A Verification Statistic for Numerical Wind Models

T. C. Kerrigan

September 1978

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ABSTRACT

A generalized wind estimate based on a limited number of uncertain field measurements is computed at each point in a given geographical region. A point-by-point comparison with a numerical model prediction of the wind field is then described. This comparison results in numerical assessments of the probability that the model succeeded in predicting the actual wind field and that the field measurements contain sufficient information on which to base such a comparison.

CONTENTS

ABSTRACT.	i
ACKNOWLEDGMENTS	iv
FIGURES	v
TABLES.	vii
I. INTRODUCTION	1
II. THE GENERALIZED WIND ESTIMATE.	3
III. THE VERIFICATION STATISTIC	8
IV. CONCLUDING REMARKS	11
REFERENCES.	15
APPENDIX A: A SIMPLE REPRESENTATION OF THE SUPERPOSITION OF ROTATIONAL VELOCITY FIELDS	A-1
APPENDIX B: CONSTRUCTION OF TWO STATE OF NATURE MODELS	B-1
APPENDIX C: A BEST GUESS FOR THE WIND VELOCITY BASED ON MEASUREMENTS MADE AT SAMPLING SITES.	C-1
APPENDIX D: A BEST ESTIMATE FOR THE PARAMETERS IN A GAUSSIAN PROBABILITY DENSITY.	D-1
APPENDIX E: A TECHNIQUE FOR QUANTIFYING THE UNCERTAINTY IMPLICIT IN THE STATE OF NATURE MODEL	E-1
APPENDIX F: CONSTRUCTION OF A GENERALIZED WIND ESTIMATE AT A SAMPLING SITE.	F-1
APPENDIX G: CONSTRUCTION OF A GENERALIZED WIND ESTIMATE AT AN ARBITRARY POINT.	G-1
APPENDIX H: A FORTRAN PROGRAM WHICH COMPUTES THE GENERALIZED WIND ESTIMATE.	H-1
APPENDIX I: AN ESTIMATE OF THE PROBABILITY THAT THE WIND MODEL HAS PREDICTED THE ACTUAL WIND VELOCITY AT A GIVEN POINT. . .	I-1
APPENDIX J: PROOFS OF SEVERAL BASIC PROPERTIES OF THE FUNCTION ESTIMATING PROBABILITY OF AGREEMENT BETWEEN PREDICTED AND ACTUAL WIND VELOCITIES	J-1

APPENDIX K:	COMPUTATION OF THE VARIANCE FOR A GAUSSIAN PROBABILITY DISTRIBUTION.	K-1
APPENDIX L:	COMPUTATIONS DEPICTING THE POTENCY OF FIELD MEASUREMENTS IN VERIFYING WIND MODEL PREDICTIONS.	L-1

FIGURES

B1	A Depiction of the Sites and Sample Wind Data Listed in Table B1. The tails of the velocity vectors coincide with the sampling sites. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.).	B-4
B2	Contours for f_1 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-5
B3	Contours for f_2 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-6
B4	Contours for f_3 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-7
B5	Contours for f_5 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-8
B6	Contours for f_6 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-9
B7	Contours for f_1 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-5
B8	Contours for f_2 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-6
B9	Contours for f_3 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-7
B10	Contours for f_5 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-8
B11	Contours for f_6 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.)	B-9

- C1 A Depiction of a Wind Field Based on the Monopole Model for f .
The tails of the vectors coincide with the locations having the
indicated wind velocities. The coordinates of the border corners
are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$ C-4
- C2 A Depiction of a Wind Field Based on the Self-consistent Model
for f . The tails of the vectors coincide with the locations
having the indicated wind velocities. The coordinates of the
border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$ C-4
- E1 Contours for d Based on the Monopole Model for f . The
coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$,
 $(-4.,-4.)$, $(4.,-4.)$ E-3
- E2 Contours for d Based on the Self-consistent Model for f .
The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$,
 $(-4.,-4.)$, $(4.,-4.)$ E-3
- L1 Contours for ψ Based on the Monopole Model for f . The
coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$,
 $(-4.,-4.)$, $(4.,-4.)$ L-3
- L2 Contours for ψ Based on the Self-consistent Model for f .
The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$,
 $(-4.,-4.)$, $(4.,-4.)$ L-3

TABLES

B1	A List of the Sites and Sample Wind Data Used in Illustrating the Computation of f	B-3
L1	A List of the Sites and Sample Wind Data Used in Illustrating the Computation of ψ	L-2

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A VERIFICATION STATISTIC FOR NUMERICAL WIND MODELS

T. C. Kerrigan

I. INTRODUCTION

A primary requirement for using numerical models to prospect for wind energy is that numerical predictions and field measurements of the wind agree. Settling on an adequate definition of agreement has turned out to be a very thorny problem, however. The purpose of this document is to construct a statistic which can serve as the basis for this definition. This statistic quantifies the comparison between computed and measured values of the wind and so should prove useful in assessing the performance of numerical models in predicting field measurements.

Specifically, we envision the following exercise. We consider a geographical region R in which prospecting for wind energy is of interest. A modeler devises a numerical scheme for predicting the horizontal ground-level wind field $v(r)$ for r ranging in R . The model may require information about terrain features, atmospheric stability, the wind at the boundary or at certain interior points of R , etc. On a particular occasion, we determine this information and in addition measure the wind velocity $v(r_i)$ at several locations $r_1, \dots, r_n$ in R . The modeler then uses the information he requires to compute u , the numerical estimate for v . Our purpose is to devise a scheme for comparing $\{u(r)\}_{r \in R}$ and $\{v(r_i)\}_{i=1}^n$ in order to accept or reject the model as a useful prospecting tool.

Any realistic comparison scheme is complicated at the onset by several intrinsic features of numerical modeling and field measurements. We now turn attention to these in light of their inevitable influence on our statistic's final form.

A numerical model presumably predicts motions no smaller than its grid size (or equivalent measure of its ability to resolve). An anemometer, however, can generally sense motions on a much smaller scale. In comparing numerical predictions and field measurements, some account must be taken of systematic deviations in the flow due to terrain features smaller than those

included in the model. Further, turbulence induces an uncertainty in the experimental determination of the wind. In comparing numerical predictions and field measurements, some account must also be taken of these random fluctuations in the wind field.

Major events such as the passage of fronts may not occur simultaneously throughout R . As a result, the interpretation of measurements made during these events tends to be ambiguous. So for the present we insist that field measurements be limited to periods in which large scale motions are in a quasi-steady state. This restriction is not essential, however, and we shall discuss the consequences of dropping it once the time-independent mathematical theory has been developed.

The usefulness of a numerical model should be assessed based on its performance on several independent occasions and for several different geographical regions. A model might do well under certain meteorological conditions and poorly under others, or do well for one kind of terrain and poorly for another. Clearly, the appreciation of such variations is central in assessing a model's general performance.

In addition, although using numerical models to prospect for wind energy entails the computation of long-term averages of wind functionals, unless we obtain agreement between numerical predictions and field measurements on a time to time basis, agreement of averages might be only fortuitous and of doubtful value in establishing a model's reliability. Despite the importance of this consideration, this document does not deal with the problem of statistical analysis it raises. Instead, it deals with the statistical analysis of a model's performance during a single measurement episode in a fixed geographical region. In fact, the analysis of this simple case will constitute the underpinnings of the analysis of more complex cases.

Finally, devising a scheme for estimating the wind throughout R based on measurements at $r_1, \dots, r_n$ is evidently unavoidable. Viewed simplistically, the sites at which analysis of a model's predictions becomes of interest may differ from those at which field measurements are originally

made. Repeating the field measurements with more or relocated anemometers may be inconclusive since v during a second measurement episode is generally different from v during the first. Estimating the wind throughout R based on the original measurements gives us a way of avoiding the wasted time and expense of this kind of inconclusive repetition.

We proceed to construct a statistic based on the above considerations. We shall refer to it nominally as the verification statistic. It will hopefully serve as a basis for assessing the performance of numerical models with field measurements.

Development of our verification statistic is undertaken in the context of horizontal ground-level wind velocity in order to make the methodology more concrete. Sufficient generality has been retained, however, so that constructing verification statistics for quantities such as turbulence intensity, persistence of wind speed and direction, etc., is transparent.

Finally, although the following may seem idealistic, developing a simple quantitative performance analysis is essential for the intelligent use of numerical models in general.

II. THE GENERALIZED WIND ESTIMATE

The object of this section is to construct a generalized estimate of the horizontal ground-level wind field in R . According to the preceding section, this estimate will be based on a limited number of uncertain field measurements. We naturally insist that its structure account for the shortcomings in these measurements. In fact, we shall represent the wind at each point r in R by a formal Gaussian probability density $p(r, w)$. This representation is indicated because of the ineluctable interpretation of the Gaussian parameters in terms of the modeling and measurement features described above: the mean velocity represents a best guess for the wind and the variance matrix embodies the uncertainty in this best guess implicit in the measurement shortcomings. We shall use this generalized wind estimate as the basis for constructing the verification statistic in the next section.

We first turn attention to constructing a best guess $v(r)$ for the wind

at r based on field measurements $v_1, \dots, v_n$ made at sampling sites $r_1, \dots, r_n$. Although $v(r)$ is not the generalized wind estimate we seek, it does play an important part in its construction.

For simplicity, we construct $v(r)$ by fitting $v_1, \dots, v_n$ in a least-squares (L_2) sense by a flow consisting of a superposition of translations and rotations. The following remark reduces the computations considerably. Let $c_1, \dots, c_m$ denote the centers of these rotations and $\omega_1, \dots, \omega_m$ the corresponding angular velocities. Then there almost always exist c and ω such that

$$\sum_{j=1}^m \omega_j (r - c_j) \times \hat{k} = \omega (r - c) \times \hat{k} \quad .$$

(Appendix A). Thus, we may confine attention to fitting these data by a flow of the form $u + \omega(r - c) \times \hat{k}$.

