ORO 5126-158
U. of Md. PP #82-153

NUCLEAR DISSIPATION AS DAMPING OF COLLECTIVE MOTION
IN THE TIME-DEPENDENT RPA AND EXTENSIONS OF IT

Ph.D. Dissertation by

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July 1982

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by

Constantine P. Yannouleas

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Dissertation submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1982

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Title of Dissertation: "Nuclear Dissipation as Damping of Collective Motion
in the Time-Dependent RPA and Extensions of It"

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2. NUCLEAR DISSIPATION AS DAMPING OF COLLECTIVE MOTION IN THE TIME-DEPENDENT RPA. I: THE MICROSCOPIC MODEL, C. Yannouleas, M. Dworzecka and J. J. Griffin, Nucl. Phys. A379 (1982) 256-286.

Professional Publications (cont.):

B. Papers presented in scientific meetings:

1. MICROSCOPIC NUCLEAR DISSIPATION MECHANISM AS DAMPING OF COLLECTIVE MOTION IN THE SECOND RPA, C. Yannouleas, M. Dworzecka and J. J. Griffin, Proc. Int. Work. on Gross Properties of Nuclei and Nuclear Excitations X (Hirschegg, Austria, January 18-23, 1982), p. 163.
2. THE "WALL FORMULA" FOR NUCLEAR DISSIPATION AS CLASSICAL LIMIT OF THE DAMPING OF COLLECTIVE MOTION IN THE TIME-DEPENDENT RPA, C. Yannouleas, M. Dworzecka and J. J. Griffin, Proc. Work. on Nuclear Dynamics (Granlibakken, California, February 22-26, 1982), p. 40.
3. TIME-DEPENDENT SCHRÖDINGER CALCULATIONS OF INDEPENDENT PARTICLE MATTER FLOW IN A ONE-DIMENSIONAL BOX, C. Yannouleas and J. J. Griffin, APS Bull. 23 (1978) 537.
4. A WALL FORMULA INCLUDING SUPERFLUIDITY EFFECTS, C. Yannouleas and M. Dworzecka, APS Bull. 24 (1979) 592.
5. MICROSCOPIC DERIVATION OF THE WALL FORMULA, C. Yannouleas, M. Dworzecka and J. J. Griffin, APS Bull. 26 (1981) 592.
6. NUCLEAR DISSIPATION AS DAMPING OF THE ONE-PHONON COLLECTIVE STATE IN THE SECOND RPA, C. Yannouleas, M. Dworzecka and J. J. Griffin, APS Bull. 27 (1982) 575.
7. NONLINEAR QUANTAL FRICTION FROM A MICROSCOPIC MODEL, M. Dworzecka, C. Yannouleas and J. J. Griffin, APS Bull. 27 (1982) 576.

ABSTRACT

Title of Dissertation: "Nuclear Dissipation as Damping of Collective Motion in the Time-Dependent RPA and Extensions of It"

Constantine P. Yannouleas, Doctor of Philosophy, 1982

Dissertation directed by: James J. Griffin, Professor of Physics, and Maria Dworzecka, Assistant Professor of Physics
Department of Physics and Astronomy

We have formulated a nonperturbative, microscopic dissipative process in the limit of an infinite mean free path. This process does not require any statistical assumptions. It attributes the damping of the collective motion to real transitions from the collective state to degenerate, more complicated nuclear states. The dissipation is described through wave packets which, initialized with the collective state, solve an approximate Schrödinger equation within extended subspaces, larger than the original subspace of the undamped motion.

When the simple EPA is used, this process associates the dissipation with the escape width for direct particle emission. When the Second RPA is used, it associates the dissipation with the spreading width for transitions to the 2p-2h components of the nuclear compound states. The energy loss rate for sharp n-phonon initial states is proportional to the total collective energy, unlike classical dissipation, where it is proportional to the kinetic part only. The classical dissipation, however, is obtained for coherent, multiphonon, initial packets. These multiphonon packets describe the damping of the mean field oscillations, and allow a theoretical connection with the Vibrating Potential Model, and thereby with models of one-body dissipation, in particular, the Swiatecki "Wall Formula". The present model contrasts with linear response theories which relate to the adiabatic cranking model.

Canonical coordinates for the collective degree of freedom are explicitly introduced. This allows the construction of a nonlinear frictional Hamiltonian which provides a connection with quantal friction.

The dissipation process developed here is properly *reversible* rather than *irreversible*, in the sense that it is described by an approximate Schrödinger equation which honors time reversibility, rather than by a coarse grained master equation which violates it. Thus, the present theory contrasts with transport theories which extend the TDHF equations by the addition of a time *irreversible* Boltzmann "collision" term.

NUCLEAR DISSIPATION AS DAMPING OF COLLECTIVE MOTION
IN THE TIME-DEPENDENT RPA AND EXTENSIONS OF IT

Dedication

Στοὺς γονεῖς μου
Παναγιώτη καὶ Ἑλένη Γιαννουλέα

To my parents
Panayotis and Helen Yannouleas

by
Constantine P. Yannouleas

Dissertation submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1982

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ACKNOWLEDGEMENTS

I wish to express deep appreciation to my advisors,
Prof. James J. Griffin and Dr. Maria Dworzecka, for their
excellent guidance during the course of my research.

Many thanks are due to Loretta Robinette for her great
performance in typing this manuscript.

Finally, I wish to thank my wife, Brenda, for her
understanding, patience and encouragement.

The support of the U. S. Department of Energy for this
work is gratefully acknowledged

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I. INTRODUCTION

I.1 Historical Background

A growing body of evidence about dissipative phenomena in nuclei is being supplied by experimental studies of the deep inelastic heavy-ion collisions,¹⁻⁵ of fission,^{6,7} and of the widths of giant vibrations.⁸⁻¹⁰ As a result, considerable effort¹⁻³⁵ has been devoted toward the formulation of both phenomenological and microscopic models for nuclear dissipation. The intent of these models is to explain how the energy of the collective degree of freedom is converted into excitation energy in intrinsic degrees of freedom.

The first formulae applied in the analysis of nuclear dissipative phenomena were based on the assumption that nuclear dissipation proceeds primarily through the "two-body/short mean free path" viscosity of Navier-Stokes hydrodynamics,⁷ as in the case of ordinary fluids. This "two-body/short mean free path" hydrodynamic viscosity is always a bulk phenomenon, and results from an irreversible collision term¹¹ in the equation of motion for the one-body density matrix. In the framework of nuclear physics, several authors¹²⁻¹⁸ have proposed both heuristic and microscopic statistical transport theories, which yield such a collision term. The physical processes underlying this collision term are the binary collisions¹¹⁻¹⁸ between individual nucleons. Binary collisions are processes describing the decay of an initial single-particle excitation to other more complicated 2p-1h excitations degenerate with the original single-particle state.^{11-20,36,37} The mean free path, λ_{mfp} , is proportional to the lifetime, $1/\Gamma_{sp}$, of a single-particle excitation^{12b,20,36,37} according to the relation:

$$\lambda = \frac{1}{\Gamma_{sp}} \sqrt{\hbar^2 \epsilon_{sp} / 2\mu}, \quad (I.1.1)$$

where Γ_{sp} is the single-particle decay width, ϵ_{sp} the single-particle energy and μ the particle mass.* One expects the influence of the

