})_{av} = (\boldsymbol{p}^*\boldsymbol{p}^*)_{av} = \sigma \underline{\boldsymbol{I}} \tag{1.44.2}$$

$$(\boldsymbol{p}\boldsymbol{p}g)_{av} = (\boldsymbol{p}\boldsymbol{p})_{av}g = \sigma \underline{\boldsymbol{I}}g \tag{1.44.3}$$

$$((\boldsymbol{p}\,\boldsymbol{p} - \boldsymbol{p}^*\boldsymbol{p}^*):\underline{\boldsymbol{k}})_{av} = (\boldsymbol{p}\,\boldsymbol{p} - \boldsymbol{p}^*\boldsymbol{p}^*)_{av}:\underline{\boldsymbol{k}} = 0 \tag{1.44.4}$$

and finally

$$(\boldsymbol{p}\boldsymbol{p}(\boldsymbol{p}\,\boldsymbol{p} - \boldsymbol{p}^*\boldsymbol{p}^*))_{av}:\underline{\boldsymbol{k}} = 2\sigma^2\underline{\boldsymbol{k}} - \sigma^2 \underline{\boldsymbol{I}}\, Tr\, \underline{\boldsymbol{k}} \tag{1.44.5}$$

Here g, $\boldsymbol{h} = \boldsymbol{L}g$ and $\underline{\boldsymbol{k}} = \boldsymbol{L}\boldsymbol{L}g$ are independent of θ. Furthermore, $\underline{\boldsymbol{k}}$ is a symmetric two-tensor since

$$\alpha(\underline{\boldsymbol{k}}) = \alpha(\boldsymbol{L}\boldsymbol{L}g) = \boldsymbol{I}\cdot\boldsymbol{L}^*g = 0 \tag{1.45}$$

With these identities, a short calculation yields

$$(\boldsymbol{p}f_1)_{av} = (\boldsymbol{p}\boldsymbol{p}\cdot\boldsymbol{h})_{av} = (\boldsymbol{p}\boldsymbol{p})_{av}\cdot\boldsymbol{L}g = \sigma\boldsymbol{L}g \tag{1.46.1}$$

and

$$(\boldsymbol{p}\boldsymbol{p}f_2^+)_{av} = (\boldsymbol{p}\boldsymbol{p}\,\frac{1}{4}(\boldsymbol{p}\,\boldsymbol{p} - \boldsymbol{p}^*\boldsymbol{p}^*):\underline{\boldsymbol{k}})_{av} + (\boldsymbol{p}\boldsymbol{p}k)_{av} = \frac{1}{2}\sigma^2(\boldsymbol{L}\boldsymbol{L} - \frac{1}{2}\boldsymbol{I}\boldsymbol{L}\cdot\boldsymbol{L})g + \underline{\boldsymbol{I}}\sigma k \tag{1.46.2}$$

which, together with (1.34.1)-(1.34.5) and (1.33.1)-(1.33.2) comprise the equation of motion to lowest order.

Define the operation of performing the species sum and momentum integrals as

$$\langle \mathcal{L} \rangle = 2\pi \sum_s \int d\sigma \; dq \; \mathcal{L} g, \tag{1.47}$$

where $\mathcal{L}$ is an operator. Adapting (1.46.1)-(1.46.2) and indicating the gyrophase-averaged moments, $\boldsymbol{s}^{(3)}$ and $\boldsymbol{T}^{(3)}$ in the format (1.34.1)-(1.34.5) can be represented as

$$\boldsymbol{s} = (\langle \sigma \boldsymbol{L} \rangle + \chi_0 \boldsymbol{E}^{*} B) + O(\varepsilon^2) \tag{1.48.1}$$

$$S = \langle q \rangle + O(\varepsilon^2) = S_\mu + O(\varepsilon^2) \tag{1.48.2}$$

$$\begin{aligned} \boldsymbol{t} = & (\langle \frac{\sigma}{m} \rangle + \frac{B^2}{2\mu_0}) \boldsymbol{I} \\ & + \varepsilon^2 (\langle \frac{\sigma^2}{2m} (\boldsymbol{L}\boldsymbol{L} - \frac{1}{2} \boldsymbol{I} \boldsymbol{L}\cdot\boldsymbol{L}) \rangle + \mathcal{K}_\sigma \boldsymbol{I} + \frac{B^2}{2\mu_0} (\boldsymbol{\tau}^{*}\boldsymbol{\tau}^{*} - \boldsymbol{\tau}\boldsymbol{\tau}) + \frac{\chi_0}{2} (\boldsymbol{E}^{*}\boldsymbol{E}^{*} - \boldsymbol{E}\boldsymbol{E}) + O(\varepsilon^4) \end{aligned} \tag{1.48.3}$$

$$\boldsymbol{T} = \boldsymbol{u} = (\langle \frac{\sigma q}{m} \boldsymbol{L} \rangle - \frac{1}{\mu_0} B^2 \boldsymbol{\tau}) + O(\varepsilon^2) \tag{1.48.4}$$

$$U = \langle \frac{q^2}{m} \rangle - \frac{B^2}{2\mu_0} + O(\varepsilon^2) \tag{1.48.5}$$

where $\mathcal{K}_\sigma$ appears due to the occurence of a constant of the gyrophase integration. Finally, the equation of motion for the gyroelastic fluid can by expressed in the concise form

$$\nabla\cdot((\langle \frac{\sigma}{m} \rangle + \frac{B^2}{2\mu_0}) \underline{I} = 0 + O(\varepsilon^2) \tag{1.49.1}$$

at lowest order.

$$\partial_t \langle q \rangle + \nabla\cdot(\langle \frac{\sigma q}{m} \boldsymbol{L} \rangle - \frac{1}{\mu_0} B^2 \boldsymbol{\tau}) + \partial_z (\langle \frac{q^2}{m} \rangle - \frac{B^2}{2\mu_0}) = U + O(\varepsilon^2) \tag{1.49.2}$$

at order ϵ^1 (relative to lowest order) and

$$\partial_t(\prime\ \sigma\mathbf{L} + \chi_0 \mathbf{E}^* B) + \nabla\cdot\left(\frac{\sigma^2}{2m}(\mathbf{LL} - \frac{1}{2}\,\mathbf{I}\,\mathbf{L}\cdot\mathbf{L}) + \frac{B^2}{2\mu_0}(\boldsymbol{\tau}^*\boldsymbol{\tau}^* - \boldsymbol{\tau}\,\boldsymbol{\tau}) + \frac{\chi_0}{2}(\mathbf{E}^*\mathbf{E}^* - \mathbf{E}\,\mathbf{E}) + \mathcal{K}_0 \mathbf{I}\right) + \partial_z\left(\frac{\sigma q}{m}\,\mathbf{L} - \frac{1}{\mu_0}B^2\boldsymbol{\tau} \right) = 0 + O(\epsilon^2) \tag{1.49.3}$$

at order ϵ^2 (relative to lowest order.)

The Isorrhopic State

To this point in the analysis, no mention has been made of symmetry. The equation of motion outlined above is in fact applicable without qualification. However, a crucial juncture in the analysis has been reached. A unique opportunity to simplify matters has presented itself.

The general gyroelastic system may possess a certain symmetry called isorrhopy. In the gyroelastic ordering a system is isorrhopic for all time; the system once isorrhopic remains isorrhopic naturally. Distortions of the network of contours due to motions of the isorrhopic gyroelastic fluid do not destroy the symmetry. This allows the decoupling of phenomena which involve perturbations of the distribution function along fluid contours from those which do not: drift type modes from fluid-type modes. (The question of stability against anisorrhopic perturbations will not be addressed in this study).

In the following, I will specify the conditions under which a gyroelastic system is isorrhopic. Then I will proceed to show the system indeed remains so. Return for a moment to the inception of the present line of reasoning–the Vlasov equation in the form (1.12). Again factors of ϵ will be deleted for clarity.

To lowest order,(1.12) portrayed in the current vernacular is

$$e\,\frac{B}{m}\,\mathbf{p}\cdot\nabla^*_{\mathbf{p}} g = (\partial_t + \frac{q}{m}\,\partial_z + e\mathcal{E}\partial_q)g + (\frac{\mathbf{p}}{m}\cdot\nabla + \epsilon(-\frac{q}{m}\,B\boldsymbol{\tau}^*\cdot\nabla_{\mathbf{p}} + \frac{\mathbf{p}}{m}\cdot B\boldsymbol{\tau}^*\partial_q + \mathbf{E}\cdot\nabla_{\mathbf{p}}))\mathbf{p}\cdot\mathbf{L}g \tag{1.50}$$

Gyrophase-average this equation according to the prescription given in (1.42). Resort to the definition of the operator $\mathbf{L}$ (1.36) to resolve the result into the form of the electric-drift kinetic equation

$$D_e = \sigma\partial_\sigma \nabla\cdot\mathbf{v}_e + \frac{\sigma}{eB^2m}\nabla^*B\cdot\nabla g - \frac{q}{m}\partial_\chi g - e(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau)\partial_q g$$
$$+ \frac{\sigma}{eB}\partial_\chi B(\partial_q - q\partial_\sigma)g \tag{1.51}$$

where $\partial_\chi = \partial_z + \tau\cdot\nabla$ is closely related to the directional derivative along a magnetic field line and $D_e = \partial_t + \mathbf{v}_e\cdot\nabla$ is the operator of time differentiation along an electric drift trajectory.

Next define the quantity

$$P = \frac{B^2}{2\mu_0} + \frac{\sigma}{m} \tag{1.52}$$

and convert its gradient

$$\nabla P = \frac{1}{\mu_0}B\nabla B + \frac{\sigma}{m}\nabla \tag{1.53}$$

by using Ampere's law (1.5.4) to evaluate ∇B. The result is then

$$\nabla P = \eta\mathbf{E} - JB\tau^* \tag{1.54}$$

As it turns out, it is possible to specify a condition *ab-initio* that will insure that η and J will vanish to lowest order identically, if initially. I'll proceed under this presumption, to be justified presently. But first, differentiate P along an electric drift trajectory and with the help of Faraday's law (1.5.2) in the form

$$D_e = \partial_t B + \mathbf{v}_e\cdot\nabla B = -\nabla\cdot\mathbf{E}^* + \frac{\mathbf{E}^*}{B}\cdot\nabla B = -B\,\nabla\cdot\mathbf{v}_e \tag{1.55}$$

deduce that

$$D_e P = \frac{1}{\mu_0}BD_e B + \frac{\sigma}{m}D_e$$
$$= -\frac{1}{\mu_0}B^2\nabla\cdot\mathbf{v}_e + \nabla\cdot\mathbf{v}_e\left(-2\frac{\sigma}{m} + \frac{\sigma^2}{m^3c^2}\right) + \frac{1}{B^2}\nabla^*B\cdot\nabla\left(\frac{\sigma^2}{em^2}\right.$$
$$\left. - (\mathcal{E} - \mathbf{v}_e^*\cdot B\tau)\frac{e\sigma q}{m^2}\right) \tag{1.56}$$

For convenience, define the quantity γ by

$$\gamma P = \frac{1}{\mu_0}B^2 + \left(2\frac{\sigma}{m} - \frac{\sigma^2}{m^3c^2}\right) \tag{1.57}$$

and notice that (1.56) can be abbreviated as

$$D_e P + \gamma P\, \nabla\cdot \mathbf{v}_e = \psi \tag{1.58}$$

where

$$\psi = \frac{\sigma^2}{m^2 e B^2} \nabla^* B\cdot\nabla + \frac{2q\sigma}{m^2 B}\partial_\chi B - \frac{q\sigma}{m^2}\partial_\chi - \frac{eq\sigma}{m^3 c^2}(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau) \tag{1.59}$$

Now the quantity ψg plays a very important role in the theory of gyroelastic systems. This is due to the remarkable fact that *if* ψg, J *and* η *vanish everywhere ab-initio, then they vanish identically in time.* It should be emphasized here that by *vanish everywhere* is meant at lowest order in the gyroelastic scaling.

Now separate g according to the following prescription:

$$g(q) = g^+(q) + g^-(q) \tag{1.60}$$

where $g^+(q) = g^+(-q)$ and $g^-(q) = -g^-(-q)$ are the even and odd parts of g expressed as a function of parallel momentum. Multiply (1.5.2) by $e_s q$, sum over species and integrate over momentum variables q and σ. When g^- vanishes, there then results that

$$(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau)\sum_s e_s^2 \int d\sigma\, dq\, \frac{1}{m}\left(1 - \frac{q^2}{m^2c^2}\right)g^+$$
$$= \sum_s e_s \int d\sigma\, dq\, \frac{1}{m^2}\left(q^2\partial_\chi + \frac{\sigma - q^2}{B}\partial_\chi B\right)g^+ \tag{1.61}$$

It is clear from this relation that if g^-, $\partial_\chi g^+$ and $\partial_\chi B$ vanish the electric field parallel to the field lines then also vanishes:

$$(g^-,\ \partial_\chi g^+,\ \partial_\chi B) = 0 \ \Rightarrow\ (\mathcal{E} - \mathbf{v}_e^*\cdot B\tau) = 0 \tag{1.62}$$

With this last item in mind, it is easily shown that the initialization consisting of the following conditions satisfies $\psi g = 0$:

$$g^- = 0 + O(\varepsilon^2) \tag{1.63.1}$$

$$\eta = 0 + O(\varepsilon^4) \tag{1.63.2}$$

$$\partial_\chi B = 0 + O(\varepsilon^4) \tag{1.63.3}$$

$$\partial_\chi g = 0 + O(\varepsilon^4) \tag{1.63.4}$$

$$\nabla^* B \cdot \nabla g = 0 + O(\varepsilon^4) \tag{1.63.5}$$

For the moment, assume the system is enclosed by rigid and conducting boundaries. The electric field $\boldsymbol{E}$ is then perpendicular to the boundary there. From (1.58) and the initial conditions which assure $\psi = 0$ at $t = 0$, there derives

$$D_e P + \gamma P\, \nabla\cdot\boldsymbol{v}_e = 0 \tag{1.64}$$

Integrate this relation over a crossection at constant z of the bounded system to find

$$D_e P \int_R dx\; dy\; \frac{1}{\gamma P} = -\int_R dx\; dy\; \nabla\cdot\boldsymbol{v}_e = \oint_{\partial R} d\boldsymbol{x}\cdot\boldsymbol{v}_e^* = -\oint_{\partial R} d\boldsymbol{x}\cdot\frac{\boldsymbol{E}}{B} = 0 \tag{1.65}$$

Since $\gamma P > 0$ this implies that, at $t = 0$ at least,

$$D_e P = 0 \tag{1.66}$$

so that

$$\nabla\cdot\boldsymbol{v}_e = 0 \tag{1.67}$$

Furthermore, from (1.51),(1.62),(1.63.1)–(1.63.5) and (1.67), at $t = 0$

$$D_e g = 0 \tag{1.68}$$

The commutator of D_e and ∂_χ will be helpful in completing the proof. It is evaluated as follows:

$$[D_e, \partial_\chi] = [\partial_t + \boldsymbol{v}_e\cdot\nabla, \partial_z + \tau\cdot\nabla] = (D_e\tau - \partial_\chi \boldsymbol{v}_e)\cdot\nabla \tag{1.69}$$

but, since

$$\partial_t\tau = \frac{1}{B}(\partial_t(B\tau) - \tau\partial_t B) = \frac{1}{B}(\partial_z E^* - \nabla^*\mathcal{E} + \tau\nabla\cdot(\mathbf{v}_e B))$$

$$\partial_t\tau = \frac{1}{B}(\partial_z(\mathbf{v}_e B) - \nabla^*(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau) + \nabla^*(\mathbf{v}_e^*\cdot B\tau) + B\tau\nabla\cdot\mathbf{v}_e + \tau\nabla B\cdot\mathbf{v}_e$$

$$\partial_t\tau = \frac{1}{B}(B\partial_z\mathbf{v}_e - \mathbf{v}_e\nabla\cdot(B\tau) - \nabla^*(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau) + \nabla^*\mathbf{v}_e\cdot B\tau^* + \nabla^*(B\tau)^*\cdot\mathbf{v}_e$$

$$+ B\tau\nabla\cdot\mathbf{v}_e - B^2\tau\nabla\frac{1}{B}\cdot\mathbf{v}_e)$$

$$\partial_t\tau = \partial_z\mathbf{v}_e + \tau\cdot\nabla\mathbf{v}_e - \mathbf{v}_e\frac{1}{B}\nabla\cdot(B\tau) - \frac{1}{B}\nabla^*(B\tau)\cdot\mathbf{v}_e^* - \frac{1}{B}\nabla^*(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau)$$

$$\partial_t\tau = \partial_\chi\mathbf{v}_e - \mathbf{v}_e\cdot\nabla\tau - \frac{1}{B}\nabla^*(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau) \tag{1.70}$$

thus

$$[D_e, \partial_\chi] = -\frac{1}{B}\nabla^*(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau)\cdot\nabla \tag{1.71}$$

and by (1.62) and (1.69)

$$D_e\tau = \partial_\chi\mathbf{v}_e \tag{1.72}$$

Now apply D_e to (1.63.1)–(1.63.5) at $t = 0$. The maintenance of the condition specified as an initialization hinges on the following argument: First

$$D_e g^- = \sigma\partial_\sigma g^-\ \nabla\cdot\mathbf{v}_e + \frac{\sigma}{eB^2m}\nabla^*B\cdot\nabla g^- - \frac{q}{m}\partial_\chi g^+ - e(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau)\partial_q g^+$$

$$+\frac{\sigma}{mB}\partial_\chi B(\partial_q - q\partial_\sigma)g^+ = 0 \tag{1.73.1}$$

since $D_e g^-$ is odd in q and by (1.62) and (1.63.1)–(1.63.5); then

$$D_e\eta = 0 \tag{1.73.2}$$

by (1.63.1)–(1.63.5); then

$$D_e\partial_\chi B = \partial_\chi D_e B - \frac{1}{B}\nabla^*(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau)\cdot B = -\partial_\chi(B\nabla\cdot\mathbf{v}_e) = 0 \tag{1.73.3}$$

by (1.62), (1.67) and (1.72); then

$$D_e\partial_\chi g = \partial_\chi D_e g - \frac{1}{B}\nabla^*(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau)\cdot\nabla g = 0 \tag{1.73.4}$$

by (1.62),(1.68) and (1.72); and finally

$$D_e(\nabla^* B\cdot\nabla g) = (\nabla^* D_e B - \nabla^* \mathbf{v}_e\cdot\nabla B)\cdot\nabla g + (\nabla D_e g - \nabla\mathbf{v}_e\cdot\nabla g)\cdot\nabla^* B$$
$$= -\nabla^*(B\ \nabla\cdot\mathbf{v}_e)\cdot\nabla g = 0 \tag{1.73.5}$$

by (0.17.2) and (1.67). This is sufficient to assure that at $t = 0$

$$D_e(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau) = 0 \tag{1.74}$$

and as a finishing touch

$$D_e(\psi g) = 0 \tag{1.75}$$

To summarize the argument: Choose at $t = 0$ that $g^- = 0$, $\partial_\chi g^+ = 0$ and $\partial_\chi B = 0$. There results that $(\mathcal{E} - \mathbf{v}_e^*\cdot B\tau) = 0$. Further choose that at $t = 0$, $\nabla^* B\cdot\nabla g^+ = 0$ so that $\psi g = 0$ and $\nabla\cdot\mathbf{v}_e = 0$ result. Then as a consequence of these choices, it happens that $D_e\psi g = 0$. The argument is then iterated. By induction

$$D_e^{n-1}(g^-,\ \eta,\ \partial_\chi B,\ \partial_\chi g,\ \nabla^* B\cdot\nabla g) = 0 \tag{1.76}$$

for $n \cdot 1$ implies

$$D_e^n(\psi g) = 0 \tag{1.77}$$

Since all physical quantities are representable as analytic functions of time the initialization as specified and all its consequences are self preserving.

Conditions (1.63.1)–(1.63.4) are satisfied trivially in the two-dimensional gyroelastic system since ∂_χ, τ and $\mathcal{E}$ all vanish identically in that case. As was shown by Newcomb[22] the isorrhopic condition in that limit reduces to (1.63.5) alone. Gyroelastic isorrhopy requires that (at constant z) to lowest order the magnetic field B and the distribution function g are constant along the same contours (see fig. 1.)

Conditions (1.63.3) and (1.63.4) are introduced by the additional freedom allowing small variations in the z direction. These conditions state that isorrhopy requires B and g to be constant along magnetic field lines as well as alc. isorrhopes (the intersections of isorrhopic surfaces with a constant z-plane.) One concludes that in the three-dimensional isorrhopic gyroelastic system (*1*) *the magnetic field lines lie within surfaces of simultaneously constant B and g and* (*2*) *these surfaces move as a comoving network so as to preserve the isorrhopy of the configuration.*

In addition to B and g, any quantity depending solely on B and/or g (such as ρ_m, the mass density) is also constant on an isorrhopic surface. It is thus expedient to represent any such quantity as a function of a single variable, say s, which labels isorrhopic surfaces. This is the representation I will choose: B and g are freely specified definite functions of $s = s(\mathbf{x},z)$; s represents the comoving network of isorrhopic surfaces which evolves according to the dynamics of the system. The quantity s may have physical significance also; for example it may represent the volume contained within the (closed) isorrhopic surface.

Condition (1.63.5) can be expressed deliberately as

$$\nabla^{*}B\cdot\nabla g = B_s g_s \nabla^{*}s\cdot\nabla s = 0 \tag{1.78}$$

Let's revamp the equation of motion (1.49.1)-(1.49.3) now, using some more physically suggestive symbols for the moments indicated by the symbols $\langle\ \rangle$ in the notation introduced with (1.47). Define the quantities

$$K(s,z) = \langle \frac{\sigma^2}{2e^2m} \rangle \tag{1.79.1}$$

$$B(s,z)M(s,z) = \langle \frac{\sigma^2}{e}\partial_\sigma \rangle \tag{1.79.2}$$

$$p_\parallel(s,z) = \langle \frac{q^2}{m} \rangle \tag{1.79.3}$$

$$p_\perp(s,z) = \langle \frac{\sigma}{m} \rangle \tag{1.79.4}$$

$$\rho_m(s,z) = \langle \partial_\sigma(\sigma m) \rangle \tag{1.79.5}$$

and notice that

$$\frac{1}{B^2}\nabla\cdot\nabla\cdot K = \frac{1}{B}\nabla\cdot(\frac{1}{B}\nabla\cdot K)$$

$$\frac{1}{B^2}\nabla\cdot\nabla\cdot K = \nabla\cdot(\frac{1}{B^2}\nabla\cdot K) - {}^{1}\nabla\cdot\frac{1}{B}\nabla\cdot K$$

$$\frac{1}{B^2}\nabla\cdot\nabla\cdot K = \nabla\cdot(\nabla\cdot sK_s\frac{1}{B^2}) + \frac{1}{B^3}\nabla\cdot B\nabla\cdot K$$

$$\frac{1}{B^2}\nabla\cdot\nabla\cdot K = \nabla\cdot\nabla\cdot(\int dK\frac{1}{B^2}) + \frac{1}{B^3}\nabla\cdot B\nabla\cdot K$$

$$\frac{1}{B^2}\nabla\cdot\nabla\cdot K = \nabla\cdot\nabla\cdot\mathcal{J} + \frac{1}{B^3}\nabla\cdot B\nabla\cdot K \tag{1.80}$$

With the help of the above definitions and (1.80), the equation of motion can be cast in the following form:

$$\nabla(p_\perp + \frac{B^2}{2\mu_0}) = 0 \tag{1.81.1}$$

at lowest order and

$$\partial_t S_\mu + \partial_x p_\parallel = 0 \tag{1.81.2}$$

at order ε (relative to lowest order) reveal no new information. The system has been constructed so that these relations are satisfied (identically.)

$$\rho D_e \mathbf{v}_e - Q\partial_x \tau - X_e \nabla\cdot s\cdot\nabla\mathbf{v}_e - Y_e\nabla\cdot s\cdot\nabla\nabla\cdot s + \nabla\kappa = 0 \tag{1.81.3}$$

at next order (ε^2 relative to lowest order) contains the essential dynamics of the isorrhopic gyroelastic system. In this last expression we have introduced the following symbols:

$$\rho = \rho_m + \chi_0 B^2 \tag{1.82.1}$$

$$Q = \frac{1}{\mu_0}B^2 + p_\perp - p_\parallel \tag{1.82.2}$$

$$X_e = -\frac{1}{2B^2}(MB^2)_s \tag{1.82.3}$$

$$Y_e = -\frac{1}{B^3}B_s K_s \tag{1.82.4}$$

$$\kappa = \mathcal{K}_a - \frac{1}{2}(\mathcal{G}_s \nabla^2 s + \mathcal{G}_{ss}(\nabla s)^2 - Y_e(\nabla s)^2$$
$$- Q\tau^2 + \rho \boldsymbol{v}_e^2 - X_e \nabla^* s \cdot \boldsymbol{v}_e + \frac{1}{2}\nabla^* \cdot M\boldsymbol{v}_e) \tag{1.82.5}$$

The quantities X_e and Y_e are called the gyroscopic and quasielastic force coefficients. The gyroscopic force is first order in the fluid velocity, $F_g = X\,\nabla^* s \cdot \nabla \boldsymbol{v}$ and has been called a reactive viscous force though unlike a viscous drag it is non-dissipative in nature. The net rate of work done by the action of the gyroscopic force in a region bounded by an isorrhope is given by

$$2\int dx\ dy\ X\nabla^* s \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{v} = \int dx\ dy\ \nabla \cdot X\nabla^* s + \oint d\boldsymbol{x}^* . \nabla^* s\ X\boldsymbol{v}^2$$
$$= \int dx\ dy\ (X\nabla \cdot \nabla^* s + X\nabla s \cdot \nabla^* s) = 0 \tag{1.83}$$

The gyroscopic force does no net work; it arises as fluid motion distorts isorrhopes as a kind of isorrhope rigidity. The gyroscopic force is a consequence of the differential motion of segments of isorrhopes.

The quasielastic force on the other hand depends on the instantaneous state of distortion of the comoving network of isorrhopic surfaces, $F_q = Y\,\nabla^* s \cdot \nabla \nabla^* s$. The network exhibits a kind of elasticity.

A closed system of equations describing the behavior of isorrhopic gyroelastic fluids can now be assembled. By definition, the isorrhopic surfaces move with the fluid velocity; this is stated

$$D_e s = 0 \tag{1.84}$$

The flow field is incompressible; (the isorrhopic condition enforces incompressibility)

$$\nabla \cdot \boldsymbol{v}_e = 0 \tag{1.85}$$

The parallel electric field vanishes identically due also to the isorrhopic condition, so (1.72) holds; differentiation along an electric drift trajectory commutes with differentiation along a field line

$$D_e \tau = \partial_\chi \boldsymbol{v}_e \tag{1.86}$$

Self-consistency of the ordering scheme requires that the equation of motion at lowest order and order ε' (relative to lowest order), (1.81.1) and (1.81.2), be satisfied. The equation of motion at the next order (ε^2 relative to lowest order) completes the system:

$$\rho D_e \mathbf{v}_e - Q\partial_\chi \tau - X_e \nabla^* s \cdot \nabla \mathbf{v}_e - Y_e \nabla^* s \cdot \nabla\nabla^* s + \nabla\kappa = 0 \qquad (1.87)$$

These are six nonlinear partial differential equations in six unknowns: $\tau, \mathbf{v}_e, s$ and κ. The system is closed. The solution however is representation dependent, the fluid velocity in the isorrhopic gyroelastic system may be redefined. The velocity of a fluid element is indistinct in so far as all fluid elements on a given isorrhope are equivalent. Certain velocity fields which carry fluid points along isorrhopes may be superimposed on the comoving network without changing the character or form of the dynamics of the system. The resulting velocity sum is called the representative velocity; the transformation is a change of representation.

This representation dependence is familiar in the study of electromagnetic phenomena. The electric and magnetic fields are components of an entity which transform covariantly under Lorentz transformations. Electric and magnetic forces are representation dependent. Gyroscopic and quasielastic forces are exemplars of magnetic-like and electric-like forces in this respect.

Changes of representation leave the dynamical equations of the system invariant in form. The force coefficients transform as a single covariant entity under changes of representation as do the electric and magnetic fields under Lorentz transformations. The analogy, though not formal, is not entirely hypothetical. The change of representation is a form of *local* Lorentz transformation (applied in the tangent bundle of an isorrhopic manifold.) The generator of the group of such transformations is an object called a permutator.

The permutator is a symmetry operator; it leaves invariant the action integral for the isorrhopic system. This is its quintessential feature. The next chapter will examine the Lagrangian of the isorrhopic gyroelastic system and further detail some properties of the group of permutators.

CHAPTER 2

The Action Principle

The Generalized Lagrangian

On the basis of the previous discussion of representation dependence, all subscipts $_e$ referring to the particular representation in which the fluid velocity is the electric drift velocity are droppped. The generalized Lagrangian appropriate for the isorrhopic gyroelastic system is then given by

$$\mathscr{L} = \frac{1}{2}\left(\rho \mathbf{v}^2 - X\mathbf{v}\cdot\nabla^{*}s - Y\nabla^{*}s\cdot\nabla^{*}s - Q\boldsymbol{\tau}\cdot\boldsymbol{\tau}\right) \tag{2.1}$$

There are several illuminating exercizes which can be carried out with the expression for the Lagrangian in hand. Foremost among these is to demonstrate (by a variational method) that the Euler–Lagrange equation which arises naturally as a consequence of the minimization of the associated action integral is the one previously derived by other means, namely (1.81.3). However, before a segue into the realm of the inscrutable a few words are in order on the variational notation I will use.