We next hypothesize the existence of "weights" $f_1, \dots, f_n$ for this L_2 scheme. They carry the physical information on which this best guess is based (the influence of atmospheric stability, terrain, etc.). Specifically, each f_k is a function of $r_1, \dots, r_n, v_1, \dots, v_n$ and r , which takes values between 0 and 1. We interpret the value of this function as a numerical measure of the importance of v_k in estimating the wind at r . We shall refer to $f \equiv (f_1, \dots, f_n)$ in the following as the state of nature (Appendix B).

Assume for the moment that f has been specified. Set $|f| \equiv \sum_{k=1}^n f_k$, $\bar{f}_k \equiv f_k/|f|$ and $v(r) \equiv u + \omega(r - c) \times \hat{k}$. Define F by

$$F(u, \omega, c) \equiv \left[\sum_{k=1}^n \bar{f}_k |v_k - v(r_k)|^2 \right]^{1/2} .$$

The unique values for u, ω and c at which F assumes its minimum are precisely the values for which $v(r)$ is a best L_2 estimate of the wind at r .

The following is the result of computing these values (Appendix C).

Let v_{ki} and r_{ki} denote the i^{th} components of v_k and r_k . Define

$$C_i \equiv \sum_{k=1}^n \bar{f}_k r_{ki} ,$$

$$U_i \equiv \sum_{k=1}^n \bar{f}_k v_{ki} ,$$

$$C_i \cdot C_j \equiv \sum_{k=1}^n \bar{f}_k r_{ki} r_{kj} ,$$

$$U_i \cdot C_j \equiv \sum_{k=1}^n \bar{f}_k v_{ki} r_{kj} ,$$

and

$$W \equiv \frac{(U_2 \cdot C_1 - U_1 \cdot C_2) - (U_1 \cdot C_2 - U_1 \cdot C_2)}{(C_1^2 - C_1 \cdot C_1) + (C_2^2 - C_2 \cdot C_2)} .$$

Then our best estimate for the wind at r is given by

$$v(r) = U + W(r - C)x\hat{k} ,$$

where $C \equiv (C_1, C_2)$ and $U \equiv (U_1, U_2)$.

We have thus developed an explicit estimate for the wind at r in terms of $r_1, \dots, r_n, v_1, \dots, v_n, f$ and r . We shall refer to this expression in the following as the L_2 extrapolation at r based on $r_1, \dots, r_n, v_1, \dots, v_n$ and f .

As indicated at the beginning of this section, $v(r)$ does not have a sufficiently rich structure to serve as our generalized wind estimate. We rely on it heavily in the following, however, in constructing a more suitable candidate.

First, for fixed r, U, W and C , the function

$$v_r(s) \equiv U + W(s - C)x\hat{k}$$

may be interpreted as an estimate of the wind velocity throughout R based on $v_1, \dots, v_n$ at $r_1, \dots, r_n$. Since $v_r(r) = v(r)$, the uncertainty with which $v(r)$ estimates the wind at r is at least as great as the uncertainty with which $v_r(s)$ estimates the wind at $r_1, \dots, r_n$. A method has been devised for fitting these "errors" $\{v_r(r_k) - v_k\}_{k=1}^n$ by a Gaussian probability density

$$q(r, w) = \frac{\sqrt{|Q|}}{2\pi} \exp \left(-\frac{1}{2} w^* Q w \right),$$

where Q is a 2×2 positive definite quadratic form (Appendices D, E). This fit should be viewed as nothing deeper than a convenient mathematical device for quantifying the uncertainty in the L_2 extrapolation at r based on $r_1, \dots, r_n, v_1, \dots, v_n$ and f .

Second, because of turbulent fluctuations and terrain features beyond the power of the model to resolve, we must necessarily associate a distribution of wind velocities with each sampling site. As above, we represent these distributions by Gaussian probability densities

$$p_k(w) = \frac{\sqrt{|P_k|}}{2\pi} \exp \left(-\frac{1}{2} (w - \bar{v}_k)^* P_k (w - \bar{v}_k) \right),$$

$k = 1, \dots, n$, where $\bar{v}_k$ is the center of the k^{th} density and P_k is a 2×2 positive definite quadratic form (Appendices D, F).

Finally, we combine $p_1, \dots, p_n$ and q according to the following to obtain the generalized wind estimate p . We first specify f . We next compute $\bar{v}(r)$, the L_2 extrapolation at r based on $r_1, \dots, r_n, \bar{v}_1, \dots, \bar{v}_n$ and f . $\bar{v}(r)$ serves as the mean wind velocity for p . We then compute Q based on the "errors" $\{\bar{v}_r(r_k) - \bar{v}_k\}_{k=1}^n$, where $\bar{v}_r$ is the analogue of v_r obtained by substituting $\bar{v}_1, \dots, \bar{v}_n$ for $v_1, \dots, v_n$. And finally, we combine $P_1, \dots, P_n$ and Q according to Appendix G to obtain the 2×2 positive definite quadratic form P such that

$$p(r, w) = \frac{\sqrt{|P|}}{2\pi} \exp \left(-\frac{1}{2} (w - \bar{v}(r))^* P (w - \bar{v}(r)) \right).$$

Thus, $\bar{v}(r)$ and P completely specify the generalized wind estimate at r .

We conclude this section by describing several features of Appendices E and G which deserve more prominent exposure.

First, we recognize that small values for the components of f should imply a large uncertainty in the estimate of the wind at r . In quantifying this notion, we have interpreted each f_k as the fraction of information in the measurement v_k which is helpful in predicting the wind at r . By assuming that the information contributed by each v_k is in some sense independent, we can then invoke a formal probabilistic argument to devise a modification of Q which accounts for this interpretation of f . Using f in this way clearly extends its significance beyond that constituted by the L_2 extrapolation.

Second, the complicated dependence of $v(r)$ on $v_1, \dots, v_n$ has made an uncompromised assessment of the contribution of each P_k to P impossible. A tractable approximation has been obtained, however, by linearizing $v(r)$ in these arguments. Apparently the accuracy of this approximation should suffer from a strong dependence of f on $v_1, \dots, v_n$, a consequence of sharply varying terrain for example. Since extrapolation schemes fail in general under these circumstances, we assume that linearizing has not inordinately handicapped ours.

Third, once the above modifications have been implemented, the rule for combining $P_1, \dots, P_n$ and Q to get P derives from the convolution theory for sums of independent random variables. The Gaussian characterizations are especially attractive in this context since the parametric description for P is obtained from those for $P_1, \dots, P_n$ and Q by arithmetic operations.

We have thus constructed a formal distribution of velocities for each r in R based on $r_1, \dots, r_n, p_1, \dots, p_n$ and f . This construction has been cast in the form of a computer program in Appendix H in order to render it more palpable. We now proceed to the next section where we use this generalized wind estimate as a basis for testing the predictions of numerical models.

III. THE VERIFICATION STATISTIC

The object of this section is to construct a statistic for testing the accuracy of numerical wind predictions.

We first pick a state of nature f and a function $g(r)$ with the following properties:

$$1) \quad g(r) \geq 0 \text{ for } r \text{ in } R,$$

$$2) \quad \int_R g(r) dr = 1 \quad .$$

The purpose of g is to assign relative weights to the points in R according to interest. That is, if interest in the performance of a model in predicting the wind at r is high, then $g(r)$ should have a correspondingly high relative value.

We next choose n sampling sites in R . Determining the number and location of these will be more conveniently taken up at the end of this section. We note in the meantime, however, that in some sense f determines the uncertainty and g determines the usefulness of the generalized wind estimate. Thus $r_1, \dots, r_n$ should be chosen with regard to f and g in order to maximize the amount of useful information obtained from the actual field measurements.

We then collect the information needed to run the model and to establish the densities $p_1, \dots, p_n$. The model produces a wind vector $u(r)$ for each r in R . The procedure established in the preceding section produces a formal Gaussian probability density $p(r, w)$ for the wind velocity at each r in R .

Our analysis of u using p now proceeds according to the following program. Fix r for the present. Then choose ϵ such that the model succeeds at r if and only if the predicted wind is within ϵ of the actual wind there. Since we have at best a probabilistic description of the actual wind, namely p , we shall have to settle for assessing the performance of the model in probabilistic terms. Let $D_\epsilon(u)$ denote the disk with center u and radius ϵ , and define

$$x_{\varepsilon, u}(x) \equiv \begin{cases} 1 & \text{for } x \text{ in } D_{\varepsilon}(u) \\ 0 & \text{otherwise} \end{cases}$$

Then, formally, the probability that the model prediction succeeds at r is given by

$$\int_{D_{\varepsilon}(u)} p(x) dx = \int p(x) x_{\varepsilon, u}(x) dx.$$

In order to estimate this integral, we approximate $x_{\varepsilon, u}$ by a bell-shaped curve with center u , variance matrix $E \equiv 2\varepsilon^{-2}I$ and integral $\pi\varepsilon^2$:

$$\pi\varepsilon^2 \frac{\sqrt{|E|}}{2\pi} \exp \left(-\frac{1}{2} (x - u)^* E (x - u) \right).$$

Then according to Appendix I, our estimate $\phi(r)$ of the probability that the model succeeds in predicting the wind at r is given by

$$\begin{aligned} \phi(r) &= \pi\varepsilon^2 \frac{1}{2\pi} \sqrt{\frac{|P||E|}{|P+E|}} \exp \left(-\frac{1}{2} (\bar{v} - u)^* \left[E - E(P + E)^{-1}E \right] (\bar{v} - u) \right) \\ &= \sqrt{\frac{|P|}{|P+E|}} \exp \left(-\frac{1}{2} (\bar{v} - u)^* \left[E - 4\varepsilon^{-4}(P + E)^{-1} \right] (\bar{v} - u) \right). \end{aligned}$$

Approximating the probability of the model's success by ϕ is certainly no more indelicate than approximating the actual wind by p . In addition, ϕ has certain essential properties in common with $\int p(x) x_{\varepsilon, u}(x) dx$:

- 1) $\lim_{\varepsilon \rightarrow \infty} \frac{\phi(\varepsilon)}{\pi\varepsilon^2} = p(\bar{v} - u)$,
- 2) $\phi(\varepsilon_1) \leq \phi(\varepsilon_2)$ in case $\varepsilon_1 \leq \varepsilon_2$,
- 3) $\lim_{\varepsilon \rightarrow \infty} \phi(\varepsilon) = 1$.