* According to Refs. 36,37, μ in (I.1.1) should be the "effective mass".

collision term to be significant in the case of short mean free paths and high enough nuclear temperatures arising, probably, at the later stages of deep inelastic heavy-ion reactions.^{12b}

However, for nuclear temperatures up to a few MeV (≈ 4.5 MeV (according to Ref. 20)), the mean free path for bound nucleons is large compared with the nuclear dimensions because of the Pauli exclusion principle.^{11,12b,19,20} This long mean free path suggests that nuclear dissipation proceeds mainly through processes which require, as a necessary precondition, the mediation of the collective degree of freedom, a situation very different from the case of ordinary viscosity. In a phenomenological approach, the dissipative exchange of energy between nuclear collective and intrinsic degrees of freedom may then be viewed as proceeding through collisions of independent nucleons with the time-varying part of the nuclear mean field,²¹⁻²⁸ or, in the case of a leptodermous system, with a moving boundary of the system.^{21,22,26} This approach has been schematized by replacing the self-consistent time-varying mean field by an externally driven one-body potential which contains a Fermi gas of nucleons, leading to the "one-body/long mean free path" theories such as the piston model proposed by Gross,²⁷ the wall formula proposed by Swiatecki,^{21,22,26} and the linear response theory of Hofmann and Siemens.²⁸

Still a third approach to nuclear dissipation does not address the question of long versus short mean free path, but concentrates instead on formulating frictional nonlinear Schrödinger equations by inquiring about the proper quantization rules to be imposed upon classical friction.^{32-35,38,39} A special problem related to this approach is the quantum mechanics of the damped harmonic oscillator.³⁸

A common limitation of all the proposed theories until now is that they do not address the problem of what constitutes collective, as opposed to intrinsic, motion, but instead assume that the proper collective and intrinsic degrees of freedom are known. The problem of making an explicit separation of the nuclear degrees of freedom into collective and intrinsic coordinates--by which the theory would encompass the physical fact that the collective motion is nothing more nor less than a coherent motion of the individual nucleons themselves--is not addressed. The present research provides a theoretical formulation in which this deficiency is rectified.

I.2 Scope of the Dissertation

Within the current research context outlined above, the purpose of the research described herein was to formulate a microscopic model for nuclear dissipation which:

- (i) takes into account the long mean free path of bound nucleons with respect to the nuclear dimensions (in particular, this microscopic model describes nuclear dissipation in the limit of infinite mean free path);
- (ii) addresses the question of the proper explicit separation of the nuclear degrees of freedom into collective and intrinsic coordinates;
- (iii) provides a connection with phenomenological theories; specifically, the nonlinear quantal friction,³²⁻³⁵ the Swiatecki wall formula,^{21,22,26} and the linear response theories.²⁸

The present microscopic model utilizes a time-dependent version of the usual stationary RPA^{40,41} and extensions⁴²⁻⁴⁴ of it. This time-dependent RPA describes the damping of the collective motion through appropriate wave packets which, initialized with the collective state, evolve in time within extended subspaces larger than the restricted original subspace of the undamped collective motion.

Chapter II describes the damping of the one-phonon RPA collective state. This one-phonon RPA state is defined within a restricted subspace, S_R , of discrete 1p-1h states. Since it is defined within a subspace of the full Hilbert space only, the collective state is not an eigenstate of the complete Hamiltonian, and, therefore, it can decay in time through real transitions to other more complicated nuclear states degenerate

with the collective state. These transitions result from residual couplings between the restricted subspace and the more complicated states within an additional subspace, S_A . When S_A contains additional states of the 1p-1h structure only, the energy dissipation rate is associated with the escape width, $\Gamma^\dagger$, for direct particle emission to the continuum. When S_A contains additional states of the 2p-2h structure as well, the energy dissipation rate is associated--besides $\Gamma^\dagger$ --with the spreading width $\Gamma^\dagger$ to nuclear compound states.

When S_A contains additional states of the 1p-1h structure only, the equations of motion in the extended subspace, $S = S_R + S_A$, remain inside the simple RPA framework. Since the simple RPA averages out the two-body forces into effective one-body interactions, the dissipation process described by this time-dependent version of the simple RPA can properly be characterized as one-body dissipation. When S_A contains 2p-2h states as well, the equations of motion in the extended subspace are described by extensions of the simple RPA, referred to as Second RPA. The couplings between 1p-1h and 2p-2h states are due to residual two-body interactions, which persist beyond the averaging of the simple RPA. However, observe that the class of residual two-body interactions as utilized by the present model, do not generate binary collisions between individual particles in the sense of a collision term, that is real transitions from single-particle initial excitations to 2p-1h states degenerate with the original single-particle state. The lifetime of the single-particle excitations entering in the Second RPA is infinite, and, correspondingly, the mean free path remains infinite.* Rather, these residual two-body

*This infinite mean free path limit is not expected to be realistic in all cases. The present model needs, therefore, to be generalized in order to account for a long but finite mean free path. Such a generalization, however, is beyond the scope of the present dissertation.

interactions, as utilized by the present model, account for couplings between nucleons and the collective phonons. In the case of a leptodermous and incompressible system, a phenomenological viewpoint would consider the present treatment of the two-body residual forces as describing interactions of nucleons with a moving surface of the system.* Thus, even when applied in the framework of the Second RPA, the present microscopic model relates to the "one-body/long mean free path" phenomenological theories, instead of to the "two-body/short mean free path" theories where dissipation is always a bulk phenomenon. However, because of the residual two-body forces in the Second RPA, the use of the term "one-body" to characterize all long mean free path dissipative phenomena is misleading. A better term is particle-phonon** dissipation. This term contrasts with the particle-particle dissipation of ordinary viscosity.

Repeated application of the RPA collective operator--defined only in the restricted subspace, S_R --upon the ground state generates an infinite set of sharp n-phonon states. Chapter III examines the decay of these sharp n-phonon states. The main result here is that the energy loss rate is proportional to the total collective energy.

Chapter IV describes the damping of the time-varying oscillations of the nuclear mean field. Such mean field oscillations require initial wave packets formed by linear superposition of the sharp n-phonon states. In this chapter, explicit coordinate and momentum variables for the collective degree of freedom are introduced. The collective energy of the system is calculated as the sum of the collective kinetic and collective potential energy parts of a nonlinear frictional Hamiltonian con-

*The nucleons interacting with the moving surface, however, are no longer independent.

**In the case of a leptodermous and incompressible system, another term can be particle-surface dissipation.

structed to prescribe the same time behavior for the collective coordinates as does the exact Hamiltonian in the RPA and extensions of it. The energy dissipation rate is found proportional to the kinetic part of the total collective energy. This is also the case with classical dissipation, but contrasts with the dissipation of the sharp n-phonon initial states. The infinite lifetime of either the simple or Second RPA single-particle states is reflected in the fact that the diagonal elements of the one-body density matrix (the occupation probabilities of the single-particle levels) are, here, time-independent. To describe the damping of the mean field oscillations, the present dissipative process involves exclusively the off-diagonal density matrix elements, unlike the case with the transport theories¹²⁻¹³ where the diagonal elements solely contribute to the dissipation process.