There are two variational operators which will be convenient in various contexts: δ and Δ. The first of these operators will be referred to as the Eulerian variation; the second will be referred to as the Lagrangian variation. The action of either operator is to denote the variation or small change in a quantity (that on which they act) caused by an infinitesmal virtual displacement of the system, $\boldsymbol{\xi}(\mathbf{x},z,t)$. The Eulerian variation δ is a fixed point variation:

it denotes the small change induced by ξ at a fixed space-time point. The Lagrangian variation Δ is the small change induced by ξ as observed from the vantage of the displaced fluid point. The two operators are related through

$$\Delta = \delta + \xi\cdot\nabla \tag{2.2}$$

It will prove helpful to become familiar with a few common identities regarding the usage of these operators. For example:

$$\Delta \mathbf{x} = \xi \tag{2.3.1}$$

$$\Delta \mathbf{v} = D\xi \tag{2.3.2}$$

$$\Delta s = 0 \tag{2.3.3}$$

The virtual displacement does not change the labeling of isorrhopes. It is required to be divergenceless:

$$\nabla\cdot\xi = 0 \tag{2.3.4}$$

There is a certain analogy between the algebra of virtual displacements operators and time differentiation operators:
$\delta \leftrightarrow \partial_t$ and $\Delta \leftrightarrow D$. With the further replacement $\xi \leftrightarrow \mathbf{v}$ the commutation relations for the two sets of operators are identical. For example

$$[\Delta,\nabla] = \Delta\nabla - \nabla\Delta = -\nabla\xi\cdot\nabla \tag{2.4.1}$$

and

$$[\Delta,\nabla^*] = \Delta\nabla^* - \nabla^*\Delta = -\nabla^*\xi\cdot\nabla \tag{2.4.2}$$

correspond to (0.17.1) and (0.17.2). It should come as no surprise then, in view of (1.72), that

$$\Delta\tau = \partial_x\xi \tag{2.5}$$

Applying (2.2) to these relations we can find easily that

$$\delta \mathbf{x} = 0 \tag{2.6.1}$$

$$\delta \boldsymbol{v} = D\xi - \xi\cdot\nabla\boldsymbol{v} \tag{2.6.2}$$

$$\delta s = -\xi\cdot\nabla s \tag{2.6.3}$$

and

$$\delta \tau = \partial_x \xi - \xi\cdot\nabla\tau \tag{2.6.4}$$

Now apply the variational method to the action integral $\mathfrak{I}$ straightaway. Write the action integral

$$\mathfrak{I} = \int dx\ dy\ dz\ dt\ \mathscr{L} \tag{2.7}$$

using the generalized lagrangian (2.1). The Euler-Lagrange equation results from the requirement that the variation $\Delta\mathfrak{I}$ vanish:

$$\Delta\mathfrak{I} = \int dx\ dy\ dz\ dt\ \Delta\mathscr{L} \tag{2.8}$$

Calculate the terms in $\Delta\mathfrak{I}$ using the above identities as follows:

$$\begin{aligned}
&\Delta \frac{1}{2}\rho\boldsymbol{v}^2 = \rho\boldsymbol{v}\cdot\Delta\boldsymbol{v} = \rho\boldsymbol{v}\cdot D\xi \\
&\Delta \frac{1}{2} X\boldsymbol{v}\cdot\nabla^* s = \frac{1}{2}(\Delta\boldsymbol{v}\cdot\nabla^* s + \boldsymbol{v}\cdot\Delta\nabla^* s) = \frac{1}{2} X(D\xi\cdot\nabla^* s + \nabla^* s\cdot\nabla\xi\cdot\boldsymbol{v}) \\
&\Delta \frac{1}{2} Y(\nabla^* s\cdot\nabla^* s) = Y\nabla^* s\cdot\Delta\nabla^* s = Y\nabla^* s\cdot\nabla\xi\cdot\nabla^* s \\
&\Delta \frac{1}{2} Q\tau\cdot\tau = Q\tau\cdot\Delta\tau = Q\tau\cdot\partial_x\xi
\end{aligned} \tag{2.9}$$

The varied action integral can be expressed then as

$$\Delta\mathfrak{I} = \int dx\ dy\ dz\ dt\ \left(\rho\boldsymbol{v}D\xi - \frac{1}{2} X(D\xi\cdot\nabla^* s + \nabla^* s\cdot\nabla\xi\cdot\boldsymbol{v}) - Y(\nabla^* s\cdot\nabla\xi\cdot\nabla^* s) - Q\tau\cdot\partial_x\xi\right) \tag{2.10}$$

An integration by parts may be carried out with the help of the identity

$$\int dx\ dy\ \boldsymbol{a}\cdot\nabla\boldsymbol{c}\cdot\boldsymbol{b} = \oint d\boldsymbol{x}^*\cdot\boldsymbol{ab}\cdot\boldsymbol{c} - \int dx\ dy\ \boldsymbol{c}\cdot(\nabla\cdot\boldsymbol{ab}) \tag{2.11}$$

The result can be arranged in the form

$$\Delta\mathfrak{J} = -\int dx\, dy\, dz\, dt\, (\rho D\mathbf{v} - X\nabla^* s\cdot\nabla\mathbf{v} - Y\nabla^* s\cdot\nabla\nabla^* s - Q\partial_x \tau)\cdot\boldsymbol{\xi}$$
$$+ \int dx\, dy\, dz\, \mathcal{L}_{\mathbf{v}}\cdot\boldsymbol{\xi}\,|_t + \int dx\, dy\, dt\, \mathcal{L}_{\tau}\cdot\boldsymbol{\xi}\,|_z$$
$$+ \int dt\, dz \oint d\mathbf{x}^*\cdot(\nabla^* s\mathcal{L}_{\nabla}{}^* s + \tau\mathcal{L}_{\tau})\cdot\boldsymbol{\xi} \tag{2.12}$$

where the subscriping of $\mathcal{L}$ indicates (partial) differentiation with respect to the subscript two-vector. The boundary terms can be made to vanish by imposing side conditions limiting the class of admissible displacements. Choosing the boundary to coincide with an isorrhope and to be fixed these conditions are expressed as

$$\boldsymbol{\xi}\cdot d\mathbf{x}^* = 0$$
$$\boldsymbol{\xi}(t_1) = \boldsymbol{\xi}(t_2)$$
$$\boldsymbol{\xi}(z_1) = \boldsymbol{\xi}(z_2) \tag{2.13}$$

The minimization condition reduces to the vanishing of

$$\Delta\mathfrak{J} = -\int dx\, dy\, dz\, dt\, (\rho D\mathbf{v} - X\nabla^* s\cdot\nabla\mathbf{v} - Y\nabla^* s\cdot\nabla\nabla^* s - Q\partial_t \tau)\cdot\boldsymbol{\xi} \tag{2.14}$$

for $\boldsymbol{\xi}$ satisfying the side conditions (2.13), but otherwise arbitrary.

Now we are prepared to check the Euler–Lagrange equation with the previously derived equation of motion: substituting the equation of motion (1.81.3) into (2.14) the variation of the action integral becomes

$$\Delta\mathfrak{J} = \int dx\, dy\, dz\, dt\; \boldsymbol{\xi}\cdot\nabla\kappa = -\int dx\, dy\, dz\, dt\; \kappa\nabla\cdot\boldsymbol{\xi} + \int dz\, dt \oint d\mathbf{x}^*\cdot\kappa\boldsymbol{\xi} = 0 \tag{2.15}$$

The Lagrangian (2.1) reproduces the equation of motion. The scalar κ containing the unknown constant of the gyrophase integration is seen to play the role of a Lagrange multiplier associated with the incompressibility constraint.

There exist certain $\boldsymbol{\xi}$ which do not satisfy the side conditions (2.13), but which can be employed to unveil constants of the fluid motion or induce conservation laws of one sort or another. These $\boldsymbol{\xi}$ are symmetry operators. Consider for example an infinitesmal time translation $\boldsymbol{\xi} = -\mathbf{v}\tau$ of the system where τ is a time infinitesmal. The action of a virtual displacement of this sort on a quantity F can be written

$$F(t-\tau) = F(t) - \tau\partial_t F(t) = F + \delta F \tag{2.16}$$

The correspondence $\delta \Leftrightarrow -\tau\partial_t$ is recognized (the time differential operator serves as a generator or propagator). The Eulerian variation of the action integral is then

$$\delta \mathfrak{I} = \int dx\ dy\ dz\ dt\ \delta\mathcal{L} = \int dx\ dy\ dz\ dt\ (-\tau\partial_t\mathcal{L}) = -\tau \int dx\ dy\ dz\ \mathcal{L}\ \Big|_t \tag{2.17}$$

But, since $\delta\mathfrak{I} = \Delta\mathfrak{I}$

$$\begin{aligned}\delta\mathfrak{I} &= \int dx\ dy\ dz\ dt\ \delta\mathcal{L} = \int dx\ dy\ dz\ dt\ (\Delta\mathcal{L} - \xi\cdot\nabla\mathcal{L}) \\ &= \Delta\mathfrak{I} + \int dx\ dy\ dz\ dt\ \mathcal{L}\nabla\cdot\xi - \int dz\ dt \oint d\mathbf{x}^{*}\cdot\mathcal{L}\xi = \Delta\mathfrak{I}\end{aligned} \tag{2.18}$$

the expression (2.17) can be equated to (2.12). Substituting the symmetry operator $\xi = -\mathbf{v}\tau$ in the resulting equation then representing the differences as differentials reveals the law of conservation of energy in the isorrhopic gyroelastic sytem:

$$\begin{aligned}&\int dx\ dy\ dz\ (\mathcal{L} - \mathcal{L}_{\mathbf{v}}\cdot\mathbf{v})\Big|_t - \int dx\ dy\ dt\ (\mathcal{L}_{\tau}\cdot\mathbf{v})\Big|_z = 0 \\ &\partial_t \int dx\ dy\ (-\frac{1}{2}\ (\rho\mathbf{v}^2 + Y(\nabla^{*}s)^2 + Q\tau^2)) + \partial_z \int dx\ dy\ Q\tau\cdot\mathbf{v} = 0 \\ &\partial_t H + \partial_z \Gamma_{\parallel H} = 0\end{aligned} \tag{2.19}$$

The time translation operation $\xi = -\mathbf{v}\tau$ is representation dependent; the conservation law it induces reflects this fact. Return to this in a moment; first turn to the issue of permutation symmetry.

As previously discussed, the permutator is the symmetry operator which is associated with exchange invariance, isrrhopy. Though the exact form of the permutator is yet to be determined, we do know already that owing to its being a symmetry operator, the permutator must leave the action integral invariant.

It is essential that the permuted systems be relevantly indistinguishable. Any distinction concerning the identity of fluid elements must be irrelevant with regard to the dynamical evolution of the system. The Eulerian variation δ induced by a permutator ζ (applied to any quantity relevant to the action) must therefore vanish.

Owing also to its being a symmetry operation, ζ must induce a variation of the action integral which vanishes, so in addition to

$$\delta s = -\zeta\cdot\nabla s = 0$$

$$\delta\nabla^* s = \nabla^*\delta s = 0$$

$$\delta\tau = \partial_t\zeta - \zeta\cdot\nabla\tau = 0$$

$$\delta\mathbf{v} = D\zeta - \zeta\cdot\nabla\mathbf{v} \tag{2.20}$$

and $\delta\mathcal{F} = 0$, also $\delta\mathbf{J} = \Delta\mathbf{J} = 0$.

A short calculation employing the commutation relations

$$[D,\nabla^*] = -\nabla^*\mathbf{v}\cdot\nabla \tag{2.21.1}$$

and

$$[\partial_t,\nabla^*] = -\nabla^*\tau\cdot\nabla \tag{2.21.2}$$

affirms that the correct form for the permutator is

$$\zeta = \omega(s)\,\nabla^* s \tag{2.22}$$

where $\omega(s)$ is an arbitrary function of s. The permutator is a vector field whose integral curves are isorrhopes. The permutator is the infinitesmal generator of the group of permutations: it permutes the identity of adjacent fluid elements. The group is a continuous group of infinite dimension. Since an entire function, namely $\omega(s)$, is necessary to specify the generator, the group of permutations is not a Lie group.

Now that an expression of the form of the permutator is known, the behavior of the system under changes of representation can be examined in detail. The transformation

$$\mathbf{v} \to \mathbf{v}' = \mathbf{v} - \zeta = \mathbf{v} - \omega(s)\,\nabla^* s \tag{2.23}$$

evokes a change of representation. The calculations leading to the following results are straightforward and not dissimilar to those first presented by Newcomb[21-22] with regard to the two-dimensional system.

The generating function $\omega(s)$ induces the transformation

$$\mathbf{v} \to \mathbf{v}' = \mathbf{v} - \omega\nabla^* s$$

$$X \to X' = X - 2p\omega$$

$$Y \to Y' = Y + \omega X - p\omega^2 \tag{2.24}$$

Two important invariants of the above transformation are

$$\Phi = X^2 + 4\rho Y \to \Phi' = \Phi = X'^2 + 4\rho Y' \tag{2.25.1}$$

the gyroelasticity (gyroelastic modulus) which like any elasticity is a nonnegative quantity, and

$$v^{(0)} = v - \frac{X}{2\rho}\nabla\cdot s \to v^{(0)\prime} = v^{(0)} = v' - \frac{X'}{2\rho}\nabla\cdot s \tag{2.25.2}$$

the canonical velocity.

Certain representations are distinguished by some unique feature and thus are raised above the sea of anonymity with a name. For example the representation in which the gyroscopic force vanishes is called the canonical representation

$$X = X^{(0)} = 0 \;,\; Y = Y^{(0)} = \frac{\Phi}{4\rho} \;,\; v = v^{(0)} \tag{2.26}$$

Two representations related to one another through

$$\hat{X} = -X \;,\; \hat{Y} = Y \;,\; \hat{v} = v - \frac{X}{\rho}\nabla\cdot s \tag{2.27}$$

are said to be dual to one another. The canonical representation is its own dual.

The characteristic representations are a pair of duals in which the quasielastic force vanishes. For this pair of duals

$$X^{*} = \hat{X}^{*} = \Phi^{\frac{1}{2}} \;,\; Y^{*} = \hat{Y}^{*} = 0 \;,\; \hat{v}^{*} = v^{*} = v^{(0)} \pm \frac{\Phi^{\frac{1}{2}}}{2\rho}\nabla\cdot s \tag{2.28}$$

We will adhere to the notation introduced here for referring to these special representations.

Since ζ is a symmetry operation it can be shown to induce a conservation law (as previously time translation symmetry was used to derive the conservation law for energy). Acting on (2.12) with $\xi = \zeta$ there results

$$\begin{aligned}\Delta\mathfrak{I} = 0 &= \int dx\, dy\, dz\, \mathcal{L}_{v}\cdot\zeta\,|_t + \int dx\, dy\, dt\, \mathcal{L}_{\tau}\cdot\zeta\,|_z \\ &= -\int ds\, \omega\, \Big(\int dz\, \rho\mathcal{G}|_t - \int dt\, Q\mathcal{F}|_z \Big)\end{aligned} \tag{2.29}$$

where

$$\mathscr{C} = \int \boldsymbol{v}^{(0)} \cdot d\boldsymbol{x} \ , \ \mathscr{T} = \oint \boldsymbol{\tau} \cdot d\boldsymbol{x} \tag{2.30}$$

Since $\omega(s)$ is a freely chosen (generating) function, the expression in parentheses in (2.29) vanishes for each s. The law of conservation of circulation in the isorrhopic gyroelastic system can then be put in differential form as

$$\partial_t(\rho\mathscr{C}) - \partial_z(Q\mathscr{T}) = 0 \tag{2.31}$$

In the two-dimensional system $\mathscr{T} = 0$ and consequently $\rho\mathscr{C}$ is a constant of the motion on each isorrhope. Since ρ is obviously a constant since $Ds = 0$ the circulation is a constant of the motion.

Return to the discussion following (2.19) concerning the representation dependence of the conservation law induced by a time translation symmetry operation. It is easily seen that accommodating the transformation

$$\boldsymbol{\xi} \to \boldsymbol{\xi}' = - \boldsymbol{v}\tau - \boldsymbol{\zeta} \tag{2.32}$$

the law of conservation of energy becomes

$$\partial_t H' + \partial_z \Gamma'_{H} = 0 \tag{2.33.1}$$

where

$$H' = H + \int ds\ \omega\ \rho\mathscr{C} \tag{2.33.2}$$

and

$$\Gamma'_{;H} = \Gamma_{;H} + \int ds\ \omega\ Q\mathscr{T} \tag{2.33.3}$$

The action of a change of representation on the law of conservation of energy is simply to add some multiple of the law of conservation of circulation.

Geometrical Optics

In the geometrical optics limit the study of linear waves in the isorrhopic gyroelastic medium is simplified by the local nature of the dispersion relation. The wave field is made up of locally plane waves which propagate according to local laws. For the local condition to be valid, the medium must be uniform on the scale of a wavelength and unchanging during a period of oscillation of the wave.

Begin by writing the generalized Lagrangian in the characteristic representation(s) directly as

$$\mathcal{L}^{\pm} = \frac{1}{2}(\rho \mathbf{v}^{\pm 2} \mp \tfrac{1}{2} Q \mathbf{v}^{\pm}\cdot\nabla^{\bullet} s - Q\boldsymbol{\tau}\cdot\boldsymbol{\tau}) \tag{2.34}$$

and in the symmetrized form

$$\mathcal{L}^{+} = \mathcal{L}^{-} = \frac{1}{2}(\rho \mathbf{v}^{+}\cdot\mathbf{v}^{-} - Q\boldsymbol{\tau}\cdot\boldsymbol{\tau}) \tag{2.35}$$

where $\mathbf{v}^{+}$ and $\mathbf{v}^{-}$ are the characteristic representative velocities defined in (2.28). The equation of motion can likewise be symmetrized as

$$\rho D^{\mp}\mathbf{v}^{\pm} - Q\partial_{\chi}\boldsymbol{\tau} + \nabla\kappa = 0 \tag{2.36}$$

where $D^{\pm}$ is the operator of time differentiation along a $\mathbf{v}^{\pm}$ fluid trajectory. The simplicity of the symmetrized equation of motion here is the reason for the choice of representation.

To study the behavior of linear waves in the geometrical optics limit the equation of motion is varied (linearized), then a certain ansatz is taken for $\boldsymbol{\xi}$, the displacement or wave field, such that the conditions outlined above apply.

Since the operation of Eulerian variation δ and ∇ commute, the following congruency holds:

$$\delta(\rho D^{\mp}\mathbf{v}^{\pm} - Q\partial_{\chi}\boldsymbol{\tau}) = -\,\delta\nabla\kappa = -\,\nabla\delta\kappa \sim 0 \tag{2.37}$$

Furthermore, since $\delta + \boldsymbol{\xi}\cdot\nabla = \Delta$, the variation of the equation of motion can be restated as the congruency

$$\Delta(\rho D^{\tau}\mathbf{v}^{\tau} - Q\partial_{\chi}\boldsymbol{\tau}) - \boldsymbol{\xi}\cdot\nabla(\rho D^{\tau}\mathbf{v}^{\tau} - Q\partial_{\chi}\boldsymbol{\tau}) = -\delta\nabla\kappa = -\nabla\delta\kappa \sim 0 \tag{2.38}$$

Taking account of the relation

$$\Delta\partial_{\chi}\tau = \partial_{\chi}\partial_{\chi}\xi \tag{2.39}$$

the linearized equation of motion can be seen to satisfy

$$\rho D^{\tau}D^{\tau}\boldsymbol{\xi} - Q\partial_{\chi}\partial_{\chi}\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla(\rho D^{\tau}\mathbf{v}^{\tau} - Q\partial_{\chi}\boldsymbol{\tau}) \sim 0 \tag{2.40}$$

In curled form this congruency becomes the equation

$$\nabla\cdot(\rho D^{\tau}D^{\tau}\boldsymbol{\xi} - Q\partial_{\chi}\partial_{\chi}\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla(\rho D^{\tau}\mathbf{v}^{\tau} - Q\partial_{\chi}\boldsymbol{\tau}))^{*} = 0 \tag{2.41}$$

since, of course, $\nabla\cdot\nabla^{*}\delta\kappa = 0$.

Now choose the ansatz for the displacement wave field to have the form

$$\boldsymbol{\xi} = \nabla^{*}(\mathcal{A}e^{i\varphi}) \tag{2.42}$$

where φ is a rapidly oscillating phase and $\mathcal{A}$ is a slowly (in space) varying amplitude. To keep the scaling intact, take the phase to be $\varphi \sim 1/\sqrt{\epsilon}$. (This assures that the wave oscillations obey the stated conditions for geometrical optics to apply, yet they remain far slower than the gyromotion of single particles, so the medium can still be described accurately as gyroelastic.)

Further define canonical wave variables as follows (note these variables are new and have nothing whatever to do with variables these symbols have been used to designate in previous chapters—ω is not a generating function):

$$\begin{aligned}
\boldsymbol{\kappa} &= \nabla\varphi = \textit{two-wavevector} \\
k &= \partial_{z}\varphi = \textit{parallel wavevector} \\
\omega &= -\partial_{t}\varphi = \textit{frequency} \\
\sigma &= -D\varphi = -\partial_{t}\varphi - \nabla\varphi\cdot\boldsymbol{\kappa} = \omega - \boldsymbol{\kappa}\cdot\mathbf{v} = \textit{proper frequency}
\end{aligned} \tag{2.43}$$

along with the obvious specializations (see (2.28))

$$\sigma^{(0)} = -D^{(0)}\varphi = \omega - \boldsymbol{\kappa}\cdot\mathbf{v}^{(0)} = \textit{canonical proper frequency} \tag{2.44.1}$$

and

$$\sigma^{\tau} = -D^{\tau}\varphi = \omega - \boldsymbol{\kappa}\cdot\mathbf{v}^{\tau} = \textit{characteristic proper frequencies} \tag{2.44.2}$$

Inserting the ansatz (2.42) into the linearized equation of motion (2.41) there results (to order $1/\epsilon^2$) the local characteristic equation (dispersion relation) in the form

$$\sigma^+\sigma^- = (\omega - \boldsymbol{\kappa}\cdot\boldsymbol{v}^+)(\omega - \boldsymbol{\kappa}\cdot\boldsymbol{v}^-) = \frac{Q}{\rho}(k + \boldsymbol{\kappa}\cdot\boldsymbol{\tau})^2 \tag{2.45}$$

The dispersion relation is quadratic in ω and therefore is solved by two values of ω, say ω_+ and ω_-

$$\omega_\pm = \boldsymbol{\kappa}\cdot\boldsymbol{v}^{(0)} \pm \Omega \tag{2.46}$$

where

$$\Omega^2 = \frac{\mathbb{Q}}{4\rho^2}(\boldsymbol{\kappa}\cdot\nabla^{\bullet}s)^2 + \frac{Q}{\rho}(k + \boldsymbol{\kappa}\cdot\boldsymbol{\tau})^2 \tag{2.47}$$

The solutions $\omega_\pm$ restrict to two the possible canonical proper frequencies; call them $\sigma_\pm^{(0)}$

$$\sigma_\pm^{(0)} = \omega_\pm - \boldsymbol{\kappa}\cdot\boldsymbol{v}^{(0)} = \pm\,\Omega \tag{2.48}$$

Expressed as

$$\mathcal{D}(\}\boldsymbol{\kappa},k\{,\sigma^{(0)}) = (\sigma^{(0)} - \frac{\mathbb{Q}^{\frac{1}{2}}}{2\rho}(\boldsymbol{\kappa}\cdot\nabla^{\bullet}s))(\sigma^{(0)} + \frac{\mathbb{Q}^{\frac{1}{2}}}{2\rho}(\boldsymbol{\kappa}\cdot\nabla^{\bullet}s)) - \frac{Q}{\rho}(k + \boldsymbol{\kappa}\cdot\boldsymbol{\tau})^2 = 0 \tag{2.49}$$

it is apparent that the dispersion relation (2.45) admits the scaling

$$\mathcal{D}(\lambda\}\boldsymbol{\kappa},k\{,\lambda\sigma^{(0)}) = \lambda^2\mathcal{D}(\}\boldsymbol{\kappa},k\{,\sigma^{(0)}) = 0 \tag{2.50}$$

so that $\mathcal{D}$ is homogeneous of degree 2. The dispersion relation is a homogeneous function of its arguments $(\}\boldsymbol{\kappa},k\{,\sigma^{(0)})$ due to the fact that the gyroelastic medium is nondispersive to high frequency waves.

It can be shown that homogeneous functions $\mathcal{D}(\}\boldsymbol{\kappa},k\{,\sigma)$ of degree n conform so as to satisfy

$$\sigma\partial_\sigma\mathcal{D} + \boldsymbol{\kappa}\cdot\partial_{\boldsymbol{\kappa}}\mathcal{D} + k\partial_k\mathcal{D} = n\,\mathcal{D} \tag{2.51}$$

This relation, known as Euler's theorem, can be used to eliminate $\partial_{\sigma(\bullet)}\mathcal{D}$ from the expression for the total derivative of $\mathcal{D}$ with respect to wavevector. This accomplished, impose the restriction (2.48) to find

$$\}\mathbf{v}_\pm, V_\pm\{ = (\partial_\kappa \sigma^{(0)}, \partial_k \sigma^{(0)})\big|_{\sigma^{(0)}=\sigma_\pm^{(0)}}$$

$$= \pm \frac{1}{\Omega} \Big\} \frac{Q}{4\rho^2} (\kappa\cdot\nabla^* s)\nabla^* s + \frac{Q}{\rho} (k + \kappa\cdot\tau)\tau \ , \ \frac{Q}{\rho} (k + \kappa\cdot\tau) \Big\{ \tag{2.52}$$

From this result it is manifest that

$$\frac{1}{\sigma^{(0)}} \ \}\kappa, k\{ \cdot \}\mathbf{v}_\pm, V_\pm\{ \ \equiv \ \}\mathbf{k}, k\{ \cdot \}\mathbf{u}, U\{ \ = 1 \tag{2.53}$$

The quantity $\}\mathbf{k}, k\{ \ \equiv \ \}\kappa, k\{/\sigma^{(0)}$, the normal slowness (3-vector), is closely related to the usual phase velocity (same direction, reciprocal in magnitude). The vector $\}\mathbf{u}, U\{ \ \equiv \ \}\mathbf{v}_\pm, V_\pm\{$ is closely related to the group velocity (in a local rest frame.) The resonance condition (2.53) states that a wave packet moves in resonance with its own wave fronts (as expected in a non-dispersive medium.) In other words, the ray trajectories (space-time trajectories of velocity $\}\mathbf{v}_\pm, V_\pm\{$) lie wholly within the phase planes (space-time hyperplanes of constant φ) when viewed in the local rest frame (in this case the frame moving with the fluid at the canonical representative velocity.) The ray constancy of φ can be restated for a frame at rest relative to a fixed point as

$$D^R_\pm \varphi = (\partial_t + \mathbf{v}^R_\pm\cdot\nabla)\varphi = (\partial_t + \mathbf{v}^{(0)}\cdot\nabla + \mathbf{v}^R_\pm\cdot\nabla + V^R_\pm \partial_z)\varphi$$

$$= -\,\omega_\pm + \mathbf{v}^{(0)}\cdot\kappa + \mathbf{v}^R_\pm\cdot\kappa + V^R_\pm k = -\,\sigma^{(0)}_\pm + \sigma^{(0)}_\pm = 0 \tag{2.54}$$

The general ray velocities $\}\mathbf{v}^R_\pm, V^R_\pm\{$ introduced here can also be recognized as the $\}\kappa, k\{$-space gradients of ω. This circumstance will be used shortly to construct a canonical Hamiltonian wave theory. First let's step back and take a look at the global picture thus far.

The propagation of waves in a non-dispersive medium can be described with the use of an artifice called the ray surface or its dual counterpart, the normal slowness surface. Points on the ray surface are related in one-to-one correspondence with tangent planes of the normal slowness surface and vice-versa. Before proceeding to construct these surfaces a minor reorientation will prove helpful.

As is clear from (2.52), ray trajectories remain within isorrhopic surfaces. The natural coordinate basis to which to refer a ray or normal slowness surface in the isorrhopic gyroelastic system is composed of local tangents to magnetic field lines and isorrhopes.

To avoid confusion as to which surfaces are being referred to in the following, let us agree to refer always to the comoving network of (nested) isorrhopic surfaces as the isorrhopic manifold $\mathcal{M}$, the configuration space of the gyroelastic system. Refer to the space-time ray surface as the ray conoid and its dual as the normal slowness conoid.

Choose the axes of an orthonormal basis ($\mathbf{e}_{i'}$) in the tangent space of $\mathcal{M}$ at a point $\mathcal{P}$, $\mathcal{TM}(\mathcal{P})$, to be oriented (to lowest order) as follows: let $\mathbf{e}_{x'}$ be directed tangent to the local isorrhope, then performing an infinitesmal rotation $\delta\omega = \boldsymbol{\tau}^{*}$ bring $\mathbf{e}_{z'}$ into alignment with the local magnetic field line. In this (primed) coordinate system $\boldsymbol{\tau} = 0$. Choose $\mathbf{v}^{(0)} = 0$ for simplicity and the dispersion relation (2.54) evaluates to

$$\Omega^2 = \frac{\mathbb{Q}}{4\rho^2}(\boldsymbol{\kappa}\cdot\nabla^{*}s)^2 + \frac{Q}{\rho}k'^2 = (\sigma^{(0)})^2 \tag{2.55}$$

where k' is the wavevector component along $\mathbf{e}_{z'}$. This is the equation for the normal slowness surface in the space dual to $\mathcal{TM}(\mathcal{P})$, $\{\kappa', k'\}$-space at $\mathcal{P}$. Waves passing through $\mathcal{P}$ with wavevectors in a given direction will be moving with normal slowness magnitude corresponding to the point on the slowness surface selected by that direction. The normal slowness surface (in the space dual to $\mathcal{TM}(\mathcal{P})$) is an elliptical cylinder whose axis passes through the origin in the direction $\mathbf{e}_{y'}$ (see figure 2.)