Proofs of these may be found in Appendix J.

We define the ϕ -statistic by the integral

$$\phi \equiv \int_R \phi(r)g(r)dr \quad .$$

Loosely speaking, it is the probability that the model succeeds in predicting the wind in R on the occasion of the field measurements.

We conclude the development of the ϕ -statistic by remarking that ϵ may depend on $\bar{v}$, P , u and other variables as well as on r . Thus, there are many statistics ϕ ; in fact, one for each triple (f, g, ϵ) . In any case, however, specification of ϵ as well as f and g is emphatically not the responsibility of the statistician.

Unfortunately, ϕ by itself does not carry sufficient information to serve as the verification statistic. The difficulty is that small values of ϕ are ambiguous. Loosely speaking, they originate in large uncertainties in the generalized wind estimate and in large separations between $\bar{v}$ and u . According to the preceding section, if the uncertainties are large, then the information in the field measurements is not sufficient to distinguish between successful and unsuccessful wind predictions. In this case, conclusions about the performance of a numerical model are impossible. On the other hand, if uncertainties are small and separations are large, then we may conclude that the model prediction is unsuccessful. Thus, we evidently need a second statistic ψ which measures the uncertainty in the generalized wind estimate and so helps to remove this ambiguity. The pair (ϕ, ψ) should then constitute a potent verification statistic.

Our definition of ψ is based on the following observation. Let U denote a random variable with probability density given by $p(r, w)$, let E denote expectation with respect to p and take $P \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Then

$$E(U - \bar{v}(r)) \cdot (U - \bar{v}(r)) = \frac{a + c}{ac - b^2} \quad .$$

The proof of this fact is given in Appendix K. Now define ψ by

$$\psi(r) \equiv \sqrt{\frac{a + c}{ac - b^2}} \quad .$$

Thus, $\psi(r)$ is a simple scalar measure of the uncertainty in the generalized wind estimate at r . Finally, define Ψ by

$$\Psi \equiv \int_R \psi(r)g(r)dr \quad .$$

Loosely speaking, Ψ is the uncertainty in the generalized wind estimate averaged over R . Several contours of $\psi(r)$ have been plotted for the purpose of illustration (Appendix L). As in the case of the ϕ -statistic, other measures of this uncertainty are certainly possible.

The numerical values of ϕ and Ψ constitute an implicit comparison between predicted and measured values of the wind. The model prediction is in some sense successful in case ϕ is large, it is unsuccessful in case ϕ and Ψ are both small and the analysis is simply inconclusive in case ϕ is small and Ψ is large. The distinction between large and small, of course, depends on the level of performance required of the model and the discretion of the person who interprets the test. Deeper insight can be obtained by repeating the calculation of ϕ and Ψ using the same field measurements and varying choices for f , g and ϵ .

We conclude this section with several comments on choosing the sampling sites. Clearly, we should choose them to minimize Ψ , the pertinent measure of the uncertainty in the field measurements. Mathematically, this problem is exceedingly easy to state and exceedingly difficult to solve. Three general principles are intuitively obvious, however. Choosing physically realistic f should help in minimizing Ψ , the sites should be dispersed near the places in R where g tends to be large, and Ψ should decrease as n increases.

IV. CONCLUDING REMARKS

In concluding, we first restate our view that the (ϕ, Ψ) -statistic is a compelling solution to the verification problem. First, model verification certainly falls within the domain of statistics since it attempts to base an inference on a limited number of random samples. Second, we have

been guided by the experience of statistical decision theory in perceiving the need to construct a generalized wind estimate throughout R [2]. The form of this estimate is based on the uncertainty in the wind measurements at the sampling sites themselves, and success in extending this estimate throughout R in terms of the smallness of ψ is a forthright indication of the representativeness of the field measurements. Some definite indication of representativeness is essential in order to assert that agreement between field measurements and model predictions at the sampling sites has any bearing on model performance elsewhere in R . Third, the (ϕ, ψ) -statistic is a variant of the Bayesian procedures of statistical decision theory in which a device corresponding to the state of nature is unavoidable. In effect, f is the unprovable hypothesis which enables us to include field measurements and other physical information in making a best guess for the generalized wind estimate. Fourth, the need to evaluate the potency of the ϕ -statistic is obvious. So, even though interpreting the pair (ϕ, ψ) is more difficult, it is certainly more significant than interpreting ϕ alone. And fifth, the formal objects from which ϕ and ψ are constructed are both mathematically simple and physically comprehensive.

We also restate our view that the construction of the (ϕ, ψ) -statistic may be repeated for elements of a large class of numerical models. We provide the following outline in order to suggest the general circumstances under which the construction applies. Let w denote the quantity under study. Then

- 1) Identify R .
- 2) Devise an L_2 extrapolation scheme based on a simple physical model for w .
- 3) Specify f and g .
- 4) Locate the sampling sites $r_1, \dots, r_n$.
- 5) Simultaneously measure w at these sites and determine whatever information is necessary to run the model.

- 6) Characterize the measurements at each r_k by a formal Gaussian probability density p_k .
- 7) Use the mean values $\bar{w}_1, \dots, \bar{w}_n$ associated with $p_1, \dots, p_n$ to extrapolate a mean $\bar{w}(r)$ for each r in R .
- 8) Characterize the uncertainty in this extrapolation by a formal Gaussian probability density q .
- 9) Construct a generalized estimate $p(r)$ for w at r , a formal Gaussian probability density based on the mean $\bar{w}(r)$ and a combination of the uncertainties represented by $p_1, \dots, p_n$ and q .
- 10) Run the model to obtain a numerical prediction for w .
- 11) Choose ϵ , a measure of closeness within which the prediction must come to be called successful.
- 12) Compute ϕ , the estimated probability that the model prediction succeeds on a global basis.
- 13) Compute ψ , a measure of the potency of ϕ -test.

Obviously, not all numerical models are subject to this analysis. Nevertheless, it may be possible to extend it by returning to the remark about large scale nonstationary events made in Section I. The hypothesis that the deficiencies in the extrapolation technique and the uncertainty in the field measurements are accounted for in the generalized wind estimate may extend out of hand to such events. We expect only that data need be taken throughout these events, so that data trends be fully represented by $p_1, \dots, p_n$.

Several major tasks stand in the way of bringing this project to completion:

- 1) We need to devise a realistic procedure for constructing f .
- 2) We need to devise a technique for assessing the long-term performance of a model based on a sequence of the form

$$\left\{ (\phi, \psi)_j \right\}_{j=1}^m.$$

- 3) We need to devise a technique for evaluating a model's ability to predict the distribution of wind energy in R based on a sequence of the form $\{(\phi, \psi)_j\}_{j=1}^m$.

Although the completion of 2) and 3) does not seem particularly difficult, it should await the completion of 1) to keep this development from becoming just a mathematical pastime.

Despite the endless possibilities for mathematical refinement, we are ultimately faced with the shortcoming of statistics in general. In particular, the choice to accept or reject a model based on a sequence of the form $\{(\phi, \psi)_j\}_{j=1}^m$ must ultimately be made outside the domain of mathematics altogether. The legitimate role of the (ϕ, ψ) -statistic is not to make this choice, but to cast it in a concise quantitative form.

Finally, this document offers two significant products besides the (ϕ, ψ) -statistic itself: a numerical wind model in the form of the generalized wind estimate and a comprehensive discussion of the intrinsic difficulties in model verification.

REFERENCES

- [1] R. V. Hogg and A. T. Craig, Introduction to Mathematical Statistics. Macmillan Company, New York, Second Edition, 1965.
- [2] L. Weiss, Statistical Decision Theory. McGraw-Hill Book Co., Inc., 1961.

APPENDIX A

A SIMPLE REPRESENTATION OF THE
SUPERPOSITION OF ROTATIONAL VELOCITY FIELDS

APPENDIX A

A SIMPLE REPRESENTATION OF THE SUPERPOSITION OF ROTATIONAL VELOCITY FIELDS

Before stating the principal result in this section, we first note that $(v_1, v_2, 0) \times \hat{k} = (v_2, -v_1, 0)$. Since the third component participates only trivially in this vector cross product, we shall feel free to suppress it altogether in our horizontal wind analysis.

Lemma: Let c_j and ω_j , $j = 1, \dots, m$, denote centers and angular velocities characterizing rotations in the plane. Then the superposition of these motions can be written as

$$\sum_{j=1}^m \omega_j (r - c_j) \times \hat{k} = \omega (r - c) \times \hat{k} ,$$

$$\text{where } \omega = \sum_{j=1}^m \omega_j \text{ and } c = \left(\sum_{j=1}^m \omega_j c_j \right) / \omega .$$

Pf:

$$\begin{aligned} & \sum_{j=1}^m \omega_j (r - c_j) \times \hat{k} \\ &= \sum_{j=1}^m \omega_j r \times \hat{k} - \sum_{j=1}^m \omega_j c_j \times \hat{k} \\ &= \left(\sum_{j=1}^m \omega_j \right) \left(r - \left(\sum_{j=1}^m \omega_j c_j \right) / \left(\sum_{j=1}^m \omega_j \right) \right) \times \hat{k} \quad \blacksquare \end{aligned}$$

APPENDIX B

CONSTRUCTION OF TWO STATE OF NATURE MODELS

APPENDIX B

CONSTRUCTION OF TWO STATE OF NATURE MODELS

We offer two models for f for the purpose of illustration. Before describing them, however, we touch on several important general considerations. First, a realistic procedure for constructing f should be devised. Second, at least two components of f should be nonzero at each r in R in order to avoid degeneracy in the L_2 wind estimate. Third, the inevitable failure in trying to distinguish a best model for f is a consequence of the fact that f is an essential hypothesis in this mathematical endeavor—we shall not be able to prove that a particular f is best except by adjoining additional assumptions.