Chapter V provides a description of dissipation complete, from the collective viewpoint, within the collective coordinate subspace, and thereby makes the connection with the phenomenological nonlinear quantal friction.³²⁻³⁵

Chapter VI focuses on the connection with the phenomenological Vibrating Potential Model^{40a,45} for the nuclear collective motion, and thereby with the Swiatecki wall formula²¹ for nuclear dissipation.

Chapter VII contrasts the present RPA model to the linear response theories²⁸ for nuclear dissipation.

Chapter VIII presents a summary and concluding remarks.

We note that the dissipation mechanism presented in this research does not require any assumptions of a statistical nature. The decay it describes is the behavior of an approximate solution of truly reversible dynamical equations of motion. In particular, the dissipative process here is a reversible, rather than irreversible process, properly so

called in that it is described by an approximate Schrödinger equation which honors time reversibility, rather than by a coarse grained master equation which violates it.⁴⁵ Thus, in this respect also, the present theory contrasts with the transport theories,¹²⁻¹⁸ which extend the TDHF equations by the addition of a time irreversible Boltzmann "collision" term.

The present theory is rather analogous to the reversible Landau damping of plasma oscillations as described by the many-body quantum theory of ionized gases.^{47,48} Besides reversibility, the analogy consists in that both the collective plasmon mode and its decay by Landau damping are described by using the RPA approximation^{47,48} and extensions of it.⁴⁸ However, the present model describes a finite system initialized at zero temperature, whereas Landau damping theory describes an infinite medium at finite temperature. Landau first demonstrated the Landau damping⁴⁹ of the plasmon mode in an electron gas by solving the *collisionless* Boltzmann (or Vlasov) equation using the Laplace transform method. Later, Van Kampen⁵⁰ and Dawson⁵¹ provided interpretations of the Landau damping closer in spirit to our model for nuclear dissipation, which ascribes the damping mechanism to the time evolution of special wave packets.

Since the present model is microscopic and quantal, it can describe dissipation in extreme quantal cases which have no classical analog, i.e., the damping of sharp α -phonon initial states. This contrasts with the phenomenological "one-body/long mean free path" approaches,²¹⁻²⁸ where the phenomenology itself depends upon the existence of classical analogs in order to introduce collective variables and, thereby, the time variations of the nuclear mean field.

Finally, observe that the present RPA treatment is nonperturbative in time, unlike the model of Gross,²⁷ the linear response theories of Hofmann and Siemens,²⁸ and the Koonin et al.²² approach to the wall formula, all three of which utilize a time perturbative method.

Appendix A summarizes some of the simple RPA and Second RPA formalism used in Ch. II.

Appendix B presents an alternative mathematical derivation of the results of Ch. II for the case when S_A has a dense energy spectrum, based on a direct solution of the time-dependent RPA equations of motion.

Appendix C provides some explicit formalism in support of Sec. II.3, and Appendix D analyzes the conditions for the validity of the approximation (II.3.18).

Appendix E calculates some matrix elements used in Ch. IV in the undamped mean field case.

Appendix F calculates some matrix elements used in Ch. IV in the description of damped mean field oscillations.

Finally, Appendix G calculates the proportionality coefficient γ (Eq. (VI.2.14)) in the large radius/infinite square well shell model potential limit, and Appendix H discusses the exact way to count the number of states in a spherical Fermi gas.

II. THE DAMPING OF THE ONE-PHONON COLLECTIVE STATE

II.1 Introduction

The microscopic model developed in Ch. II describes the initial collective state as a one-phonon coherent state obtained by the diagonalization of the complete many-body Hamiltonian within a restricted one-particle one-hole subspace of the full Hilbert space only into which, it is presumed, the excitation process envisaged is able to excite the ground state. Since it is defined in a subspace, the collective state is not an eigenstate of the complete Hamiltonian, and, therefore, it can decay in time as a result of residual couplings between the restricted subspace and the rest of the Hilbert space.

In Sec. II.2 we give a general formulation of such a microscopic dissipative process. First, we consider the exact problem, specified by the complete Hamiltonian and its eigenstates in the full Hilbert space. Then we describe the damping of the collective motion by the time evolution of the wave packet which solves formally the initial value problem defined by the complete, exact, time-dependent many-body Schrödinger equation initialized with the one-phonon collective state. Under appropriate conditions, the probability amplitude for a dissipative system to be found in the collective state at a later time, as well as its collective energy, is expected to, and does (on the average) exhibit a monotonic decay. This decay is interpreted as describing the "dissipation" of the initial collective state into more complicated intrinsic states.

In Secs. II.3, II.4 we turn to a practically solvable special situation which allows an explicit calculation of these probability amplitude and collective energy, and demonstration of their decay. Here, we assume the collective state to be the one-phonon coherent RPA state which results from the solution of the RPA equations of motion in the appropriate sub-

space, S_R , of discrete 1p-1h states. On the other hand, the exact physical many-body eigenstates (which are impossible to determine in practice) are replaced by simplified but nontrivial "eigenstates" constructed by solving the linearized equations of motion,^{40,41} in an extended subspace, S , larger than the subspace, S_R . The additional subspace, $S_A = S - S_R$, may include 1p-1h, 2p-2h, 3p-3h, etc., states in a general treatment.

In Sec. II.3 we have opted for the simplest possible additional subspace, S_A ; that is, one which adds only 1p-1h states to S_R . Thus, the many-body eigenstates in the extended subspace, S , as well as the wave packet whose time-evolution describes the damping of the collective motion, remain within the usual RPA framework. For a finite- A nucleus, the additional subspace, S_A , is then formed with the continuum of unbound single-particle states. In this particular case, the dissipative mechanism is simply the transition from a bound state to an outgoing state, and the energy dissipation rate is associated with the escape width, $\Gamma^\dagger$.

In Sec. II.4 we examine a more general case. Here, we use the Second RPA,⁴²⁻⁴⁴ and, therefore, include in S_A 2p-2h as well as 1p-1h states. For a finite- A nucleus then, the energy dissipation rate is associated with the spreading width, $\Gamma^\dagger$, for transitions to more complicated nuclear compound states.

II.2. The General Microscopic Dissipative Mechanism

The collective state, $|c\rangle$, is obtained by the diagonalization of the Hamiltonian within an "appropriate"* subspace, S_R , of the full Hilbert space. Formally, $|c\rangle$ is defined by the action of the collective creation operator, $Q_c^\dagger$, on the ground state, $|0\rangle_{S_R}$, as follows:

$$|c\rangle = Q_c^\dagger |0\rangle_{S_R} \quad (\text{II.2.1})$$

The ground state, $|0\rangle_{S_R}$, includes all the $Q_c^\dagger$ correlations describable within S_R since it satisfies the equation:

$$Q_c |0\rangle_{S_R} = 0, \quad (\text{II.2.2})$$

where Q_c is the corresponding collective annihilation operator.

