A MONTE-CARLO STUDY OF HOT SU(2) GLUE

L. McLerran* and B. Svetitsky**

Stanford Linear Accelerator Center
Stanford University
Stanford, California 94305
U.S.A.

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Non-perturbative studies of non-Abelian gauge theories have received widespread attention in recent months.¹⁻² In particular, SU(2) and SU(3) Yang-Mills theories in the absence of fermions have been studied numerically as lattice gauge theories.¹⁻³ Monte-Carlo integrations of Feynman path integrals have been evaluated to obtain expectation values of quantum operators. These same numerical techniques are well known to condensed matter theorists, and have been developed and applied to study spin systems.

Ben Svetitsky and I have been using Monte-Carlo integration techniques to study SU(2) Yang-Mills theory at finite temperature.⁴ This same problem has been independently and simultaneously studied by J. Kuti, J. Polonyi, and K. Szlachanyi⁵ and is also being considered by J. Engels, F. Karsch, and H. Satz.⁶

The starting point for this analysis is the Feynman path integral representation for the expectation values of operators at finite temperature,

$$\langle O \rangle = \frac{\text{Tr } e^{-\beta H}}{\text{Tr } e^{-\beta H}} = \frac{\int (\prod_x dU(x)) O(U) \exp - \int_0^\beta dt \int_V d^3x \mathcal{L}(U(x))}{\int (\prod_x dU(x)) \exp - \int_0^\beta dt \int_V d^3x \mathcal{L}(U(x))} \quad (1)$$

The set of variables U in this integration represent the fields, the operator $O(U)$ is a functional of these fields, $\mathcal{L}(U(x))$ is the Euclidean space Lagrange density, β is the inverse temperature, and V is the spatial volume. The space-time metric $g^{\mu\nu}$, is Euclidean.

To evaluate these integrals, the fields may be described on a four-dimensional discrete Euclidean space-time grid. The grid is chosen with lattice spacings (grid spacings) a_t in the time direction and a_s in the space direction. The number of lattice sites (grid points) is N_t in the time direction and N_s in the space direction. The inverse temperature and volume are given by

$$\beta = N_t a_t \quad (2)$$

$$V^{1/3} = N_s a_s$$

The continuum limit corresponds to $N_t, N_s \rightarrow \infty$ and $a_t, a_s \rightarrow 0$, with $N_t a_t$ and $N_s a_s$ held fixed. The limit $a_t, a_s \rightarrow 0$ means that a_t and a_s are taken to be small compared to a scale size typical of the operator being evaluated.

For renormalizable field theories such as Yang-Mills gauge theories, this zero lattice size limit is singular. The bare coupling constant g_0 must be adjusted simultaneously with the lattice size to allow for a finite limiting value for physically measurable quantities. For non-Abelian gauge theories, the bare coupling constant g_0 should approach zero as the lattice spacings vanish. A convenient physical quantity to hold fixed while this limit is taken is the string tension. The string tension is a measure of the strength of interaction between a quark antiquark pair.

K. Wilson has derived a form of the Lagrange density for non-Abelian gauge theories on a lattice. This Lagrange density is gauge invariant, and in the continuum limit yields the continuum Lagrange density appropriate for Yang-Mills fields. This Lagrange density is conveniently expressed in terms of the link variable

$$U^\mu(x) \equiv e^{i\tau \cdot A^\mu(x)} \quad (3)$$

where the matrix τ^a is a generator of SU(N),

$$[\tau^a, \tau^b] = i f^{abc} \tau^c \quad (4)$$

in the fundamental representation.

With the lattice spacing in the direction $\hat{e}^\mu$ denoted by a^μ , the Lagrange density may be expressed in terms of link variables as

$$\mathcal{L}(x) = \frac{1}{2} \sum_{\mu \neq \nu} \frac{1}{2} \text{tr} \left\{ 1 - U^\mu(x) U^\nu(x+a^\mu) U^{\mu-1}(x+a^\nu) U^{-1}(x) \right\} \quad (5)$$

This Lagrangian is invariant under the gauge transformation

$$U^\mu(x) = V(x+a^\mu) U^\mu(x) V^{-1}(x) \quad (6)$$

where $V(x)$ represents a matrix which is a member of the gauge group

$$V(x) = e^{i\tau \cdot \Lambda(x)} \quad (7)$$

The Lagrangian density of Eq. (4) is defined on a discrete set of space-time points. The functional integral of Eq. (1) may now be attempted since this integral is now properly defined. The Monte-Carlo integration methods developed by Creutz, Jacobs and Rebbi,¹ and Creutz² may be employed in this evaluation.