THE MONOPOLE MODEL

Set

$$f_k(r) \equiv \frac{1}{2} \left(1 + \left(\hat{u}_k \cdot \hat{v}_k \right)^2 \right) \exp \left(-\alpha_k \frac{|r - r_k|}{|v_k|} \right),$$

where $\hat{u}_k \equiv u_k/|u_k|$ and $\hat{v}_k \equiv v_k/|v_k|$. The importance of v_k in predicting v at r depends on the measurement v_k , the relative positions of r and r_k , a distinguished direction u_k , and a rate α_k at which confidence decreases as separation from the measurement increases. The model shows no confidence in the contribution of v_k to v in case $v_k = 0$. $u_k = r - r_k$ in featureless terrain and f_k is symmetric with respect to the wind's blowing parallel and antiparallel to u_k . The most awkward feature of this model is that it has v_i contributing to the determination of v_j , although v_i and v_j are actually established without the participation of any f .

THE SELF-CONSISTENT MODEL

We first develop a technique for estimating the importance of v_i in predicting the wind at r_j . For fixed j , define the "weight" w_k by

$$w_k \equiv \begin{cases} |r_k - r_j|^{-2}, & k \neq j \\ 0, & k = j \end{cases}.$$

We substitute w_k for f_k in the L_2 estimate of the wind described in Appendix C. Let $v'(r)$ denote this estimate at r based on the data $\{v_1, \dots, v_n\} - \{v_j\}$ and let $v''(r)$ denote this estimate at r based on the data $\{v_1, \dots, v_n\} - \{v_i, v_j\}$. Define

$$f_{ij} \equiv \begin{cases} \exp\left(-\alpha_{ij} \frac{|v'(r_j) - v_j|}{|v''(r_j) - v_j|}\right), & i \neq j \\ 1, & i = j \end{cases}$$

We now use these values to define f_i at r :

$$f_i(r) \equiv \frac{\sum_{j=1}^n f_{ij} |r - r_j|^{-2}}{\sum_{j=1}^n |r - r_j|^{-2}}$$

The ratio of $|v'(r_j) - v_j|$ to $|v''(r_j) - v_j|$ is a measure of the relative improvement in the estimate of the measured value v_j by including the datum v_i . Thus, f_{ij} may be viewed as a measure of the contribution of v_i in predicting v_j and $f_i(r)$ may be viewed as an interpolation of this contribution. Unlike the monopole model, this one incorporates all values $\{v_1, \dots, v_n\}$ in constructing f_i at r . Further, the physical effects giving rise to a particular set $\{v_1, \dots, v_n\}$ are included in f implicitly. The most awkward feature of this model is that $\lim_{r \rightarrow \infty} f_k(r) \neq 0$. This behavior is unimportant near the sampling sites, however.

Assuming the sampling sites and wind velocities given in Table B1 and Figure B1, we have produced contours of the components of f for the state of nature models described in this section. Contours for the monopole model are given in Figures B2 through B6, where $u_k = r - r_k$ and $\alpha_k = 0.5$. Contours for the self-consistent model are given in Figures B7 through B11, where $\alpha_{ij} = 2.5$. Contours for the missing f_k may be obtained by symmetry.

Table B1

A List of the Sites and Sample Wind Data
Used in Illustrating the Computation of f

Site Number	Site Coordinates	Wind Velocity Coordinates
1	(1.,0.)	(2.,0.)
2	(0.,1.)	(0.5,-0.5)
3	(-1.,0.)	(2.,0.)
4	(0.,-1.)	(0.5,0.5)
5	(2.,2.)	(0.25,-0.25)
6	(-2.,2.)	(0.25,-0.25)
7	(-2.,-2.)	(0.25,0.25)
8	(2.,-2.)	(0.25,0.25)

Figure B1. A Depiction of the Sites and Sample Wind Data Listed in Table B1. The tails of the velocity vectors coincide with the sampling sites. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.).

Figure B2. Contours for f_1 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.).

Figure B7. Contours for f_1 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.).

Figure B3: Contours for f_2 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.).

Figure B8: Contours for f_2 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.).

Figure B4: Contours for f_3 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$.

Figure B9: Contours for f_3 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$.

Figure B5: Contours for f_5 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$.

Figure B10: Contours for f_5 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$.

Figure B6: Contours for f_6 Based on the Monopole Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.).

Figure B11: Contours for f_6 Based on the Self-consistent Model for the State of Nature. The coordinates of the border corners are (4.,4.), (-4.,4.), (-4.,-4.), (4.,-4.).

APPENDIX C

A BEST GUESS FOR THE WIND VELOCITY
BASED ON MEASUREMENTS MADE AT SAMPLING SITES

APPENDIX C
A BEST GUESS FOR THE WIND VELOCITY
BASED ON MEASUREMENTS MADE AT SAMPLING SITES

Recall the definitions

$$v(r) \equiv u + \omega(r - c) \times \hat{k}$$

and

$$F(u, \omega, c) \equiv \left[\sum_{k=1}^n \bar{f}_k |v_k - v(r_k)|^2 \right]^{1/2}.$$

The object of this section is to compute the unique values of u , ω and c at which F assumes its minimum. We proceed with the conventional L_2 analysis. Let r_{ki} and v_{ki} denote the i^{th} components of r_k and v_k . Then

$$\begin{aligned} v_k - v(r_k) \\ = (v_{k1} - u_1 - \omega(r_{k2} - c_2), v_{k2} - u_2 + \omega(r_{k1} - c_1)) \end{aligned},$$

so

$$\begin{aligned} F^2(u, \omega, c) \\ = \sum_{k=1}^n \bar{f}_k \left[(v_{k1} - u_1 - \omega(r_{k2} - c_2))^2 + (v_{k2} - u_2 + \omega(r_{k1} - c_1))^2 \right] \end{aligned}$$

F assumes its minimum at the unique solution to the system

$$\frac{\partial F^2}{\partial u_1} = \frac{\partial F^2}{\partial u_2} = \frac{\partial F^2}{\partial \omega} = \frac{\partial F^2}{\partial c_1} = \frac{\partial F^2}{\partial c_2} = 0.$$

Computing these derivatives and substituting

$$c_i \equiv \sum_{k=1}^n \bar{f}_k r_{ki},$$

$$u_i \equiv \sum_{k=1}^n \bar{f}_k v_{ki},$$

$$c_i \cdot c_j \equiv \sum_{k=1}^n \bar{f}_k r_{ki} r_{kj} ,$$

and

$$u_i \cdot c_j \equiv \sum_{k=1}^n \bar{f}_k v_{ki} r_{kj} ,$$

we obtain the following system:

$$\begin{aligned} u_1 - u_1 - \omega(c_2 - c_2) &= 0 , \\ u_2 - u_2 + \omega(c_1 - c_1) &= 0 , \\ \omega(c_1^2 + c_2^2) - 2\omega(c_1 c_1 + c_2 c_2) \\ + (u_1 - u_1)c_2 - (u_2 - u_2)c_1 + \omega(c_1 \cdot c_1 + c_2 \cdot c_2) \\ - (u_1 \cdot c_2 - u_1 c_2) + (u_2 \cdot c_1 - u_2 c_1) \\ &= 0, \end{aligned}$$

$$\omega u_2 - \omega u_2 + \omega^2 c_1 + \omega^2 c_1 = 0 ,$$

$$\omega u_1 - \omega u_1 - \omega^2 c_2 + \omega^2 c_2 = 0 .$$

Although these last two equations are redundant, considerable simplification is still possible. Assume for the moment that $\omega \neq 0$. Then

$$c_1 = c_1 + \frac{(u_2 - u_2)}{\omega} ,$$

and

$$c_2 = c_2 - \frac{(u_1 - u_1)}{\omega} .$$

Substituting these expressions into the third equation and solving for ω , we obtain

$$\omega = \frac{(U_2 \cdot C_1 - U_2 C_1) - (U_1 \cdot C_2 - U_1 C_2)}{(C_1^2 - C_1 \cdot C_1) + (C_2^2 - C_2 \cdot C_2)} .$$

Therefore, the best L_2 fit by a superposition of translations and rotations to the data $v_1, \dots, v_n$ at $r_1, \dots, r_n$ is given by

$$\begin{aligned} v(r) &= u + \omega(r - c) \times \hat{k} \\ &= u + \omega \left(r - \left(c + \frac{(U - u) \times \hat{k}}{\omega} \right) \right) \times \hat{k} \\ &= u + \omega(r - c) \times \hat{k} - \omega \frac{U - u}{\omega} \times \hat{k} \times \hat{k} \\ &= U + \omega(r - c) \times \hat{k} , \end{aligned}$$

where $C \equiv (C_1, C_2)$ and $U \equiv (U_1, U_2)$.

This technique can clearly provide an estimate for the wind velocity at each r in R . We illustrate this point by plotting the two wind fields implied by the sample data and states of nature given in Figures B1 through B11. The wind field based on the monopole model for f is depicted in Figure C1 and the wind field based on the self-consistent model for f is depicted in Figure C2. The lengths of the arrows indicate relative speed only. The scales in Figures B1, C1 and C2 are equal, however.

Figure C1: A Depiction of a Wind Field Based on the Monopole Model for f . The tails of the vectors coincide with the locations having the indicated wind velocities. The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$.

Figure C2: A Depiction of a Wind Field Based on the Self-consistent Model for f . The tails of the vectors coincide with the locations having the indicated wind velocities. The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$.