Since it is defined in a subspace, this collective state, $|c\rangle$, is an eigenstate not of the complete exact Hamiltonian H , but only of an approximate Hamiltonian, H_0 , which omits the couplings between S_R and the rest of the Hilbert space. H_0 satisfies exactly the commutation relation:

$$[H_0, Q_c^\dagger] = \omega_c Q_c^\dagger \quad (\text{II.2.3})$$

in the subspace S_R .

We turn now to the relation between the approximate solution defined above and the exact problem specified by the Hamiltonian, H , and its

*This "appropriate" subspace is to be chosen on physical grounds; e.g., for giant vibrational states it may be the 1p-1h subspace into which the excitation process envisaged is able to excite the assumed ground state into a coherent collective motion. In such a case one is led to RPA solutions which are considered in the present work. However, the present formulation can certainly accommodate more general assumptions about the collective state.

eigenstates, $|\nu\rangle$. Each such eigenstate can be presented by the action of a particular operator, $\hat{Q}_\nu^\dagger$, on the exact many-body correlated ground state, $|0\rangle_{EX}$; i.e.,

$$|\nu\rangle = \hat{Q}_\nu^\dagger |0\rangle_{EX} \quad (\text{II.2.4a})$$

with

$$\hat{Q}_\nu |0\rangle_{EX} = 0. \quad (\text{II.2.4b})$$

The creation operators, $\hat{Q}_\nu^\dagger$, and the eigenstates, $|\nu\rangle$, obey the many-body time-independent Schrödinger equation:

$$H|\nu\rangle = (E_{0,EX} + \hbar\omega_\nu)|\nu\rangle; \quad (\text{II.2.5a})$$

or

$$[H, \hat{Q}_\nu^\dagger] |0\rangle_{EX} = \hbar\omega_\nu \hat{Q}_\nu^\dagger |0\rangle_{EX} \quad (\text{II.2.5b})$$

Here, $E_{0,EX}$ is the energy of the exact physical ground state, $|0\rangle_{EX}$, and $\hbar\omega_\nu$ is the excitation energy of the eigenstate, $|\nu\rangle$.

We focus next on the initial value problem specified by the time-dependent many-body Schrödinger equation

$$i\hbar \frac{d}{dt} |\phi(t)\rangle = H |\phi(t)\rangle \quad (\text{II.2.6a})$$

when initialized with the collective state, $|c\rangle$, that is when

$$|\phi(t=0)\rangle = |c\rangle. \quad (\text{II.2.6b})$$

If the collective state, $|c\rangle$, is of the form

$$|c\rangle = \sum_\nu b_\nu |\nu\rangle \quad (\text{II.2.7})$$

when expanded in the set of exact eigenstates, then the time-dependent solution is

$$|\phi(t)\rangle = e^{-iHt/\hbar} |c\rangle = e^{-iE_{0,EX}t/\hbar} \sum_\nu b_\nu e^{-i\omega_\nu t} |\nu\rangle. \quad (\text{II.2.8})$$

Because of (II.2.8) and (II.2.7), the probability amplitude $a_c(t)$, for the system under consideration to be found in the collective state, $|c\rangle$, at a later time t is equal to

$$a_c(t) = e^{iE_{0,EX}t/\hbar} \langle c | \phi(t) \rangle = \sum_\nu |b_\nu|^2 e^{-i\omega_\nu t}. \quad (\text{II.2.9})$$

If the wave packet (II.2.8) is to describe a process which can properly be called a "damping" of the collective motion, then the probability amplitude, $a_c(t)$, should exhibit--at least on the average over a certain time interval--a monotonic time decay. The fact that trigonometric sums of the form (II.2.9) may in special cases behave approximately as time-decaying exponential functions has been utilized in atomic^{52,53} and molecular⁵⁴ physics and in the analysis of nuclear strength function phenomena.⁵⁵ Therefore, it is not surprising that the expression (II.2.9) provides a basis for the description of the damping of collective states.

We turn our attention now to the collective energy, $E_c(t)$, of the system. The collective state creation operator, $Q_c^\dagger$, is approximately a boson operator in that it satisfies approximately⁵⁶ within the restricted subspace, S_R , the boson commutation relation:

$$[Q_c, Q_c^\dagger] = 1. \quad (\text{II.2.10})$$

Because of the commutation relations (II.2.3) and (II.2.10), the Hamiltonian, H_0 , contains the following term which can be interpreted as the collective Hamiltonian, H_c :⁵⁶

$$H_c = \hbar\omega_c Q_c^\dagger Q_c. \quad (\text{II.2.11})$$

The collective energy, $E_c(t)$, of the system at any time t is here defined* to be the expectation value of the collective Hamiltonian, H_c , with the wave packet, $|\phi(t)\rangle$ (Eq. (II.2.8)), which describes the time evolution of the system, i.e.,

*This choice is unique only in the limit of vanishing coupling between S_R and the rest of the Hilbert space; nevertheless, we retain it until Sec. IV.4 where the full discussion on the appropriate way to specify the collective Hamiltonian is deferred.

$$E_c(t) = \pi\omega_c \langle \phi(t) | Q_c^\dagger Q_c | \phi(t) \rangle. \quad (\text{II.2.12})$$

If the wave packet (II.2.8) is to describe a process which can properly be called a "damping" of the collective motion, then in addition to the probability amplitude, $a_c(t)$ (Eq. (II.2.9)), the collective energy, $E_c(t)$, should exhibit a time decay. In this case, it is physically plausible to interpret the rate of decrease in the collective energy, $E_c(t)$, as the energy dissipation rate, $\dot{E}_{\text{diss}}(t)$, from the collective degree represented by the collective state, $|\phi\rangle$, in the restricted subspace, S_R , to the intrinsic individual degrees of freedom represented by the states in the rest of the full Hilbert space. Thus one has:

$$\dot{E}_{\text{diss}}(t) = \frac{d}{dt} E_c(t). \quad (\text{II.2.13})$$

II.3. The Damping of the One-Phonon Collective State

in the Time-Dependent RPA

A. Approximate Solution of the Initial Value Problem in RPA

Because the exact physical many-body eigenstates cannot be determined in practice, we approximate them with the solutions of the linearized equations of motion^{40,41,56} in an extended subspace S larger than the initial S_R . The additional subspace, $S_A = S - S_R$, might include, e.g., additional lp-lh states, as well as 2p-2h, 3p-3h, etc., states in a general treatment. However, in this section we restrict ourselves to the simple RPA and therefore we keep only additional lp-lh states.* Then the usual RPA approximate solutions in the extended subspace, S , are given by

$$|v\rangle_{\text{RPA}} = \hat{Q}_{v,\text{RPA}}^\dagger |0\rangle_S \quad (\text{II.3.1a})$$

where

$$\begin{aligned} \hat{Q}_{v,\text{RPA}}^\dagger = & \sum_{m1} \{ Y_{m1}(\omega_v) a_m^\dagger a_1 - Z_{m1}(\omega_v) a_1^\dagger a_m \} \\ & + \sum_{\tilde{m}\tilde{1}} Y_{\tilde{m}\tilde{1}}(\omega_v) a_{\tilde{m}}^\dagger a_{\tilde{1}} - Z_{\tilde{m}\tilde{1}}(\omega_v) a_{\tilde{1}}^\dagger a_{\tilde{m}} \}. \end{aligned} \quad (\text{II.3.1b})$$

The solutions (II.3.1) are here formally analogous to the exact eigensolutions of Sec. II.2 in that they are defined in the "complete" subspace taken here to be the full lp-lh subspace of the RPA

For this reason, the notation of Sec. II.2 is utilized here also. The indices** $(m,1)$ denote the lp-lh states of the original subspace, S_R , while the indices** $(\tilde{m},\tilde{1})$ denote the lp-lh states of the additional subspace, S_A .