Transforming the expressions for the ray velocities given in (2.52) to the primed frame it can be shown that

$$\frac{(\mathbf{v}^R_{\pm}\cdot\mathbf{e}_{x'})^2}{\frac{\mathbb{Q}}{4\rho^2}(\nabla^{*}s)^2} + \frac{(V^{R\prime}_{\pm})^2}{\frac{Q}{\rho}} \equiv \frac{u'^2}{a^2} + \frac{U'^2}{b^2} = 1 \tag{2.56}$$

Since (2.53) must be satisfied and since no restriction is placed on $(\boldsymbol{\kappa}\cdot\mathbf{e}_{y'})$, $(\mathbf{v}^R_{\pm}\cdot\mathbf{e}_{y'})$ must vanish. The ray surface is therefore an ellipse in the x'–z' plane in $\mathcal{TM}(\mathcal{P})$ (see figure 3.) Waves passing through $\mathcal{P}$ moving with ray velocity in a given direction will be moving with ray velocity magnitude corresponding to the point on the ray surface selected by that direction.

The actual shapes of these surfaces depend on $\mathcal{P}$. As can be seen in figures 2 and 3 the shapes are symmetrical. In the two-dimensional case dealt with by Newcomb[21-22] these surfaces become degenerate. On the ray surface, the plus and minus waves are represented by the two points at the ends of the ray ellipse. These points correspond to two lines on the normal slowness surface. The notation distinguishing between plus and minus waves simply refers to the two sides of the surfaces in the three-dimensional case, which is to say that there exist a continuum of dual characteristic pairs, not just one single pair. The ± distinction seems somewhat pleonastic in the three-dimensional case, and will henceforth be dropped.

Geometrical optics can be systematized in Hamitonian formalism with the wave Hamiltonian function

$$\mathcal{H} = \mathcal{H}(\{\boldsymbol{\kappa}, k\}; \{\mathbf{x}, z\}; t) = \boldsymbol{\kappa}\cdot\mathbf{v}^{(0)} + \Omega - \omega \tag{2.57}$$

A short calculation yields the following canonical equations:

$$\begin{aligned} \partial_{\kappa} \mathcal{H} &= D^R \mathbf{x} = \mathbf{v}^R \\ \partial_{k} \mathcal{H} &= D^R z = V^R \\ \partial_{\mathbf{x}} \mathcal{H} &= - D^R \boldsymbol{\kappa} \\ \partial_{z} \mathcal{H} &= - D^R k \end{aligned} \tag{2.58}$$

where we've used (2.52) in the form

$$\{\mathbf{v}^R, V^R\} = \partial_{\{\kappa, k\}}\omega \tag{2.59}$$

and (2.43). Proceed to evaluate $D^R(k + \boldsymbol{\kappa}\cdot\boldsymbol{\tau})$ using the previously derived results

$$D^R\boldsymbol{\tau} = \partial_{\chi}\mathbf{v}^{(0)} + (\mathbf{v}^R - \mathbf{v}^{(0)})\cdot\nabla\tau + V^R\partial_{z}\tau \tag{2.60}$$

and

$$\partial_{\chi}\nabla^{*}s = \nabla^{*}s\cdot\nabla\boldsymbol{\tau} \tag{2.61}$$

The calculation discloses

$$D^R(k + \boldsymbol{\kappa}\cdot\boldsymbol{\tau}) = 0 \tag{2.62}$$

$(k + \boldsymbol{\kappa}\cdot\boldsymbol{\tau})$ is a ray constant, as was discovered of the wave phase (2.54) φ.

An obvious corollary to (2.62) is that the quantity $(k + \boldsymbol{\kappa}\cdot\boldsymbol{\tau})$ vanishes everywhere along a ray trajectory if it vanishes anywhere along that ray trajectory.

Ray trajectories remain always within isorrhopic surfaces, so s is also a ray constant

$$D^R s = 0 \tag{2.63}$$

A slightly more involved though straightforward calculation reveals yet another ray constant:

$$D^R(\boldsymbol{\kappa}\cdot\nabla^{*}s) = 0 \tag{2.64}$$

The projection of the two-wavevector on the isorrhope (at $\mathcal{P}$) is a ray constant. Using (2.62), (2.63) and (2.64) it can be shown that

$$D^R\Omega = 0 \tag{2.65}$$

and

$$D^R V^R = 0 \tag{2.66}$$

as well. By (2.65) the dispersion relation is a ray invariant; by (2.66) the z component of the ray velocity is a ray constant.

To the set of canonical equations above, (2.58), should now be added

$$\begin{aligned} \partial_\omega \mathcal{H} &= - D^R t = - 1 \\ \partial_t \mathcal{H} &= D^R \omega = D^R(\kappa \cdot \mathbf{v}^{(0)}) = 0 \end{aligned} \tag{2.67}$$

A complete prescription for evolving the wave field is contained in (2.58) and (2.67).

The curled linearized equation of motion (2.41) with the replacement (2.42) for the displacement field was used to derive the dispersion relation (2.48). The dispersion relation was in fact multiplied by the largest factor, the phase considered to be rapidly oscillating. The next largest term is a conservation law:

$$\partial_t \mathcal{N} + \nabla^{(3)} \cdot \Gamma_{\mathcal{N}}^{(3)} = 0 \tag{2.68}$$

where the conserved quantity

$$\mathcal{N} = \sigma^{(0)} (\kappa A)^2 \tag{2.69}$$

is the density of wave action (quanta) per unit mass. The flux of wave action

$$\Gamma_{\mathcal{N}}^{(3)} = \}\mathbf{v}^R, V^R\{ \mathcal{N} \tag{2.70}$$

is composed of wave action carried along at the ray velocity.

Accordingly, the conclusion to be drawn is that under the conditions of geometrical optics (outlined previously) wave action (quanta) can neither be emitted nor absorbed. A localized perturbation or wave packet behaves somewhat like a particle: initially propagating in the neighborhood of a particular ray trajectory, it will continue to be identified with that ray in perpetuity.

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CHAPTER 3

Motions Of A Gyroelastic Screwpinch

The Equilibria

In previous chapters some properties of a general gyroelastic system were considered. It was found that a particular class of initialization gives the gyroelastic system a self-preserving symmetry called isorrhopy. This symmetry was associated with certain so-called gyroscopic and quasielastic forces and was further asserted to cause drift modes of oscillation and fluid modes of oscillation to decouple one from another. The symmetry operator associated with isorrhopy, called a permutator, is the generator of a group of infinitesmal transformations called permutations; finite transformations called changes of representation are generated from these permutations.

Transformation properties of the gyroscopic and quasielastic forces under changes of representation were reviewed it being emphasized that the forces transform as components of a single covariant entity. This is the reason for the term gyroelastic. The isorrhopic gyroelastic system was then made the focus of attention and a closed system of equations describing its behavior was derived.

Next it was asserted that a certain fluid Lagrangian would give rise to the same closed system of equations and by a standard variational technique this was shown to be the case. Symmetry operations were applied to the action integral and demonstrated to induce conservation laws. One such symmetry operation was a permutation of a time translated system.

Following the specialization of the general nonlinear system to the isorrhopic case, an examination of wave propagation in the geometrical optics limit was undertaken. The view thus presented was one of a slowly developing nonlinear system supporting a rapidly oscillating, short wavelength, small amplitude wave field. To lowest order the wave field is found to behave according to local conditions fixed by the configuration of the (isorrhopic gyroscopic) system.

In this chapter the configuration is specialized to a particular class of screwpinch equilibria for the purpose of studying linear global motions and stability of these equilibria. The system is first linearized and the small amplitude Lagrangian is presented in the screwpinch-specialized coordinate system. The linearized equation of motion is found quite generally to have singularities over specific ranges of the parameters which map the solution space called continua. The solutions in these continua are generalized functions also called distributions.

Distributions are linear functionals. Although they are not members of the Hilbert space of possible motions of the equilibrium, they do play an essential role in constructing an invertible integral transform to evolve arbitrary perturbations of the equilibrium which are possible motions.

To construct the screwpinch equilibria to be studied, take the isorrhopic surfaces to be circular cylinders, so that $s = \boldsymbol{x}^2/2$ where $\boldsymbol{x}$ is a two-vector directed from the axis of symmetry. It follows that $\nabla^* s = \boldsymbol{x}^*$. Further choose the velocity field to be represented as

$$\boldsymbol{v} = -\,\Omega(s)\ \boldsymbol{x}^* \tag{3.1}$$

where $\Omega(s)$ is (topologically) the fluid angular velocity on an isorrhope. It is evident that Ω transforms under changes of representation according to

$$\Omega(s) \rightarrow \Omega'(s) = \Omega(s) + \omega(s) \tag{3.2}$$

and that the system of isorrhopic surfaces does not deform as a result of the steady flow. The steady state hypothesis, $\partial_t = 0$, then requires that the equilibrium state satisfy

$$\begin{aligned}
&\rho\ \boldsymbol{v}\cdot\nabla\boldsymbol{v} - X\ \nabla^* s\cdot\nabla\boldsymbol{v} - Y\ \nabla^* s\cdot\nabla\nabla^* s - Q\ \partial_\chi \tau + \nabla\kappa = 0 \\
&\boldsymbol{v}\cdot\nabla\tau - \partial_\chi \boldsymbol{v} = 0 \\
&\boldsymbol{v}\cdot\nabla s = 0 \\
&\nabla\cdot\boldsymbol{v} = 0
\end{aligned} \tag{3.3}$$

In view of (3.1), the equation of motion for the steady flow equilibrium assumes the form

$$(\rho\Omega^2 + X\Omega - Y)\, \nabla^* s \cdot \nabla\nabla^* s - Q\, \partial_\chi \tau + \nabla\kappa = 0 \tag{3.4}$$

The quantity in parentheses in (3.4) is a representational invariant. It can be evaluated by noting that in the canonical representation (see (2.26))

$$X^{(0)} = 0 \;,\; Y^{(0)} = \frac{\text{[illegible]}}{4\rho} \;,\; \mathbf{v}^{(0)} = -\,\Omega^{(0)} \mathbf{x}^* \tag{3.5}$$

so that

$$(\rho\Omega^2 + X\Omega - Y) = \rho\left(\Omega^{(0)} - \frac{\text{[illegible]}}{2\rho}\right)\left(\Omega^{(0)} + \frac{\text{[illegible]}}{2\rho}\right) = \rho\Omega^+\Omega^- \tag{3.6}$$

where the identification $\mathbf{v}' = -\,\Omega' \mathbf{x}^*$ has been made (see (2.28).) Another useful representation is the null representation, designated by a subscript $_0$ and defined by the condition $\Omega_0 = 0$. In the null representation it is found that

$$X_0 = 2\rho\Omega^{(0)} \;,\; Y_0 = -\,\rho\Omega^+\Omega^- \;,\; \mathbf{v}_0 = 0 \tag{3.7}$$

With this description of the equilibria, let us now turn to an analysis of the linearized system. The linearized equation of motion derived previously is reproduced here as

$$\rho\, D^+ D^+ \xi - Q\, \partial_\chi \partial_\chi \xi - \xi \cdot \nabla(\rho\, D^+ \mathbf{v}^+ - Q\, \partial_\chi \tau) + \nabla\kappa = 0 \tag{3.8}$$

For convenience, represent the displacement ξ in terms of its contravariant components α and β

$$\alpha = \xi \cdot \nabla s = \xi \cdot \mathbf{x}$$
$$\beta = \xi \cdot \nabla\theta = -\,\frac{1}{2s}\, \xi \cdot \mathbf{x}^* \tag{3.9}$$

so that

$$\xi = \frac{1}{2s}\, \alpha \mathbf{x} - \beta \mathbf{x}^* \tag{3.10}$$

Using the identities (in the null representation,

$$\boldsymbol{x} = \nabla s \ , \ \boldsymbol{x}^* = \nabla^* s = -2s\nabla\theta$$
$$\boldsymbol{x}_t = \boldsymbol{v} = 0$$
$$\boldsymbol{x}_s = \frac{1}{2s}\boldsymbol{x}$$
$$\boldsymbol{x}_\theta = -\boldsymbol{x}^* \ . \ \boldsymbol{x}_{\theta\theta} = \boldsymbol{x}^{**} = -\boldsymbol{x}$$
$$\boldsymbol{v}_\theta^{(0)} + \boldsymbol{v}^{(0)*} = \boldsymbol{v}\theta + \boldsymbol{v}^{z*} = 0 \tag{3.11}$$

(where subscripting refers to partial differentiation with respect to the subscript) the linearized equation of motion can be developed as

$$\mathcal{A}\,\boldsymbol{x} + \mathcal{B}\,\boldsymbol{x}^* + \nabla\kappa = 0 \tag{3.12.1}$$

where

$$\begin{aligned}\mathcal{A} = \ & \rho\,\Big(\frac{1}{2s}\,\alpha_{tt} + \Omega^+\Omega^-\big(\frac{1}{2s}\,\alpha_{\theta\theta} - 2\beta_\theta\big) + 2\Omega^{(0)}\big(\frac{1}{2s}\,\alpha_{t\theta} - \beta_t\big)\Big) \\ & - Q\,\Big(\frac{1}{2s}\,\alpha_{zz} + \nu^2\big(\frac{1}{2s}\,\alpha_{\theta\theta} - 2\beta_\theta\big) + 2\nu\big(\frac{1}{2s}\,\alpha_{z\theta} - \beta_z\big)\Big) \\ & + \alpha\,(\rho\Omega^+\Omega^- - Q\nu^2)_s\end{aligned} \tag{3.12.2}$$

and

$$\begin{aligned}\mathcal{B} = -\rho\,\Big(\beta_{tt} + \Omega^+\Omega^-\big(\beta_{\theta\theta} + \frac{1}{s}\,\alpha_\theta\big) + 2\Omega^{(0)}\big(\beta_{t\theta} + \frac{1}{2s}\,\alpha_t\big)\Big) \\ + Q\,\Big(\beta_{zz} + \nu^2\big(\beta_{\theta\theta} + \frac{1}{s}\,\alpha_\theta\big) + 2\nu\big(\beta_{z\theta} + \frac{1}{2s}\,\alpha_z\big)\Big)\end{aligned} \tag{3.12.3}$$

The symbol ν has been introduced and is defined by

$$\tau = -\nu\boldsymbol{x}^* \tag{3.13}$$

In the gyroelastic system isorrhopy guarantees that $\tau(\mathcal{P})$ exists in the tangent space $\mathcal{TM}(\mathcal{P})$ and this is what inspires the choice (3.13). Also, since in equilibrium the isorrhopic surfaces are cylinders, $\nu_z = 0$.

The displacement field ξ is required to be divergenceless $\nabla\cdot\xi = 0$, so there exists a homomorphism represented by

$$\xi = \nabla^*\Psi = \frac{m}{\sqrt{(2s)}}\,\Psi\mathbf{e}_s + \sqrt{(2s)}\,\Psi_s\mathbf{e}_\theta \tag{3.14}$$

where Ψ is called the stream function. An identification with (3.10) reveals the association

$$\Psi_s = -\beta \ , \ \Psi_\theta = \alpha \tag{3.15}$$

Certain portions of the linear analysis that follows are more conveniently or succinctly cast in terms of the stream function, other portions will be done in terms of the displacement field or its components. It should be recognized that either representation is entirely equivalent to the other.

Curling the linearized equation of motion (3.12.1) to get rid of the gradient, there results the expression

$$\mathcal{J}_s(2s\beta, -\alpha) = \mathcal{J}_\theta(\frac{1}{2s}\alpha, \beta) \tag{3.16.1}$$

or equivalently

$$\mathcal{J}_s(-2s\Psi_s, -\Psi_\theta) = \mathcal{J}_\theta(\frac{1}{2s}\Psi_\theta, -\Psi_s) \tag{3.16.2}$$

where $\mathcal{J}$ is given by an operator relation by

$$\begin{aligned}\mathcal{J}(x,y) = &- \rho \ (x_{tt} + 2\Omega^{(0)} x_{t\theta} + \Omega^+\Omega^- x_{\theta\theta} - 2\Omega^{(0)} y_t - \Omega^+\Omega^- y_\theta) \\ &+ Q \ (x_{zz} + 2\nu x_{z\theta} + \nu^2 x_{\theta\theta} - 2\nu y_z - \nu^2 y_\theta)\end{aligned} \tag{3.16.3}$$

The appropriate small amplitude Lagrangian for the linearized system is given by

$$\begin{aligned}\mathcal{L} = \ &\frac{1}{2} \rho \ (\frac{1}{2s} \alpha_t^2 + 2s\beta_t^2 + 2\Omega^{(0)}(\frac{1}{2s} \alpha_\theta\alpha_t + 2s\beta_\theta\beta_t + 2\alpha\beta_t) \\ &\qquad + \Omega^+\Omega^-(\frac{1}{2s} \alpha_\theta^2 - 2\alpha\alpha_s + 2s\alpha_s^2)) \\ &- \frac{1}{2} Q \ (\frac{1}{2s} \alpha_z^2 + 2s\beta_z^2 + 2\nu(\frac{1}{2s} \alpha_\theta\alpha_z + 2s\beta_\theta\beta_z + 2\alpha\beta_z) \\ &\qquad + \nu^2(\frac{1}{2s} \alpha_\theta^2 - 2\alpha\alpha_s + 2s\alpha_s^2)) \\ &+ \underline{\mathcal{L}}\end{aligned} \tag{3.17.1}$$

where

$$\underline{\mathcal{L}} = (\frac{1}{4s} \alpha^2 \ P(s))_s \tag{3.17.2}$$

Notice the auxilliary $\underline{\mathcal{L}}$ satisfies the Euler equation

$$\frac{d}{ds}(\frac{\partial\underline{\mathcal{L}}}{\partial\alpha_s}) - \frac{\partial\underline{\mathcal{L}}}{\partial\alpha} = 0 \tag{3.18}$$

identically.

The (small amplitude) action integral is then minimized by solutions of the (small amplitude) Euler equations

$$\frac{d}{dz}\left(\frac{\partial\mathcal{L}}{\partial\alpha_z}\right) + \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\alpha_t}\right) + \frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\alpha_s}\right) + \frac{d}{d\theta}\left(\frac{\partial\mathcal{L}}{\partial\alpha_\theta}\right) - \frac{\partial\mathcal{L}}{\partial\alpha} = 0 \tag{3.19.1}$$

and

$$\frac{d}{dz}\left(\frac{\partial\mathcal{L}}{\partial\beta_z}\right) + \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\beta_t}\right) + \frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\beta_s}\right) + \frac{d}{d\theta}\left(\frac{\partial\mathcal{L}}{\partial\beta_\theta}\right) - \frac{\partial\mathcal{L}}{\partial\beta} = 0 \tag{3.19.2}$$

A short calculation exposes the curled linear equation of motion (3.16.1) as no more than the sum of derivatives of these Euler equations for α and β. The identification made is

$$(3.16.1) = \frac{d}{d\theta}(3.19.1) - \frac{d}{ds}(3.19.2) \tag{3.19.3}$$

Since the equilibrium structure is independent of (θ, z, t) this partial differential equation in (s, θ, z, t) is seperable and reducible to four second order ordinary differential equations. Each of three of these have constant coefficients and are solved by complex exponentials. The fourth, the equation in s, will henceforth be referred to as the Euler equation (though it is of course only one of four.)

The appropriate ansatz for the stream eigenfunction Ψ is then a helical-harmonic complex exponential with an s-dependent amplitude:

$$\Psi_{km}^{(n)} = \psi_{km}^{(n)}(s)\ exp(i(kz + m\theta - \omega_{km}^{(n)}t)) \tag{3.20}$$

Equivalently

$$\begin{aligned}\xi_{km}^{(n)} &= \frac{1}{2s}\,\alpha_{km}^{(n)}\ \mathbf{x} - \beta_{km}^{(n)}\ \mathbf{x}^* \\ &= \left(\frac{im}{2s}\,\psi_{km}^{(n)}(s)\ \mathbf{x} + (\psi_{km}^{(n)}(s))_s\ \mathbf{x}^*\right)\ exp(i(kz + m\theta - \omega_{km}^{(n)}t))\end{aligned} \tag{3.21}$$

is the form of the displacement. Any solution to the partial differential equation (3.16.2) (or (3.8)) for given boundary conditions and initial conditions, can be created by a superposition of eigensolutions of the form (3.20) (or (3.21).) The determination of a proper inner product on this solution space and related questions will be dealt with later. First we'll tackle the Euler equation itself, exploring some of the more intriguing cloisters of arcane mathematics along the way to an integral theorem.

Inserting (3.20) into (3.16.2) the linear equation of motion for the stream function amplitude assumes the form

$$(f\psi_s)_s = g\psi \tag{3.22.1}$$

where the coefficient functions f and g are given by

$$f = 2s(\rho(\omega^2 - 2m\omega\Omega^{(0)} + m^2\Omega^+\Omega^-) - Q(k + \iota m)^2) \tag{3.22.2}$$

and

$$g = \frac{m^2}{2s}(\rho(\omega^2 - 2m\omega\Omega^{(0)} + m^2\Omega^+\Omega^-) - Q(k + \iota m)^2) + \frac{d}{ds}(\rho(-2m\omega\Omega^{(0)} + m^2\Omega^+\Omega^-) - Q(2k\iota m + \iota^2 m^2)) \tag{3.22.3}$$

and where $\Omega^{(0)}$, $\Omega^{\pm}$, Q, ι and ρ are all functions of s specified by the particular equilibrium chosen.

Detour for a moment back to the linear equation of motion in the form (3.12.1) and rewrite that equation for a mode of definite energy (dropping all attached subscripts and superscripts k, m, n for simplicity) as

$$\rho(\frac{1}{2s}\alpha\boldsymbol{x} - \beta\boldsymbol{x}^*)_{tt} + \mathcal{A}'\boldsymbol{x} + \mathcal{B}'\boldsymbol{x}^* + \nabla\kappa = 0 \tag{3.23}$$

or equivalently as

$$\rho\xi_{tt} - F(\xi) = 0 \tag{3.24}$$

Now follow the eclectic procedure outlined by Frieman and Rotenberg[6] for obtaining an energy integral: multiply (3.24) by $\xi^\dagger$ († indicates complex conjugate) and integrate over volume so that there results

$$-\int ds\, d\theta\, dz\, \rho\, \xi^\dagger\cdot\xi_{tt} = \omega^2\int ds\, d\theta\, dz\, \rho\, \xi^\dagger\cdot\xi = -\int ds\, d\theta\, dz\, \xi^\dagger\cdot F(\xi) \tag{3.25}$$

Noting that

$$\int ds\, d\theta\, dz\, \xi^\dagger\cdot\nabla\kappa = \int ds\, d\theta\, dz\, (\nabla\cdot\kappa\xi^\dagger - \kappa\nabla\cdot\xi) = \int dz \oint \xi^\dagger\cdot d\boldsymbol{x}^*\kappa = 0 \tag{3.26}$$

it is seen that there remains the relation

$$\omega^2\int ds\, d\theta\, dz\, \rho\, \xi^\dagger\cdot\xi = \int ds\, d\theta\, dz\, (\alpha^\dagger\mathcal{A}' - \beta^\dagger 2s)$$
$$\omega^2 N = \delta W \tag{3.27}$$

where N is a positive definite normalization. The relation (3.27) is a quadratic form in ω which can be reexpressed in terms of the stream eigenfunction amplitude ψ as

$$a\omega^2 + b\omega + c = 0 \tag{3.28.1}$$

where

$$a = \int ds\ \rho\ (2s\psi_s^2 + \frac{m^2}{2s}\psi^2) \tag{3.28.2}$$

$$b = -\int ds\ (2s\psi_s^2(2m\rho\Omega^{(0)}) + \psi^2(\frac{m^2}{2s}\ 2m\rho\Omega^{(0)} + (2m\rho\Omega^{(0)})_s)) \tag{3.28.3}$$

and

$$c = \int ds\ (2s\psi_s^2(m^2\rho\Omega^+\Omega^- - Q(k + m\nu)^2) + \psi^2(\frac{m^2}{2s}(m^2\rho\Omega^+\Omega^- - Q(k + m\nu)^2) + (m^2\rho\Omega^+\Omega^- - Q(2m\nu k + m^2\nu^2))_s)) \tag{3.28.4}$$

It is clear from (3.28.1) that if $\Omega^{(0)} = 0$, ω^2 is a real quantity (if $\Omega^{(0)} \neq 0$, ω^2 may be complex.) An integral theorem for the latter case (canonically rotating) is thus inherently more complicated than for the former case (canonically static). Notice that the canonical angular frequency $\Omega^{(0)}$ is a representational invariant, so this choice is not arbitrary, but deliberate.

Shear in the canonical velocity profile can cause Helmholtz-type instabilities and this can be seen to arise due to the presence of the derivative in the expession for the coefficient b above. However, a sufficiently large value of the gyroelastic modulus Q can stabilize such instabilities.

The superposition of eigenmodes (the integral transform) can be accomplished by summation over the discrete spectra and integration over the continua for each κ,m. Since all eigenvalues of the spectra for the static equilibria are confined to the real and imaginary axes, the task of finding them and including them in the integral transform in that case is simplified (over the rotating equilibrium.) Also, the static case admits to analysis by an energy principle, which will be presented in succeeding chapters.

The next step towards an integral theorem is to choose a particular equilibrium to study in detail. The choice is made so as to avoid needless complexity yet retaining as many nontrivial features of the general theory as possible. For example, since one object of the study is to examine the stability of gyroelastic systems, we should pick an equilibrium with a non-zero gyroelastic modulus-preferably one which would be unstable in the absence of gyroelasticity.

It has been shown previously (3.28.1) that $\Omega^{(0)} \neq 0$ complicates the integral theorem. To avoid possible obfuscation of the central issue (for this exercize) therefore, choose the equilibrium without canonical rotation. Choose the material pressure to be isotropic, $p_\parallel = p_\dagger$ and comparable to B^2 so that the equilibrium has a high average β (ratio of material to magnetic pressure.)

It is well known that without gyroelasticity the Suydam criterion[19,35] supplies a necessary condition for stability for the columnar pinch. In the notation used in this tome the Suydam criterion is given by

$$(\frac{d}{ds}\ ln\ \nu)^2 > \frac{2}{s}\frac{d}{ds}\ ln\ Q \tag{3.29}$$

According to this condition, if the axial current density is chosen to be uniform so that ν is constant, a finite-β pinch will be unstable: somewhere Q will be an increasing function of s.

With these considerations in mind the following equilibria are chosen for closer scrutiny:

The magnetic pressure is to be linear in s

$$\begin{aligned} B^2(s) &= B_0^2(1 - \beta(1 - \frac{s}{s_0})) \equiv B_0^2u^2(s) \qquad & s < s_0 \\ B^2(s) &= B_0^2 & s > s_0 \end{aligned} \tag{3.30}$$

Let the variable s be scaled by s_0 so that the edge of the plasma is located at (new variable) $\underset{\sim}{s} = 1$. The material pressure is then determined to be

$$p_\dagger = p_\parallel = \frac{1}{2}B_0^2\beta(1-\underset{\sim}{s}) \tag{3.31}$$

where β is a parameter which can be varied over the range $0 < \beta < 1$ and roughly signifies the average β ratio of the equilibrium. The mass density ρ is constant

$$\rho = \rho_0 \tag{3.32}$$

and the axial current density is also constant, J_0, so that ν is given by

$$\nu = \frac{J_0}{2B_0\sqrt{(1-\beta(1-\underset{\sim}{s}))}} \equiv \frac{J_0}{2B_0u(\underset{\sim}{s})} \tag{3.33}$$

Furthermore, choose the gyroelastic modulus to be constant and the equilibrium to be static so that

$$\Omega^{+}\Omega^{-} = (\Omega^{(0)})^2 - \Omega_g^2 = -\Omega_g^2 \tag{3.34}$$

where the *gyroelastic frequency* is defined by

$$\Omega_g^2 = \frac{\Phi}{4\rho_0^2} = constant \tag{3.35}$$

Finally, choose an equilibrium in which there is no circulation so that $\Omega^{(0)} = 0$.

The Euler equation (3.22.1) can now be written explicitly for this class of equilibrium, in non-dimensionalized form, as

$$(f\psi_{\underset{\sim}{s}})_{\underset{\sim}{s}} = g\psi \tag{3.36.1}$$

with the coefficient functions f and g given by

$$f = 2\underset{\sim}{s}(\underset{\sim}{\omega}^2 - \underset{\sim}{\Omega}_g^2 - (\underset{\sim}{\kappa}u+1)^2) \tag{3.36.2}$$

and

$$g = \frac{m^2}{2\underset{\sim}{s}}(\underset{\sim}{\omega}^2 - \underset{\sim}{\Omega}_g^2 - (\underset{\sim}{\kappa}u+1)^2) - \frac{\underset{\sim}{\kappa}\beta}{u} \tag{3.36.3}$$

and where the nondimensionalized variables are defined as follows:

$$\frac{k_0}{m} = \frac{\mu_0 J_0}{2\,B_0} = \frac{\tau_0}{\sqrt{(2s)}} \;;\; V_{a0}^2 = \frac{B_0^2}{\mu_0\rho_0} \;;\; \frac{\omega_0}{m} = k_0 V_{a0}$$
$$\underset{\sim}{s} = \frac{s}{s_0} \;;\; \underset{\sim}{\kappa} = \frac{k}{k_0} \;;\; \underset{\sim}{\omega}^2 = \frac{\omega^2}{\omega_0^2} \;;\; \underset{\sim}{\Omega}_g^2 = \frac{\Omega_g^2}{\omega_0^2}$$
$$u^2 = 1-\beta(1-\underset{\sim}{s}) \tag{3.37}$$

Equivalently, the Euler equation can be written as a second order ordinary differential equation for the radial component of ξ, $\xi\cdot e_s = \xi$, by using the incompressibility constraint. The result is

$$(f\xi_{\underset{\sim}{s}})_{\underset{\sim}{s}} = g\xi \tag{3.38.1}$$

where the coefficient functions f and g (not the same as those given above) are given by

$$f = 2\underset{\sim}{s}^2(\underset{\sim}{\omega}^2 - \underset{\sim}{\Omega}_g^2 - (\underset{\sim}{\kappa}u+1)^2) \tag{3.38.2}$$

and

$$g = \left(\frac{m^2-1}{2}\right)(\omega^2 - \Omega_g^2 - (\kappa u+1)^2) + s\kappa^2\beta \tag{3.38.3}$$

The same normalization has been used here as above.