APPENDIX D

A BEST ESTIMATE FOR THE PARAMETERS
IN A GAUSSIAN PROBABILITY DENSITY

APPENDIX D

A BEST ESTIMATE FOR THE PARAMETERS IN A GAUSSIAN PROBABILITY DENSITY

Let $\{u_k\}_{k=1}^n$ denote a set of n independent random variables. Assume that each has a probability density given by

$$\frac{\sqrt{|Q|}}{2\pi} \exp \left(-\frac{1}{2} u^* Q u \right) ,$$

where $Q \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is a positive definite quadratic form. Let u_k denote the result of an experiment to measure U_k . The object of this section is to develop a best estimate for the parameters a , b and c in terms of $u_1, \dots, u_n$.

The definition of this best estimate derives from the following point of view. We first recall that the probability density for the vector $(U_1, \dots, U_n)$ is the product of the probability densities for the individual components. Accordingly, we define the likelihood function L for a , b and c by

$$L(a,b,c) \equiv \left(\frac{\sqrt{|Q|}}{2\pi} \right)^n \exp \left(-\frac{1}{2} \sum_{k=1}^n u_k^* Q u_k \right) .$$

(See [1].) It may be interpreted as a measure of the relative probability that the data $u_1, \dots, u_n$ are n independently chosen samples from a collection whose distribution is parameterized by a , b and c . Our best estimate for the triple (a,b,c) is that at which L assumes its maximum.

The remainder of this section is devoted to finding this triple by solving the system

$$\frac{\partial L}{\partial a} = \frac{\partial L}{\partial b} = \frac{\partial L}{\partial c} = 0 .$$

Let u_{ki} denote the i^{th} component of u_k , and define

$$\alpha \equiv \frac{1}{n} \sum_{k=1}^n u_{k1}^2 ,$$

$$\beta \equiv \frac{1}{n} \sum_{k=1}^n u_{k1} u_{k2} \quad ,$$

and

$$\gamma \equiv \frac{1}{n} \sum_{k=1}^n u_{k2}^2 \quad .$$

Then a straightforward differentiation yields the system

$$\left(\frac{nc}{2|Q|} - \frac{n}{2} \alpha \right) \exp \left(- \frac{1}{2} \sum_{k=1}^n u_k^* Q u_k \right) = 0 \quad ,$$

$$\left(- \frac{nb}{|Q|} - n\beta \right) \exp \left(- \frac{1}{2} \sum_{k=1}^n u_k^* Q u_k \right) = 0 \quad ,$$

$$\left(\frac{na}{2|Q|} - \frac{n}{2} \gamma \right) \exp \left(- \frac{1}{2} \sum_{k=1}^n u_k^* Q u_k \right) = 0 \quad ,$$

which implies that a, b and c satisfy $a = \gamma|Q|$, $b = -\beta|Q|$, and $c = \alpha|Q|$.

Combining these equations, we obtain

$$ac - b^2 = (\alpha \gamma - \beta^2) |Q|^2$$

or

$$|Q| = (\alpha \gamma - \beta^2)^{-1}$$

Thus,

$$a = \gamma(\alpha \gamma - \beta^2)^{-1} \quad ,$$

$$b = -\beta(\alpha \gamma - \beta^2)^{-1} \quad ,$$

and

$$c = \alpha(\alpha \gamma - \beta^2)^{-1} \quad .$$

Therefore, our best estimate of the probability density fitting the data $u_1, \dots, u_n$ is specified by taking

$$Q = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}^{-1} \quad .$$

APPENDIX E

A TECHNIQUE FOR QUANTIFYING THE
UNCERTAINTY IMPLICIT IN THE STATE OF NATURE MODEL

APPENDIX E

A TECHNIQUE FOR QUANTIFYING THE UNCERTAINTY IMPLICIT IN THE STATE OF NATURE MODEL

Appendix D presents a technique which can be used to characterize the uncertainty represented by the "errors" $\{u_k \equiv v(r_k) - v_k\}_{k=1}^n$ by a quadratic form Q . The technique should be extended in this case, however, in order to account for additional uncertainty implicit in f . The object of this section is to describe such an extension.

According to Appendix D,

$$Q = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}^{-1},$$

where

$$\begin{aligned} \alpha &\equiv \frac{1}{n} \sum_{k=1}^n u_{k1}^2, \\ \beta &\equiv \frac{1}{n} \sum_{k=1}^n u_{k1} u_{k2}, \\ \gamma &\equiv \frac{1}{n} \sum_{k=1}^n u_{k2}^2. \end{aligned}$$

Our first step is to replace these values for α , β and γ by

$$\begin{aligned} \alpha &\equiv \sum_{k=1}^n \bar{f}_k u_{k1}^2, \\ \beta &\equiv \sum_{k=1}^n \bar{f}_k u_{k1} u_{k2}, \\ \gamma &\equiv \sum_{k=1}^n \bar{f}_k u_{k2}^2 \end{aligned}$$

in order to account for the relative contribution of the various errors.

Our second step is based on the following formalism. We interpret $f_i(r)$ as the fraction of independent information in the wind measurement at r_i which is available for estimating the wind at r . Let X denote a probability space with probability measure P . Assume that to each f_i we have associated an event S_i in X such that $f_i = P(S_i)$ and that the

$\{S_i\}_{i=1}^n$ are mutually independent. Then the fraction of information represented by f is equal to the probability that some elementary event is in the union of the $\{S_i\}_{i=1}^n$:

$$\begin{aligned}
 & P\left(\bigcup_{i=1}^n S_i\right) \\
 &= 1 - P\left(\left(\bigcup_{i=1}^n S_i\right)^*\right) \\
 &= 1 - P\left(\bigcap_{i=1}^n S_i^*\right) \\
 &= 1 - \prod_{i=1}^n P(S_i^*) \\
 &= 1 - \prod_{i=1}^n (1 - f_i) .
 \end{aligned}$$

The width of the Gaussian density characterized by Q should vary inversely with the fraction of information represented by f . Therefore, our final expressions for α , β and γ are

$$\begin{aligned}
 \alpha &\equiv d^2 \sum_{k=1}^n \bar{f}_{k1}^2 , \\
 \beta &\equiv d^2 \sum_{k=1}^n \bar{f}_{k1} \bar{f}_{k2} , \\
 \gamma &\equiv d^2 \sum_{k=1}^n \bar{f}_{k2}^2 ,
 \end{aligned}$$

where $d \equiv \left[1 - \prod_{i=1}^n (1 - f_i) \right]^{-1}$.

Contours for d are given in Figures E1 and E2. Figure E1 is based on the monopole model for f given in Figures B2 through B6; Figure E2 is based on the self-consistent model for f given in Figures B7 through B11.

Figure E1: Contours for d Based on the Monopole Model for f . The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$.

Figure E2: Contours for d Based on the Self-consistent Model for f . The coordinates of the border corners are $(4.,4.)$, $(-4.,4.)$, $(-4.,-4.)$, $(4.,-4.)$.

APPENDIX F

CONSTRUCTION OF A GENERALIZED WIND
ESTIMATE AT A SAMPLING SITE

APPENDIX F

CONSTRUCTION OF A GENERALIZED WIND ESTIMATE AT A SAMPLING SITE

This appendix describes a technique for constructing a generalized wind estimate at each sampling site. The principal difficulty is assessing the influence of small scale terrain features and turbulent fluctuations on wind measurements.

The following recommendations are indicated. First, measurements should be taken at a minimum of three anemometers which have pairwise separations comparable to the scale the model claims to resolve. Three is the smallest number of anemometers needed to simultaneously resolve the longitudinal and lateral wind variations for winds of arbitrary direction. We hypothesize that the similarity among these measurements is produced by the wind component which the model should succeed in predicting, and that the differences are produced in part by small scale terrain features which influence the wind at different anemometers differently. Second, the sampling period should be long enough for estimating the distribution of turbulent fluctuations at each anemometer, yet short enough to avoid the failure of the stationarity assumption for large scale wind flow.

We regard the set of these measurements $\{w_j\}_{j=1}^m$ taken at the i^{th} site as independent data, which we characterize by a mean $\bar{v}_i$ and a variance matrix P_i . $\bar{v}_i$ is defined by

$$\bar{v}_i \equiv \frac{1}{m} \sum_{j=1}^m w_j \quad ,$$

and P_i is obtained by applying the technique developed in Appendix D to the data $\{u_j \equiv w_j - \bar{v}_i\}_{j=1}^m$. We then take the corresponding Gaussian probability density

$$p_i(w) = \frac{\sqrt{|P_i|}}{2\pi} \exp \left(-\frac{1}{2} (w - \bar{v}_i)^* P_i (w - \bar{v}_i) \right)$$

as our generalized wind estimate at r_i . P_i includes a best guess $\bar{v}_i$ for the wind velocity at r_i in the absence of turbulence and terrain effects and also a measure P_i of the uncertainty in this guess as a result of these effects.

Finally, attention must be paid to at least one finer experimental point. Field measurements may require adjustment to compensate for differences between the heights above the ground at which the measurements are taken and the height above the ground at which the model predicts the wind. This adjustment depends on meteorological influences requiring considerable expertise to interpret.