*For a finite-A nucleus a physically meaningful additional lp-lh subspace, S_A , is then formed with the continuum of the unbound single-particle states when the discrete lp-lh spectrum is used to define the collective state subspace, S_R .

**Throughout this paper we use the indices m, n, p, q to denote particle states and the indices $1, j, k, l$ to denote hole states. Indices $\alpha, \beta, \gamma, \delta$ will denote both particle and hole states.

All the 1p-1h states are defined with respect to the correlated ground state, $|0\rangle_S$.

The correlated ground state, $|0\rangle_S$, is analogous to the exact physical ground state, $|0\rangle_{EX}$, of Sec. II.2. The RPA annihilation operators, $\hat{Q}_{\nu,RPA}$, given by the adjoint Eq. (II.3.1b) satisfy the equation

$$\hat{Q}_{\nu,RPA}|0\rangle_S = 0 \quad \text{for all } \nu's. \quad (II.3.1c)$$

In RPA the commutator of the exact many-body Hamiltonian, H , with the RPA creation operator $\hat{Q}_{\nu,RPA}^\dagger$ is approximately equal to $M\omega_{\nu,RPA}\hat{Q}_{\nu,RPA}^\dagger$:^{40,41,56}

$$[H, \hat{Q}_{\nu,RPA}^\dagger] \approx M\omega_{\nu,RPA}\hat{Q}_{\nu,RPA}^\dagger \quad (II.3.2a)$$

or

$$H|\nu\rangle_{RPA} \approx (E_{0,S} + M\omega_{\nu,RPA})|\nu\rangle_{RPA}. \quad (II.3.2b)$$

We solve now the initial value problem posed in Sec. II.2 (Eq. (I.2.6)) in the simple RPA. The formal exact solution (II.2.8) can be rewritten in the following way:

$$\begin{aligned} |\phi(t)\rangle &= e^{-iHt/M}|\psi\rangle = \sum_{\nu} b_{\nu} e^{-iHt/M}|\nu\rangle \\ &= \sum_{\nu} b_{\nu} \left\{ \sum_{\alpha=1}^{\infty} \frac{(-1)^{\alpha}}{\alpha!} \left(\frac{iHt}{M} \right)^{\alpha} \right\} |\nu\rangle. \end{aligned} \quad (II.3.3)$$

We evaluate the expression (II.3.3) in the RPA. We use

the approximation (II.3.1) for the states $|\nu\rangle$ and Eqs. (II.3.2) in conjunction with the operator identity

$$[A, B^n] = \sum_{s=0}^{n-1} B^s [A, B] B^{n-s-1} \quad (II.3.4)$$

to obtain

$$H^{\alpha}|\nu\rangle_{RPA} \approx (E_{0,S} + M\omega_{\nu,RPA})^{\alpha}|\nu\rangle_{RPA}, \quad \alpha = 1, 2, 3, \dots \quad (II.3.5)$$

Thence we obtain finally the following RPA-approximate expression for the wave packet (II.3.3):

$$|\phi(t)\rangle \approx e^{-iE_{0,S}t/M} \sum_{\nu} b_{\nu,RPA} e^{-i\omega_{\nu,RPA}t} |\nu\rangle_{RPA}. \quad (II.3.6)$$

Since the remainder of the discussion will be based entirely upon this RPA, we omit henceforth the subscript, RPA.

B. The RPA Eigenstates. $|\nu\rangle$, in the Extended Subspace, S

The amplitudes $Y_{m1}(\omega_{\nu})$, $Z_{m1}(\omega_{\nu})$, $Y_{\tilde{m}1}(\omega_{\nu})$, $Z_{\tilde{m}1}(\omega_{\nu})$ and the frequencies ω_{ν} determining the states $|\nu\rangle$ in Eq. (II.3.1b) obey the following RPA equation of motion (see Appendix A for more details):

$$\begin{pmatrix} A & B & C & D \\ -B^* & -A^* & -D^* & -C^* \\ C & D & E & 0 \\ -D^* & -C^* & 0 & -E \end{pmatrix} \begin{pmatrix} Y(\omega_{\nu}) \\ Z(\omega_{\nu}) \\ \tilde{Y}(\omega_{\nu}) \\ \tilde{Z}(\omega_{\nu}) \end{pmatrix} = M\omega_{\nu} \begin{pmatrix} Y(\omega_{\nu}) \\ Z(\omega_{\nu}) \\ \tilde{Y}(\omega_{\nu}) \\ \tilde{Z}(\omega_{\nu}) \end{pmatrix}. \quad (II.3.7)$$

Here the submatrices A, B express the couplings among the states (m, i) of the initial subspace, S_R , and are given by

$$A_{m1,nj} = \delta_{mn}\delta_{ij}(\epsilon_m - \epsilon_i) + V_{mj}i_n; \quad (II.3.8a)$$

$$B_{m1,nj} = V_{mij}, \quad (II.3.8b)$$

while the submatrices C and D describe the coupling between the subspaces S_R and S_A , and have the form:

$$C_{m1,\tilde{n}j} = V_{mj}i_{\tilde{n}}; \quad D_{m1,\tilde{n}j} = V_{\tilde{m}ij}. \quad (II.3.8c)$$

In writing Eq. (II.3.7) we assumed that the couplings among the states $(\tilde{m}, i)$ of the additional subspace, S_A , can be neglected. As a result, the matrix block, $\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$, is purely diagonal, and the submatrix E is of the form

$$E_{\tilde{m}1,\tilde{n}j} = \delta_{\tilde{m}\tilde{n}}\delta_{ij}(\epsilon_{\tilde{m}} - \epsilon_{\tilde{n}}). \quad (II.3.8d)$$

When the submatrices C and D are set equal to zero, the subspaces S_R and S_A decouple, and Eq. (II.3.7) has simple eigenvector solutions of the form:

$$\begin{pmatrix} 0 \\ 0 \\ \delta_{mn} \delta_{\alpha\beta} \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{mn} \delta_{\alpha\beta} \end{pmatrix} \text{ and/or } \begin{pmatrix} y(\omega_a) \\ z(\omega_a) \\ 0 \\ 0 \end{pmatrix}, \quad (\text{II.3.9})$$

where a runs over the dimensions of the initial subspace, S_R , and the vectors, $\begin{pmatrix} y(\omega_a) \\ z(\omega_a) \end{pmatrix}$, are eigensolutions of the RPA equations of motion in the subspace, S_R :

$$\begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} y(\omega_a) \\ z(\omega_a) \end{pmatrix} = \omega_a \begin{pmatrix} y(\omega_a) \\ z(\omega_a) \end{pmatrix}. \quad (\text{II.3.10})$$

The physical RPA states, $|a\rangle$, in the subspace S_R are

$$|a\rangle = Q_a^\dagger |0\rangle_{S_R} \quad (\text{II.3.11a})$$

where

$$Q_a^\dagger = \sum_{mi} \{ y_{mi}(\omega_a) a_{m1}^\dagger - z_{mi}(\omega_a) a_{i1}^\dagger \}. \quad (\text{II.3.11b})$$

The collective state, $|c\rangle$, is here taken to be the coherent RPA solution--the solution of Eqs. (II.3.10) labelled by $a = c$. Again, by analogy with the more general formal development of Sec. II.2, the same notation is utilized here also.