The boundary condition appropriate for the fixed boundary system (plasma in contact with conducting wall) is that $\xi = 0$ at the outer boundary and ξ is bounded at the origin. The equivalent conditions for ψ are that ψ must vanish both at the origin and at the boundary.

Stripped naked of the trappings tethering it to its physical origin like tracks in the snow the problem now slips through the portcullis to abstraction on a mathematical odyssey whose object is the appointed encounter with an integral theorem.

The Point Spectrum

The Euler equation for the equilibria chosen (referred to henceforth as the standard case) given by (3.36.1) is to be solved on the domain $0 < s < 1$ with the boundary conditions $\psi(0) = \psi(1) = 0$. All quantities are suitably normalized according to (3.37) and the normalized signifier is henceforth dropped. If the coefficient function f vanishes on the interval $0 < s < 1$ the Euler equation is said to be singular and the point(s) at which this occurs is (are) called singular point(s). In the event that such points do exist, standard integration techniques fail and caution must be exercized to arrive at the correct solution.

First, consider the case in which no internal singularities occur. The only singularity is then at the origin of coordinates $s = 0$. This point is a regular singular point and a series solution about it may be developed as follows: Rewrite the Euler equation (3.36.1) in the canonical form

$$\psi_{ss} + \frac{1}{s}p(s)\,\psi_s + \frac{1}{s^2}q(s)\,\psi = 0 \tag{3.39}$$

and take the form of the solution ψ to be representable by an infinite series of the form

$$\psi = \sum_{i=0}^{i=\infty} \psi_i s^{i+r} \tag{3.40}$$

Near the origin the coefficient functions f and g are given by the known series

$$p = \sum_{i=0}^{i=\infty} p_i s^i \ , \ q = \sum_{i=0}^{i=\infty} q_i s^i \tag{3.41}$$

Substituting these series into the differential equation results in a recursion relation for the coefficients of the solution series ψ_i. The zeroth recursion relation, called the indicial equation, is given by

$$r(r-1) + p_0 r + q_0 = 0 \tag{3.42}$$

The recursion relation can be developed by induction to give the i^{th} coefficient as

$$\psi_i = -\frac{\sum_{l=0}^{i-1} \psi_l (p_{i-l}(r-l) + q_{i-l})}{(r+i)(r+i-1) + (r+i)p_0 + q_0} \tag{3.43}$$

The coefficient function series p and q can be expressed in terms of the f and g series (3.36.2) as

$$p = s\,\frac{f_s}{f} \ , \ q = -\,s^2\,\frac{g}{f} \tag{3.44}$$

Some algorithms useful in computing the coefficients of the series p and q above are: The exponent series:

$$\left(\sum_{i=0}^{i=\infty} a_i x^i\right)^n = \sum_{i=0}^{i=\infty} c_i x^i \tag{3.45.1}$$

where

$$c_0 = a_0^n \ ; \ c_l = \frac{1}{l a_0} \sum_{i=1}^{i=l} (in-l+i) a_i c_{l-i} \tag{3.45.2}$$

the quotient series:

$$\frac{\sum_{i=0}^{i=\infty} b_i x^i}{\sum_{i=0}^{i=\infty} a_i x^i} = \frac{1}{a_0} \sum_{i=0}^{i=\infty} c_i x^i \tag{3.46.1}$$

where

$$c_l = b_l - \frac{1}{a_0} \sum_{i=1}^{i=l} c_{l-i} a_i \tag{3.46.2}$$

and lastly the product series:

$$\left(\sum_{i=0}^{i=\infty} a_i x^i\right)\left(\sum_{i=0}^{i=\infty} b_i x^i\right) = \sum_{i=0}^{i=\infty} c_i x^i \tag{3.47.1}$$

where

$$c_l = \sum_{i=0}^{i=l} a_i b_{l-i} \tag{3.47.2}$$

The series representation of the solution developed about the regular singular point at $s = 0$ is convergent out to the nearest singularity in the complex s-plane. Once the solution is known in a neighborhood of the origin a standard numerical scheme can be employed to integrate the Euler equation repeatedly for various values of ω. A shooting method is devised to direct the search for values of ω (given values of $\kappa, m, \beta, \Omega_g^2$) for which the eigenvalue equation

$$\psi_{km}(s=1, \omega_{km}^n) = 0 \tag{3.48}$$

is satisfied. These values of ω comprise the discrete or point spectra for the equilibrium.

All succeeding discussions of spectra will be referred to a particular two dimensional space whose coordinates are the normalized axial wavevector or helicity κ for abscissa and the eigenvalue related function

$$\lambda = \omega^2 - \Omega_g^2 \tag{3.49}$$

for ordinate. In this space the point spectra are representable as curves or trajectories. Each value of κ selects a spectrum of values of λ (for given values of m and β.) As κ is varied continuously through a range of values, each point in a given spectrum traces out a curve connecting it with points in neighboring spectra labeled with the same radial and azimuthal mode numbers (see figs. 4,8.)

Figure 5 depicts the solutions of the Euler equation for chosen values of β and κ. As the radial and azimuthal mode numbers increase the energy associated with the displacement becomes localized near a point s_0. Refer to this point as the annihilation point; the reason for the nomenclature will be clarified presently. Figure 6 graphs the level curves (at constant z) of a few of the eigensteamfunctions in figure 5. The motion of the fluid is tangent to these streamlines.

Figure 4 shows the placement in κ-λ space of the eigenvalue trajectories corresponding to the three lowest eigenvalues for $m=1$. Figure 8 is an enlargement of the area enclosed by the larger box in figure 4. The shaded regions in figure 4 correspond to the continua- the darker shading represents the degenerate continua and the lighter shading represents the non-degenerate continua.

For given values of β and κ a certain critical value $\lambda = \lambda_L$ marks the lower limit of the continuum. For a range of values of λ greater than λ_L, $\lambda_L < \lambda < \lambda_U$, there exist one or more internal singularities of the Euler equation. In the event that such singularities appear, the spectra are no longer discrete. If only one singularity occurs in the interval $(0<s<1)$ for a given pair of values $(\kappa$-$\lambda)$ then there exists one and only one solution to the Euler equation (at that point in κ-λ space.) If more than one singularity occurs in the interval $(0<s<1)$ then there exist a number of independent solutions to the Euler equation for that point in κ-λ space—a number equal to the number of singularities. In this latter case, the generalized function space is said to be degenerate, of dimension equal to the number of singularities. The shaded regions in figures 4,8,13 correspond to these continua- the darker shading covers the degenerate continua and the lighter shading covers the non-degenerate continua. We will address certain questions concerning the structure of the point spectra first, then turn to an in depth examination of the continua.

It is expedient to ascertain the dimension of the nonsingular function space and the distribution of eigenvalues in the point spectra. Such information can facilitate the search algorithm. It will be necessary to properly normalize the nonsingular eigenfunctions and this too will be done presently.

First, to determine the dimension of the nonsingular eigenfunction space, examine the behavior of the eigenfunction in the vicinity of the continuum lower edge, say $\lambda = \lambda_L - \delta\lambda$, and near the point $s = s_0$ which becomes the singular point for $\lambda = \lambda_L + \delta\lambda$. The number of zeroes of the eigenfunction in this neighborhood is closely related to the number of eigenfunctions with eigenvalue $\lambda < \lambda_L - \delta\lambda$.

Rewrite the Euler equation in the form

$$(2s\Delta(s)\psi_s)_s = (\frac{m^2}{2s}\Delta(s) - \frac{\kappa\beta}{u})\psi \tag{3.50}$$

with

$$\Delta(s) \equiv \lambda - (\kappa u+1)^2 \tag{3.51}$$

To clarify the situation, consider three separate but contiguous ranges of κ; call them cases I, II and III.

Case I:

Examine first the behavior of the Euler solutions near the point s_0 where

$$\Delta(s_0) = 0 \tag{3.52.1}$$

and

$$\Delta_s(s_0) = 0 \tag{3.52.2}$$

so that

$$s_0 = 1 - \frac{1}{\beta}(1 - \frac{1}{\kappa^2}) \tag{3.53}$$

Such points exist–s_0 is necessarily positive by construction– only in a range of helicities

$$-\frac{1}{\sqrt{(1-\beta)}} < \kappa < -1 \tag{3.54}$$

Near the continuum edge and near $\varepsilon = (s - s_0) = 0$ the Euler equation has the form

$$\psi_{\varepsilon\varepsilon} + \frac{2}{\varepsilon}\psi_\varepsilon = \frac{1}{\varepsilon^2}(-\frac{2}{s_0\beta\kappa^2} + \frac{m^2\varepsilon^2}{4s_0^2})\ \psi = 0 + O(\varepsilon) \tag{3.55}$$

The indicial equation for the series solution of ψ near $\varepsilon = 0$ has the roots

$$r = \frac{1}{2}(-1 \pm \sqrt{D}) \tag{3.56}$$

where

$$\mathscr{D} = 1 - \frac{8}{s_0 \beta \kappa^2} \tag{3.57}$$

(and where s_0 is given by (3.53).) Solutions of (3.55) are of the form

$$\Psi \sim \varepsilon^{\tau} f(\varepsilon) \tag{3.58}$$

where $f(0)$ is finite.

If $\mathscr{D}$ is negative, τ is complex and (3.58) becomes infinitely oscillatory near $\varepsilon = 0$. For the range of κ (3.54) the entire range of values of $\mathscr{D}$ is negative

$$1 - \frac{8}{\beta} > \mathscr{D} \geq -7 \tag{3.59}$$

This being the case, thare are denumerably infinitely many zeroes of ψ near the (singular) point $s = s_0$ and as many point spectrum eigenfunctions. The continuum lower edge in the range of helicities κ (3.54) is a *curve of accumulation*; s_0 is called an annihilation point. As λ decreases through the curve of accumulation two singularities (to the right and to the left of s_0) approach one another then touch and vanish. For λ below the curve of accumulation there exist a denumerably infinite set of eigenvalues λ.

Figure 7 shows the solutions to the (reduced radial) Euler equation for values of κ–λ chosen so that s_0 has the value .5. The annihilation point is most easily detected for low radial mode number in high β equilibria (upper right hand corner of figure 7.) The larger plasma pressure thus tends to concentrate the displacement energy even for the lowest frequency modes.

Case II:

For the range of helicities

$$\kappa < -\frac{1}{\sqrt{(1-\beta)}} \tag{3.60}$$

there appears a singularity at $s = 0$ near the continuum lower edge. Following the procedure outlined above, the indicial equation is again found to be solved by roots τ given by (3.56) but with $\mathscr{D}$ now given by

$$\mathscr{D} = 1 + m^2 + \frac{2}{\kappa\sqrt{(1-\beta)} + 1} \tag{3.61}$$

For values of κ in the range (3.60) such that

$$\kappa(1-\beta) > -\frac{(m^2+3)}{(m^2+1)} \tag{3.62}$$

$\mathcal{D}$ is again negative and the continuum lower edge is again a *curve of accumulation*. For $m = 1$ the *curve of accumulation* extends to the right of a critical (lower bound) helicity $\kappa_c(m = 1)$

$$\kappa_c(m=1) = -\frac{2}{\sqrt{(1-\beta)}} \tag{3.63}$$

For $m > 1$ this critical helicity is less negative. As larger values of azimuthal mode number m are considered it is found that the critical helicity approaches closer and closer to the limit

$$\kappa = -\frac{1}{\sqrt{(1-\beta)}} \tag{3.64}$$

There all values of m cause $\mathcal{D}$ to be negative, as already determined (this point also falling within the purview of case I.)

Beyond (to the left of) the critical helicity for a given m there are a finite number of eigenvalues in the point spectrum adjacent to and below the continuum lower edge.

Case III:

Next examine the point $\kappa = -1$ and the range of κ extending to its right. By the same procedure followed in cases I and II, it is found that the point $\kappa = -1$ is included in case I and is thus a point of accumulation. The behavior of the point spectra in the vicinity of this point is quite unusual; so unusual, in fact, that the point has been squelched from the slime of anonymity and nailed to notoriety with the name *Kruskal-Shafranov limit*. To lay bare some of the enigma of this point to scrutiny, transform the Euler equation using the change of variables

$$\varepsilon = \frac{s - 1}{\kappa + 1} \tag{3.65}$$

Near $\varepsilon = 0^+$ the Euler equation is solved by

$$\psi = a\ I_0(\frac{\varepsilon}{2}) + b\ K_0(\frac{\varepsilon}{2}) \tag{3.66}$$

hyperbolic Bessel functions which are not oscillatory. However, in the limit as ε approaches infinity, the solutions are again infinitely oscillatory. Within a range which is vanishingly small, yet nevertheless finite, to the right of

$\kappa = -1$, a countably infinite number of eigenvalue trajectories impact the continuum lower edge. That each trajectory impacts at a different point is a corollary of a general theorem stated simply by Goedbloed and Sakanaka[9]. In the interest of completeness it is reproduced along with its proof below. Let ψ satisfy the self-adjoint equation

$$(f\psi_s)_s = g\psi \tag{3.67}$$

with boundary conditions

$$\psi(0) = \psi(1) = 0 \tag{3.68}$$

Define the function $R(\omega^2, n)$ by

$$\psi(s = R(\omega^2, n), \omega^2) = 0 \tag{3.69}$$

where n is the number of zeroes of $\psi(s,\omega^2)$ in the interval $0 < s < R(\omega^2, n)$. R is an increasing function of (radial mode number) n. For fixed ω^2, successive zeroes of the eigenfunction will be labelled by successively higher values of n.

Theorem. $R(\omega^2, n)$ is monotonic in ω^2 for nonsingular regions of the spectrum (point spectra uninterupted by continua.) Thus either

$$\frac{\partial R}{\partial\omega^2} > 0 \quad or \quad \frac{\partial R}{\partial\omega^2} < 0 \tag{3.69.1}$$

Proof: Suppose $\partial_{\omega^2} R|_{\omega_0^2} = 0$, then we could analytically continue $R(\omega^2, n)$ in the complex ω^2 plane near ω_0^2 as

$$R(\omega^2, n) = R(\omega_0^2, n) + \frac{1}{2}\frac{\partial^2 R}{\partial(\omega^2)^2}\Big|_{\omega_0^2}(\omega^2 - \omega_0^2)^2 + \cdots \tag{3.70}$$

Choosing $\omega^2 = \omega_0^2 + i\delta^2$ so that $(\omega^2 - \omega_0^2)^2 < 0$ we find a real zero of ψ for complex ω^2, contradicing the self-adjointedness property (all eigenvalues ω^2 are real.) Now fix R to the wall position $s = 1$: ω^2 will be monotonic in n.

Corollary (a): No singular (point) subspectrum eigenvalue trajectory $\omega^2(\kappa, n)$ for fixed n can intersect any other, since by intersecting, $\omega^2(\kappa, n) = \omega^2(\kappa, n')$ which by the above theorem cannot happen.

Corollary (b): At most one eigenvalue trajectory labelled with fixed n can intersect the continuum at any fixed κ.

The point $\kappa = 0$ alone remains uncharacteristic of the spectral properties unearthed so far. The point $\kappa = 0$ is unique. A superficial inspection of the Euler equation (3.50) reveals the fact that when κ vanishes ψ solves

$$(\lambda-1)((2s\psi_s)_s = \frac{m^2}{2s}\psi) \tag{3.71}$$

Any ψ satisfying the boundary conditions satisfies the differential equation trivially if $\lambda = 1$. Only one value of λ is allowed. The entire spectrum shrinks to a point and any function in the Hilbert space (any function of s satisfying the boundary conditions) is an eigenfunction The two-dimensional case is specified in this space at a single point!

It remains to determine the normalization of the point spectrum eigenfunctions. The inner product can be written immediately in terms of the displacement vector as

$$N(\lambda_n) = \int ds\; d\theta\; dz\; \xi^{\dagger}(\lambda_n)\cdot\xi(\lambda_n) \tag{3.72}$$

With the complex exponential part suitably normalized this relation can be rewritten involving only the (vector) amplitudes $\xi(\lambda_n)$

$$N(\lambda_n) = \int_0^1 ds\; \xi(\lambda_n)\cdot\xi(\lambda_n) = \int_0^1 ds\; ((\xi(\lambda_n)\cdot e_s)^2 + (\xi(\lambda_n)\cdot e_\theta)^2) \tag{3.73}$$

where e_s and e_θ are unit vectors in the radial and azimuthal directions. Using (3.14), (3.73) can then be reexpressed in terms of $\psi(s,\lambda_n)$ $\psi(\lambda_n)$

$$N(\lambda_n) = \int_0^1 ds\; (\frac{m^2}{2s}\,\psi^2(\lambda_n) + 2s\psi_s^2(\lambda_n)) \tag{3.74}$$

In addition, $\psi(n)$ solves the differential equation (3.50) with boundary conditions (3.48). Now write (3.50) for $\lambda = \lambda_\nu$ and for $\lambda = \lambda_{\nu'}$, where ν and ν' are not necessarily integers ($\omega^{(\nu)}$ is not necessarily an eigenvalue.) Multiply the two resulting ordinary differential equations each by the function solving the other; integrate the difference of the two equations over the s domain to get

$$\int_0^1 ds\ (\psi(\lambda_{\nu'})(2s\Delta(\lambda_\nu)\psi_s(\lambda_\nu))_s - \psi(\lambda_\nu)(s\Delta(\lambda_{\nu'})\psi(\lambda_{\nu'}))_s)$$

$$= \int_0^1 ds\ \frac{m^2}{2s}(\Delta(\lambda_\nu) - \Delta(\lambda_{\nu'}))\psi(\lambda_\nu)\psi(\lambda_{\nu'}) \tag{3.75}$$

Now integrate the left member of (3.75) by parts and use the fact that

$$\Delta(\lambda_\nu) - \Delta(\lambda_{\nu'}) = \lambda_{\nu'} - \lambda_\nu \tag{3.76}$$

to find

$$(\lambda_{\nu'} - \lambda_\nu) \int_0^1 ds\ (2s\psi_s(\lambda_{\nu'})\psi_s(\lambda_n) + \frac{m^2}{2s}\ \psi(\lambda_{\nu'})\psi(\lambda_n))$$

$$= 2s(\Delta(\lambda_\nu)\psi_s(\lambda_\nu)\psi(\lambda_n) - \Delta(\lambda_n)\psi_s(\lambda_n)\psi(\lambda_\nu))\Big|_{s=0}^{s=1} \tag{3.77}$$

Allow one value of λ_ν to approach the other, say

$$\lambda_{\nu'} - \lambda_\nu = \delta\omega^2 \tag{3.78}$$

and express the relation (3.77) for $\nu = n$, an eigenvalue (i.e. insist the solution meet the boundary condition.) The result is

$$\int_0^1 ds\ (2s\psi_s^2(\lambda_n) + \frac{m^2}{2s}\ \psi^2(\lambda_n)) = -\ 2\ \Delta(\lambda_n)\ \psi_s(\lambda_n)\ \frac{\partial}{\partial\lambda}\psi(\lambda_n)\Big|_{s=1} \tag{3.79}$$

The normalization constant $N^{(n)}$ is thus found to be

$$N(\lambda_n) = -\ 2(\lambda_n - (\kappa+1)^2)\ \psi_s(1,\lambda_n)\ \frac{\partial}{\partial\lambda}\psi(1,\lambda_n) \tag{3.80}$$

These are quantities calculated naturally by the algorithm which generates the eigenfunctions. The nonsingular functions are normalized by dividing them each by the square root of this constant.

The Continuum

The Euler equation for the chosen class of equilibria (the standard case) has the self-adjoint form (3.50) reproduced here

$$(2s\Delta\psi_s)_s = (\frac{m^2}{2s}\Delta - \frac{\kappa\beta}{u})\,\psi \tag{3.81}$$

where

$$\Delta(s) = \lambda - (\kappa u(s)+1)^2$$
$$u^2(s) = 1 - \beta(1-s)$$
$$\lambda = \omega^2 - \Omega_g^2 \tag{3.82}$$

and all variables have been normalized as prescribed in (3.37). For values of λ, κ and β such that Δ does not vanish in the interval $0 < s < 1$ the solution ψ may be generated by simply integrating the ordinary differential equation (numerically) from $s = 0^+$ to $s = 1$ as prescribed in the last section dealing with the point spectrum eigenfunctions. it was remarked there also that the set of eigenvalues (alues of λ for which the fixed boundary condition (3.48) is satisfied) might be empty, finite or denumerably infinite depending on the choice of parameters κ, m and β. These sets were called point spectra associated with the given equilibrium.

In the event that $\Delta(s)$ vanishes for a value or values of s in the range $0 < s < 1$ the Euler equation is said to be singular at the point(s) s_i where

$$\Delta(s_i) = 0 \tag{3.83}$$

Whereas the Euler equation can be eaisily integrated in the nonsingular case by any of a number of standard (numerical) techniques, the singular case is a different matter. In the neighborhood of the singular point(s) standard techniques fail miserably, preventing further integration. In this section a means is developed to overcome this difficulty and illuminate the nature of the problem solved.

It will prove helpful to recast (3.81) and the boundary conditions as an equivalent integral equation. It will also be expedient to make the substitution

$$\psi(s,\lambda) = s^{\frac{m}{2}}\, y(s,\lambda) \tag{3.84}$$

which transforms the Euler equation into

$$(2s^{m+1}\Delta y_s)_s = -\, s^m (m\Delta_s + \frac{\kappa\beta}{u})y \tag{3.85}$$

The inner product has a particularly simple representation in terms of y, the singularity at the origin in ψ is not present in y and the business of bookkeeping in the work ahead is simplified by this change of variables. These issues are to be weighed against the risk of confusion injected by the transformation. The author begs the indulgence of the reader in this regard.

Integrate (3.85) from the origin to some point s, then divide the result by $s^{m+1}\Delta(s)$ and integrate again. The singular point must be treated carefully as follows: if at any stage in the course of the calculation (integration) a quantity is to be divided by zero, a multiple of a delta function (a distribution with point support) must be added

$$\frac{1}{x} = \mathcal{P}\frac{1}{x} + \mu\ \delta(x) \tag{3.86}$$

Here $\mathcal{P}$ signifies Cauchy principle value (integral) and μ is to be determined. (3.86) is a generalized function which identifies a distribution. A distribution is a linear functional, a mapping of functions into (possibly complex) numbers. That is to say, given a function with certain nice properties, the distribution is a prescrption for returning a value. Strictly speaking (3.86) alone has no meaning–it must be used in concert with the process of integration to act on functions as a distribution.

The integral equation for $y(s)$ which results from the above procedure can be written then as

$$\begin{aligned} y(s) = &-\int_0^s d\xi\ y(\xi)g(\xi) \int_s^1 \frac{d\zeta}{\zeta^{m+1}\Delta(\zeta)} \\ &-\int_s^1 d\xi\ y(\xi)g(\xi) \int_\xi^1 \frac{d\zeta}{\zeta^{m+1}\Delta(\zeta)} + \sum_i \mu_i(\Theta(s-s_i) - \Theta(1-s_i)) \end{aligned} \tag{3.87}$$

where

$$g(s) = -\, s^{m+1}(\frac{m}{2}\,\Delta_s + \frac{\kappa\beta}{2u}) \tag{3.88}$$

and the Heaviside step function θ has been used. It is evident from (3.87) that the nonsingular eigenfunctions solve a homogeneous integral equation of the Fredholm type. The singular eigensolutions solve an inhomogeneous integral equation with one or more step-function-like inhomogeneities. The differential system is self-adjoint so the kernel in the integral system is symmetric, or easily symmetrizable.

Near the singularity the asymptotic form of the differential equation can be shown to be

$$(\varepsilon y_{\varepsilon})_{\varepsilon} = h(\varepsilon) y \tag{3.89}$$

with the definition

$$\varepsilon = s - s_0 \tag{3.90}$$

On either side of the singularity then two independent asymptotic solutions have the form

$$y_1 = \sum_{i=0}^{i=\infty} a_i \varepsilon^i \tag{3.91}$$

and

$$y_2 = \ln |\varepsilon| \sum_{i=0}^{i=\infty} a_i \varepsilon^i + \sum_{i=0}^{i=\infty} b_i \varepsilon^i \tag{3.92}$$

The Euler solution can thus be represented (asymptotically) near the singular point by

$$y^L(\varepsilon) = \alpha^L y_1 + \beta^L y_2 = \ln |\varepsilon| \sum_{i=0}^{i=\infty} c_i^L \varepsilon^i + \sum_{i=0}^{i=\infty} d_i^L \varepsilon^i \tag{3.93}$$

to the (immediate) left of the singular point and by

$$y^R(\varepsilon) = \alpha^R y_1 + \beta^R y_2 = \ln |\varepsilon| \sum_{i=0}^{i=\infty} c_i^R \varepsilon^i + \sum_{i=0}^{i=\infty} d_i^R \varepsilon^i \tag{3.94}$$

to the (immediate) right of the singular point.

That an invertible integral transform over the space of eigensolutions exists is entirely contingent on whether the eigensolutions have the properties of distributions. In particular, the inner product of a generalized (eigen)function with a function in the class of admissible motions of the system, i.e. in the Hilbert space, must be finite. This condition is sufficient to determine the behavior of the singular eigensolutions at the singularity in so far as the following conditions must obtain (with reference to (3.93) and (3.94)):

$$c_0^R - c_0^L = 0$$
$$d_0^R - d_0^L = \mu \tag{3.95}$$

There is a logarithmic singularity and a finite jump in y at the singular point. The behavior of the derivative of y (near the singular point) is thus

$$\lim_{\varepsilon \to 0} y_\varepsilon = c_0 \mathcal{P} \frac{1}{\varepsilon} + \mu \, \delta(\varepsilon) \tag{3.96}$$

That (3.95) is the case can be demonstrated by substituting y into the integral equation (3.87) being careful to use $y = y_L$ to the left of the singularity, $y = y_R$ to the right of the singularity, (3.96) at the singularity and the boundary condition $y(1) = 0$.

In the case that only one singularity appears in the interval $0 < s < 1$, the value of μ is determined so that the boundary condition is met simultaneously. The additional freedom thus afforded in the construction of the eigensolution allows any value of λ in the appropriate range to serve equally well as an eigenvalue. Call this range of λ the nondegenerate continuum for reasons which will become clear shortly.

The case in which two or more singularities appear in the interval $0 < s < 1$ is slightly more elaborate than the case in which only one appears. Since the boundary condition provides only one condition on the choice of the set of μ_i (the size of the finite jumps in the eigensolution at the singular points) there then exists a degeneracy in the function space. More than one eigensolution corresponds to one eigenvalue. If there are two singularities there are two values of μ_i to be chosen so as to satisfy the boundary condition. The boundary condition becomes a linear relation in the two values . The number of singularities corresponds to the number of degrees of freedom in the choice of the μ_i. The number of degrees of freedom in the choice of the μ_i is the degeneracy or dimensionality of the function subspace.

Although any normalized basis spanning the (degenerate) function space suffices, an orthonormalized basis is most convenient for the purpose of generating an integral theorem. A procedure for accomplishing this orthonormalization serves also as a proof of the preceeding statement concerning the dimensionality of the subspace. The next section is devoted to the description of an algorithm for generating the orthonormalized basis in the nondegenerate and in the degenerate continua.

There are two roots of (3.83) for given values of β, κ and λ. The roots are real and equal in the event that $\lambda = 0$. This occurs along the *curve of accumulation* referred to previously. As λ increases from zero the two roots separate– one decreases the other increases. To distinguish between the two roots refer to the decreasing root as occuring at $s = s_1$ and the increasing root as occuring at $s = s_2$.

It will be useful to construct solutions to (3.85) according to the following prescription:

(a) If y has an expansion near a singular point s_i

$$y(s) = A_i \sum_{j=0}^{j=\infty} a_j(s-s_i)^j + B_i\left(\ln|s-s_i| \sum_{j=0}^{j=\infty} a_j(s-s_i)^j + \sum_{j=0}^{j=\infty} b_j(s-s_i)^j\right) \tag{3.97}$$

(b) and (3.97) holds on *both* sides of the singular point, then y will be said to be *regular* at s_i.

Such solutions y are easily generated numerically by finding the coefficients a_k and b_k then determining the coefficients A_i and B_i by the behavior of the numerical solution near the singular point. Having done this the integration is simply continued on the other side of the singularity. These *regular* solutions are not eigensolutions; they will be used to build eigensolutions.

With this prescription, proceed to construct the distinct solutions y_0, y_1, y_2 with the following properties:

$$\begin{aligned} &y_0(0) = 1 \ ; \ y_0 \text{ is regular at } s_1 \text{ and } s_2 \\ &y_1(s<s_1) = 0 \ ; \ y_1(s_1) = 1 \ ; \ y_1 \text{ is regular at } s_2 \\ &y_2(s<s_2) = 0 \ ; \ y_2(s_2) = 1 \end{aligned} \tag{3.98}$$

If only one singularity falls within the interval $0 < s < 1$ then only two distinct solutions, y_0 and either y_1 or y_2, are required. In what follows, solutions y will be identified by subscripts if they are regular; all solutions y are identified with the value of λ used in their generation by $y(s,\lambda)$.

The nondegenerate continuum stream eigensolutions are assembled as follows:

$$\psi(s,\lambda) = s^{\frac{m}{2}}(y_0(s,\lambda) + \mu(\lambda)\ \Theta(s-s_i)\ y_i(s,\lambda)) = s^{\frac{m}{2}} y(s,\lambda) \tag{3.99}$$

where $i = 1,2$ according as whether the singularity is decreasing or increasing with λ as described above, and where

$$\mu(\lambda) = -\frac{y_0(1,\lambda)}{y_i(1,\lambda)} \tag{3.100}$$

The degenerate continuum stream eigensolutions are constructed in similar manner:

$$\begin{aligned} {}^{\iota}\psi(s,\lambda) &= s^{\frac{m}{2}}(y_0(s,\lambda) + {}^{\iota}\mu_1(\lambda)\ \Theta(s-s_1)\ y_1(s,\lambda) + {}^{\iota}\mu_2(\lambda)\ \Theta(s-s_2)\ y_2(s,\lambda)) \\ &= s^{\frac{m}{2}}\,{}^{\iota}y(s,\lambda) \end{aligned} \tag{3.101}$$

where the superscript ι serves to distinguish the members of the basis set in the degenerate (generalized) function subspace. Also

$${}^{\iota}\mu_2(\lambda) = -\frac{y_0(1,\lambda) + {}^{\iota}\mu_1(\lambda)\ y_1(1,\lambda)}{y_2(1,\lambda)} \tag{3.102}$$

is satisfied so that ${}^{\iota}\psi(1,\lambda) = 0$ as required. Notice the choice of the values of the μ is not yet uniquely specified. The orthonormalization of the subspace will provide this specification.