APPENDIX G

CONSTRUCTION OF A GENERALIZED WIND ESTIMATE AT AN ARBITRARY POINT

APPENDIX G

CONSTRUCTION OF A GENERALIZED WIND ESTIMATE AT AN ARBITRARY POINT

The object of this section is to devise a scheme for computing P from $P_1, \dots, P_n$ and Q in a way which observes the uncertainty interpretation. Loosely speaking, the most agreeable procedure is to compute the uncertainty at r induced by the uncertainty at r_k via the linearized L_2 extrapolation, and then to sum these uncertainties at r as if they were independent Gaussian random variables. Adopting this procedure, we see that the problem of computing P is equivalent to the problem of computing the distribution of

$$U_0 + \sum_{k=1}^n F_k U_k ,$$

where $U_0, U_1, \dots, U_n$ are independent random variables with probability densities $q, p_1, \dots, p_n$, and

$$F_k \equiv \begin{pmatrix} \frac{\partial(v(r))_1}{\partial v_{k1}} & \frac{\partial(v(r))_1}{\partial v_{k2}} \\ \frac{\partial(v(r))_2}{\partial v_{k1}} & \frac{\partial(v(r))_2}{\partial v_{k2}} \end{pmatrix}$$

Before recalling the solution to this problem, we first remark that the matrices $F_1, \dots, F_n$ are most easily computed numerically, as in Appendix H for example, and that keeping track of $\bar{v}_1, \dots, \bar{v}_n$ may be abandoned with impunity.

We proceed abstractly. Let $W_1, \dots, W_m$ denote m independent random variables. Assume that each W_j has a probability density given by

$$\frac{\sqrt{|Q_j|}}{2\pi} \exp \left(-\frac{1}{2} w^* Q_j w \right) ,$$

where Q_j denotes a 2×2 positive definite quadratic form. Let $G_1, \dots, G_m$ denote m 2×2 matrices, where we assume that each G_j is invertible for the sake of convenience. The remainder of this section is devoted to computing the probability density for the random variable

$$W \equiv \sum_{j=1}^m G_j W_j$$

in terms of $G_1, \dots, G_m$ and $Q_1, \dots, Q_m$.

We first consider the case for $m=1$.

$$\begin{aligned} & P(G_1 W_1 \in B) \\ &= P(W_1 \in G_1^{-1} B) \\ &= \int_{G_1^{-1} B} \frac{\sqrt{|Q_1|}}{2\pi} \exp\left(-\frac{1}{2} w^* Q_1 w\right) dw \\ &= \int_B \frac{\sqrt{|Q_1|}}{2\pi} \exp\left(-\frac{1}{2} (G_1^{-1} w)^* Q_1 (G_1^{-1} w)\right) |G_1^{-1}| dw \\ &= \int_B \frac{\sqrt{|Q_1|}}{2\pi |G_1|} \exp\left(-\frac{1}{2} w^* (G_1^{-1} Q_1 G_1^{-1}) w\right) dw. \end{aligned}$$

Therefore, the probability density for the random variable $G_1 W_1$ is given by

$$\frac{\sqrt{|Q_1|}}{2\pi |G_1|} \exp\left(-\frac{1}{2} w^* (G_1^{-1} Q_1 G_1^{-1}) w\right).$$

We next state the following lemma as preparation for generalizing the above result. Let F denote the Fourier transform and define $s(w) = w^* S w$, where S denotes a 2×2 positive definite quadratic form. Then

Lemma: $F\left(\frac{\sqrt{|S|}}{2\pi} \exp\left(-\frac{1}{2} s\right)\right)(\beta) = \exp\left(-2\pi^2 \beta^* S^{-1} \beta\right).$

Pf:
$$\begin{aligned} & \exp(2\pi^2 \beta^* S^{-1} \beta) F\left(\exp\left(-\frac{1}{2} s\right)\right)(\beta) \\ &= \exp(2\pi^2 \beta^* S^{-1} \beta) \int dw \exp\left(-\frac{1}{2} w^* S w\right) \exp(-2\pi i \beta \cdot w) \end{aligned}$$

$$\begin{aligned}
&= \int dw \exp \left[-\frac{1}{2} (w^* S w + 2(2\pi i \beta) \cdot w + (2\pi i \beta)^* S^{-1} (2\pi i \beta)) \right] \\
&= \int dw \exp \left[-\frac{1}{2} (\sqrt{S} w + \sqrt{S^{-1}} (2\pi i \beta))^2 \right] \\
&= \int dw \exp \left[-\frac{1}{2} w^* S w \right] \\
&= \frac{2\pi}{\sqrt{|S|}} .
\end{aligned}$$

Now consider the general case. As above, the probability density for the random variable $G_j W_j$ is given by

$$\frac{\sqrt{|Q_j|}}{2\pi|G_j|} \exp \left(-\frac{1}{2} w^* (G_j^{-1} Q_j G_j^{-1}) w \right) .$$

Recall that the Fourier transform of the density of the sum of independent random variables is the product of the Fourier transforms of the densities of these random variables. By the lemma,

$$\begin{aligned}
&F \left(\frac{\sqrt{|Q_j|}}{2\pi|G_j|} \exp \left(-\frac{1}{2} s_j \right) \right) (\beta) \\
&= \exp \left(-2\pi^2 \beta^* (G_j Q_j^{-1} G_j^*) \beta \right) ,
\end{aligned}$$

where $s_j(w) \equiv w^* (G_j^{-1} Q_j G_j^{-1}) w$. Thus, the Fourier transform of the density for W is given by

$$\begin{aligned}
&\prod_{j=1}^m \exp \left(-2\pi^2 \beta^* (G_j Q_j^{-1} G_j^*) \beta \right) \\
&= \exp \left(-2\pi^2 \beta^* \left(\sum_{j=1}^m G_j Q_j^{-1} G_j^* \right) \beta \right) .
\end{aligned}$$

Therefore, the density for W is given by

$$\frac{\sqrt{|\tilde{Q}|}}{2\pi} \exp \left(-\frac{1}{2} w^* \tilde{Q} w \right) ,$$

where $\tilde{Q} \equiv \left(\sum_{j=1}^m G_j Q_j^{-1} G_j^* \right)^{-1}$.

APPENDIX H

A FORTRAN PROGRAM WHICH COMPUTES
THE GENERALIZED WIND ESTIMATE

APPENDIX H

A FORTRAN PROGRAM WHICH COMPUTES THE GENERALIZED WIND ESTIMATE

The object of this section is to list a FORTRAN program which computes $\bar{v}(r)$, P and $\psi(r)$ from given field measurements and state of nature. (The definition of ψ may be found in Section III.) Subroutines which compute f based on the monopole and self-consistent models of Appendix B are also listed.

The organization of the program corresponds to that of Section II. By interpreting the FORTRAN variables in terms of the notation of Section II, the meaning of the program becomes self-evident:

A, B, C - the elements of P

AD, BD, CD - the elements of P_k^{-1}

ADT, BDT, CDT - the elements of $F_k P_k^{-1} F_k^*$

ALPH = α

DERIV - the subroutine which computes the elements of F_k

DUM - a dummy variable whose meaning changes throughout the program

DV1D1 }
DV1D2 } - the elements of F_k
DV2D1 }
DV2D2 }

F = f

FIJ = f_{ij}

FIT - the subroutine which computes the parameters (U_1, U_2) ,
W and (C_1, C_2) from which $\bar{v}(r)$ is constructed

GAUSS1 - the subroutine which computes the elements of P_k^{-1}

GAUSS2 - the subroutine which computes the elements of Q^{-1}

INVERS - the subroutine which inverts P^{-1}

NATFIT - the version of FIT associated with the self-consistent model for f

NATURE - the subroutine which computes f

ND - the number of wind data at each sampling site

NS - the number of sampling sites

OMEG = W

PSI = $\psi(r)$

R = r_k

(RX, RY) = r

R1R1 = $C_1 \cdot C_1$

R2R2 = $C_2 \cdot C_2$

TRNSFM - the subroutine which transforms P_k^{-1} into $F_k P_k^{-1} F_k^*$

V = $\bar{v}_k$

(VX, VY) = $\bar{v}(r)$

(VXK, VYK) = $\bar{v}(r_k)$

V1R2 = $U_1 \cdot C_2$

V2R1 = $U_2 \cdot C_1$

```

SUBROUTINE WIND(RX,RY,VX,VY,A,B,C,PSI)

COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)

DATA(NTST=0)
IF(NTST.EQ.1)GO TO 10
NTST=1

C   SITE BY SITE GAUSSIAN DATA SUMMARY*****
DO 8 K=1,20

C   THE DATA AT EACH SITE*****
READ(5,1)ND,(R(K,I),I=1,2)
1  FORMAT(15,2F10.1)
   IF(ND.EQ.0)GO TO 9
   PRINT 2,ND,(R(K,I),I=1,2)
2  FORMAT(1H0,*ND,R(K)=*,I10,8X,2E10.3)
   DO 3 J=1,ND
   READ(5,4)(DUM(J,I),I=1,2)
   PRINT 5,(DUM(J,I),I=1,2)
3  CONTINUE
4  FORMAT(2F10.1)
5  FORMAT(1H0,*V(K,J)=*,2E10.3)

C   THE MEAN WIND AT EACH SITE*****
V(K,1)=V(K,2)=0.
DO 6 I=1,2
DO 6 J=1,ND
6  V(K,I)=V(K,I)+DUM(J,I)
   X=ND
   V(K,1)=V(K,1)/X
   V(K,2)=V(K,2)/X

C   THE COVARIANCE MATRIX AT EACH SITE*****
DO 7 I=1,2
DO 7 J=1,ND
7  DUM(J,I)=DUM(J,I)-V(K,I)
   CALL GAUSS1(ND,AD(K),BD(K),CD(K))

8  CONTINUE
   NS=20
   GO TO 10
9  NS=K-1

10  CONTINUE

```

```

C      THE STATE OF NATURE*****
      CALL NATURE(NS,RX,RY)

C      THE MEAN EXTRAPOLATED WIND*****
      CALL FIT(NS,OMEG,R1,R2,V1,V2)
      VX=V1+OMEG*(RY-R2)
      VY=V2+OMEG*(R1-RX)

C      THE COVARIANCE MATRIX CHARACTERIZING ERRORS IN THE FIT**
      DO 11 K=1,NS
      VXK=V1+OMEG*(R(K,2)-R2)
      VYK=V2+OMEG*(R1-R(K,1))
      DUM(K,1)=V(K,1)-VXK
11     DUM(K,2)=V(K,2)-VYK
      CALL GAUSS2(NS,AI,BI,CI)

C      THE TOTAL UNCERTAINTY IN THE WIND*****
      DO 12 K=1,NS
      CALL TRNSFM(K,NS,RX,RY)
      AI=AI+ADT(K)
      BI=BI+BDT(K)
12     CI=CI+CDT(K)
      CALL INVERS(AI,BI,CI,A,B,C)

      PSI=SQRT((A+C)/(A*C-B*B))

      RETURN
      END