The set of vectors (II.3.9) spans the extended subspace, S . As the first step toward the solution of Eq. (II.3.7), we expand the eigenvectors--solutions of Eq. (II.3.7) in the complete basis set of (II.3.9):

$$\begin{pmatrix} Y(\omega_v) \\ Z(\omega_v) \\ \tilde{Y}(\omega_v) \\ \tilde{Z}(\omega_v) \end{pmatrix} = \sum_a f_{va} \begin{pmatrix} y(\omega_a) \\ z(\omega_a) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \tilde{Y}(\omega_v) \\ \tilde{Z}(\omega_v) \end{pmatrix} \quad (\text{II.3.12})$$

Among the vectors $\begin{pmatrix} y(\omega_a) \\ z(\omega_a) \end{pmatrix}$ we have included the vectors

$$\begin{pmatrix} y(\omega_a) \\ z(\omega_a) \end{pmatrix} = \begin{pmatrix} z^*(\omega_a) \\ y^*(\omega_a) \end{pmatrix} \quad (\text{II.3.13})$$

which correspond to negative eigenenergies, $\omega_a = -\omega_a$.⁴⁰ These vector solutions correspond to the annihilation operators, Q_a , and do not yield any physical excitations above the ground state, since

$$Q_a^\dagger |0\rangle_{S_R} = -Q_a |0\rangle_{S_R} = 0. \quad (\text{II.3.14})$$

However, they are needed to span the whole RPA subspace S since the subspaces in RPA have twice as many dimensions as the corresponding subspaces in shell-model (TDA) calculations.

The expansion coefficient with respect to a vector (II.3.13) will be denoted by $f_{v\bar{a}}$.

Substituting the expansion (II.3.12) into Eq. (II.3.7), using Eq. (II.3.10) and the RPA orthonormality properties (A.30) of the vectors (II.3.9), one gets

$$(\omega_v - \omega_b) f_{vb} = \sum_{nj} \{ K_{nj}^*(\omega_b) Y_{nj}(\omega_v) + \Lambda_{nj}^*(\omega_b) Z_{nj}(\omega_v) \} \quad (\text{II.3.15a})$$

$$-(\omega_v + \omega_b) f_{vb} = \sum_{nj} \{ \Lambda_{nj}(\omega_b) Y_{nj}(\omega_v) + K_{nj}(\omega_b) Z_{nj}(\omega_v) \} \quad (\text{II.3.15b})$$

$$(\omega_v - \epsilon_{mi}) Y_{mi}(\omega_v) = \sum_a \{ f_{va} K_{mi}(\omega_a) + f_{v\bar{a}} \Lambda_{mi}^*(\omega_a) \} \quad (\text{II.3.15c})$$

$$-(\omega_v + \epsilon_{mi}) Z_{mi}(\omega_v) = \sum_a \{ f_{va} \Lambda_{mi}(\omega_a) + f_{v\bar{a}} K_{mi}^*(\omega_a) \} \quad (\text{II.3.15d})$$

where

$$K_{mi}(\omega_a) = \sum_{nj} \{ V_{mj} y_{jn}(\omega_a) + V_{mi} z_{nj}(\omega_a) \} \quad (\text{II.3.16a})$$

$$\Lambda_{mi}(\omega_a) = \sum_{nj} \{ V_{jn} z_{mj}(\omega_a) + V_{ij} y_{nm}(\omega_a) \}. \quad (\text{II.3.16b})$$

First, we examine solutions of Eqs. (II.3.15) when the additional subspace, S_A , is spanned by a discrete set. The case of continuous S_A spectrum is examined in Sec. II.3.D. No analytic solution of Eqs. (II.3.15),

unsimplified, is available. However, closed form approximate solutions can be obtained under the "slow variation" condition, that the coupling matrix elements, $|K_{nj}(\omega_b)|$, $|A_{nj}(\omega_b)|$, and the spacing parameter, d_{nj} , vary slowly with the energies ϵ_{nj} of the spectrum of the additional subspace, S_A ; i.e.,

$$\left| \frac{\partial |K_{nj}(\omega_b)|}{\partial \epsilon_{nj}} \right| \ll 1; \quad \left| \frac{\partial |A_{nj}(\omega_b)|}{\partial \epsilon_{nj}} \right| \ll 1; \quad \left| \frac{\partial d_{nj}}{\partial \epsilon_{nj}} \right| \ll 1. \quad (\text{II.3.17a})$$

The most general solution under conditions (II.3.17a) is examined in Appendix C.* However, for the sake of simplicity in describing the damping mechanism, we will, in the main body of this paper, restrict ourselves to a physically simple case when the S_R -to- S_A coupling is "weak" in the sense that:

- (1) it results in a small shift in the collective frequency, ω_c ;
- (2) it distributes the strength of the collective state, $|c\rangle$, only over the states, $|v\rangle$, lying within an energy band around $\mathcal{H}\omega_c$ of width $\mathcal{H}\Gamma$ much smaller than the distances $|\mathcal{H}\omega_c - \mathcal{H}\omega_b|$ from the other eigenenergies in the subspace S_R .

(II.3.17b)

The approximate solutions which correspond to this simple physical case have the form:**

$$\left\{ \begin{aligned} f_{vb} &= f_{vc} \delta_{bc}; \quad f_{vb} = 0; \quad Y_{mj}(\omega_v) = f_{vc} \frac{K_{mj}(\omega_c)}{\mathcal{H}\omega_v - \epsilon_{mj}}; \quad Z_{mj}(\omega_v) = -f_{vc} \frac{A_{mj}(\omega_c)}{\mathcal{H}\omega_v + \epsilon_{mj}} \end{aligned} \right\} \quad (\text{II.3.18})$$

*This more general analytic solution can provide the basis for a theory of widths yielding results more general than the "golden rule" formula.

