HIGHER-ORDER TERMS IN A PERTURBATION EXPANSION FOR THE IMPEDANCE OF A CORRUGATED WAVE GUIDE*

S. Krinsky and R. L. Gluckstern
National Synchrotron Light Source
Brookhaven National Laboratory
Upton, N.Y. 11973

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Abstract

We consider a circular waveguide whose radius $a(z) = a(1 + \epsilon s(z))$ varies periodically with the axial coordinate z . Chataud-Moulin and Papiernik¹ have introduced a perturbation expansion in powers of ϵ for the longitudinal impedance of such a waveguide. We have reformulated the derivation of this expansion in a manner which elucidates the structure of the higher-order terms, and allows the determination of the dependence on ϵ of the resonant frequencies. For a square-wave (SW) wall distortion, there are divergent coefficients in the perturbation expansion. Hence, a special treatment is required in this case, and we present a calculation of the resonant behavior for small ϵ , using an approach which does not assume an expansion in powers of ϵ . The resulting expression for the resonant impedance involves functions singular at $\omega = 0$; however, to leading order in ϵ , the loss-factors and resonant frequencies are in agreement with the perturbation theory of ref. 1.

Derivation of the Perturbation Expansion

Chataud-Moulin and Papiernik¹ have introduced a perturbation technique for the calculation of the electromagnetic fields generated by an electron beam moving along the axis of a circular waveguide, whose radius $a(z)$ varies periodically with the axial coordinate z . The perturbation parameter ϵ is introduced via

$$a(z) = a(1 + \epsilon s(z)), \quad (1)$$

$$s(z) = \sum_{p=-\infty}^{\infty} C_p \exp(2\pi i p z / L), \quad (2)$$

where a denotes the unperturbed radius, and the shape function $s(z) = s(z+L)$ has period L . Applications of the method developed in ref. 1 can be found in the work of Krinsky² and Cooper and Morton³. Here, we reformulate the derivation of the perturbation expansion to elucidate the structure of the higher-order terms.

We begin by assuming an axial current density,

$$J_z = I(\omega) / \pi \rho^2 \delta(\rho - r) \exp(-i\omega t), \quad r = z - vt, \quad (3)$$

where cylindrical coordinates (r, ϕ, z) are employed. The electron beam radius is denoted ρ , the phase velocity v , the laboratory time t and the step function $\delta(x)$ vanishes for $x < 0$ and is unity for $x > 0$. The current density generates an azimuthal magnetic field, $H_\phi(r, z) \exp(-i\omega t)$, where $H_\phi(r, z+L) = H_\phi(r, z)$ is determined by the inhomogeneous wave equation $(\nabla^2 + k^2 - v^2/c^2)H_\phi = 0$:

$$\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r H_\phi)}{\partial r} \right) - \frac{\omega^2}{v^2} H_\phi + \frac{\partial^2 H_\phi}{\partial z^2} + \frac{2i\omega}{v} \frac{\partial H_\phi}{\partial z} = \frac{\partial J_z}{\partial r}, \quad (4)$$

together with the boundary condition¹ at $r = a(z)$:

$$\frac{\partial (r H_\phi)}{\partial r} = a'(z) \left[\frac{\partial (r H_\phi)}{\partial z} + \frac{1}{c} (r H_\phi) \right], \quad (5)$$

Here, $a'(z) = da(z)/dz$, and the partial derivative $\partial/\partial z$ is taken with $t = z/v$ held fixed. The electric fields E_z and E_r can be calculated from H_ϕ using the curl equations.