An interesting quantity to evaluate by these techniques is the free energy of an isolated quark. To determine an expression for this free energy, I shall introduce operators which create and annihilate static quarks at the position $\vec{r}$ and time t as

$$[\psi_a(\vec{r}, t), \psi_b^\dagger(\vec{r}, t)] = \delta_{ab} \quad (8)$$

with all other equal time anti-commutators vanishing. These quark fields also satisfy the equation of motion

$$\left\{ \frac{1}{i} \frac{\partial}{\partial t} - \tau \cdot A^0(\vec{r}, t) \right\} \psi(\vec{r}, t) = 0 \quad (9)$$

which may be integrated to obtain

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$$\psi(\vec{r}, t) = T e^{\int_0^t dt' \tau \cdot A^0(\vec{r}, t')} \psi(\vec{r}, 0) \quad (10)$$

In this equation, T denotes a time ordered exponential.

The free energy of an isolated quark therefore is

$$\begin{aligned} e^{-\beta F} &= \frac{1}{N} \sum_{\text{All states } s' \text{ with heavy quarks}} \langle s' | e^{-\beta H} | s' \rangle \\ &= \frac{1}{N} \sum_{\text{All states } s \text{ with no heavy quarks}} \langle s | \sum_a \psi_a(\vec{r}, 0) e^{-\beta H} \psi_a^\dagger(\vec{r}, 0) | s \rangle \\ &= \frac{1}{N} \text{Tr} \int e^{-\beta H} \psi_a(\vec{r}, 0) \psi_a^\dagger(\vec{r}, 0) \\ &= \frac{1}{N} \sum_a \langle \psi_a(\vec{r}, 0) \psi_a^\dagger(\vec{r}, 0) \rangle \end{aligned} \quad (11)$$

In these equations, the parameter N is the N of $SU(N)$, and T is the inverse temperature or alternatively the size of the four-dimensional field theory system in Euclidean time. Now upon using Eq. (10) for the time dependence of ψ , Eq. (8) for the equal time anticommutator of ψ and $\psi^\dagger$, and the definition of the Wilson line as

$$L(\vec{r}) = \frac{1}{N} \text{tr} T e^{\int_0^\beta dt' \tau \cdot A^0(\vec{r}, t')} \quad (12)$$

Equation (11) becomes⁷⁻¹⁰

$$e^{-\beta F} = \langle L(\vec{r}) \rangle \quad (13)$$

The expectation value of Eq. (13) is invariant under gauge transformations V which satisfy

$$V(\vec{r}, \beta) = V(\vec{r}, 0) \quad (14)$$

The class of allowable gauge transformations is in general, however, not restricted to periodic gauge transformations. The constraint is only that the links $U^a(\vec{r}, t)$ remain periodic under gauge transformation. This requirement forces

$$V(\vec{r}, \beta) = C V(\vec{r}, 0) \quad (15)$$

where C is an element of the center of the group of gauge transformations,

$$C = \begin{pmatrix} e^{2\pi i j/N} & & \\ & e^{2\pi i j/N} & \\ & & e^{2\pi i j/N} \end{pmatrix} \quad (16)$$

and where j is an integer.^{4,9,11-12}

The symmetry of Eqs. (15-16) is a Z_N symmetry of the gauge theory. Under this transformation, the gluon Lagrange density is unchanged. The Wilson line is changed, however, by

$$L(\vec{r}) \rightarrow \frac{1}{N} \text{tr} e^{\frac{2\pi i j}{N} \int_0^\beta dt' \tau \cdot A^0(\vec{r}, t')} \quad (17)$$

If this symmetry is dynamically realized the free energy of a quark is divergent since

$$e^{-\beta F} = \langle L(\vec{r}) \rangle = 0 \quad (18)$$

The quarks are not realizable as physical states.

This Z_N symmetry, however, need not be dynamically realized. Field configurations which are related by a Z_N gauge transformation need not be connected smoothly with finite probability by a set of finite energy field configurations. The field theory could be realized with a broken Z_N symmetry phase.

There are N broken symmetry phases parameterized by the expectation values of L . In weak coupling, the allowed values of $\langle L \rangle$ are

$$\langle L \rangle = e^{\frac{2\pi i j}{N}}, \quad j = 0, 1, \dots, N-1 \quad (19)$$

These values correspond to Z_N transformations of $\langle L \rangle = 1$, the value true for weak coupling where the quark energy is zero. As a function of $\langle L \rangle$, the free energy could have local minima past some critical value of $1/g^2$, corresponding to the onset of spontaneous symmetry breakdown (Fig. 1).

This Z_N symmetry and its realization may be illustrated by a simple example.¹¹⁻¹² Suppose all the timelike links on an equal time surface were to be flipped by a gauge transformation in the center of the group. The action would be unchanged by this transformation. The probability of this transformation occurring dynamically is, however, very small, since all equal time links must flip simultaneously. The dynamics of the field theory also may not allow for a continuous evolution through probable finite energy field configurations between these Z_N related field configurations. The Z_N symmetry may therefore be spontaneously broken.