The inner product (as described in (3.72)-(3.74)) can be expressed in terms of y using the identity

$$\psi_s = (\frac{m}{2s}\ y + y_s)s^{\frac{m}{2}} \tag{3.103}$$

as

$$({}^{\iota}\psi(\lambda), {}^{\iota'}\psi(\lambda')) = \int_0^1 ds\ 2s^{m+1}\,{}^{\iota}y_s(s,\lambda)\,{}^{\iota'}y_s(s,\lambda') \tag{3.104}$$

(the ι superscript is superfluous in the nondegenerate continuum.) Differentiating (3.101) with respect to s it is easily established that

$$\begin{aligned} {}^{\iota}y_s(s,\lambda) = y_{0s}(s,\lambda) &+ {}^{\iota}\mu_1(\delta(s-s_1)y_1(s,\lambda) + \Theta(s-s_1)y_{1s}(s,\lambda)) \\ &+ {}^{\iota}\mu_2(\delta(s-s_2)y_2(s,\lambda) + \Theta(s-s_2)y_{2s}(s,\lambda)) \end{aligned} \tag{3.105}$$

(The values of μ and s_i are understood to depend on λ.) This expression is to be substituted in (3.104) to determine the μ so as to orthonormalize the basis. The calculation is by no means trivial and success exacts close attention. First consider the case $\lambda \neq \lambda'$.

Propitious use of the differential equation (3.85) yields the relation

$$\int_a^b ds\ 2s^{m+1}\ {}^{\iota}y_s(s,\lambda)\ {}^{\iota'}y_s(s,\lambda')$$

$$= \frac{2s^{m+1}}{(\lambda-\lambda')}\ (\Delta(\lambda)\ {}^{\iota}y_s(s,\lambda)\ {}^{\iota'}y(s,\lambda') - \Delta(\lambda')\ {}^{\iota'}y_s(s,\lambda')\ {}^{\iota}y(s,\lambda))\ \Big|_a^b \tag{3.106}$$

for intervals $a < s < b$ in which no singularity of the integrand exists. Taking note of the facts that

$$\Delta(s_i,\lambda) = 0 \tag{3.107}$$

and

$$\Delta(s_i(\lambda'),\lambda) = \Delta(s_i',\lambda) = \lambda - \lambda' \tag{3.108}$$

(for ease of notation introduce the abbreviation $s_i(\lambda') \equiv s_i'$ a straightforward though tedious calculation yields the result that the inner product (3.104) vanishes identically for $\lambda = \lambda'$ as expected. The continuum eigensolutions for different λ are mutually orthogonal. In fact, the inner product (3.104) is itself a generalized function (of $\lambda.\lambda'$) which defines a distribution with point support at $\lambda = \lambda'$; the inner product vanishes on the open set excluding the point $\lambda = \lambda'$. The *strength* of the distribution (the value of its mapping of the unit function), itself a function of λ, is the quantity needed for normalizing the function space. It is necessary there fore to evaluate the integral

$$N(\lambda;\iota,\iota') = \int_{\lambda_L}^{\lambda_U} d\lambda' \int_0^1 ds\ 2s^{m+1}\ {}^{\iota}y_s(s,\lambda)\ {}^{\iota'}y_s(s,\lambda') \tag{3.109}$$

According to the previous discussion, the integral n can be reduced to an integral over the restricted range $\lambda - \delta\lambda < \lambda' < \lambda + \delta\lambda$ where $\delta\lambda$ is small, and fixed. Separate the integral over s into pieces which (respectively) include (exclude) singularities of the integrand. Clearly, the latter vanish for $\delta\lambda$ chosen small enough since the s integration produces only finite results. The only surviving contributions come from the neighborhood of the singular point(s):

$$N(\lambda;\iota,\iota') = \int_{\lambda-\delta\lambda}^{\lambda+\delta\lambda} d\lambda' \int_{s_1-\delta s_1}^{s_1+\delta s_1} ds \;\; 2s^{m+1} \;\; {}^{\iota}y_s(s,\lambda) \;\; {}^{\iota'}y_s(s,\lambda')$$

$$+ \int_{\lambda-\delta\lambda}^{\lambda+\delta\lambda} d\lambda' \int_{s_2-\delta s_2}^{s_2+\delta s_2} ds \;\; 2s^{m+1} \;\; {}^{\iota}y_s(s,\lambda) \;\; {}^{\iota'}y_s(s,\lambda') \tag{3.110}$$

It is crucial to realize the limiting process intended in (3.110) is such as to insure, for all singular points, that

$$|s_i - s_i'| < |\delta s_i| \tag{3.111}$$

so no matter what the value of $\delta\lambda$ chosen, the points $s = s_i(\lambda)$ and $s = s_i(\lambda')$ are always included in the s integral. The only remaining contributions to N can be written explicitly as follows:

$$N(\lambda;\iota,\iota') = \int_{s_1(\lambda-\delta\lambda)}^{s_1(\lambda+\delta\lambda)} ds_1 \; \frac{d\lambda'}{ds_1'}\Big|_\lambda \int_{s_1-\delta s_1}^{s_1+\delta s_1} ds \; 2s^{m+1} \; \frac{B_{01}^2}{(s-s_1)(s-s_1')} + {}^{\iota}\mu_1 {}^{\iota'}\mu_1' \delta(s-s_1)\delta(s-s_1')$$

$$+ \int_{s_2(\lambda-\delta\lambda)}^{s_2(\lambda+\delta\lambda)} ds_2 \; \frac{d\lambda'}{ds_2'}\Big|_\lambda \int_{s_2-\delta s_2}^{s_2+\delta s_2} ds \; 2s^{m+1} \; \frac{(B_{02}+{}^{\iota}\mu_1 B_{12})(B_{02}+{}^{\iota'}\mu_1' B_{12})}{(s-s_2)(s-s_2')} + {}^{\iota}\mu_2 {}^{\iota'}\mu_2' \delta(s-s_2)\delta(s-s_2') \tag{3.112}$$

Here B_{ij} is the coefficient of the logarithmic series in y_i, (3.98), at the j^{th} singular point, (3.97).

It is a result of the theory of singular integral equations a so known as the Poincare-Bertrand Theorem, that for $\zeta = \varepsilon/\xi$ and in the limit as ε, $\xi \to 0$:

$$\int_{\tau-\xi}^{\tau+\xi} d\omega \int_{\tau-\varepsilon}^{\tau+\varepsilon} d\sigma \; \frac{1}{(\sigma-\tau)(\omega-\sigma)} = \pm\, \pi^2 \tag{3.113}$$

according as $\zeta \to 0$(upper sign) or $\zeta \to \infty$(lower sign). The integrals are understood in the sense of Cauchy principle value integrals. Adapting this result to evaluate (3.112) it is found that

$$N(\lambda;\iota,\iota') = 2\beta\kappa^2\sqrt{\lambda}\Big(\frac{s_1^{m+1}}{\sqrt{\lambda-1}}(B_{01}^2\pi^2 + {}^{\iota}\mu_1 {}^{\iota'}\mu_1)$$

$$+ \frac{s_2^{m+1}}{\sqrt{\lambda+1}}((B_{02}+{}^{\iota}\mu_1 B_{12})(B_{02}+{}^{\iota'}\mu_1 B_{12})\pi^2 + {}^{\iota}\mu_2 {}^{\iota'}\mu_2) \tag{3.114}$$

In the nondegenerate continuum, (3.114) is simplified by taking $B_{12} = 0$ and either $B_{02} = 0$ or $B_{01} = 0$ according to whether the singular point is respectively decreasing or increasing with λ.

In the nondegenerate continuum, the μ_i are determined immediately by the integration algorithm (3.103). Thus $N(\lambda)$ can be evaluated directly using these values as

$$N(\lambda) = \frac{2\beta\kappa^2\sqrt{\lambda}}{\sqrt{\lambda}\mp 1}\, s_i^{m+1}\left(B^2_{0i}\pi^2 + \left(\frac{y_0(1,\lambda)}{y_i(1,\lambda)}\right)^2\right) \tag{3.115}$$

The eigensolutions (3.99) are then normalized simply by dividing through by the square root of $N(\lambda)$.

The degenerate continua subspace eigensolutions are orthonormalized by an only slightly more complicated calculation. Recall the ${}^{\iota}\mu_i$ are related linearly through (3.102). Any choice of ${}^{\iota}\mu_1$ and ${}^{\iota'}\mu_1$ determines ${}^{\iota}\mu_2$ and ${}^{\iota'}\mu_2$. This then also specifies a basis set which spans the subspace. The idea is to choose so as to orthonormalize the basis: this is accomplished by requiring both

$$N(\lambda;\iota,\iota') = N(\lambda)\ \delta(\iota,\iota') \tag{3.116}$$

and

$$N(\lambda;\iota,\iota) = N(\lambda;\iota',\iota') \tag{3.117}$$

where $\delta(\iota,\iota')$ is the Kronecker delta. The first of these requirements assures the subspace is orthogonal, the second orients the basis so that both elements have the same norm. Requirements (3.116) and (3.117) comprise an algebraic system in two unknowns whose solution results in the proper choices for ${}^{\iota}\mu_1$ and ${}^{\iota'}\mu_1$:

$$\mu_1 = \frac{-k_1 \pm \sqrt{(k_2k_3-k_1^2)}}{k_2}$$

$$k_1 = B_{02}B_{12}\pi^2Y_2^2 + Y_0Y_1$$

$$k_2 = (B_{12}^2\pi^2+C)Y_2^2 + Y_1^2$$

$$k_3 = (B_{01}^2C+B_{02}^2)\pi^2Y_2^2 + Y_0^2$$

$$Y_0 = y_0(1,\lambda)\ ,\ Y_1 = y_1(1,\lambda)\ ,\ Y_2 = y_2(1,\lambda)$$

$$C \equiv \frac{A}{B} = \frac{2s_1^{m+1}\beta\kappa^2\dfrac{\sqrt{\lambda}}{\sqrt{\lambda}-1}}{2s_2^{m+1}\beta\kappa^2\dfrac{\sqrt{\lambda}}{\sqrt{\lambda}+1}} = \left(\frac{s_1}{s_2}\right)^{m+1}\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} = \frac{(\sqrt{\lambda}+1)((\sqrt{\lambda}-1)^2-\kappa^2(1-\beta))^{m+1}}{(\sqrt{\lambda}-1)((\sqrt{\lambda}+1)^2-\kappa^2(1-\beta))^{m+1}} \tag{3.118}$$

Using these values the normalization constant (it is the same for both elements of the basis because of the chosen orientation) $N(\lambda)$ can be evaluated as

$$N(\lambda) = 2A(\frac{k_3}{k_2} + \pi^2(B_{01}^2(1 - \frac{Y_2^2}{k_2}\frac{A}{B}) + \frac{B}{A}\frac{(B_{02}Y_1 - B_{12}Y_0)^2}{k_2})) \tag{3.119}$$

The orthonormalization of the degenerate continuum subspace basis is essentially complete. There remains only to construct the eigensolutions with the prescription (3.101) using the values of $'\mu_1$ specified by (3.118), then divide the construct by the square root of $N(\lambda)$. Figures 9-13 show some of the generalized eigenstreamfunctions. These figures show the inverse tangent of some multiple of the actual generalized functions. The multiplicative factor was chosen (after orthonormalization) for ease of viewing. An interesting feature of the orthogonalized degenerate solutions (see figs. 10-11) is that they are nearly equal (or equal and of opposite sign) in the regions interior to and exterior to the singularities. There thus exist motions of the equilibrium nearly entirely contained within one of those regions.

A complete and orthonormalized set of generalized functions have now been defined. Any of the possible motions of the gyroelastic screwpinch (any motion in the Hilbert space) can be evolved by means of an invertible integral transform over the space of functions elucidated above.

The Integral Theorem

Given an initial perturbation $\xi(s,\theta,z,0)$ and its time derivative $\dot{\xi}(s,\theta,z,0)$, find the subsequent motion of the system. Since $\nabla\cdot\xi = 0$, represent ξ as

$$\xi = \nabla^{*}\Phi \tag{3.120}$$

and restrict attention to the study of the stream function Φ.

First Fourier transform Φ and expand the resulting Fourier amplitude in eigenstreamfunctions:

$$\varphi_{km}(s,0) = \frac{1}{2\pi}\int d\theta' e^{-im\theta'}\int dz' e^{-ikz'}\Phi(s,\theta',z',0) = \sum_n a_{km}(\lambda_n)\psi_{km}(s,\lambda_n) \tag{3.121}$$

For the present the sum over nonsingular subspectra and integrals over relevant continua is represented *symbolically* by Σ.

Next use the orthonormality of the $\psi_{km}(s,\lambda_n)$. Deal with a single mode of definite k and m. From this point on subscripts k and m are to be understood attached to every ψ. The following orthonormality conditions apply:

$$\int_0^1 ds\ (\frac{m^2}{2s}\ \psi(s,\lambda_n)\psi(s,\lambda_{n'}) + 2s\psi_s(s,\lambda_n)\psi_s(s,\lambda_{n'})) = \delta(n,n')$$
(3.122)

for λ_n ($\lambda_{n'}$) members of the point spectrum;

$$\int_0^1 ds\ (\frac{m^2}{2s}\ \psi(s,\lambda)\psi(s,\lambda') + 2s\psi_s(s,\lambda)\psi_s(s,\lambda')) = \delta(\lambda-\lambda')$$
(3.123)

for λ (λ') members of the nondegenerate continua;

$$\int_0^1 ds\ (\frac{m^2}{2s}\ {}^i\psi(s,\lambda)^{i'}\psi(s,\lambda') + 2s\,{}^i\psi_s(s,\lambda)^{i'}\psi_s(s,\lambda')) = \delta(\lambda-\lambda')\delta(i,i')$$
(3.124)

for members of the degenerate continua. An equivalent expression of the orthonormality of the ψ can be obtained by integrating the second term by parts in the above inner product integrals. There results

$$\int_0^1 ds\ \psi(s,\lambda_{n'})(\frac{m^2}{2s}\ \psi(s,\lambda_n) - (2s\psi_s(s,\lambda_n))_s) = \delta(n,n')$$
(3.125)

for the nonsingular eigenfunctions and analogous expressions for the singular eigenfunctions.

Project (3.121) onto an element of the complete set ψ

$$(\varphi,\psi(\lambda_n)) = \sum_{n} a(\lambda_{n'})\ (\psi(\lambda_n),\psi(\lambda_{n'})) = \sum_{n'} a(\lambda_{n'})\ \delta(n,n') = a(\lambda_n)$$
(3.126)

Likewise, expressions involving the singular functions can be constructed. (Closure can be established by an extension of this calculation.)

Finally, proceed to write the entire expansion theorem in detail as

$$\Phi(s,\theta,z,t) = \frac{1}{4\pi^2}\sum_{m=1}^{m=\infty} e^{im\theta}\int_{-\infty}^{+\infty} dk\ e^{ikz}$$

$$\times(\sum_n \psi(s,\lambda_n)\ (\psi,\mathcal{F}) + \int_{C_I} d\lambda\ \psi(s,\lambda)\ (\psi,\mathcal{F}) + \int_{C_{II}} d\lambda\ ({}^1\psi(s,\lambda)\ ({}^1\psi,\mathcal{F}) + {}^2\psi(s,\lambda)\ ({}^2\psi,\mathcal{F}))$$
(3.127)

where C_I denotes the nondegenerate continuum and C_{II} denotes the (doubly) degenerate continuum. The integral referred to as $\mathcal{I}$ in this expression is given as

$$\mathcal{I}(s,\omega(\lambda)t) = \int_0^{2\pi} d\theta\ \epsilon^{im\theta} \int_{-\infty}^{+\infty} dz\ \epsilon^{ikz}\ (\Phi(s,\theta,z,0)\cos\omega t + \dot{\Phi}(s,\theta,z,0)\ \frac{\sin\omega t}{\omega}) \tag{3.128}$$

The coup-de-grace is administered by simply evaluating

$$\xi(s,\theta,z,t) = \nabla^{*}\Phi(s,\theta,z,t) \tag{3.129}$$

CHAPTER 4

Free Boundary Layer Structure

The Magnetohydrodynamic Discontinuity

The linear motions of an isorrhopic gyroelastic equilibrium with fixed boundaries were characterized in the preceeding chapter. An invertible integral transform over the space of generalized eigenfunctions was developed to evolve arbitrary (admissible) perturbations of the equilibrium. It was found that the presence of (uniform) gyroelasticity in the fixed boundary system translates the spectra of eigenvalues. This effect one might justifiably call gyroelastic stabilization. Consequently, the static (nonrotating) fixed boundary equilibria studied can be classified as being gyroelastically stabilizable; the parameter Ω_g^2, the so called gyroelastic frequency, need only be chosen large enough to stabilize the eigenmode corresponding to the lowest value of ω^2 (see figure 14.) It was found that this lowest value of ω^2 occurs for $m = 1$ and a value of helicity κ between $-1/\sqrt{(1-\beta)} < \kappa < -1$.

The free boundary system is much richer in spectral variation; it exhibits unique and intriguing properties which could hardly be predicted on the basis of the fixed boundary results alone. As the fixed boundary constraint is relaxed, the torpor of the placid spectral pond is shattered by an eruption from the proximity of the Kruskal–Shafranov point. For free boundary modes it is found that the linear behavior (of the columnar screwpinch) for small vacuum gaps between plasma and wall is dominated by eigenmodes which have no spectral counterpart in the fixed boundary system.

In this chapter, the boundary conditions joining the isorrhopic gyroelastic fluid to a surrounding vacuum are derived. It is through this calculation that access is gained to the global dispersion relation for free boundary motions.

The task is accomplished with the aid of a mathematical construct: a boundary layer whose thickness will be subsequently allowed to shrink to an infinitesmal and whose structure is chosen for convenience. First choose the boundary layer to obey the equations of ordinary magnetohydrodynamics (no gyroelasticity.) Later, the effects of finite gyroelasticity will be reinstated. The gyroelastic ordering will be imposed presently, but not at the outset.

In the absence of finite-angular-momentum-density effects the equations of ordinary magnetohydrodynamics can be written as follows (for the moment adopt standard 3 vector notation): Equilibrium, force balance.

$$\mathbf{J}\times\mathbf{B} = \nabla p \tag{4.1}$$

Linearized equation of motion

$$-\rho\omega^2\xi = \delta\mathbf{J}\times\mathbf{B} + \mathbf{J}\times\delta\mathbf{B} - \delta\nabla p \tag{4.2}$$

Incompressibility constraint:

$$\nabla\cdot\xi = 0 \tag{4.3}$$

Ohm's law:

$$\delta\mathbf{E} = i\omega\xi\times\mathbf{B} \tag{4.4}$$

Maxwell's equations:

$$\nabla\times\delta\mathbf{B} = \mu_0\delta\mathbf{J} - i\frac{\omega}{c^2}\delta\mathbf{E} \tag{4.5}$$

$$\nabla\cdot\delta\mathbf{B} = 0 \tag{4.6}$$

$$\nabla\times\delta\mathbf{E} = i\omega\delta\mathbf{B} \tag{4.7}$$

and finally

$$\nabla\cdot\delta\mathbf{E} = \frac{1}{\chi_0}\epsilon \tag{4.8}$$

In standard manner, Fourier analyze the linear motion in θ, z, t and deal only with a mode of definite m, k, ω by using the ansatz $exp(i(m\theta+kz-\omega t))$ for all first order quantities. (The linearized equations then relate the amplitudes of the linearized quantities.) I will refer to the preceeding vector relations (4.2), (4.4), (4.5) and (4.7) by appending subscripts when only a certain component is referenced. For example, I will use $(4.2)_\theta$ to denote the θ-component of the equation of motion.

The equilibrium current is given by

$$\mu_0\mathbf{J} = -B_z'\mathbf{e}_\theta + \frac{1}{r}(rB_\theta)'\mathbf{e}_z = \nabla\times\mathbf{B}_0 \tag{4.9}$$

where $'$ indicates differentiation with respect to r, radius (Subscripts refer to components.) Stipulating $\mathbf{B}_0$ then of course determines ∇p also.

The natural boundary condition at the plasma-vacuum interface for the linearized system is that the first order pressure be continuous across the (perturbed) interface. For the free boundary system this can be written in the equivalent forms (set $\mu_0 = \chi_0 = 1$)

$$\Delta(\frac{1}{2}\mathbf{B}^2 + p) = \Delta(\frac{1}{2}\hat{\mathbf{B}}^2)$$
$$\Delta\kappa = \Delta\hat{\kappa}$$
$$[\Delta\kappa] = 0 \tag{4.10}$$

Circumflex (^) will denote quantities in the vacuum and square brackets [ ] surrounding a quantity will denote jumps in the quantity across the plasma-vacuum interface. The Lagrangian variation operator Δ, defined in Chapter II, retains its original meaning here.

Proceed by first calculating $\Delta\kappa$ (the Lagrangian pressure variation inside the perturbed boundary. Then calculate the analogous quantity in the vacuum, outside the perturbed boundary, $\Delta\hat{\kappa}$. Equating these quantities as in (4.10) will result in a relation between the radial component of the displacement and its derivative at the boundary : $\xi_r(a), \xi_r'(a)$

Expanding $\Delta\kappa$ (from (4.10)) find

$$\Delta\kappa = \mathbf{B}\cdot\Delta\mathbf{B} + \Delta p = B_z(\delta B_z + \xi_r B_z') + B_\theta(\delta B_\theta + \xi_r B_\theta') + \delta p + \xi_r p' \tag{4.11}$$

Since, in equilibrium,

$$\mathbf{e}_r\cdot\nabla(p + \frac{1}{2}\mathbf{B}^2) = \mathbf{e}_r\cdot(\mathbf{B}\cdot\nabla\mathbf{B}) \tag{4.12}$$

(4.11) can be expressed as

$$\Delta\kappa = \delta\kappa - \frac{1}{r}B_\theta^2\xi_r \tag{4.13}$$

Now use $(4.7)_\theta$, $(4.7)_z$ to write $\delta\mathbf{B}$ as

$$\delta\mathbf{B} = iB_z(k+m\frac{\tau}{r})\boldsymbol{\xi} - r(\frac{B_\theta}{r})'\xi_r\mathbf{e}_\theta - B_z'\xi_r\mathbf{e}_z \tag{4.14}$$

and eliminate δp in (4.13) with $(4.2)_z$; $\delta\kappa$ is thus represented by

$$\delta\kappa = \frac{1}{i\omega}(B_z\frac{1}{r}(r\delta E_\theta)' + i(kB_\theta - \frac{m}{r}B_z)\delta E_r - B_\theta\delta E_z') + \frac{i}{k}(j_\theta\delta B_r - \delta j_r B_\theta - \omega^2\rho\xi_z) \tag{4.15}$$

Next eliminate δj_r with $(4.5)_r$, eliminate j_θ with (4.9) and again use (4.7) to eliminate (the components of) $\delta\mathbf{B}$ in favor of (the components of) $\delta\mathbf{E}$ and its derivative. In the resulting expression, apply (4.4) to replace $\delta\mathbf{E}$ and finally replace $(r\xi_r)'$ using (4.3). There remains an epiphanous expression of $\Delta\kappa$ in which the only infinitesmals appearing are the components of $\boldsymbol{\xi}$:

$$\Delta\kappa = \frac{i}{k}B_z^2(\xi_\theta\tau\frac{\omega^2}{c^2} + \xi_z((k+m\frac{\tau}{a})^2 - \frac{\rho\omega^2}{B_z^2})) - \frac{1}{a}B_\theta^2\xi_r \tag{4.16}$$

All quantities in (4.16) are to be evaluated at $r = a$ and as usual we use the symbol τ to represent $\tau = B_\theta/B_z$.

It is significant to point out here that ξ_z has previously not appeared in any calculation. That's alright. Don't fret. Eliminate it. The method follows along the lines of the preceeding calculation. Eliminate δp by taking the proper linear combination of the transverse components of the linearized equation of motion $(4.2)_\theta$,$(4.2)_z$. Systematically replace $\delta\mathbf{B}$ and then $\delta\mathbf{E}$ as in the calculation leading to (4.16). The result (evaluated at $r = a$) is

$$\xi_z = \xi_\theta\frac{k\,\mathcal{Y} + B_z^2(k+m\frac{\tau}{a})\frac{\omega^2}{c^2}}{\frac{m}{a}\mathcal{Y} + B_z^2\tau(k+m\frac{\tau}{a})\frac{\omega^2}{c^2}} + \xi_r\frac{2iB_z^2k(k+m\frac{\tau}{a})\frac{\tau}{a}}{\frac{m}{a}\mathcal{Y} + B_z^2\tau(k+m\frac{\tau}{a})\frac{\omega^2}{c^2}} \tag{4.17}$$

where the quantity $\mathcal{Y}$ is defined as

$$\mathcal{Y} = \rho\omega^2 - B_z^2(k+m\frac{T}{a})^2 \tag{4.18}$$

With (4.17) to express ξ_z and neglecting all but the lowest order terms (in the gyroelastic ordering), (4.16) can be revamped to become

$$\Delta\kappa = -i\frac{a}{m}\xi_\theta(\rho\omega^2 - B_z^2(k+m\frac{T}{a})^2) + \xi_r B_z^2 2\frac{T}{m}(k+m\frac{T}{a}) - B_z^2\frac{T^2}{a}\xi_r \tag{4.19}$$

where ρ in the total (electromagnetic) mass density

$$\rho = \rho_m + \frac{B_z^2}{c^2} \tag{4.20}$$

Another useful result can be produced by substituting (4.17) for ξ_z in (4.3). To lowest order, the result is the familiar

$$\xi_\theta = \frac{i}{m}(a\xi_r' + \xi_r) \tag{4.21}$$

as expected in the gyroelastic ordering.

Forging ahead, proceed to calculate $\Delta\hat{\kappa}$, the Lagrangian variation of the magnetic pressure in the vacuum. Maxwell's equations must be solved in the (helically rippled) annular domain between the plasma and the wall. Once having found the fields in this region, $\Delta\hat{\kappa}$ can be evaluated as

$$\Delta\hat{\kappa} = \hat{\mathbf{B}}\cdot\Delta\hat{\mathbf{B}} = \hat{\mathbf{B}}_0\cdot(\delta\hat{\mathbf{B}} + \boldsymbol{\xi}\cdot\nabla\hat{\mathbf{B}}_0) \tag{4.22}$$

where $\hat{\mathbf{B}}_0$ is the vacuum equilibrium field and $\delta\hat{\mathbf{B}}$ solves the wave equation

$$(\nabla^2 - \frac{1}{c^2}\partial_t^2)\delta\hat{\mathbf{B}} = 0 \tag{4.23}$$

in the vacuum, subject to the constraint

$$\nabla\cdot\delta\hat{\mathbf{B}} = 0 \tag{4.24}$$

The system (4.23),(4.24) admits two classes of solution: (1) those in which the magnetic vector potential $\delta\hat{\mathbf{A}}$ is longitudinal, so that $\delta\hat{B}_z = 0$ and (2) those in which the vector potential is purely transverse so that $\delta\hat{B}_z \neq 0$. The wave field in the annular region is a linear combination of these solutions.

Let

$$\beta = K_m(\alpha r) + c_0 I_m(\alpha r) \tag{4.25.1}$$

and

$$\underset{\sim}{\beta} = K_m(\alpha r) + \underset{\sim}{c}_0 I_m(\alpha r) \tag{4.25.2}$$

where $c_0 \neq \underset{\sim}{c}_0$, K_m and I_m are hyperbolic Bessel functions of order m and α satisfies

$$\alpha^2 = k^2 - \frac{\omega^2}{c^2} \tag{4.26}$$

The vector potential $\delta\hat{\mathbf{A}}^{(3)}$ is then given by

$$\delta\hat{\mathbf{A}}^{(3)} = c_1\{\delta\hat{\mathbf{a}},0\} + c_2\{0,\delta\hat{A}\} \tag{4.27}$$

where the quantities $\delta\hat{\mathbf{a}}$ and $\delta\hat{A}$ are defined by

$$\delta\hat{\mathbf{a}} = \left(\frac{1}{ik\alpha r}\beta\mathbf{e}_r + \frac{1}{mk}\beta'\mathbf{e}_\theta\right)e^{i(m\theta+kz-\omega t)} \tag{4.28.1}$$

and

$$\delta\hat{A} = \frac{1}{im\alpha}\beta\mathbf{e}_z e^{i(m\theta+kz-\omega t)} \tag{4.28.2}$$

Defining the quantities

$$\delta\hat{\mathbf{b}} = -\frac{i}{m}\beta'\mathbf{e}_r + \frac{1}{\alpha r}\beta\mathbf{e}_\theta \;;\; \delta\hat{\underset{\sim}{\mathbf{b}}} = -\frac{i}{m}\underset{\sim}{\beta}'\mathbf{e}_r + \frac{1}{\alpha r}\underset{\sim}{\beta}\mathbf{e}_\theta \tag{4.29.1}$$

and

$$\delta\hat{B} = \frac{\alpha}{km}\beta \;;\; \delta\hat{\underset{\sim}{B}} = \frac{\alpha}{km}\underset{\sim}{\beta} \tag{4.29.2}$$

the fixed point variation of the vacuum magnetic field amplitude can be expressed as

$$\delta\hat{\mathbf{B}}^{(3)} = c_1\{\delta\hat{\underset{\sim}{\mathbf{b}}},\delta\hat{\underset{\sim}{B}}\} - \frac{\omega^2}{c^2\alpha^2}c_2\{\delta\hat{\mathbf{b}}^*,0\} \tag{4.30}$$

Maxwell's equations then yield the fixed point variation of the electric field $\delta\hat{\mathbf{E}}^{(3)}$ as

$$\delta\hat{\mathbf{E}}^{(3)} = \frac{\omega}{k}\left(c_1\{\delta\hat{\underset{\sim}{\mathbf{b}}}^*,0\} + \frac{k^2}{\alpha^2}c_2\{\delta\hat{\mathbf{b}},\delta\hat{B}\}\right) \tag{4.31}$$

The coefficients c_1, c_2, c_0 and $\mathfrak{c}_0$ are to be determined so as to satisfy boundary conditions at both an exterior conducting (rigid) wall bounding the vacuum region, and at the helically deformed plasma–vacuum interface.