**The mathematical conditions corresponding to the weak coupling condition (II.3.17b), which guarantee that (II.3.18) is a good approximation, are discussed in Appendix D.

where the coefficients f_{vc} and the frequencies ω_v remain to be determined. Substitution of Eq. (II.3.18) into Eq. (II.3.15a), when $b=c$, yields the following dispersion relation for the eigenenergies $\mathcal{H}\omega_v$:

$$\mathcal{H}\omega_v - \mathcal{H}\omega_c = \sum_{nj} \left\{ \frac{|K_{nj}(\omega_c)|^2}{\mathcal{H}\omega_v - \epsilon_{nj}} - \frac{|A_{nj}(\omega_c)|^2}{\mathcal{H}\omega_v + \epsilon_{nj}} \right\}. \quad (\text{II.3.19})$$

Using conditions (II.3.17a), the term $\sum_{nj} |K_{nj}(\omega_c)|^2 / (\mathcal{H}\omega_v - \epsilon_{nj})$ can be split in two terms^{53,54,57} by an expansion of $|K_{nj}(\omega_c)|^2$ about

$$K_{pk}(\omega_c)|^2: \quad \sum_{nj} \frac{|K_{nj}(\omega_c)|^2}{\mathcal{H}\omega_v - \epsilon_{nj}} \approx \sum_{nj} \frac{|K_{pk}(\omega_c)|^2 + (\epsilon_{nj} - \epsilon_{pk} - \frac{d_v}{2}) [\partial |K_{nj}(\omega_c)|^2 / \partial \epsilon_{nj}]}{\epsilon_{pk} + \tau_v - \epsilon_{nj}}, \quad (\text{II.3.20})$$

where $(\tilde{p}, \tilde{k})$ is the state $(\tilde{n}, \tilde{j})$ closest to $\mathcal{H}\omega_v$ and $d_v = d_{\tilde{p}\tilde{k}}$. The $K_{pk}(\omega_c)|^2$ has been used in (II.3.20) instead of the value of $|K_{nj}(\omega_c)|^2$ when $\epsilon_{nj} = \epsilon_{pk} + \frac{d_v}{2}$ because of their approximate equality by condition (II.3.17a). Finally, the quantity, τ_v , determines the exact position of the root $\mathcal{H}\omega_v$ between two successive energies ϵ_{nj} , and is defined by $\mathcal{H}\omega_v = \epsilon_{pk} + \tau_v$.

If now the states $(\tilde{n}, \tilde{j})$ are numerous enough, one can extend the summation to infinity. Furthermore, because of the assumed slow variation of the spacing parameter, d_{nj} , we consider the spacing as uniform around the energy ϵ_{pk} :

$$\epsilon_{nj} \approx \epsilon_{pk} + \mu d_v, \quad \mu = 0, \pm 1, \pm 2, \pm 3, \dots, \infty. \quad (\text{II.3.21})$$

Then the summation over the first term in the right-hand side of Eq. (II.3.20) can be obtained in closed form, as follows:⁵⁸

$$|K_{pk}(\omega_c)|^2 \sum_{\mu=-\infty}^{+\infty} \frac{1}{\tau_v - \mu d_v} = \frac{|K_{pk}(\omega_c)|^2}{d_v} \cot\left(\frac{\pi \tau_v}{d_v}\right). \quad (\text{II.3.22})$$

The second sum in the right-hand side of Eq. (II.3.20) results from the slow variation of the $|K_{nj}(\omega_c)|^2$ with the energies, ϵ_{nj} . It can be approximated by the sum

$$F = \sum_{nj} \frac{(\epsilon_{nj} - \epsilon_{pk} - \frac{d_v}{2}) \{ \partial |K_{nj}(\omega_c)|^2 / \partial \epsilon_{nj} \}}{\epsilon_{pk} + \frac{d_v}{2} - \epsilon_{nj}}, \quad (II.3.23a)$$

with an error, E, with a definite upper bound given by

$$E \leq \frac{d_v}{2} \frac{\partial |K_{nj}(\omega_c)|^2}{\partial \epsilon_{nj}} \frac{\pi}{d_v} \cot\left(\frac{\pi \tau_v}{d_v}\right). \quad (II.3.23b)$$

Because of conditions (II.3.17a), this upper bound is negligible compared with the term (II.3.22). Furthermore, it becomes zero as the additional spectrum of S_A becomes continuous ($d_v \rightarrow 0$). Using now the fact that

$$\sum_{nj} \frac{|K_{pk}(\omega_c)|^2}{\epsilon_{pk} + \frac{d_v}{2} - \epsilon_{nj}} = |K_{pk}(\omega_c)|^2 \sum_{\mu=-\infty}^{+\infty} \frac{1}{\frac{d_v}{2} - \mu d_v} = |K_{pk}(\omega_c)|^2 \frac{\pi}{d_v} \cot\left(\frac{\pi}{2}\right) = 0, \quad (II.3.23c)$$

we reconstitute the sum (II.3.23a) in the form

$$F = \sum_{nj} \frac{|K_{nj}(\omega_c)|^2}{\epsilon_{pk} + \frac{d_v}{2} - \epsilon_{nj}}. \quad (II.3.24)$$

The sum (II.3.24) does not depend on the exact position of the root,

$\kappa\omega_v = \epsilon_{pk} + \tau_v$, and goes over to a principal value integral when $d_v \rightarrow 0$ (continuous S_A). In this way, the dispersion relation (II.3.19)

becomes

$$\kappa\omega_v - \kappa\omega_0 = |K_{pk}(\omega_c)|^2 \frac{\pi}{d_v} \cot\left(\frac{\pi \tau_v}{d_v}\right), \quad (II.3.25a)$$

where

$$\kappa\omega_0 = \kappa\omega_c + F - G \quad (II.3.25b)$$

with

$$G = \sum_{nj} \frac{|A_{nj}(\omega_c)|^2}{\kappa\omega_v + \epsilon_{nj}}. \quad (II.3.25c)$$

Finally, the coefficients f_{vc} in Eq. (II.3.18) are calculated with the help of the normalization conditions $\langle v|v \rangle = 1$ and $\langle c|c \rangle = 1$.

One finds

$$|f_{vc}|^2 = \left[1 + \sum_{nj} \left(\frac{|K_{nj}(\omega_c)|^2}{(\kappa\omega_v - \epsilon_{nj})^2} - \frac{|A_{nj}(\omega_c)|^2}{(\kappa\omega_v + \epsilon_{nj})^2} \right) \right]^{-1}. \quad (II.3.26)$$

Due to the second-order power, the series expansion of Eq. (II.3.26) is convergent—unlike the series in Eq. (II.3.19)—so that the main contribution to it comes from the terms with nearly zero values of the difference in the denominator. Using the series⁵⁸

$$|K_{pk}(\omega_c)|^2 \sum_{\mu=-\infty}^{+\infty} \frac{1}{(\tau_v - \mu d_v)^2} = \left(\frac{|K_{pk}(\omega_c)|^2}{d_v} \right)^2 \operatorname{cosec}^2\left(\frac{\pi \tau_v}{d_v}\right), \quad (II.3.27)$$

and taking into consideration the dispersion relation (II.3.25a), one

obtains the following expression for the amplitudes f_{vc} :

$$|f_{vc}|^2 \approx \left\{ 1 + \left(\frac{|K_{pk}(\omega_c)|^2}{d_v} \right)^2 + \frac{(\kappa\omega_v - \kappa\omega_0)^2}{|K_{pk}(\omega_c)|^2} \right\}^{-1}. \quad (II.3.28)$$

These have values appreciably different from zero in a limited energy range around $\kappa\omega_c$ because of the slow variation condition (II.3.17a) and the small shift in the collective frequency ω_c . By approximating the $|K_{pk}(\omega_c)|^2$ over this limited energy range by a constant, $|K_{q\ell}(\omega_c)|^2 = \kappa^2$, and the spacing parameter d_v , by $d_{q\ell} = d$, where (q, ℓ) denotes the state (n, j) closest to $\kappa\omega_c$, one gets

$$|f_{vc}|^2 \approx \left\{ 1 + \left(\frac{\pi \kappa}{d} \right)^2 + \frac{(\kappa\omega_v - \kappa\omega_0)^2}{\kappa^2} \right\}^{-1}. \quad (II.3.29)$$

The shifted energy $\kappa\omega_0$ in (II.3.29) is calculated at the point $\kappa\omega_v \approx \epsilon_{q\ell}$.