For simplicity, we consider the limit $v \rightarrow c$, so that $v = c$ and $\omega/c \ll 1$. Then the general solution to Eq. (4) is

$$H_\phi(r, z) / I(\omega) = g(r) - \frac{\omega}{cL} \sum_{p=-\infty}^{\infty} B_p \frac{I_1(\lambda_p r)}{\lambda_p I_0(\lambda_p a)} e^{2\pi i p z / L}, \quad (6)$$

where $g(r) = r/2\pi \rho^2 (r < \rho)$, and $g(r) = 1/2\pi r (r > \rho)$,

$$\lambda_p^2 = (2\pi p/L)^2 + (2\pi p/L)(\omega/c), \quad (7)$$

and I_m is the modified Bessel function. In order to derive an infinite set of linear equations for the unknown coefficients B_p , we substitute (6) into the boundary condition (5). It is useful to define

$$\Gamma_{np} = (2\pi n/L)(\omega/c) + (2\pi p/L)(\omega/c) + (2\pi n/L)(2\pi p/L), \quad (8)$$

$$D_k(\lambda_p) = \frac{1}{L} \int_0^L dz \exp(-2\pi i k z / L) I_0(\lambda_p a(z)) / I_0(\lambda_p a) \quad (9)$$

$$= \delta_{ko} + \lambda_p^2 \sum_{m=1}^{\infty} \frac{1}{m} C_m^{(m)} C_k^{(m)} \epsilon^m, \quad (9a)$$

$$E_k = \frac{1}{L} \int_0^L dz \exp(-2\pi i k z / L) a'(z) / a(z) \quad (10)$$

$$= - (2\pi i k / L) \sum_{m=1}^{\infty} \frac{1}{m} C_m^{(m)} C_k^{(m)} \epsilon^m / m, \quad (10a)$$

where $C_k^{(m)}$ is the k -th Fourier coefficient of $[s(z)]^m$,

$$C_p^{(m)} = (a^m \lambda_p^{m-2} / m!) I_0^{(m)}(\lambda_p a) / I_0(\lambda_p a), \quad (11)$$

and $I_0^{(m)}$ is the m -th derivative of I_0 . It now follows that the coefficients B_p are determined by:

$$\sum_{p=-\infty}^{\infty} B_p D_{n-p}(\lambda_p) \Gamma_{np} / \lambda_p^2 = -i L E_n / 2\pi \quad (n = -\infty, \dots, \infty). \quad (12)$$

For vanishing electron beam radius, $\rho \rightarrow 0$, the longitudinal impedance, $Z(\omega)$, is related to the solution of Eq. (12) by

$$Z(\omega) = i Z_0 B_0, \quad (13)$$

where $Z_0 = 1/c\epsilon_0$.

Using Eqs. (9a), (10a) and (12), a perturbation expansion is derived:

$$B_p = \sum_{m=1}^{\infty} \frac{1}{m!} B_p^{(m)} \epsilon^m, \quad (14)$$

$$B_p^{(m)} = U_p^{(m)} + \sum_{l=1}^{m-1} \frac{1}{l!} \sum_{p'=-\infty}^{\infty} G_{pp'}^{(l)} B_{p'}^{(m-l)}, \quad (15)$$

with $U_p^{(m)} = (-1)^{m+1} (p/m) C_p^{(m)}$ and $G_{pp'}^{(n)} = -\Gamma_{pp'} C_{p-p}^{(n)} C_{p'}^{(m)}$.

Now defining $b(m) = U_0^{(m)} = 0$, and for $k \geq 2$,

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†University of Maryland, College Park, MD. 20742

$$b(m_1, \dots, m_k) = \sum_{p_1, \dots, p_{k-1}} G_{op_1}^{(m_1)} G_{p_1 p_2}^{(m_2)} \dots G_{p_{k-2} p_{k-1}}^{(m_{k-1})} U_{p_{k-1}}^{(m_k)}, \quad (16)$$

It is easily seen that

$$B_o^{(m)} = \sum_{(m)} b(m_1, \dots, m_k), \quad (17)$$

where the sum $\sum_{(m)}$ is over the 2^{m-1} sets of integers $\{m_j\}$, with $m_j \geq 1$ and $\sum m_j = m$. In particular, $B_o^{(2)} = b(1, 1)$, and $B_o^{(3)} = b(2, 1) + b(1, 2) + b(1, 1, 1)$.