At the conducting wall, $r = a$ and it is required that

$$\boldsymbol{n}^{(3)}\cdot\delta\hat{\boldsymbol{B}}^{(3)} = 0 \tag{4.32}$$

and

$$\boldsymbol{n}^{(3)}\times\delta\hat{\boldsymbol{E}}^{(3)} = 0 \tag{4.33}$$

be satisfied. $\boldsymbol{n} = \boldsymbol{e}_r$ is a unit normal to the wall. A cursory examination of (4.30) and (4.31) reveals that

$$\beta(\alpha a_w) = \beta'(\alpha a_w) = 0 \tag{4.34}$$

satisfies (4.32) and (4.33). This determines c_0 and c_0 so as to render

$$\beta(\alpha r) = K_m(\alpha r) - \frac{K'_m(\alpha a_w)}{I'_m(\alpha a_w)}\, I_m(\alpha r) \tag{4.35.1}$$

and

$$\beta(\alpha r) = K_m(\alpha r) - \frac{K_m(\alpha a_w)}{I_m(\alpha a_w)}\, I_m(\alpha r) \tag{4.35.2}$$

as the correct combinations of Bessel functions to be used.

To get $\Delta\hat{\kappa}$, it is yet necessary to evaluate coefficients c_1 and c_2. This can be done by integrating the equations whose linearized forms are given by (4.4) and (4.6) across a thin transition layer bounding the plasma at the plasma–vacuum interface. The result is the requirement that the normal magnetic field be continuous across the boundary

$$\boldsymbol{n}\cdot[\boldsymbol{B}] = 0 \tag{4.36}$$

and the jump in tangential electric field (due to the motion of the boundary) be given by

$$\boldsymbol{n}\times[\boldsymbol{E}] = \boldsymbol{n}\cdot\boldsymbol{v}[\boldsymbol{B}] \tag{4.37}$$

These two equations are now varied to give relations which obtain at the actual (helically deformed) interface. From (4.36) there derives

$$\Delta\boldsymbol{n}\cdot[\boldsymbol{B}] + \boldsymbol{n}_0\cdot[\Delta\boldsymbol{B}] = 0 \tag{4.38}$$

and from (4.37) find

$$\mathbf{n}_0\times[\![\mathbf{E}]\!] = \mathbf{n}_0\cdot\mathbf{v}[\![\mathbf{B}]\!] \tag{4.39}$$

The perturbed surface normal is obtained by noting

$$\mathbf{n} = -\nabla(r-r_0-\xi_r) = \mathbf{e}_r(-1+\xi_r') + \mathbf{e}_\theta(i\frac{m}{a}\xi_r) + \mathbf{e}_z ik\xi_r = \mathbf{n}_0 + \Delta\mathbf{n} \tag{4.40}$$

Form the scalar product of (4.39) with the equilibrium magnetic field $\mathbf{B}_0$. The requirement that the paralell electric field vanish within the plasma then gives the result

$$B_\theta\delta\hat{E}_\theta + B_z\delta\hat{E}_z = -i\omega\xi_r(B_z[\![B_\theta]\!] - B_\theta[\![B_z]\!]) \tag{4.41}$$

Furthermore, by (4.14), inside the plasma it is required that

$$\delta B_r = i\frac{m}{a}\xi_r B_\theta + ik\xi_r B_z \tag{4.42}$$

This in concert with (4.38) guarantees that outside the interface

$$\delta\hat{B}_r = i\frac{m}{a}\xi_r\hat{B}_\theta + ik\xi_r\hat{B}_z \tag{4.43}$$

Equations (4.41) and (4.43) form a simultaneous linear algebraic system in the coefficients c_1 and c_2 whose solution is

$$\begin{aligned} c_1 &= -\xi_r\frac{1}{\beta'}(\hat{B}_\theta\frac{m^2k^2}{\alpha^2 a} + \hat{B}_z km) \\ c_2 &= i\xi_r\frac{1}{\beta}(\hat{B}_\theta\alpha m) \end{aligned} \tag{4.44}$$

Finally, $\Delta\hat{\kappa}$ can be written as

$$\Delta\hat{\kappa} = -\xi_r\hat{B}_z^2(\frac{\beta}{\beta'}(\hat{\tau}\frac{2km}{\alpha a} + \hat{\tau}^2\frac{k^2m^2}{\alpha^3a^2} + \alpha) - \frac{\beta'}{\beta}\hat{\tau}^2\frac{\omega^2}{\alpha c^2} + \hat{\tau}^2\frac{1}{a}) \tag{4.45}$$

Enforcing the gyroelastic ordering and also taking the limit $ka_w \ll 1$ for simplicity, (4.45) reduces further to

$$\Delta\hat{\kappa} = \xi_r\hat{B}_z^2(\frac{a}{m}\Upsilon((k+m\frac{\hat{\tau}}{a})^2 - \frac{\omega^2}{c^2}) - \hat{\tau}^2\frac{1}{a}) \tag{4.46}$$

where

$$\Upsilon = - \frac{1 + (\frac{a_w}{a})^{2m}}{1 - (\frac{a_w}{a})^{2m}} \tag{4.47}$$

Equating the expressions (4.19) and (4.46) as required by (4.10) (first order pressure balance) the ordinary magnetohydrodynamic dispersion relation (within the gyroelastic ordering, but without gyroelasticity) appears as

$$\Delta\kappa = \Delta\hat{\kappa}$$

$$- i\frac{a}{m}\xi_\theta(\rho\omega^2 - B_z^2(k+m\frac{T}{a})^2 + \xi_r(2\frac{T}{m}(k+m\frac{T}{a}))) - B_z^2\xi_r T^2\frac{1}{a}$$
$$= \xi_r\hat{B}_z^2(\frac{a}{m}\Upsilon((k+m\frac{\hat{T}}{a})^2 - \frac{\omega^2}{c^2}) - \hat{T}^2\frac{1}{a}) \tag{4.48}$$

The Gyroelastic Sliding Discontinuity

The condition (4.48) characterizes the discontinuity at the interface between an MHD fluid within the gyroelastic ordering and a vacuum. However, we wish to generalize the condition to include the effect of finite gyroelasticity. To accomplish this, it is necessary to determine the character of a general sliding discontinuity between two different isorrhopic gyroelastic fluids. The sliding discontinuity allows a relative sliding of mass and fluid on opposite sides, but not across the discontinuity. It coincides with an isorrhope.

Once this task is completed, the trick will be to meld the two types of discontinuity into a single structure representing the transition layer between the gyroelastic medium and the vacuum. The result will be a global dispersion relation for motions of the free boundary gyroelastic screwpinch.

Begin this part of the analysis by projecting the nonlinear equation of motion (2.36) in the characteristic representation onto a surface normal $\boldsymbol{n}$

$$\boldsymbol{n} = \sigma\boldsymbol{e} + \zeta\boldsymbol{e}_z \tag{4.49}$$

where $\boldsymbol{e}$ is parallel to the direction of ∇s and ζ is of order ε relative to σ. The result can be cast in the form

$$\rho D^{\pm}(\boldsymbol{n}\cdot\boldsymbol{v}^{\mp}) - Q\,\partial_\chi(\boldsymbol{n}\cdot\boldsymbol{\tau}) + \boldsymbol{n}\cdot\nabla\kappa - \rho D^{\pm}\boldsymbol{n}\cdot\boldsymbol{v}^{\mp} - Q\,\partial_\chi\boldsymbol{n}\cdot\boldsymbol{\tau} = 0 \tag{4.50}$$

A discontinuity can be modeled mathematically as a limit. A region of finite thickness, a transition layer in which continuous changes occur, is caused to shrink (the properties of adjoining regions being maintained throughout.) The limit of the process is a discontinuity. Some quatities interior to the transition layer remain bounded throughout the procedure, some do not. Let us signify that a variable remains bounded by

$$F < \infty \tag{4.51}$$

In particular, it is evident that the following quantities fit the description of quantities which remain bounded:

$$s, D^{\dagger}s, \boldsymbol{n}, \nabla\boldsymbol{n}, D^{\dagger}\boldsymbol{n}, \tau, \partial_z\boldsymbol{n}, \partial_\chi\boldsymbol{n} < \infty \tag{4.52}$$

We also require that the zeroth order pressure be continuous across the discontinuity (sharp boundary effects will be discussed in Appendix II.) Noting also that owing to the isorrhopy of the configuration, $\boldsymbol{n}\cdot\tau = 0$, it can be shown from (4.50) that the quantity

$$\rho\ D^{\dagger}(\boldsymbol{n}\cdot\boldsymbol{v}^{\dagger}) + \boldsymbol{n}\cdot\nabla\kappa < \infty \tag{4.53}$$

is also bounded in the transition layer. Since also $D^{\dagger}\rho = 0$ and $\nabla\cdot\boldsymbol{v}^{\dagger} = 0$, (4.53) can be rewritten as

$$D^{\dagger}(\rho(\boldsymbol{n}\cdot\boldsymbol{v}^{\tau})) + \nabla\cdot\boldsymbol{n}\kappa - \kappa\nabla\cdot\boldsymbol{n} < \infty \tag{4.54}$$

(within the transition layer.) As an immediate consequence of this, it follows that

$$\partial_t(\rho(\boldsymbol{n}\cdot\boldsymbol{v}^{\mp})) + \nabla\cdot(\boldsymbol{v}^{\dagger}\rho\boldsymbol{n}\cdot\boldsymbol{v}^{\mp} + \boldsymbol{n}\kappa) < \infty \tag{4.55}$$

$\boldsymbol{v}^{\pm}$ differ only in their tangential components. Let $\boldsymbol{n}\cdot\boldsymbol{v}^{\dagger} = u$ and integrate (4.55) across the transition layer to find

$$-u^2[\rho] + u^2[\rho] + [\kappa] = 0 \tag{4.56}$$

The inescapable concusion is that κ is continuous across the layer

$$[\kappa] = 0 \tag{4.57}$$

This relation might be viewed as the gyroelastic generalization of (4.10), keeping in mind that the κ in the two cases are related, but not identical. The κ in (4.57) is given by (1.83.5). To be more precise, κ is given by (4.10) to lowest order ($O(\varepsilon^0)$); (1.83.5) gives the next lowest non-vanishing correction to κ. It belongs to the mystique of the gyroelastic regime that perturbations to both contributions to the pressure enter at the same order in the analysis of the boundary layer structure.

Continue the calculation by projecting the equation of motion (2.36) onto an isorrhope, noting since

$$[\rho D^t \mathbf{v}^t - Q \partial_\chi \tau + \nabla\kappa] = 0 \tag{4.58}$$

then also

$$\nabla^* s \cdot [\rho D^t \mathbf{v}^t - Q \partial_\chi \tau] = -\nabla^* s \cdot [\nabla\kappa] = -\nabla^* s \cdot \nabla[\kappa] = 0 \tag{4.59}$$

Now vary a state in which there exists such a sliding discontinuity and examine the behavior of the variation at the discontinuity. Since (4.59) holds quite generally, it holds in particular for the varied state and thus also for the (Lagrangian) variation of the state:

$$\Delta(\nabla^* s \cdot [(\rho D^t \mathbf{v}^t - Q \partial_\chi \tau)]) = \Delta[\nabla^* s \cdot (\rho D^t \mathbf{v}^t - Q \partial_\chi \tau)] = 0 \tag{4.60}$$

Making use of the identities

$$\Delta\nabla^* s = \nabla^* s \cdot \nabla\xi \ ; \ \Delta\nabla - \nabla\Delta = 0 \ ; \ \Delta s = 0 \tag{4.61}$$

the linearized jump condition can be expressed as

$$[\nabla^* s \cdot \nabla\xi \cdot (\rho D^t \mathbf{v}^t - Q \partial_\chi \tau)] + [\nabla^* s \cdot (\rho D^t D^t \xi - Q \partial_\chi \partial_\chi \tau)] = 0 \tag{4.62}$$

For the case of the steady flow equilibrium considered in chapter III, making use of the obvious condition

$$\mathbf{n} \cdot [\xi] = 0 = \nabla s \cdot [\xi] \tag{4.63}$$

(4.62) can be reduced to the form

$$\frac{i}{m}\xi_r[\rho(\omega^2 + m^2\Omega_g^2 - (\omega - \Omega^{(0)})^2) + B^2((k + m\frac{T}{a})^2 - k^2)]$$
$$+ [\xi_\theta(\rho((\omega - m\Omega^{(0)})^2 - m^2\Omega_g^2) - B^2(k + m\frac{T}{a})^2)] = 0 \tag{4.64}$$

where all quatities are to be evaluated at $r = a$.

Since ξ_r is continuous it is necessary only to distinguish ξ_θ on the two sides of the sliding discontinuity. Identify the ξ_θ in (4.48) as occuring outside the discontinuity. ((4.21) is valid on both sides of the discontinuity; it can be used to eliminate ξ_θ inside the discontinuity in favor of ξ_r and ξ_r'.) Finally then, using (4.64) in (4.48) the generalized boundary condition can be written as a condition on the logarithmic derivative of ξ_r at (immediately inside) the surface $r = a$. The result is

$$\frac{\xi_r'}{\xi_r} = -\left(\frac{1}{a}\right)\frac{\omega^2\left(\rho+m\Upsilon\frac{B^2}{c^2}\right) - B^2\left(\left(k+m\frac{\tau}{a}\right)^2(1+m\Upsilon)-2\frac{m}{a}\tau\left(k+m\frac{\tau}{a}\right)\right)}{\rho(\omega^2-m^2\Omega_g^2) - B^2\left(k+m\frac{\tau}{a}\right)^2} . \tag{4.65}$$

The boundary condition is to be used as follows: integrate the Euler equation out to the plasma-vacuum interface and compute the logarithmic derivative of the Euler solution–its value coincides with that given by the relation (4.65) in the event that the value of ω^2 is an eigenvalue (and the solution is an eigensolution.)

The trajectories thus traced in κ–λ space represent the global dispersion relation, for free boundary motions of the gyroelastic system. Figures 15 and 16 show results of this calculation. In figure 15 the vacuum gap is allowed to increase from zero producing a range of unstable modes near the *Kruskal–Shafranov* point. In this case the speed of light has been taken to be negligibly large relative to the Alfven speed. The fixed boundary (model) pinch (top pictures) appears to be gyroelasti[illegible] stabilizable only as a consequence of the fixed boundary model assumptions As the limit of a free boundary system (ie. for vanishingly small vacuum g[illegible] [illegible] pinch is not stabilized. In the latter case the range of unstable modes shinks to a point at $\kappa=-1$. Figure 16 shows that increasing the Alfven speed (relative to the speed of light) gives rise to (displacement current) effects which tend to decrease the growth rates of the unstable modes, though never actually stabilizing them.

CHAPTER 5

The Energy Principle

The Columnar Pinch

The equation of motion describing linear motions of a gyroelastic screwpinch equilibrium was derived in chapter III from a small amplitude Lagrangian. Minimizing the action integral

$$\delta \mathfrak{I} = \int dr \ r \ (f\xi'^2 + g\xi^2) = \int dr \ \mathcal{L}(\xi, \xi'; r) = 0 \tag{5.1}$$

led to the Euler-Lagrange equation

$$\left(\frac{\partial \mathcal{L}}{\partial \xi'}\right)' - \frac{\partial \mathcal{L}}{\partial \xi} = 0 \tag{5.2}$$

which can also be written as

$$(rf\xi')' - rg\xi = 0 \tag{5.3}$$

To get an energy principle for the free boundary system, simply form the scalar product of (5.3) with ξ then integrate over the domain. After an integration by parts this procedure yields

$$\int_0^a dr(rf\xi'^2 + rg\xi^2) = \Big|_0^a rf\xi'\xi \tag{5.4}$$

The coefficient functions f and g are given by

$$f = \rho r^2((\frac{\omega}{m} - \Omega^{(0)})^2 - \frac{\mathbb{Q}}{4\rho^2} - \frac{Q}{\rho}(\frac{k}{m} + \frac{\tau}{r})^2) \tag{5.5}$$

and

$$g = (m^2-1)\rho((\frac{\omega}{m} - \Omega^{(0)})^2 - \frac{\mathbb{Q}}{4\rho^2} - \frac{Q}{\rho}(\frac{k}{m} + \frac{\tau}{r})^2) - r(\rho(\frac{\omega}{m})^2 - Q(\frac{k}{m})^2)' \tag{5.6}$$

The class of equilibria chosen previously consisted of the additional restrictions:

$$p_\parallel = p_\dagger \;\; ; \; Q = B^2 \tag{5.7}$$

isotropic (material) pressure,

$$\frac{\mathbb{Q}}{4\rho^2} = \Omega_g^2 = constant \tag{5.8}$$

uniform gyroelastic frequency and

$$\Omega^{(0)} = 0 \tag{5.9}$$

zero canonical (angular) velocity. The system is thus self adjoint. Lastly, the equilibrium was chosen to have a uniform current density. With these provisos, the coefficient functions f and g can be expressed as

$$f = r^2(\rho((\frac{\omega}{m})^2 - \Omega_g^2) - B_a^2(\frac{k}{m}u + \frac{\tau_a}{a})^2) \tag{5.10}$$

and

$$g = (m^2-1)(\rho((\frac{\omega}{m})^2 - \Omega_g^2) - B_a^2(\frac{k}{m}u + \frac{\tau_a}{a})^2) - r(\rho(\frac{\omega}{m})^2 - B_a^2u^2(\frac{k}{m})^2)' \tag{5.11}$$

where $u^2(r)$ is the (normalized) profile of magnetic pressure $B^2(r)$ (subscript $_a$ indicates the variable is to be evaluated at the plasma-vacuum boundary $r = a$.) Substitute these last expressions into the energy principle (5.4); there then remains only to make use of the boundary condition derived in chapter IV to complete the calculation.

The analysis of the boundary layer disclosed

$$-\int_0^a r^3\ \xi\xi' = -\,a^3\xi^2(a)\ \frac{\xi'(a)}{\xi(a)} = a^2\xi^2\Lambda \tag{5.12}$$

where

$$\Lambda = \frac{\omega^2(\rho+m\Upsilon\frac{B_a^2}{c^2}) - B_a^2((k+m\frac{T}{a})^2(1+m\Upsilon)-2\frac{m}{a}\tau(k+m\frac{T}{a}))}{\rho(\omega^2-m^2\Omega_g^2) - B_a^2(k+m\frac{T}{a})^2} \tag{5.13}$$

Using Λ for the logarithmic derivative (of ξ) and owing to the self adjointness of the system guaranteed by (5.9) it is possible to collect all terms involving ω^2 (in (5.4).) Dividing through by the multiplier of ω^2 leaves the energy principle in the anticipated form

$$\omega^2 = \frac{\delta W(\xi,\xi')}{N(\xi,\xi')} \tag{5.14}$$

where numerator and denominator are given as

$$\delta W = \int_0^a dr\ r((\rho m^2\Omega_g^2 + B_a^2(ku+m\frac{T}{a})^2)(r^2\xi'^2 + (m^2-1)\xi^2) - B_a^2k^2u^{2\prime}r\xi^2) + a^2\xi_a^2B_a^2(k+m\frac{T}{a})((k+m\frac{T}{a})(1+m\Upsilon)-2m\frac{T}{a}) \tag{5.15}$$

and

$$N = \int_0^a dr\ r(\rho(r^2\xi'^2 + (m^2-1)\xi^2) - \rho' r\xi^2) + a^2\xi_a^2(\rho_a + m\Upsilon\ \frac{B_a^2}{c^2}) \tag{5.16}$$

Some general comments might be made here on the nature of the above result. First, it is clear that N, the normalization, is a positive definite quantity, as it must be. It follows that for instability ($\omega^2 < 0$) the potential energy due to the presence of the perturbation ξ, W, must be negative. Furthermore, since the Euler solution minimizes the value of ω^2 any trial function giving $\omega^2 < 0$ proves the presence of instability and estimates an upper bound on the value of ω^2 (for the fastest growing eigenmode.)

Secondly, it can be seen from (5.15) that any amount of gyroelastic stabilization can be nullified by simply choosing a trial function such that $\xi' = 0$ and $m = 1$. Such a perturbation would be a reasonable choice as a trial function for estimating maximum growth rates. Before analyzing the stability of the columnar pinch, however, let us digress momentarily to generalize the equilibrium slightly so as to include the tubular pinch. An analysis of the stability of the tubular pinch will then include the columnar pinch as a special case.

The Tubular Pinch[25]

Consider a gyroelastic system in which $p_{\perp} = p_{\parallel} = p$ and the equilibrium condition

$$p + \frac{1}{2}B^2 = \frac{1}{2}B_0^2 \tag{5.17}$$

is satisfied. The geometry is as shown. Take the equilibrium configuration to obey

$$B(a_i) = B(a_e) = B_0 \tag{5.18}$$

so that

$$p(a_i) = p(a_e) = 0 \tag{5.19}$$

Allow an axial current density to flow in the central conductor as well as in the plasma so that the poloidal magnetic field B_θ is given by

$$\begin{aligned}
B_\theta &= B_{\theta i}\frac{a_i}{r} && a_{wi} < r < a_i \\
B_\theta &= B_{\theta i}\frac{a_i}{r} + B_{\theta p} && a_i < r < a_e \\
B_\theta &= \left(B_{\theta i}\frac{a_i}{a_e} + B_{\theta p}(a_e)\right)\frac{a_e}{r} \equiv B_{\theta e}\frac{a_e}{r} && a_e < r
\end{aligned} \tag{5.20}$$

For convenience of notation, define the following quantities:

$$\tau_i \equiv \tau(a_i) = \frac{B_\theta(a_i)}{B_0} = \frac{B_{\theta i}}{B_0} \tag{5.21}$$

and

$$\tau_e \equiv \tau(a_e) = \frac{B_\theta(a_e)}{B_0} = \frac{B_{\theta e}}{B_0} \tag{5.22}$$

By a calculation entirely equivalent to that leading to the boundary condition for the columnar pinch presented in chapter IV, it is found that the condition which must be satisfied at the inner vacuum–plasma interface is

$$\Delta\hat{\kappa}_i = \Delta\kappa_i \tag{5.23}$$

where

$$\Delta\hat{\kappa}_i = \xi_i B_0^2\left(-\frac{a_i}{m}\lambda_i\left(\left(k+m\frac{T_i}{a_i}\right)^2-\frac{\omega^2}{c^2}\right) - \frac{T_i^2}{a_i}\right) \tag{5.24}$$

and

$$\begin{aligned}\Delta\kappa_i = \quad & \xi_i' \frac{a_i^2}{m^2}\left(\rho(\omega^2-m^2\Omega_g^2) - B_0^2\left(k+m\frac{T_i}{a_i}\right)^2\right) \\ & + \xi_i\left(\frac{a_i}{m^2}\left(\rho\omega^2 - B_0^2\left(k+m\frac{T_i}{a_i}\right)^2\right) + B_0^2 2\frac{T_i}{m}\left(k+m\frac{T_i}{a_i}\right) - B_0^2\frac{T_i^2}{a_i}\right)\end{aligned} \tag{5.25}$$

The logarithmic derivative of ξ at $r = a_i$ is thus required (for ξ an eigenmode) to satisfy

$$\frac{\xi_i'}{\xi_i} = -\frac{1}{a_i}\left(\frac{\omega^2\left(\rho-m\lambda_i\frac{B_0^2}{c^2}\right) - B_0^2\left(\left(k+m\frac{T_i}{a_i}\right)^2(1-m\lambda_i)-2m\frac{T_i}{a_i}\left(k+m\frac{T_i}{a_i}\right)\right)}{\rho(\omega^2-m^2\Omega_g^2) - B_0^2\left(k+m\frac{T_i}{a_i}\right)^2}\right) \tag{5.26}$$

where

$$1 \leq \lambda_i = \frac{1+\left(\frac{a_{wi}}{a_i}\right)^{2m}}{1-\left(\frac{a_{wi}}{a_i}\right)^{2m}} \leq \infty \tag{5.27}$$

At the exterior plasma–vacuum interface the mate to (5.26) is given by

$$\frac{\xi_e'}{\xi_e} = -\frac{1}{a_e}\left(\frac{\omega^2\left(\rho+m\Upsilon_e\frac{B_0^2}{c^2}\right) - B_0^2\left(\left(k+m\frac{T_e}{a_e}\right)^2(1+m\Upsilon_e)-2m\frac{T_e}{a_e}\left(k+m\frac{T_e}{a_e}\right)\right)}{\rho(\omega^2-m^2\Omega_g^2) - B_0^2\left(k+m\frac{T_e}{a_e}\right)^2}\right) \tag{5.28}$$

where

$$1 \leq \Upsilon_e = \frac{(\frac{a_{we}}{a_e})^{2m} + 1}{(\frac{a_{we}}{a_e})^{2m} - 1} \leq \infty \tag{5.29}$$

By following the procedure outlined previously for generating an energy principle for the columnar pinch in the form (5.14) the analogous relation for the tubular pinch can be formed. The result is

$$\omega^2 = \frac{\delta W(\xi,\xi')}{N(\xi,\xi')} \tag{5.30}$$

where now the potential energy due to the presence of the perturbation is

$$\begin{aligned}\delta W = \int_{a_i}^{a_e} dr\ r\ &((\rho m^2\Omega_g^2 + B_0^2(ku + m\frac{\Upsilon_e}{a_e}h)^2)(r^2\xi'^2 + (m^2-1)\xi^2) - rB_0^2k^2u^{2\prime}\xi^2) \\ &+ a_e^2\xi_e^2B_0^2(k^2 - m^2\frac{\Upsilon_e^2}{a_e^2} + m\Upsilon_e(k + m\frac{\Upsilon_e}{a_e})^2) \\ &- a_i^2\xi_i^2B_0^2(k^2 - m^2\frac{\Upsilon_i^2}{a_i^2} - m\lambda_i(k + m\frac{\Upsilon_i}{a_i})^2)\end{aligned} \tag{5.31}$$

and the normalization is

$$\begin{aligned}N = \int_{a_i}^{a_e} dr\ r(\rho(r^2\xi'^2 + (m^2-1)\xi^2) - \rho' r\xi^2) \\ + a_e^2\xi_e^2(\rho_e + m\Upsilon_e\frac{B_0^2}{c^2}) - a_i^2\xi_i^2(\rho_i - m\lambda_i\frac{B_0^2}{c^2})\end{aligned} \tag{5.32}$$

The two functions u and h in (5.31) describe the particular equilibrium profiles chosen as follows: the axial field is to be specified as

$$B(r) = B_0 u \tag{5.33}$$

In order to satisfy (5.18) and (5.19) u must obey

$$u(a_i) = u(a_e) = 1 \tag{5.34}$$

and

$$0 < u(r) < 1 \qquad\qquad a_i < r < a_e \tag{5.35}$$

The poloidal field is given by (5.20); this defines h as follows:

$$B_\theta = B_0 \frac{\tau_e}{a_e} r h(r) \qquad\qquad a_i < r < a_e \tag{5.36}$$

The only requrements on $h(r)$ are that

$$h(a_e) = 1 \tag{5.37}$$

and

$$h(a_i) = \frac{\tau_i a_e}{\tau_e a_i} \tag{5.38}$$

Otherwise, the particular form of h is determined by the choice of axial current density profile (within the plasma.)

Stability Of The Gyroelastic Screwpinch

As discussed earlier in connection with the energy principle for the columnar pinch, gyroelastic stabilization can be completely nullified by choosing a rigid body displacement for a trial function, with $m = 1$. This is true for the tubular pinch as well. Consider the energy principle (5.30) for such a displacement to get an estimate of the maximum growth rate to which the unstable system will be prone. The perturbation potential energy, W, given by (5.31), can be represented for this trial function as

$$\begin{aligned}\delta W_{trial} = \int_{a_e}^{a_e} dr\ r^2(-B^2k^2)'\xi_0^2 \\ + a_e^2\xi_e^2B_0^2(k^2-m^2\frac{\tau_e^2}{a_e^2} + m\Upsilon_e(k+m\frac{\tau_e}{a_e})^2) \\ - a_i^2\xi_i^2B_0^2(k^2-m^2\frac{\tau_i^2}{a_i^2} - m\lambda_i(k+m\frac{\tau_i}{a_i})^2)\end{aligned} \tag{5.39}$$

where ξ_0 will be specified presently.

Two questions arise immediately with regards to (5.39). First, can ξ_i and ξ_e differ? Second, can ξ be constant and meet the boundary conditions? The second question can be dispensed with easily by choosing a trial function which meets the boundary conditions differing from constancy only in a thin region of thickness ε near the boundary. The contribution to W from these regions can be made as small as is desired by choosing ε small. The boundary contributions remain unchanged.

In response to the first query, consider a trial function which is uniform on the domains $a_i < r < a_s-\varepsilon$ and $a_s+\varepsilon < r < a_e$, but has a different value on the two domains, say ξ_i and ξ_e. The result of substituting this trial function in (5.39) is a positive contribution at $r = a_s$ of order ε^{-1}. Clearly, the case $\xi_0 = \xi_i = \xi_e$ is least stable. Therefore, choose $\xi_0 = 1$. The energy principle is then simply

$$\omega^2 \leqslant \frac{- B_0^2k^2 \int_{a_i}^{a_s} dr\ r^2u^{2\prime} + a_e^2B_0^2(k^2-m^2\frac{\tau_e^2}{a_e^2} + m\Upsilon_e(k+m\frac{\tau_e}{a_e})^2) - a_i^2B_0^2(k^2-m^2\frac{\tau_i^2}{a_i^2} - m\lambda_i(k+m\frac{\tau_i}{a_i})^2)}{-\int_{a_i}^{a_s} dr\ r^2\rho' + a_e^2(\rho_e+ m\Upsilon_e\frac{B_0^2}{c^2}) - a_i^2(\rho_i- m\lambda_i\frac{B_0^2}{c^2})} \tag{5.40}$$

Now select an example. First examine the standard columnar pinch case dealt with (as a fixed boundary system) in chapter III.