```

```

      SUBROUTINE GAUSS1(ND,AI,BI,CI)

      COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)

      AI=BI=CI=0.
      DO 1 J=1,ND
      X1=DUM(J,1)
      X2=DUM(J,2)
      AI=AI+X1*X1
      BI=BI+X1*X2
1     CI=CI+X2*X2
      X=ND
      Y=1./X
      AI=Y*AI
      BI=Y*BI
      CI=Y*CI

      RETURN
      END

```

```

      SUBROUTINE NATURE(NS,RX,RY)

      COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)

C      THE MONOPOLE MODEL FOR F*****

      ALPH=0.5
      DO 1 K=1,NS
      UX=RX-R(K,1)
      UY=RY-R(K,2)
      BU=SQRT(UX*UX+UY*UY)
      BV=SQRT(V(K,1)**2+V(K,2)**2)
      TUV=UX*V(K,1)+UY*V(K,2)
      TR=SQRT((RX-R(K,1))**2+(RY-R(K,2))**2)
1      F(K)=0.5*(1.+(TUV/(BU*BV))**2)*EXP(-ALPH*(TR/BV))

      RETURN
      END

```

```

SUBROUTINE NATURE(NS,RX,RY)

COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)
DIMENSION ALPH(20,20),FIJ(20,20)

C   THE SELFCONSISTENT MODEL FOR F*****

DATA(NTST=0)
IF(NTST.EQ.1)GO TO 3
NTST=1

C   COMPUTE FIJ*****
EPS=0.0001
DO 1 I=1,NS
  FIJ(I,I)=1.
  DO 1 J=1,NS
    1  ALPH(I,J)=2.5
    DO 2 I=1,NS
      DO 2 J=1,NS
        IF(I.EQ.J)GO TO 2
        CALL NATFIT(NS,0,J,EPS,OMEG,R1,R2,V1,V2)
        VX=V1+OMEG*(R(J,2)-R2)
        VY=V2+OMEG*(R1-R(J,1))
        X=SQRT((VX-V(J,1))**2+(VY-V(J,2))**2)
        CALL NATFIT(NS,I,J,EPS,OMEG,R1,R2,V1,V2)
        VX=V1+OMEG*(R(J,2)-R2)
        VY=V2+OMEG*(R1-R(J,1))
        Y=SQRT((VX-V(J,1))**2+(VY-V(J,2))**2)
        FIJ(I,J)=EXP(-ALPH(I,J)*(X/Y))
    2  CONTINUE
  3  CONTINUE

C   COMPUTE F*****
DO 5 I=1,NS
  X=Y=0.
  DO 4 J=1,NS
    W=1./((RX-R(J,1))**2+(RY-R(J,2))**2+EPS)
    X=X+FIJ(I,J)*W
  4  Y=Y+W
5  F(I)=X/Y

RETURN
END

```

```

SUBROUTINE NATFIT(NS,I,J,EPS,OMEG,R1,R2,V1,V2)

COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)
DIMENSION WT(20)

TOT=0.
DO 2 K=1,NS
  IF(K.EQ.I)GO TO 1
  IF(K.EQ.J)GO TO 1
  WT(K)=1./((R(K,1)-R(J,1))**2+(R(K,2)-R(J,2))**2+EPS)
  TOT=TOT+WT(K)
GO TO 2
1  WT(K)=0.
2  CONTINUE
  W=1./TOT
DO 3 K=1,NS
3  WT(K)=W*WT(K)

R1=R2=V1=V2=0.
R1R1=R2R2=V1R2=V2R1=0.
DO 4 K=1,NS
  W=WT(K)
  R1=R1+W*R(K,1)
  R2=R2+W*R(K,2)
  V1=V1+W*V(K,1)
  V2=V2+W*V(K,2)
  R1R1=R1R1+W*R(K,1)**2
  R2R2=R2R2+W*R(K,2)**2
  V1R2=V1R2+W*V(K,1)*R(K,2)
4  V2R1=V2R1+W*V(K,2)*R(K,1)
  X=(V2R1-V2*R1)-(V1R2-V1*R2)
  Y=(R1*R1-R1R1)+(R2*R2-R2R2)
  OMEG=X/Y

RETURN
END

```

SUBROUTINE FIT(NS,OMEG,R1,R2,V1,V2)

COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)

```
      X=0.  
      DO 1 K=1,NS  
1      X=X+F(K)  
      Y=1./X  
      R1=R2=V1=V2=0.  
      R1R1=R2R2=V1R2=V2R1=0.  
      DO 2 K=1,NS  
      W=Y*F(K)  
      R1=R1+W*R(K,1)  
      R2=R2+W*R(K,2)  
      V1=V1+W*V(K,1)  
      V2=V2+W*V(K,2)  
      R1R1=R1R1+W*R(K,1)**2  
      R2R2=R2R2+W*R(K,2)**2  
      V1R2=V1R2+W*V(K,1)*R(K,2)  
2      V2R1=V2R1+W*V(K,2)*R(K,1)  
      X=(V2R1-V2*R1)-(V1R2-V1*R2)  
      Y=(R1*R1-R1R1)+(R2*R2-R2R2)  
      OMEG=X/Y  
  
      RETURN  
      END
```

SUBROUTINE GAUSS2(NS,AI,BI,CI)

COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)

```
      X=0.  
      Y=1.  
      AI=BI=CI=0.  
      DO 1 K=1,NS  
      Z=F(K)  
      X=X+Z  
      Y=Y*(1.-Z)  
      X1=DUM(K,1)  
      X2=DUM(K,2)  
      AI=AI+Z*X1*X1  
      BI=BI+Z*X1*X2  
1      CI=CI+Z*X2*X2  
      W=1./(X*(1.-Y)**2)  
      AI=W*AI  
      BI=W*BI  
      CI=W*CI  
  
      RETURN  
      END
```



```

SUBROUTINE TRNSFM(K,NS,RX,RY)

COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)

CALL DERIV(K,1,NS,RX,RY,DV1D1,DV2D1)
CALL DERIV(K,2,NS,RX,RY,DV1D2,DV2D2)

X11=DV1D1*AD(K)+DV1D2*BD(K)
X12=DV1D1*BD(K)+DV1D2*CD(K)
X21=DV2D1*AD(K)+DV2D2*BD(K)
X22=DV2D1*BD(K)+DV2D2*CD(K)

ADT(K)=X11*DV1D1+X12*DV1D2
BDT(K)=X11*DV2D1+X12*DV2D2
CDT(K)=X21*DV2D1+X22*DV2D2

RETURN
END

```

```

SUBROUTINE DERIV(K,I,NS,RX,RY,DV1DI,DV2DI)

COMMON R(20,2),V(20,2),DUM(20,2),F(20),
1AD(20),BD(20),CD(20),ADT(20),BDT(20),CDT(20)

D=0.5

V(K,I)=V(K,I)-D
CALL NATURE(NS,RX,RY)
CALL FIT(NS,OMEG,R1,R2,V1,V2)
VX1=V1+OMEG*(RY-R2)
VY1=V2+OMEG*(R1-RX)

V(K,I)=V(K,I)+2.*D
CALL NATURE(NS,RX,RY)
CALL FIT(NS,OMEG,R1,R2,V1,V2)
VX2=V1+OMEG*(RY-R2)
VY2=V2+OMEG*(R1-RX)

X=0.5/D
DV1DI=X*(VX2-VX1)
DV2DI=X*(VY2-VY1)
V(K,I)=V(K,I)-D

RETURN
END

```

```
SUBROUTINE INVERS(A,B,C,AI,BI,CI)
```

```
X=1./(A*C-B*B)
```

```
AI=X*C
```

```
BI=-X*B
```

```
CI=X*A
```

```
RETURN
```

```
END
```

APPENDIX I

AN ESTIMATE OF THE PROBABILITY THAT THE WIND
MODEL HAS PREDICTED THE ACTUAL WIND VELOCITY AT A GIVEN POINT

APPENDIX I

AN ESTIMATE OF THE PROBABILITY THAT THE WIND MODEL HAS PREDICTED THE ACTUAL WIND VELOCITY AT A GIVEN POINT

Define functions α and β by

$$\alpha(x) \equiv \frac{\sqrt{|A|}}{2\pi} \exp \left(-\frac{1}{2} (x-a)*A(x-a) \right) ,$$

and

$$\beta(x) \equiv \frac{\sqrt{|B|}}{2\pi} \exp \left(-\frac{1}{2} (x-b)*B(x-b) \right) ,$$

where A and B are positive definite quadratic forms. The object of this section is to evaluate their integral

$$\int \alpha(x) \beta(x) dx .$$

We first state a lemma which constitutes a two-dimensional version of "completing the square." Define

$$L(x) \equiv (x-a)*A(x-a) + (x-b)*B(x-b)$$

Then

Lemma: $L(x) = (x-c)*(A+B)(x-c) + \gamma$,

where $c = (A+B)^{-1}(Aa+Bb)$,

and $\gamma = a*Aa + b*Bb - (Aa+Bb)*(A+B)^{-1}(Aa+Bb)$.