The $O(\epsilon^2)$ contribution to the impedance is

$$Z(\omega)/Z_o = - (2\pi\omega/cL) \sum_{p=-\infty}^{\infty} |e p_p|^2 Q_p, \quad (18)$$

where we have introduced $Q_p \equiv Q_p^{(1)}$. The resonant frequencies ω_p , therefore, correspond to the poles of Q_p , i.e. $I_o(\omega_p) = 0$. If at a given frequency $\bar{\omega}$, one has $Q_{\bar{p}}$ nearly infinite and dominating all other $Q_p (p \neq \bar{p})$, then in the neighborhood of $\bar{\omega}$ we might expect,

$$Z(\omega)/Z_o \approx \sum_{m=2}^{\infty} \delta_m(\bar{\omega}) \epsilon^m / \left(Q_{\bar{p}}^{-1} - \sum_{n=1}^{\infty} \delta_n(\bar{\omega}) \epsilon^n \right). \quad (19)$$

Comparing coefficients of $(Q_{\bar{p}})^{-k} \epsilon^n$ ($1 \leq k \leq n-1$) in Eqs.

(17) and (19), one determines $\delta_m(\bar{\omega})$ and $\delta_m(\bar{\omega})$.

Clearly, $\delta_m(\bar{\omega})$ is just the coefficient of $Q_{\bar{p}}^{-m} \epsilon^m$. To write an expression for $\delta_m(p)$, we define

$$Q_p^{-1} \delta_p(m) = G_{pp}^{(m)}, \text{ and for } k \geq 2,$$

$$Q_p^{-1} \delta_p(m_1, \dots, m_k) = \sum_{p_1, \dots, p_{k-1}} G_{pp_1}^{(m_1)} G_{p_1 p_2}^{(m_2)} \dots G_{p_{k-2} p_{k-1}}^{(m_{k-1})} G_{p_{k-1} p}^{(m_k)}, \quad (20)$$

where the sum is over $p_1 \neq p, \dots, p_{k-1} \neq p$. Then, we find

$$\delta_m(p) = \sum_{(m)} \delta_p(m_1, \dots, m_k). \quad (21)$$

As we have done before in Eq. (17), we use $\sum_{(m)}$ to denote the summation over the 2^{m-1} sets of integers $\{m_j\}$, with $m_j \geq 1$ and $\sum m_j = m$. To be explicit, we write $\delta_1(p) = \delta_p(1)$, $\delta_2(p) = \delta_p(2) + \delta_p(1, 1)$, and $\delta_3(p) = \delta_p(3) + \delta_p(2, 1) + \delta_p(1, 2) + \delta_p(1, 1, 1)$. Knowledge of $\delta_m(p)$ determines the variation of the resonant frequencies with ϵ , as long as Eq. (19) is a valid approximation, that is, until the resonant frequency corresponding to some $p \neq \bar{p}$ becomes degenerate with that corresponding to $\bar{p}$.

Square Wave Wall Distortion

Let us now consider the square wave (SW) perturbation of the wall studied by Keil and Zotter.⁵ We take the shape function $g(z)$ equal to unity for $0 < z < g$, and to zero for $g < z < L$. Its Fourier coefficient,

$$C_p^{SW} = (2\pi p)^{-1} [\exp(2\pi i p g/L) - 1] \quad (22)$$

falls off slowly at large p , resulting in the divergence of the coefficients in the perturbation expansion (14). This divergence indicates that the impedance considered as a function ϵ is singular at the origin for the SW perturbation.

In order to make sense of the perturbation expansion, it is necessary to introduce a large p cutoff $p_{\max}$ and to perform a partial summation including leading terms in $p_{\max}$ for all orders of ϵ . After completing the summation, $p_{\max}$ is allowed to go to infinity and a finite result is obtained. This procedure has been applied in ref. 4 to treat the low frequency limit of $Z(\omega)/\omega$. For a narrow cavity with $g \ll L$, $g \ll a\epsilon$, $\epsilon \ll 1$, the well-known linear behavior $Z(\omega)/\omega \approx -iZ_o g \epsilon / 2\pi c$ was obtained. For a cavity with g/L of order unity, examination of the higher-order terms indicated that $Z(\omega)/\omega \approx (-iZ_o g \epsilon / \pi^2 c) \ln(L/a\epsilon) + O(\epsilon^2)$.