Case I

The standard columnar pinch is specified by

$$\begin{aligned} \rho &= \rho_0 \\ u^2 &= 1 - \beta(1-(\frac{r}{a_e})^2) \ ; \ u^{2\prime} = 2\beta\frac{r}{a_e^2} \\ a_i &= 0 \end{aligned} \tag{5.41}$$

Applying (5.41) to (5.40) there results (with $m = 1$)

$$\omega^2 \leqslant \frac{\delta W_{trial}}{N_{trial}} = \frac{B_0^2}{\rho_0}\ \frac{k^2(1-\frac{\beta}{2}) - \frac{\tau_e^2}{a_e^2} + \Upsilon_e(k+\frac{\tau_e}{a_e})^2}{1 + \Upsilon_e\frac{B_0^2}{\rho_0c^2}} \tag{5.42}$$

To find an estimate for the range of k for which instability occurs, set the right hand member of (5.42) to zero and solve for the two roots $k_\pm$ of the quadratic. These two roots bound the unstable range of k values (estimated by the chosen trial function.) It is thus estimated that modes with $m = 1$ and k in the range defined by

$$k_\pm = \frac{\tau_e\Upsilon_e}{a_e(1 + \Upsilon_e - \frac{\beta}{2})}\left(- 1 \pm \frac{(\frac{\beta}{2}(\Upsilon_e-1) + 1)^{\frac{1}{2}}}{\Upsilon_e}\right) \tag{5.43}$$

are unstable. Clearly, as Υ_e increases without limit the range shrinks to zero width about the Kruskal-Shafranov point

$$\frac{ka_e}{\tau_e} = -1 \tag{5.44}$$

The estimate of maximum growth rate ω_m^2 occurs for the value of k minimizing (5.42)

$$k_m = -\frac{\Upsilon_e \tau_e}{a_e(1 + \Upsilon_e - \frac{\beta}{2})} \tag{5.45}$$

Evaluating ω_m^2 there results

$$\omega_m^2 = - \frac{\frac{B_0^2\tau_e^2}{\rho_0 a_e^2}}{1 + \Upsilon_e \frac{B_0^2}{\rho_0 c^2}} \left(1 - \frac{\Upsilon_e(1 - \frac{\beta}{2})}{(\Upsilon_e + 1 - \frac{\beta}{2})}\right) \tag{5.46}$$

It is significant to notice that as Υ_e increases (as the outer wall approaches the plasma-vacuum boundary) k_m approaches the Kruskal-Shafranov point and the maximum growth rate goes to

$$\lim_{\Upsilon_e \to\infty} \omega_m^2 = - \frac{\frac{B_0^2\tau_e^2}{\rho_0 a_e^2}}{1 + \Upsilon_e \frac{B_0^2}{\rho_0 c^2}} \frac{\beta}{2} \tag{5.47}$$

Note that this growth rate is proportional to the parameter β.

CaseII

Let us now turn to the tubular pinch. After an integration by parts (5.40) becomes

$$\omega^2 \leq \frac{2 B_0^2 k^2 \int_{a_i}^{a_e} dr\, r u^2 - B_0^2 k^2 (a_e^2 - a_i^2) + a_e^2 B_0^2 (k^2 - m^2 \frac{\tau_e^2}{a_e^2} + m\Upsilon_e (k + m\frac{\tau_e}{a_e})^2) - a_i^2 B_0^2 (k^2 - m^2 \frac{\tau_i^2}{a_i^2} - m\lambda_i (k + m\frac{\tau_i}{a_i})^2)}{2 \int_{a_i}^{a_e} dr\, r\rho + (a_e^2 m \Upsilon_e + a_i^2 m \lambda_i)\frac{B_0^2}{c^2}} \tag{5.48}$$

For simplicity, normalize the variables as follows: take

$$s = \frac{r^2}{a_e^2} \; ; \; s_i = \frac{a_i^2}{a_e^2} \; ; \; \eta = \frac{\tau_e a_i}{\tau_i a_e} \; ; \; \kappa = \frac{k a_e}{\tau_e} \tag{5.49}$$

and let

$$\rho_0 = \int_{s_i}^{1} ds \, \rho \tag{5.50}$$

Define α by

$$\int_{s_i}^{1} ds \, u^2 = \alpha(1 - s_i) \tag{5.51}$$

where $0 < \alpha < 1$. In terms of these variables, the energy principle for the tubular pinch, (5.48), assumes the form

$$\omega^2 \leqslant \frac{a\kappa^2 + b\kappa + c}{d} \tag{5.52}$$

where

$$\begin{aligned} a &= 1 - (1-s_i)(1-\alpha) + \Upsilon_e + s_i(\lambda_i - 1) \\ b &= 2\Upsilon_e + 2\eta s_i \lambda_i \\ c &= \Upsilon_e - 1 + \eta^2 s_i (1+\lambda_i) \\ d &= 1 + \frac{B_0^2}{\rho_0}(\Upsilon_e + s_i \lambda_i) \end{aligned} \tag{5.53}$$

Proceeding as before, estimate the minimum value of ω^2. The value of κ for which this occurs is

$$\kappa_m = -\frac{b}{2a} \tag{5.54}$$

so that ω_m^2 is given by

$$\omega_m^2 = \frac{-\frac{b^2}{4a} + c}{d} = \frac{-\frac{(\Upsilon_e + \eta s_i \lambda_i)^2}{\alpha(1-s_i) + \Upsilon_e + s_i \lambda_i} + \Upsilon_e - 1 + \eta^2 s_i(\lambda_i + 1)}{1 + \frac{B_0^2}{\rho_0 c^2}(\Upsilon_e + s_i \lambda_i)} \tag{5.55}$$

For the tubular pinch ω_m^2 is not necessarily negative; indeed there do exist stabilized cases for regions of the parameter space mapped by η, Υ_e, λ_i, α, s_i and Ω_g^2.

REFERENCES

1. I.B.Bernstein, E.A.Frieman, M.D.Kruskal, and R.M.Kulsrud, *Proc. Roy. Soc.* A244 (1958), 17.
2. K.M.Case, *Ann. Phys.* 7 (1959), 349.
3. J.P.Freidberg, *Phys. Fluids* **13** (1970), 1812.
4. J.P.Freidberg, *Phys. Fluids* **15** (1972), 1102.
5. J.P.Freidberg and J.A.Wesson, *Phys. Fluids* **13** (1970), 1009.
6. E.Frieman and M.Rotenberg, *Rev. Mod. Phys.* **32** (1960), 898.
7. J.P.Goedbloed, *Physica* **53** (1971), 501.
8. J.P.Goedbloed, *Phys. Fluids* **16** (1973), 1927.
9. J.P.Goedbloed and P.H.Sakanaka, *CCO-3077-29,MF-70* (1973).
10. A.N.Kaufman, *Phys. Fluids* 3 (1960), 610.
11. N.A.Krall and L.D.Pearlstein, in *Plasma Physics and Controlled Nuclear Fusion Research* 1 (IAEA, Vienna, 1966), 735.
12. M.Kruskal, *J. Math. Phys.* 3 (1962), 806.
13. M.Kruskal and M.Schwarzchild, *Proc. Roy. Soc* 223A (1954), 348.
14. M.Kruskal and .Tuck, *Proc. Roy. Soc.* 245A (1958), 222.
15. L.Landau, *J. Phys. U.S.S.R.* 10 (1946), 25.
16. H.R.Lewis and K.R.Symon, *J. Math. Phys.* 20 (1979), 413.
17. H.R.Lewis and L.Turner, *Nuc. Fusion* 16 (1976), 993.
18. W.A.Newcomb, *Ann. Phys.* 3 (1958), 347.
19. W.A.Newcomb, *Ann. Phys.* 10 (1960), 232.
20. W.A.Newcomb, in "Dynamics of Fluids and Plasmas" (S.I.Pai, Ed.), p. 405 Academic Press, New York (1966)
21. W.A.Newcomb, *Ann. Phys.* 72 (1972), 29.
22. W.A.Newcomb, *Ann. Phys.* **81**(1973), 231.
23. W.A.Newcomb, "Notes On Magnetohydrodynamics" *unpublished*.
24. W.A.Newcomb "Notes On Guiding Center Theory" *unpublished*.
25. W.A.Newcomb and A.N.Kaufman, *Phys. Fluids* 4 (1961), 314.
26. L.D.Pearlstein and N.A.Krall, *Phys. Fluids* **9** (1966), 2231.
27. L.D.Pearlstein and J.P.Freidberg, *Phys. Fluids* 21 (1978), 1218.
28. K.V.Roberts and J.B.Taylor, *Phys. Rev. Letters* 8 (1962), 197.
29. D.C.Robinson, *Plasma Physics* 13 (1971), 439.
30. M.N.Rosenbluth, N.A.Krall, and N.Rostoker, *Nucl. Fusion, Suppl.* Pt. 1 (1962), 143.
31. M.N.Rosenbluth and A.Simon, *Phys. Fluids* **8** (1965), 1300.
32. C.E.Seyler, *Phys. Fluids* **22** (1979), 2324.
33. C.E.Seyler and J.P.Freidberg, *Phys. Fluids* **23** (1980), 331.
34. V.D.Shafranov, *Zh. Tekh. Fiz.* 40 (1970), 241; (*Sov. Phys. Tech. Phys.* **15** (1970), 175.)

35. B.R.Suydam, in *Proceedings Of The Second International Conference On The Peaceful Uses Of Atomic Energy* 31 (1958), 157.
36. R.J.Tayler, *Proc. Phys. Soc.* B70 (1957), 31.
37. R.J.Tayler, *Proc. Phys. Soc.* B70 (1957), 1049.
38. L.Turner, *Phys. Fluids* 20 (1977), 654.
39. L.Turner, *Phys. Fluids* 20 (1977), 662.
40. N.G.VanKampen, *Physica* 21 (1955), 949.

BIBLIOGRAPHY

B1. E.Hille, *Lectures On Ordinary Differential Equations*, Addison–Wesley, London (1969).

B2. E.L.Ince, *Ordinary Differential Equations*, Dover, New York (1944).

B3. S.G.Mikhlin, *Integral Equations*, Pergamon Press, New York (1957).

B4. N.I.Muskhelishvili, *Singular Integral Equations*, Noordhoff, Groningen (1953).

B5. B.W.Roos, *Analytic Functions and Distributions in Physics and Engineering*, Wiley, New York (1969).

B6. L.Schwartz, *Mathematics For The Physical Sciences*, Hermann, Paris (1966).

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APPENDIX I

The Poincare–Bertrand Theorem

The Plemelj formulae are used in the following derivation. If

$$F(z) = \frac{1}{2\pi i} \oint_{\mathscr{C}} d\sigma \; f(\sigma) \; \frac{1}{(\sigma - z)} \tag{A1.1}$$

for z not on the contour $\mathscr{C}$, then

$$F^{+}(\tau) - F^{-}(\tau) = f(\tau) \tag{A1.2}$$

and

$$F^{+}(\tau) + F^{-}(\tau) = \frac{1}{\pi i} \oint_{\mathscr{C}}^{0} d\sigma \; f(\sigma) \; \frac{1}{(\sigma - \tau)} \tag{A1.3}$$

or, equivalently

$$F^{+}(\tau) = + \frac{1}{2} f(\tau) + \frac{1}{2\pi i} \oint_{\mathscr{C}}^{0} d\sigma \; f(\sigma) \; \frac{1}{(\sigma - \tau)} \tag{A1.4}$$

and

$$F^{-}(\tau) = - \frac{1}{2} f(\tau) + \frac{1}{2\pi i} \oint_{\mathscr{C}}^{0} d\sigma \; f(\sigma) \; \frac{1}{(\sigma - \tau)} \tag{A1.5}$$

where the $\pm$ indicates a limit is to be taken as the point z (off the contour) approaches the point τ (on the contour) from inside (outside) the contour. Inside and outside are understood as left and right respectively if $\mathscr{C}$ is not closed. This is equivalent to using the contours $\mathscr{C}^0 + \mathscr{C}^{\pm}$ in the accompanying figure. If the singularity z is on the contour $\mathscr{C}$ is understood to be the contour $\mathscr{C}^0$ giving the principal part integral.

Define the following functions:

$$\Psi(z) = \oint^0 d\omega \oint^0 d\sigma\ \varphi(\omega)\ \frac{1}{(\omega-\sigma)(\sigma-z)} \tag{A1.6}$$

and

$$\Phi(z) = \oint^0 d\sigma\ \frac{1}{(\sigma-z)} \oint^0 d\omega\ \varphi(\omega)\ \frac{1}{(\omega-\sigma)} \tag{A1.7}$$

It can be shown that for z not on $\mathscr{C}$ $\Phi(z) = \Psi(z)$ since the singularity at τ has been removed. By partial fractions we write

$$\Psi(z) = \oint^0 d\omega\ \frac{1}{(\omega-z)} \oint^0 d\sigma\ \varphi(\omega)\ \left(\frac{1}{(\sigma-z)} - \frac{1}{(\sigma-\omega)}\right) = \oint^0 d\omega\ \psi(\omega,z)\ \frac{1}{(\omega-z)} \tag{A1.8}$$

Using the Plemelj formulae it can be shown that

$$\psi^+(\omega,\tau) - \psi^-(\omega,\tau) = 2\pi i\varphi(\omega) \tag{A1.9}$$

since

$$\oint d\sigma\ \varphi(\omega)\ \frac{1}{(\sigma-\omega)} \tag{A1.10}$$

doesn't ‹ end on the manner in which z approaches τ. Also

$$\psi^+(\omega,\tau) + \psi^-(\omega,\tau) = 2(c-\tau) \oint^0 d\sigma\ \varphi(\omega)\ \frac{1}{(\omega-\sigma)(\sigma-\tau)} \tag{A1.11}$$

Apply the Plemelj formulae (A1.4) and (A1.5) to (A1.8) thus obtaining

$$\Psi^+(t) = +\pi i\psi^+(\tau,\tau) + \oint^0 d\omega\ \psi^+(\omega,\tau)\ \frac{1}{(\omega-\tau)} \tag{A1.12}$$

and

$$\Psi^-(t) = -\pi i\psi^-(\tau,\tau) + \oint^0 d\omega\ \psi^-(\omega,\tau)\ \frac{1}{(\omega-\tau)} \tag{A1.13}$$

Add (A1.12) and (A1.13) and use (A1.9) and (A1.11) to cast the sum as

$$\Psi^{+}(\tau) + \Psi^{-}(\tau) = -\,2\pi^2\varphi(\tau) + 2 \oint^{0} d\omega \oint^{0} d\sigma\ \varphi(\omega)\ \frac{1}{(\omega-\sigma)(\sigma-\tau)} \tag{A1.14}$$

From the Plemelj formulae and (A1.7) it is found additionally that

$$\Phi^{+}(\tau) + \Phi^{-}(\tau) = 2 \oint^{0} d\sigma\ \frac{1}{(\sigma-\tau)} \oint^{0} d\omega\ \varphi(\omega)\ \frac{1}{(\omega-\sigma)} \tag{A1.15}$$

Finally, recognize that since $\Phi(z)$ and $\Psi(z)$ can be equated so also can the left hand sides of (A1.14) and (A1.15). The result is the Poincare-Bertrand Theorem

$$\pi^2\varphi(\tau) = \oint^{0} d\omega \oint^{0} d\sigma\ \varphi(\omega)\ \frac{1}{(\omega-\sigma)(\sigma-\tau)} - \oint^{0} d\sigma\ \frac{1}{(\sigma-\tau)} \oint^{0} d\omega\ \varphi(\omega)\ \frac{1}{(\omega-\sigma)} \tag{A1.16}$$

No particular restriction has been placed on the contour $\mathscr{C}$ in the above. In general the contribution of each term on the right in (A1.16) will depend on the particular contour chosen. We have a particular contour in mind, however, for use in determining the normalization constants for the singular generalized functions of Chapter III. The result is summarized in equation (3.113).

A more modern form of the above theorem can be stated simply in terms of the δ distribution as

$$\int d\sigma\ \frac{1}{(\omega-\sigma)(\sigma-\tau)} = \pi^2\delta(\omega-\tau) \tag{A1.17}$$

The advent of distribution theory was indeed an advance in mathematics.

APPENDIX II

The Sharp Boundary Screwpinch

Consider the case of a uniform plasma equilibrium with $p(r) = p_0$ inside a transition layer near the edge of the plasma. In the transition layer the pressure is brought smoothly to zero and the fields adjust so as to assure pressure balance in the equilibrium state. In the absence of gyroelastic effects the boundary condition is that for an MHD discontinuity such as was discussed in Chapter IV, namely equation (4.48).

In the transition layer (in the sharp boundary model) there exist gradients in quantities such as ρ, p, B, K, M and Q so that in general the canonical velocity $\mathbf{v}^{(0)}$ and the gyroelastic modulus $\mathbf{\Phi}$ do not vanish. Turner[38] uses the condition that the $\mathbf{e}_r\mathbf{e}_r$ stress at either side of the transition layer balance. Consider an alternative view in which $\Delta\kappa$ is to be determined near the outer edge of the transition layer by integrating the Euler equation through the layer. Balancing the result with $\Delta\hat{\kappa}$ (the generalized pressure in the vacuum) will then provide a dispersion relation. (The result is at variance with Turner's[38] due to a nonnegligible contribution to the balance of the $\mathbf{e}_\theta\mathbf{e}_\theta$ stress.) Thus we allow for contributions to the stress balance arising due to the presence of the layer itself. This is analogous to allowing for the presence of surface tension in dealing with a soap film bubble.

The Euler equation is as given in Chapter III, equations (3.22.1)–(3.22.3). As the layer is imagined to grow thinner so that the smooth transition of quantities approaches discontinuous behavior, those quantities proportional to gradients become large, of order h^{-1} where h is the layer thickness. Those quantities in this category are specifically $\Omega^{(0)}$, Ω_g (thus $\mathbf{\Phi}$) and Q'.

Integrating the Euler equation across the transition layer there results the jump condition

$$[f\xi'] = (m^2-1)a^2(-2m\omega\langle\rho\Omega^{(0)}\rangle + m^2\langle\rho\Omega^+\Omega^-\rangle)\xi\frac{h}{a} - a^2(\omega^2[\rho] - k^2[Q]) \tag{A2.1}$$

where

$$\langle\rho\Omega^{(0)}\rangle = \frac{1}{h}\int_{a-h}^{a} dr\ \rho\Omega^{(0)} \tag{A2.2}$$

and

$$\langle\rho\Omega^+\Omega^-\rangle = \frac{1}{h}\int_{a-h}^{a} dr\ \rho\Omega^+\Omega^- \tag{A2.3}$$

Sliding discontinuities of the variety discussed in Chapter IV occurring on either side of the transition layer annihilate each other (in the absence of gyroelasticity for $r < a-h$ owing to the uniform pressure equilibrium chosen.) The solution to the Euler equation for $r < a-h$ is well known (see Turner[38] for example) and the dispersion relation which obtains is

$$m(\rho\omega^2 - B^2q^2) + B^2(q^2-k^2) - \hat{B}^2(\hat{q}^2(1+m\Upsilon) - k^2 - 2m\frac{\hat{T}}{a}q - m\Upsilon\frac{\omega^2}{c^2})$$
$$+ (m^2-1)(-2m\omega\langle\rho\Omega^{(0)}\rangle + m^2\langle\rho\Omega^+\Omega^-\rangle)\frac{h}{a} = 0 \tag{A2.4}$$

where

$$q = k + m\frac{T}{a}\ ;\ \hat{q} = k + m\frac{\hat{T}}{a} \tag{A2.5}$$

and Υ was defined in Chapter IV.

There remains only to evaluate $\langle\rho\Omega^{(0)}\rangle$ and $\langle\rho\Omega^+\Omega^-\rangle$. For simplicity, choose $\Omega^+\Omega^-$ to vanish within the layer so that $\Omega_\sigma^2 = (\Omega^{(0)})^2$.

Choosing the gyrophase independent distribution function to be Maxwellian a short calculation yields the result

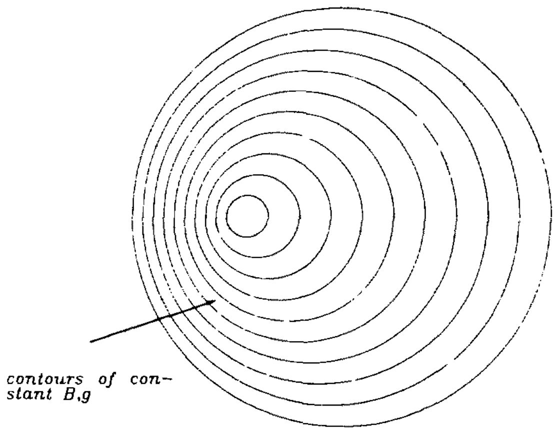
$$\langle\rho\Omega^{(0)}\rangle\frac{h}{a} = \frac{m}{2a^2e}\,\frac{p_0}{B_0\sqrt{(1-\beta_0)}} \tag{A2.6}$$

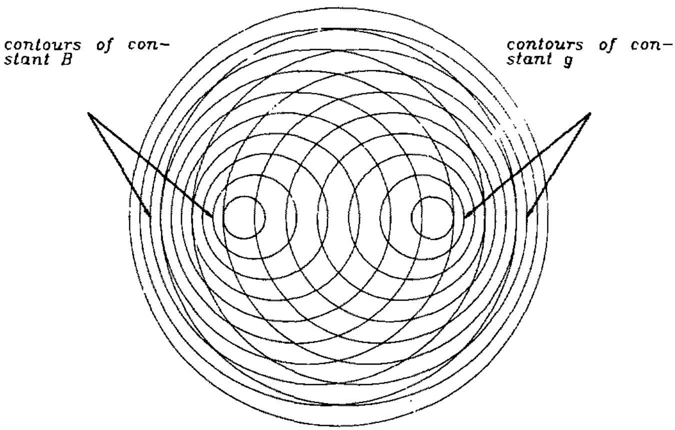
where

$$\beta_0 = 1 - \frac{B^2(r<a-h)}{B_0^2} \tag{A2.7}$$

Using this expression in (A2.4) yields a dispersion relation in agreement with that of Pearlstein and Freidberg[27].

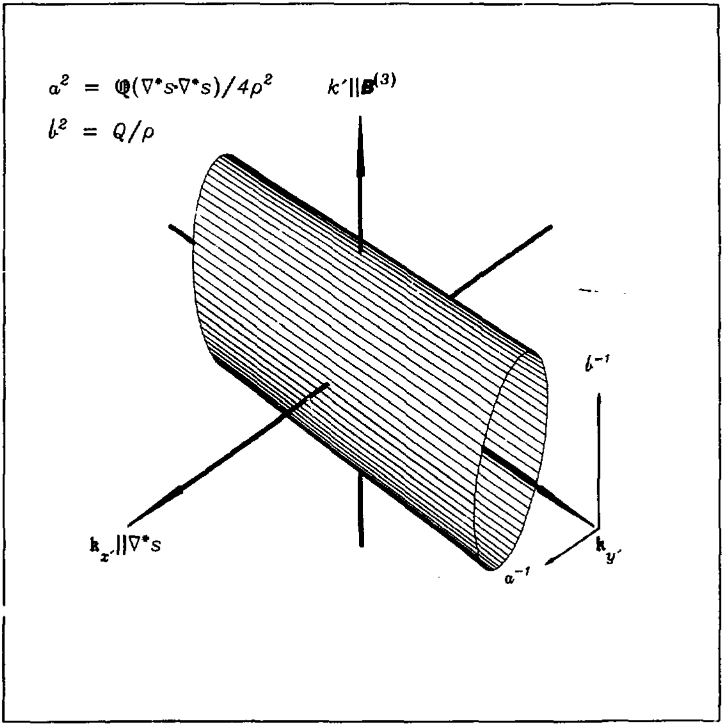
ISORRHOPIC FLUID CONTOURS





ANISORRHOPIC FLUID CONTOURS

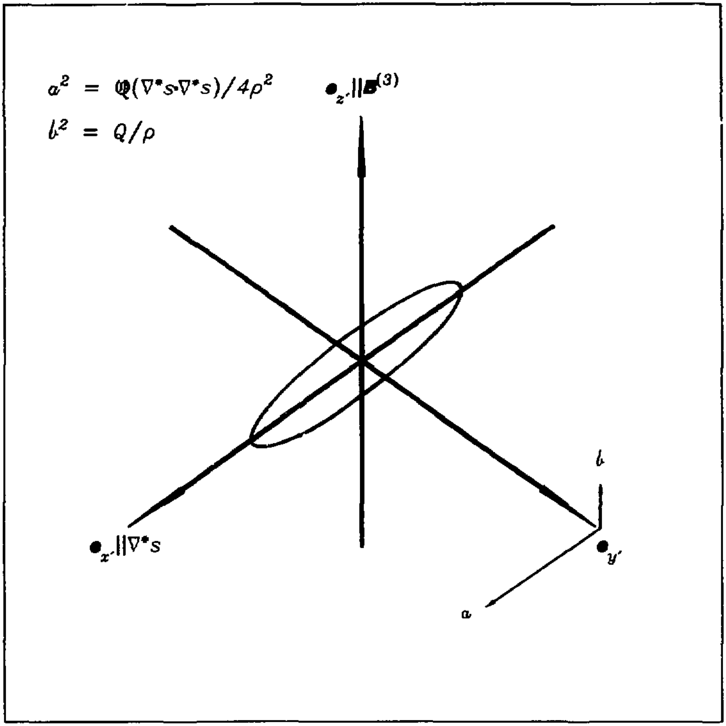
FIGURE 1



Normal Slowness Surface

in the space dual to $\mathcal{TM}(\mathcal{P})$

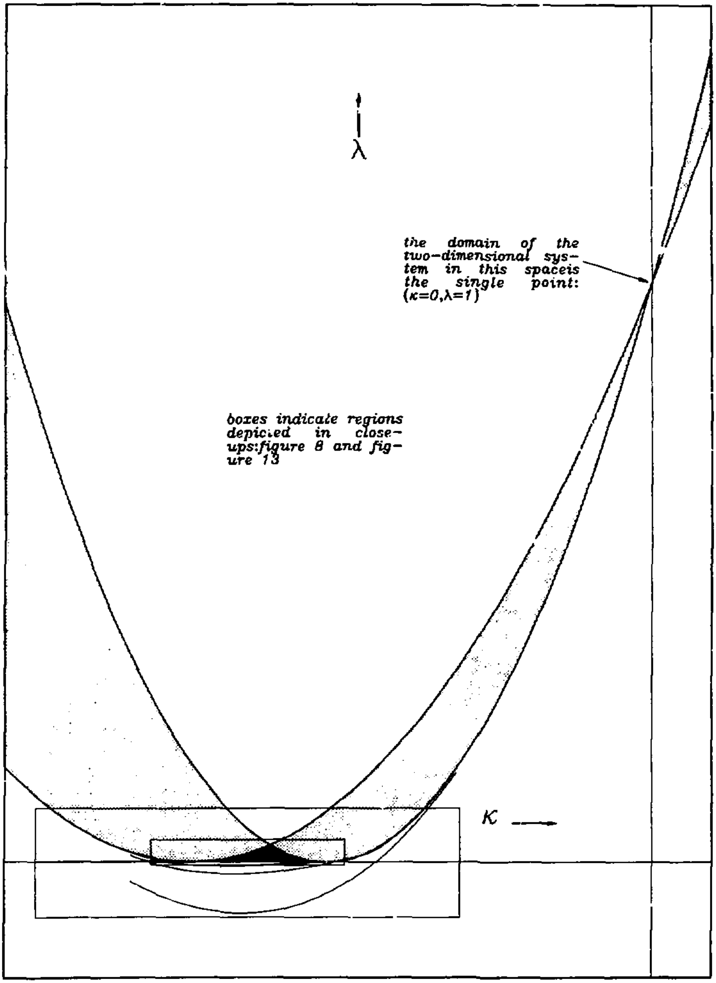
FIGURE 2



Ray Surface

in the space $\mathcal{TM}(\mathcal{P})$

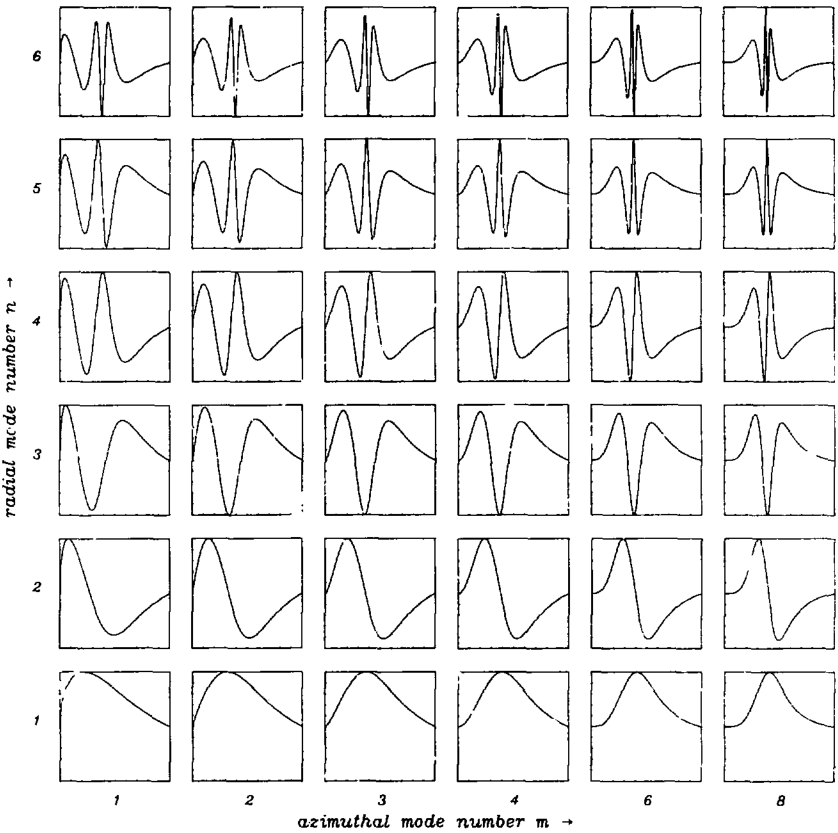
FIGURE 3



Global Dispersion Relation

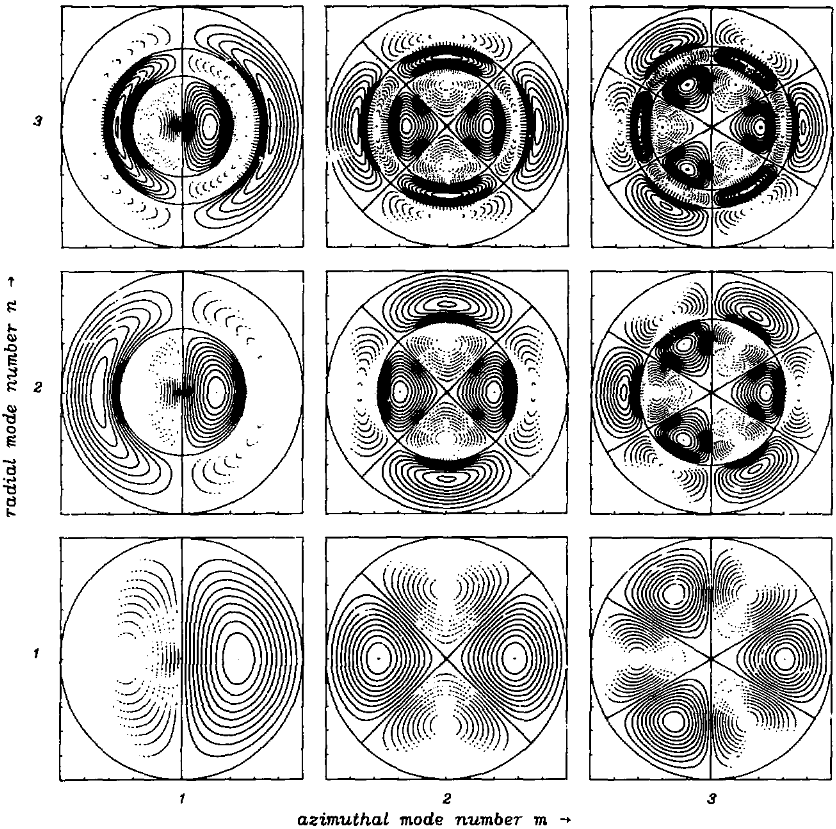
Fixed Boundary Motions Of The Gyroelastic Screwpinch