Pf: $\nabla L(x) = 2A(x-a) + 2B(x-b)$,

so $\nabla L(x) = 0$ implies that

$$x = (A+B)^{-1}(Aa+Bb).$$

Thus, L has the form

$$L(x) = (x-c)*(A+B)(x-c) + \gamma ,$$

where c is given above. We may evaluate γ by equating the constant terms in the above expressions for $L(x)$. Thus

$$\begin{aligned}
& \gamma \\
&= a^*Aa + b^*Bb - c^*(A+B)c \\
&= a^*Aa + b^*Bb - (Aa+Bb)^*(A+B)^{-1}(Aa+Bb). \quad \blacksquare
\end{aligned}$$

Theorem: $\int \alpha(x) \beta(x) dx$

$$= \frac{1}{2\pi} \sqrt{\frac{|A||B|}{|A+B|}} \exp \left(-\frac{1}{2} (a-b)^* \left[B - B(A+B)^{-1}B \right] (a-b) \right) .$$

Pf: $\int \alpha(x) \beta(x) dx$

$$\begin{aligned}
&= \frac{\sqrt{|A|}}{2\pi} \frac{\sqrt{|B|}}{2\pi} \int \exp \left(-\frac{1}{2} L(x) \right) dx \\
&= \frac{\sqrt{|A|}}{2\pi} \frac{\sqrt{|B|}}{2\pi} \int \exp \left(-\frac{1}{2} L(x+a) \right) dx
\end{aligned}$$

By the lemma,

$$\begin{aligned}
&L(x+a) \\
&= x^*Ax + (x-(b-a))^*B(x-(b-a)) \\
&= (x-c')^*(A+B)(x-c') + \gamma' ,
\end{aligned}$$

where $c' = (A+B)^{-1}B(b-a)$,

and $\gamma' = (b-a)^*B(b-a) - (b-a)^*B^*(A+B)^{-1}B(b-a)$.

Thus,

$$\begin{aligned}
&\int \alpha(x) \beta(x) dx \\
&= \frac{\sqrt{|A|}}{2\pi} \frac{\sqrt{|B|}}{2\pi} \exp \left(-\frac{1}{2} \gamma' \right) \int \exp \left(-\frac{1}{2} x^*(A+B)x \right) dx \\
&= \frac{1}{2\pi} \sqrt{\frac{|A||B|}{|A+B|}} \exp \left(-\frac{1}{2} (a-b)^* \left[B - B(A+B)^{-1}B \right] (a-b) \right) . \quad \blacksquare
\end{aligned}$$

APPENDIX J

PROOFS OF SEVERAL BASIC PROPERTIES OF THE FUNCTION
ESTIMATING PROBABILITY OF AGREEMENT BETWEEN
PREDICTED AND ACTUAL WIND VELOCITIES

APPENDIX J

PROOFS OF SEVERAL BASIC PROPERTIES OF THE FUNCTION ESTIMATING PROBABILITY OF AGREEMENT BETWEEN PREDICTED AND ACTUAL WIND VELOCITIES

The object of this section is to prove the properties indicated for ϕ in Section III:

1) To prove that $\lim_{\varepsilon \rightarrow 0} \frac{\phi(\varepsilon)}{\pi \varepsilon^2} = p(\bar{v}-u)$,

first note that $(I+A)^{-1} = \sum_{k=0}^{\infty} (-1)^k A^k$ in case the eigenvalues of A are all less than 1 in absolute value:

$$\begin{aligned} & (I+A) \left(\sum_{k=0}^{\infty} (-1)^k A^k \right) \\ &= \sum_{k=0}^{\infty} (-1)^k A^k + \sum_{k=0}^{\infty} (-1)^k A^{k+1} \\ &= I + \sum_{k=1}^{\infty} (-1)^k A^k - \sum_{k=0}^{\infty} (-1)^{k+1} A^{k+1} \\ &= I. \end{aligned}$$

Next note that

$$\begin{aligned} & 4\varepsilon^{-4} (P+E)^{-1} \\ &= 4\varepsilon^{-4} \left(E \left(I + \frac{\varepsilon^2}{2} P \right) \right)^{-1} \\ &= 2\varepsilon^{-2} \left(I + \frac{\varepsilon^2}{2} P \right)^{-1} \\ &= 2\varepsilon^{-2} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\varepsilon^2}{2} P \right)^k , \end{aligned}$$

for sufficiently small ε . Thus,

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \left[E - 4\varepsilon^{-4} (P+E)^{-1} \right] \\
&= \lim_{\varepsilon \rightarrow 0} \left[E - 2\varepsilon^{-2} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\varepsilon^2}{2} P \right)^k \right] \\
&= \lim_{\varepsilon \rightarrow 0} \left[2\varepsilon^{-2} I - 2\varepsilon^{-2} I - 2\varepsilon^{-2} (-1) \frac{\varepsilon^2}{2} P + \varepsilon^2 o(\varepsilon) \right] \\
&= P.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \frac{\phi(\varepsilon)}{\pi \varepsilon^2} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \sqrt{\frac{|P||E|}{|P+E|}} \exp \left(-\frac{1}{2} (\bar{v}-u)^* \left[E - E(P+E)^{-1} E \right] (\bar{v}-u) \right) \\
&= \frac{\sqrt{|P|}}{2\pi} \exp \left(-\frac{1}{2} (\bar{v}-u)^* P (\bar{v}-u) \right) \\
&= p(\bar{v}-u) .
\end{aligned}$$

2) To prove that $\phi(\varepsilon_1) \leq \phi(\varepsilon_2)$ in case $\varepsilon_1 \leq \varepsilon_2$, it is enough to prove that $\frac{\partial \phi(\varepsilon)}{\partial \varepsilon} \geq 0$:

$$\begin{aligned}
& \frac{\partial \phi(\varepsilon)}{\partial \varepsilon} \\
&= \int p(x) \frac{\partial}{\partial \varepsilon} \left[\pi \varepsilon^2 \frac{\sqrt{|E|}}{2\pi} \exp \left(-\frac{1}{2} (u-x)^* E (u-x) \right) \right] dx \\
&= \int p(x) \frac{\partial}{\partial \varepsilon} \exp \left(-\frac{(x-u) \cdot (x-u)}{\varepsilon^2} \right) dx
\end{aligned}$$

APPENDIX K

COMPUTATION OF THE VARIANCE FOR A
GAUSSIAN PROBABILITY DISTRIBUTION

APPENDIX K

COMPUTATION OF THE VARIANCE FOR A GAUSSIAN PROBABILITY DISTRIBUTION

Let U denote a random variable with probability density given by

$$p(w) = \frac{\sqrt{|P|}}{2\pi} \exp \left(-\frac{1}{2} (w-\bar{v})^* P (w-\bar{v}) \right) ,$$

where $P \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is a positive definite quadratic form, and let E denote expectation with respect to p . Then

Proposition: $E (U-\bar{v}) \cdot (U-\bar{v}) = \frac{a+c}{ac-b^2} .$

Pf: $E (U-\bar{v}) \cdot (U-\bar{v})$

$$\begin{aligned} &= \int dx \int dy \frac{\sqrt{|P|}}{2\pi} (x^2+y^2) \exp \left(-\frac{1}{2} (ax^2+2bxy+cy^2) \right) \\ &= \frac{\sqrt{|P|}}{2\pi} \int dx \int dy (-2) \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial c} \right) \exp \left(-\frac{1}{2} (ax^2+2bxy+cy^2) \right) \\ &= \frac{\sqrt{|P|}}{2\pi} (-2) \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial c} \right) \int dx \int dy \exp \left(-\frac{1}{2} (ax^2+2bxy+cy^2) \right) \\ &= \frac{\sqrt{|P|}}{2\pi} (-2) \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial c} \right) \frac{2\pi}{\sqrt{|P|}} \\ &= -2 \sqrt{|P|} \left(\frac{-1}{2} \right) |P|^{-3/2} (a+c) \\ &= \frac{a+c}{ac-b^2} . \end{aligned}$$

APPENDIX L

COMPUTATIONS DEPICTING THE POTENCY OF FIELD
MEASUREMENTS IN VERIFYING WIND MODEL PREDICTIONS

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The object of this section is to present contours of ψ for the purpose of illustration. They have been computed using the state of nature models depicted in Figures B2 through B11, the computer program listed in Appendix H, and the site locations and corresponding sample wind data listed in Table L1. Note that $r_1, \dots, r_n$ and $\bar{v}_1, \dots, \bar{v}_n$ coincide with the corresponding data given in table B1. Contours for ψ based on the monopole model for f are given in Figure L1 and contours for ψ based on the self-consistent model for f are given in Figure L2.

Table L1

A List of the Sites and Sample Wind Data
Used in Illustrating the Computation of ψ

Site Number	Site Coordinates	Wind Velocity Coordinates
1	(1.,0.)	(1.9,0.0) (2.1,0.0) (2.0,-0.1) (2.0,0.1)
2	(0.,1.)	(0.4,-0.5) (0.6,-0.5) (0.5,-0.6) (0.5,-0.4)
3	(-1.,0.)	(1.9,0.0) (2.1,0.0) (2.0,-0.1) (2.0,0.1)
4	(0.,-1.)	(0.5,0.4) (0.5,0.6) (0.4,0.5) (0.6,0.5)
5	(2.,2.)	(0.25,-0.3) (0.25,-0.2) (0.2,-0.25) (0.3,-0.25)
6	(-2.,2.)	(0.25,-0.3) (0.25,-0.2) (0.2,-0.25) (0.3,-0.25)
7	(-2.,-2.)	(0.25,0.2) (0.25,0.3) (0.2,0.25) (0.3,0.25)
8	(2.,-2.)	(0.25,0.2) (0.25,0.3) (0.2,0.25) (0.3,0.25)

Figure L1: Contours for ψ Based on the Monopole Model for f . The coordinates of the border corners are $(4., 4.)$, $(-4., 4.)$, $(-4., -4.)$, $(4., -4.)$.

Figure L2: Contours for ψ Based on the Self-consistent Model for f . The coordinates of the border corners are $(4., 4.)$, $(-4., 4.)$, $(-4., -4.)$, $(4., -4.)$.

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