Such a symmetry breakdown may be studied in finite temperature Monte-Carlo calculations. The lattice may be chosen as a rectangle with $N_t < N_s$, and the lattice spacings as $a_t = a_s$.⁴ (Other choices of N_t , N_s , a_t and a_s are also possible and have been considered by J. Kuti *et al.*)⁵

The lattice spacing may be varied by changing the coupling strength g , since varying the coupling constant varies the force between a static quark-antiquark pair separated by a fixed number of lattice sites. On dimensional grounds, however, the string tension which characterizes the strength of this force must be

Figure 1

An effective potential as a function of $\langle L \rangle$ for SU(2) where $\langle L \rangle$ is real and $-1 \leq \langle L \rangle \leq 1$.

proportional to $1/a$. The string tension and the lattice spacing a are related, therefore, in a calculable way.

Since the temperature is $N_t a$, the variation of g also yields a variation of the temperature. The graph of Fig. 2 shows the behavior of the order parameter $\langle L \rangle$ as a function of $1/g_0^2$. This variation is for a lattice with $N_t = 3$, $N_s = 7$. The Z_N symmetry breaking transition occurs at $1/g_0^2 = 2.15$. This coupling corresponds to a temperature $T \sim 200$ MeV if the string tension is determined from a Regge trajectory slope of $\alpha = 1 \text{ GeV}^{-2}$. This Regge trajectory has nothing directly to do with the observed slope in hadronic interactions, since our calculations are valid only for SU(2) gauge theories without light fermions.

The entropy, action, and energy densities show no evidence of discontinuities at the transition. This lack of discontinuities is not a reliable indicator of the order of the phase transition, however, since the finite temperature contributions to these quantities are difficult to isolate, due to quartic divergences in the corresponding quantities in the vacuum. The residual finite temperature contributions may be evaluated with marginal statistical validity.

Long-range fluctuations in the order parameter L are apparent above and below the phase transition. This fact, together with the lack of evidence of discontinuities in the entropy, action, and energy densities, suggests that the phase transition is of second order. This interpretation is strengthened by strong coupling analysis of lattice gauge theories. Ben Svetitsky and I have computed the critical index for $\langle L(\vec{o}) \rangle$ and find

$$\langle L(\vec{o}) \rangle \sim (T - T_c)^{\gamma} \quad (20)$$

Figure 2

The order parameter $\langle L \rangle$ as a function of $1/g_0^2$ for SU(2) in a lattice where $N_t = 3$, $N_s = 7$.

where $\gamma = .21 \pm .01$ for $N_t = 3$, $N_s = 7$.

The energy of separation of a two quark system is

$$e^{-BV(\vec{r})} = \langle L(\vec{r}) L(\vec{o}) \rangle \quad (21)$$

For SU(2), the quarks are in a real representation, and there is no difference between this system and a quark antiquark system. For SU(N), the energy of a quark-antiquark system is

$$e^{-BV(\vec{r})} = \langle L(\vec{r}) L^\dagger(\vec{o}) \rangle \quad (22)$$

where

$$L^\dagger \equiv \frac{1}{N} \text{tr} \left(P e^{\int_0^{\beta} dt \text{tr} \cdot A^0(\vec{r}, t)} \right)^\dagger \quad (23)$$

Using the generalizations of Eqs. (21)-(23) for static multi-quark configurations, it is possible to construct an expression for the energy of a multi-quark state. The Z_N transformation of Eq. (17) may then be applied to conclude that the only states which may have finite energy are those with the number of quarks minus the number of antiquarks equal to an integer multiple of N . These states are states possessing singlet configurations of quarks. In the confined phase where the Z_N symmetry is manifest, only singlet configurations of quarks may have finite energy.

Another interesting fact is that near the phase transition, the numerical value of $\langle L \rangle$ may remain constant for a very large number of iterations and then change

Figure 3

The Wilson line on a $N_t = 1$, $N_s = 5$ lattice as a function of the number of runs through the lattice.

sign (Fig. 3). (Recall that for $SU(2)$, $\langle L \rangle$ is 1, -1 in weak coupling.) As the critical temperature is approached, these changes in sign occur more often. This circumstance suggests that the confinement-deconfinement transition may be generated by the formation of domains with different values of $\langle L \rangle = \pm 1$. Near the transition, the system may begin to fill up with bubbles, bubbles having different values of $\langle L \rangle$ inside and outside.

This bubble formation may be realized by instantons.¹¹⁻¹² The instantons tunnel between field configurations of differing values of $\langle L \rangle$. The instantons correspond to bubble surfaces which separate domains characterized by different values of $\langle L \rangle$. It would be interesting to determine how well a dilute instanton gas calculation would reproduce the Monte-Carlo evaluation of $\langle L \rangle$.

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- * Present address: Department of Physics FM-15, University of Washington, Seattle, Washington 98195, U.S.A.
 ** Present address: Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, California 93106, U.S.A.
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