Here we shall present a more direct treatment⁶ of the SW distortion, focussing our attention on the resonant behavior at high frequencies. Our objective shall be to make contact with the perturbation theory results of Eqs. (19-21). In the "tube region" ($0 \leq z \leq L$, $r \leq a$) we express the EM-fields in terms of coefficients B_p using Eq. (6). In the "cavity region" ($0 \leq z \leq g$, $a \leq r \leq b \approx a(1+\epsilon)$), the fields are written in terms of coefficients β_n ,

$$H_p(r, z, t)/I(\omega) = \sum_{n=0}^{\infty} \beta_n [T_1(\psi_n r)/\psi_n T_0(\psi_n a)] f_n(z) \exp(-i\omega t). \quad (23)$$

The functions $f_n(z) = (2/g)^{1/2} \cos(n\pi z/g)$, $n \geq 1$, and $f_0(z) = (1/g)^{1/2}$ are orthonormal on $0 \leq z \leq g$. We have defined,

$$\psi_n^2 = (n\pi/g)^2 - (\omega/c)^2, \quad (24)$$

and $T_k(\psi_n r) = (-1)^k K_k(\psi_n r) - I_k(\psi_n r) K_0(\psi_n b)/I_0(\psi_n b)$, for $k = 0, 1$.

The relation between B_p and β_n follows from continuity of E_z along ($0 \leq z \leq g$, $r = a$):

$$B_p = (-c/\omega) \sum_{n=0}^{\infty} \beta_n R_{np}, \quad (25)$$

where

$$R_{np} = \int_0^g dz f_n(z) \exp(-i\omega z/c - 2\pi i p z/L). \quad (26)$$

Continuity of H_p results in an equation determining β_n ,

$$S_m^{-1} \beta_m (-aL)^{-1} \sum_{p=-\infty}^{\infty} Q_p^* \sum_{n=0}^{\infty} \beta_n R_{np} = (2\pi a)^{-1} R_{m0}^*, \quad (27)$$

with $S_m = \psi_m T_0(\psi_m a)/T_1(\psi_m a)$. Once β_n is known, the impedance is calculated from Eqs. (13) and (25). The relation of Eq. (27) to a variational principle for the impedance will be discussed in the Appendix.

As in the analysis leading to Eq. (19), let us suppose that at a frequency $\bar{\omega}$, a single $Q_{\bar{p}}$ is much larger than all other $Q_p (p \neq \bar{p})$. Then neglecting all $p \neq \bar{p}$ in the summation over p in Eq. (27), we are left with a separable equation, which is easily solved, yielding

$$\frac{Z(\omega)}{Z_o} = \frac{-c/\omega}{2\pi a} \left[\sum_{n=0}^{\infty} \beta_n |R_{n0}|^2 + \frac{|\sum_{n=0}^{\infty} \beta_n R_{n0}^*|^2}{aL Q_{\bar{p}}^{-1} - \sum_{n=0}^{\infty} \beta_n |R_{n\bar{p}}|^2} \right]. \quad (28)$$

The summations in Eq. (28) define functions of ϵ singular at the origin. To see that the sums are convergent, note that for fixed ϵ , and $\psi_n \rightarrow \infty$, one has $S_n = -\psi_n \tanh(\psi_n \epsilon)$, and that for large n , R_{np} decreases quadratically.

From Eq. (28), we can see that the perturbation expansion of Eqs. (14-21) corresponds to an invalid interchange of the order of the summation over powers of ϵ , and the summation over n . Expanding S_n (for n fixed) in powers of ϵ , the coefficients of all the higher-order terms are divergent. However, the lowest-order terms in the numerator and denominator are finite and in agreement with perturbation theory. To see this, note that for fixed n , and $\epsilon \rightarrow 0$, one has $S_n = -\frac{1}{2} \psi_n^2 \epsilon$, and use:

$$\sum_{n=0}^{\infty} \psi_n^2 R_{no} R_{np}^* = 2\pi(\omega/c) p C_p^{SH}, \quad (29)$$

$$\sum_{n=0}^{\infty} \psi_n^2 |R_{np}|^2 = g \Lambda_p^2, \quad (30)$$

where C_p^{SH} was defined in Eq. (22) and Λ_p^2 in Eq. (7).