FIGURE 4



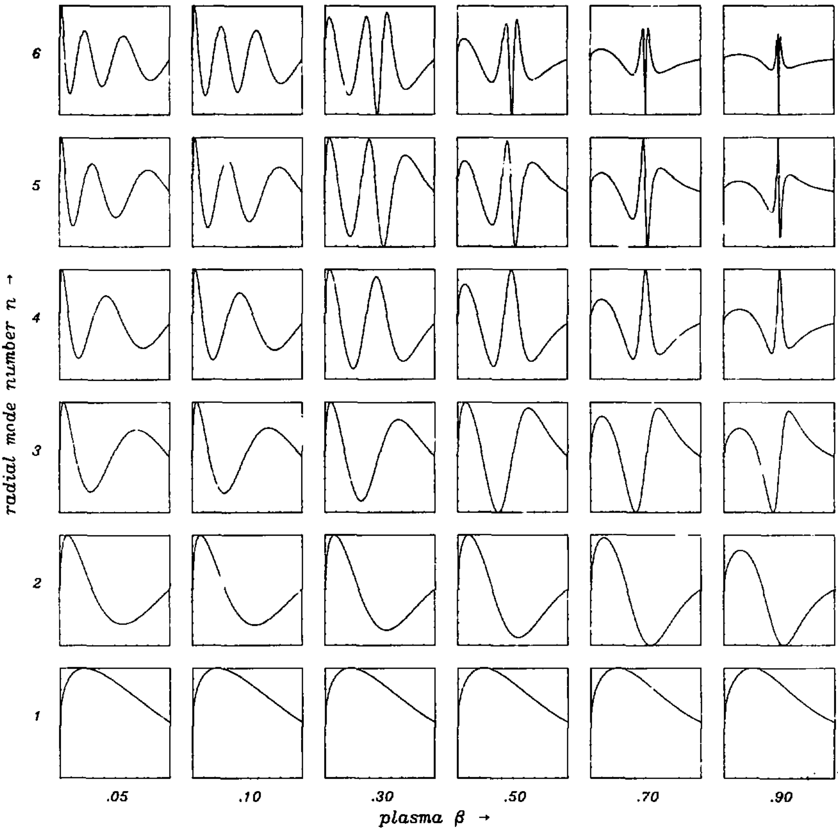
$\psi^{n}_{\kappa m}$ Eigenstreamfunctions Of A Common Helicity

FIGURE 5



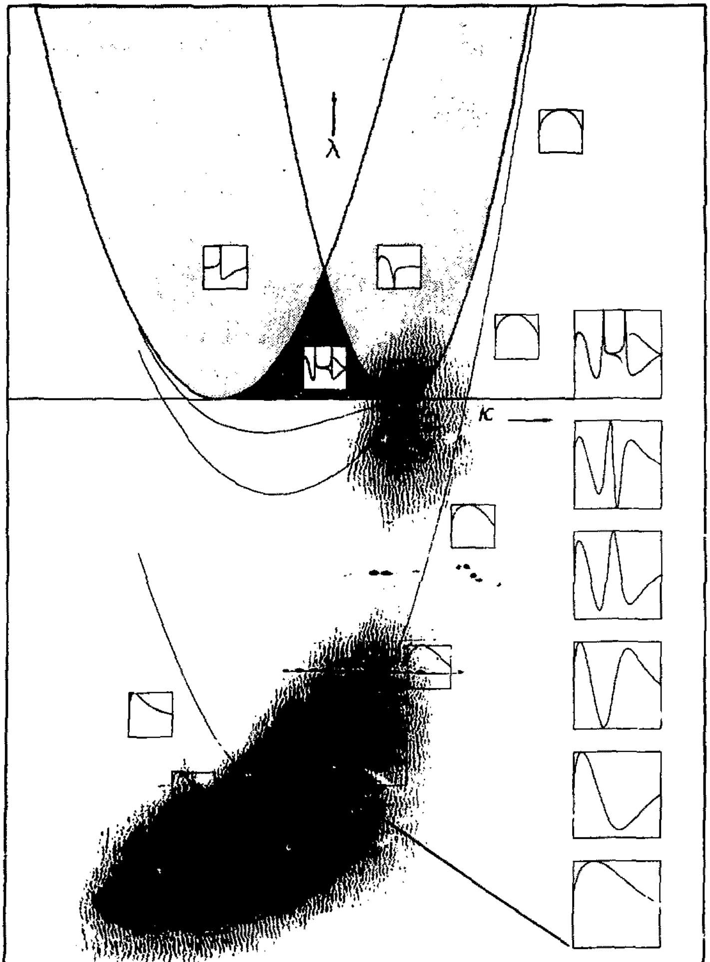
$\psi^{n}_{\kappa m}$ **Eigenstreamfunctions Of A Common Helicity**

FIGURE 6



$\psi^{n}_{\kappa m}$ Eigenstreamfunctions Of A Common Annihilation Point

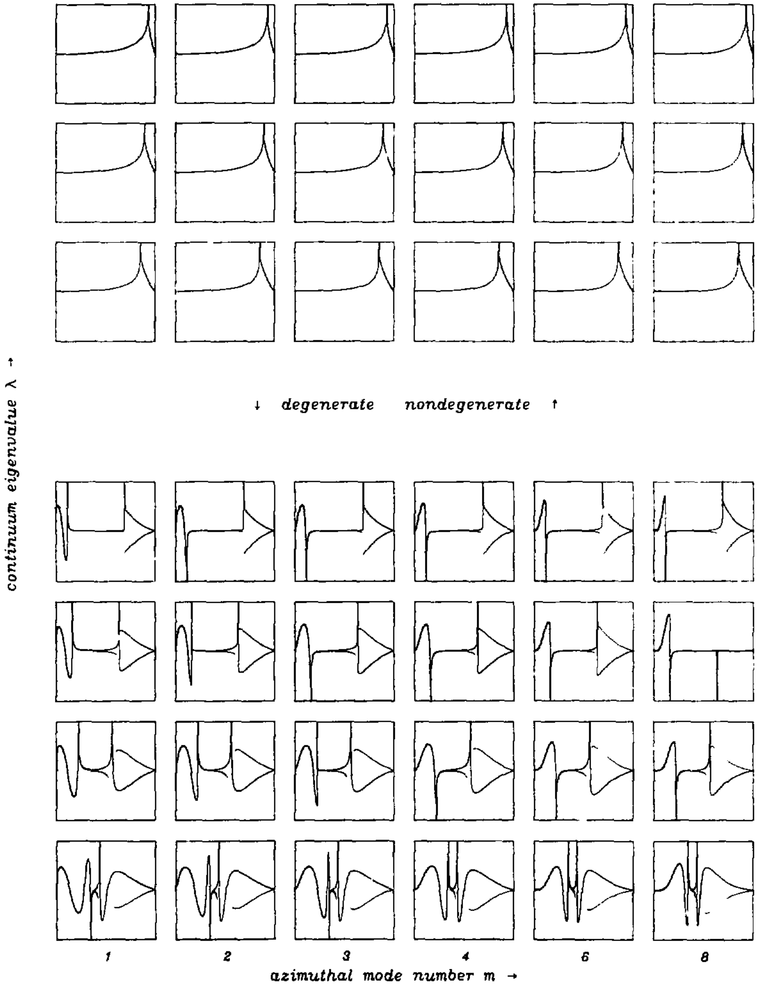
FIGURE 7



Global Dispersion Relation

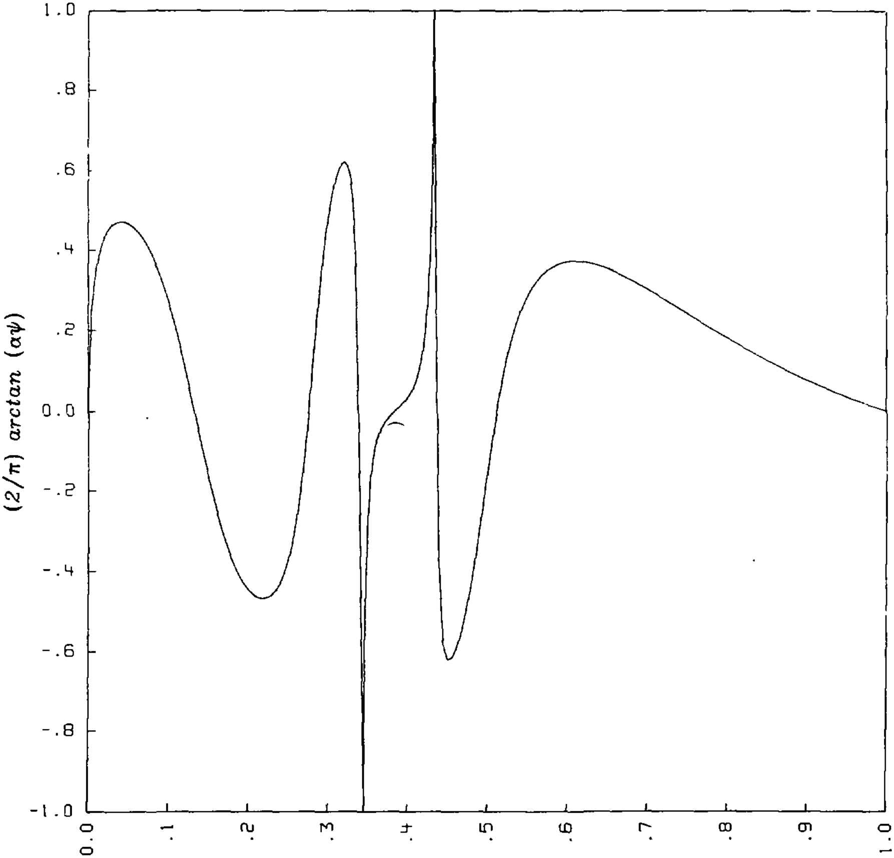
Fixed Boundary Motions Of The Gyroelastic Screwpinch

FIGURE 8



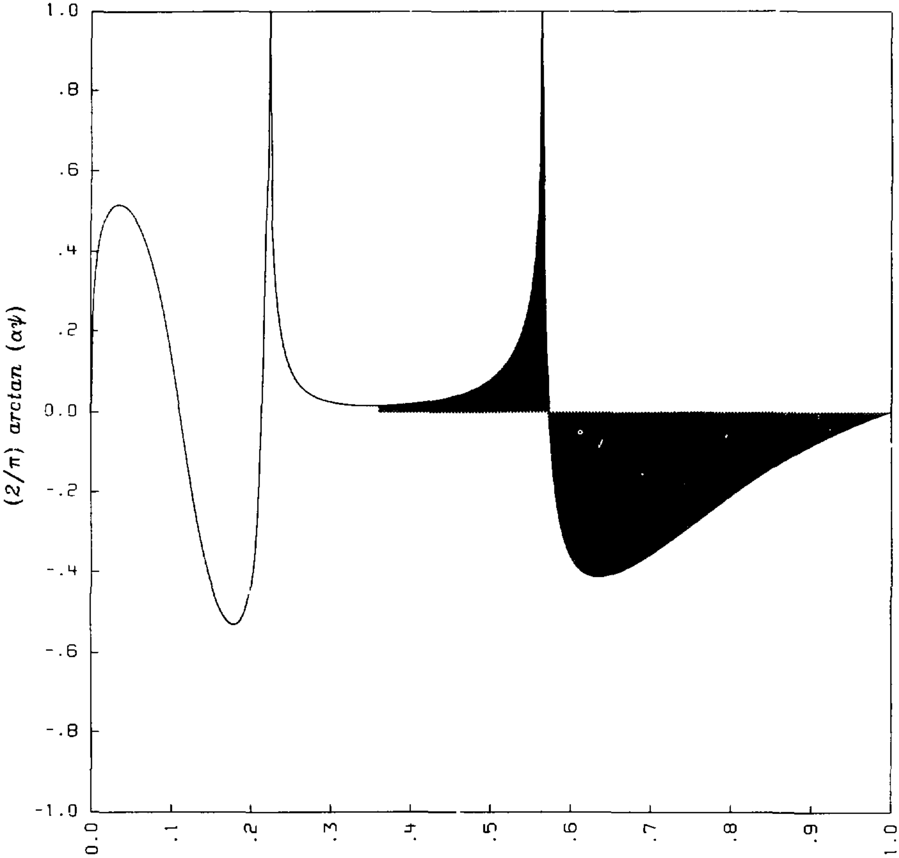
$\psi^{\nu}_{\kappa m}$ Singular Eigenstreamfunctions Of A Common Helicity

FIGURE 8



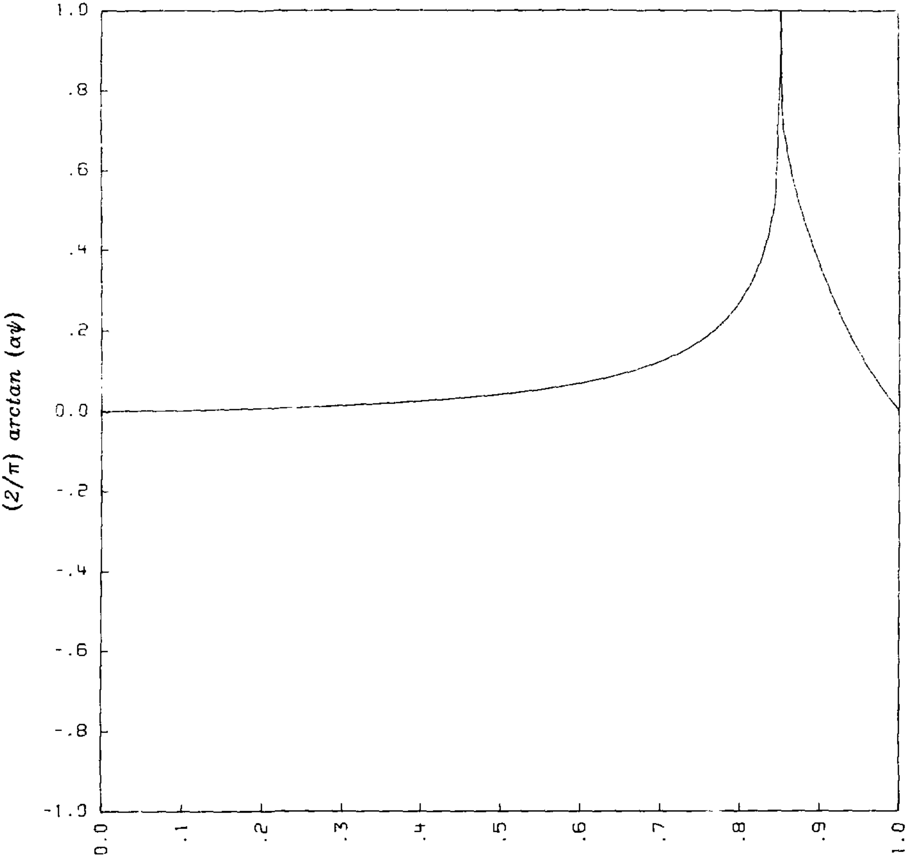
Singular Eigenstreamfunctions (*degenerate*)

FIGURE 10



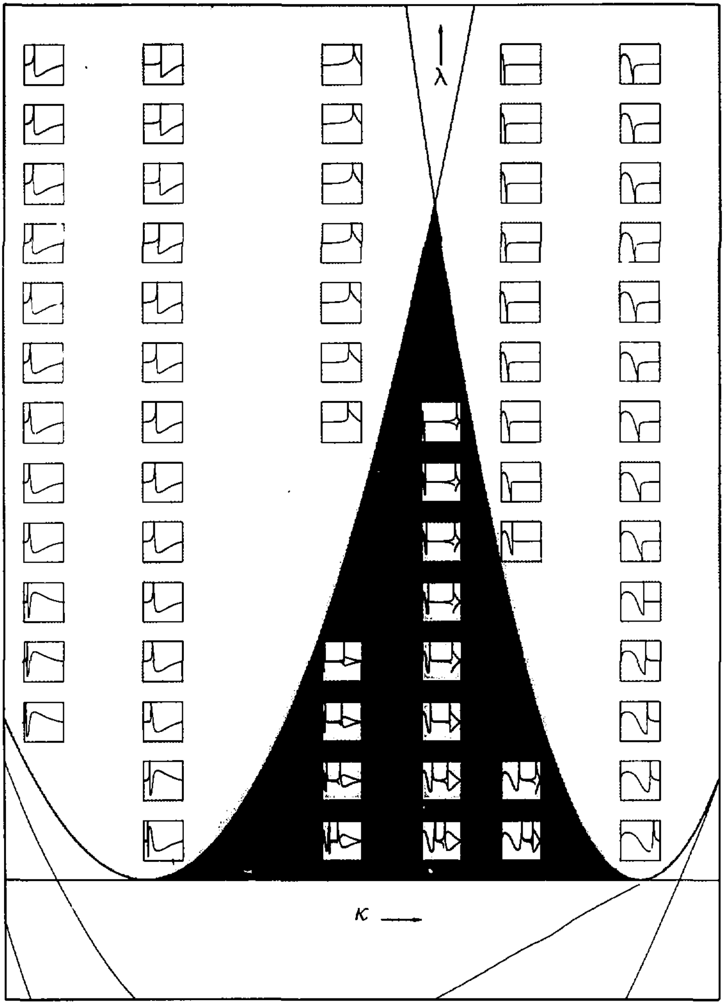
Singular Eigenstreamfunctions (*degenerate*)

FIGURE 11



Singular Eigenstreamfunction (*nondegenerate*)

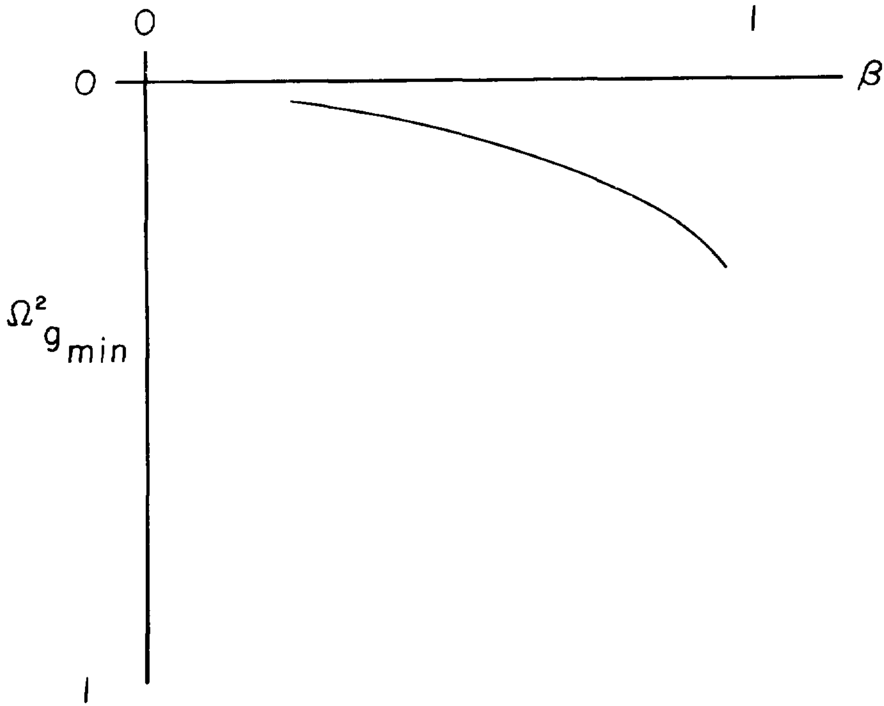
FIGURE 12



Global Dispersion Relation

ContinuaOf A Fixed Boundary Gyroelastic Screwpinch

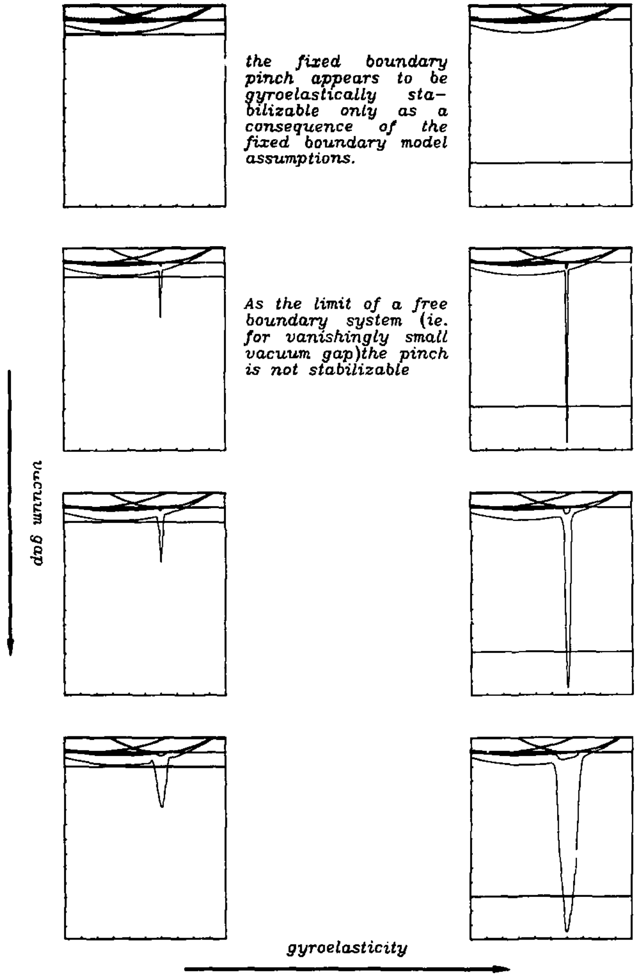
FIGURE 13



Minimum Gyroelasticity For Stabilization

Of A Fixed Boundary Gyroelastic Screwpinch

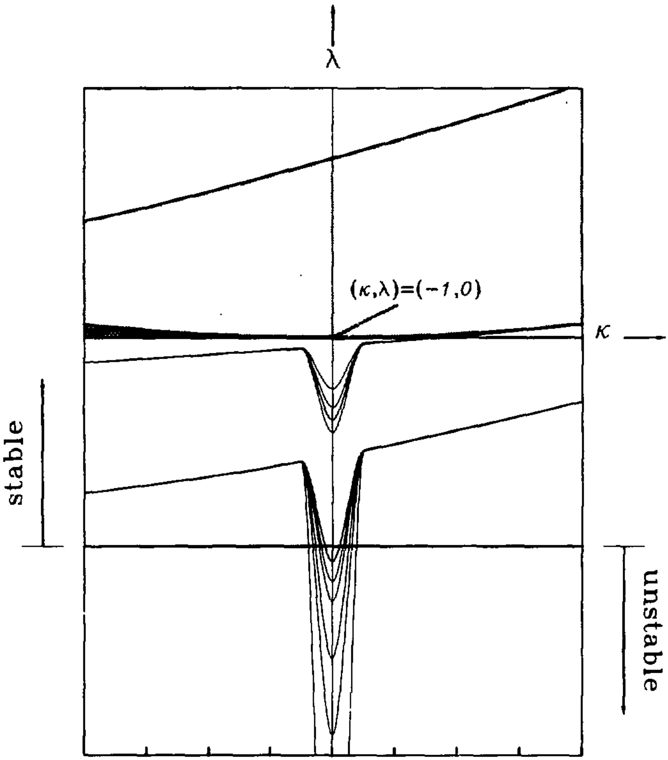
FIGURE 14



Global Dispersion Relation

Free Boundary Motions Of A Gyroelastic Screwpinch

FIGURE 15

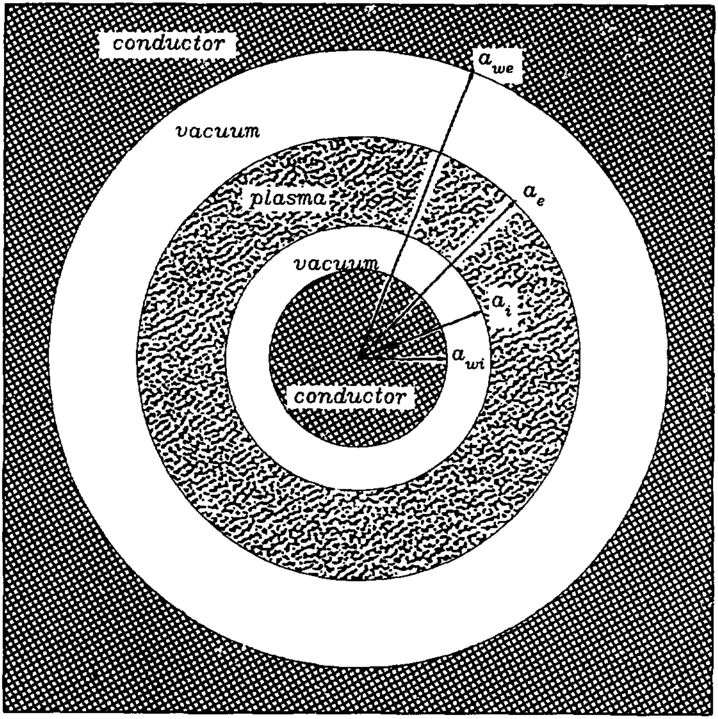


the graph depicts the lowest two eigenmode trajectories for various values of the Alfven speed; the vacuum gap is small

Global Dispersion Relation

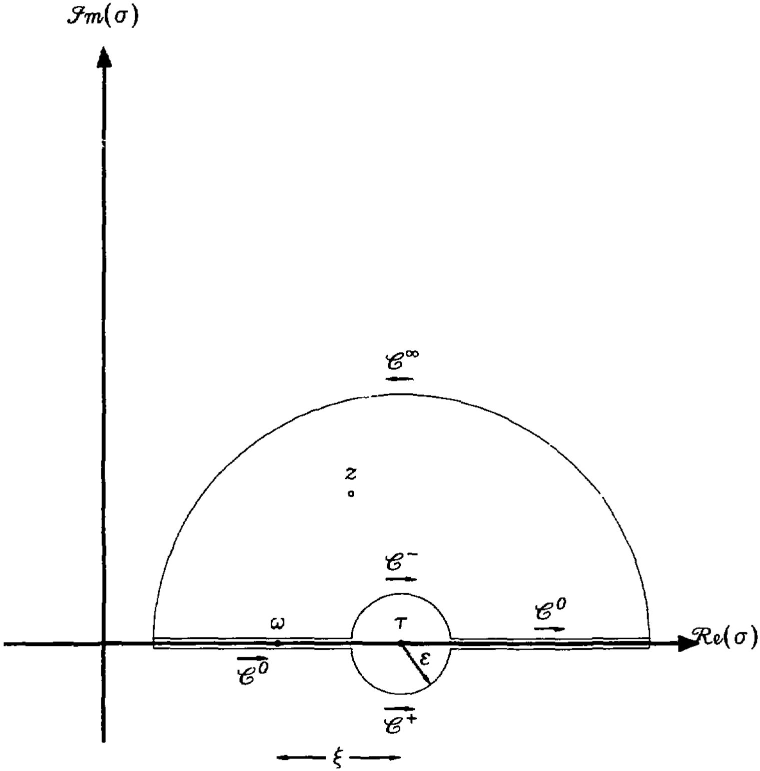
Free Boundary Motions Of The Gyroelastic Screwpinch

FIGURE 16



Tubular Screwpinch Geometry

FIGURE 17



Poincare–Bertrand Theorem

contours in the σ–plane used in Appendix I

FIGURE 18

U. S. Government Printing Office: 1981/2–789-002/5545