At the present time, we have not accomplished the full asymptotic analysis for $\epsilon \rightarrow 0$ of the sums appearing in Eq. (28). We do believe, however, that the proper treatment of the large n portion of the summations will not change the leading terms just presented, which agreed with perturbation theory. An indication of the nature of the singularity at $\epsilon=0$ is obtained by using the approximation $S_n \approx -\psi_n \tanh(\psi_n \epsilon)$ for large n ,

and converting the summation to an integral. In this manner one finds the singular term $\epsilon^{-1} \text{sgn}(\epsilon)$.

Sinusoidal Wall Distortion

For a sinusoidal wall distortion, corresponding to a shape function $s(z) = \cos(2\pi z/L)$, the coefficients of all powers of ϵ in the perturbation expansion (14) are finite. By iterating the recursion relations (15) on the computer, we have numerically evaluated the coefficients in the expansion of $Z(\omega)/\omega$ in the low frequency limit. Depending on the value of the ratio a/L , we have obtained between 20 and 30 coefficients, before encountering a loss of precision. These numerical results indicate that $\epsilon=0$ is a regular point, and that the singularities nearest to the origin, determining the radius of convergence, are located at

$$(2\pi a c/L)^2 = -1. \quad (31)$$

Pade' approximants⁷ have been used to analytically continue the perturbation expansion into the entire interval $0 < \epsilon < 1$. In Fig. 1, we plot the ratio of the [10/10] Pade' approximant to the result of second-order perturbation theory.

References

1. M. Chatard-Moulin and Papiernik, Proc. Particle Accelerator Conference, San Francisco, 1979 [IEEE Trans. Nucl. Sci. 26, 3526 (1979)]
2. S. Krinsky, Proc. Xth International Conference in High Energy Accelerators, Geneva, 1980, p 576.
3. R. K. Cooper and P. L. Morton, SLAC Informal Report, April 1980, and to be published.
4. A more detailed discussion of some of the work described in this paper can be found in: S. Krinsky and R. L. Gluckstern, BNL 28373.
5. E. Keil and B. Zotter, Particle Accelerators 3, 11 (1972).
6. See e.g. K. Bane and B. Zotter, Proc. Xth International Conference on High Energy Accelerators, Geneva, 1980 p 581.
7. G. A. Baker, Pade' Approximants, Academic Press, New York, 1975.

Appendix - Variational Principle

Define

$$\beta(z) = \exp(-i\omega z/c) \sum_{n=0}^{\infty} \beta_n f_n(z), \quad (A1)$$

then Eq. (27) is equivalent to the integral equation

$$\int_0^g dz' K(z, z') \beta(z') = (2\pi a)^{-1}, \quad (0 < z < g) \quad (A2)$$

with the Hermitean kernel

$$K(z, z') = \exp(-i\omega(z-z')/c) \sum_{n=0}^{\infty} S_n^{-1} f_n(z) f_n(z') \quad (A3)$$

$$-(aL)^{-1} \sum_{p=-\infty}^{\infty} Q_p \exp [2\pi i p(z-z')/L].$$

From Eqs. (25) and (13), the impedance is given by

$$Z(\omega)/iZ_0 = - (c/\omega) \int_0^g dz \beta(z). \quad (A4)$$

Now using the integral equation (A2), we can rewrite this as

$$Z(\omega)/iZ_0 = - \frac{c/\omega}{2\pi a} X^{-1}, \quad (A5)$$

with

$$X = \frac{\int_0^g \int_0^g dz dz' \beta^*(z) K(z, z') \beta(z')}{\int_0^g \beta^*(z) dz \int_0^g \beta(z') dz'} \quad (A6)$$

The original integral equation (A2) is equivalent to the variational condition $\delta X = 0$, when $\beta(z)$ and $\beta^*(z)$ are varied independently. Hence, if one has an approximate solution to the integral equation, the error in the impedance is quadratic in the deviation of $\beta(z)$ from the exact solution.

Figure 1. Ratio of the [10/10] Pade' approximant for $Z(\omega)/\omega$ ($\epsilon \rightarrow 0$) to the second-order result, for $2\pi a/L = 2/3, 1, 2$, and 3 .