C. The Probability Amplitude, $a_c(t)$ —Case of Discrete Additional Subspace, S_A

With the approximate solution (II.3.18) and (II.3.29) the collective state operator, $Q_c^\dagger$, can be expanded in the creation operators $Q_v^\dagger$ —of the eigenstates, $|v\rangle$, in the extended subspace, S —as follows:⁴⁰

$$Q_c^\dagger = \sum_v f_{vc}^* Q_v^\dagger. \quad (\text{II.3.30})$$

Therefore, the expansion coefficients b_v for the wave packet $|\phi(t)\rangle$ of Eq. (II.3.6), which describes the damping of the collective motion in RPA, are

$$\begin{aligned} b_v &= \langle v | c \rangle = {}_S \langle 0 | \hat{Q}_v Q_c^\dagger | 0 \rangle_{S_R} = \sum_\mu f_{\mu c}^* {}_S \langle 0 | \hat{Q}_v \hat{Q}_\mu^\dagger | 0 \rangle_{S_R} \\ &= \sum_\mu f_{\mu c}^* {}_S \langle 0 | [\hat{Q}_v, \hat{Q}_\mu^\dagger] | 0 \rangle_{S_R} \approx \sum_\mu f_{\mu c}^* \delta_{\mu v} = f_{vc}^*. \end{aligned} \quad (\text{II.3.31})$$

In Eq. (II.3.31) we have calculated the commutators approximately by replacing the correlated ground state in the extended space S by the RPA ground state in S_R , and utilizing the boson approximation (cf. Appendix Eq. (A.29e) of the RPA method. Thus, the wave packet $|\phi(t)\rangle$ of Eq. (II.3.6) becomes

$$|\phi(t)\rangle \approx e^{-iE_0 S t / \hbar} \sum_v f_{vc}^* e^{-i\omega_v t} |v\rangle. \quad (\text{II.3.32})$$

Then, taking into consideration that $|c\rangle = |\phi(t=0)\rangle$, the probability amplitude, $a_c(t)$, for finding the system in the collective state at a later time t is

$$a_c(t) = e^{iE_0 S t / \hbar} \langle c | \phi(t) \rangle = \sum_v |f_{vc}|^2 e^{-i\omega_v t}. \quad (\text{II.3.33})$$

The probability amplitude (II.3.33) is analogous to the probability amplitude (II.2.9) of Sec. II.2 where the wave function $|\phi(t)\rangle$ was formally defined in the full Hilbert space.

We substitute now expression (II.3.29) for the coefficients f_{vc} into Eq. (II.3.33). When $(\pi\kappa)^2 \gg d^2$ (dense spectrum) one can make a first approximation⁵⁵ by replacing the sum in (II.3.33) with an integral

and neglecting the unity term in the denominator of f_{vc} ; one finds:

$$a_c(t) \approx \int_{-\infty}^{+\infty} \frac{d(\hbar\omega)}{d} \frac{e^{-i\omega t}}{(\frac{\pi\kappa}{d})^2 + [(\hbar\omega - \hbar\omega_0)^2 / \kappa^2]} = e^{-i\omega_0 t - \frac{\Gamma t}{2}} \quad (\text{II.3.34})$$

where

$$\Gamma = \frac{2\pi \kappa^2}{\hbar d}. \quad (\text{II.3.35})$$

One finds a simple exponential decay.

A better approximation consists in preserving the discrete nature of the spectrum of $\hbar\omega_v$, but assuming that the distance $(\hbar\omega_v - \hbar\omega_0)$ is equal to vd , that is,

$$\hbar\omega_v - \hbar\omega_0 \approx vd; \quad v = 0, \pm 1, \pm 2, \dots, \pm \infty. \quad (\text{II.3.36})$$

The approximation (II.3.36) becomes better as the spacing constant, d , becomes smaller. In fact, the substitution of the sum with an integral is a limiting case of the approximation (II.3.36) when $d=0$.

Then the sum in Eq. (II.3.33) can be carried out with the help of the series:⁵⁸

$$2 \sum_{\mu=1}^{\infty} \frac{\cos(\mu\xi)}{\mu^2 + \sigma^2} = \frac{\pi \cosh \sigma(\pi - \xi)}{\sigma \sinh \pi \sigma} - \frac{1}{\sigma^2}, \quad 0 < \xi < 2\pi. \quad (\text{II.3.37})$$

Using expression (II.3.24), combining Eqs. (II.3.33) and (II.3.37), and renormalizing the result of the summation in order to preserve the property $a_c(t=0) = 1$, one gets

$$a_c(t) \approx e^{-i\omega_0 t} \frac{\cosh\{\sqrt{1 + (\Gamma\tau_p/4)} \left[\frac{1}{2} \sqrt{\Gamma\tau_p} - \frac{t}{(\hbar/\kappa)} \right]\}}{\cosh\left\{\frac{1}{2} \sqrt{\Gamma\tau_p} \sqrt{1 + (\Gamma\tau_p/4)}\right\}} \quad (\text{II.3.38})$$

where

$$0 < t < \frac{2\pi\hbar}{d}.$$

Notice that the picket-fence approximation (II.3.36) transformed the trigonometric sum (II.3.33) from an almost periodic to a simple periodic function with period

$$\tau_p = \frac{2\pi\hbar}{d} \quad (\text{II.3.39})$$

which obviously becomes very long as the level spacing d becomes small. This period can be interpreted as the Poincaré recurrence time for the model under consideration.

The more complete expression (II.3.38) enables the examination of both the dense and sparse limits for the additional spectrum as follows:

(1) For S_A dense $((\pi\kappa)^2 \gg d^2$ or $\Gamma\tau_p \gg 4$), Eq. (II.3.38) yields:

$$a_c(t) \approx e^{-i\omega_0 t} \frac{\cosh(\frac{\Gamma\tau_p}{4} - \frac{\Gamma t}{2})}{\cosh(\frac{\Gamma\tau_p}{4})} = e^{-i\omega_0 t} \frac{e^{-\Gamma t/2} + e^{-\Gamma\tau_p/2} e^{\Gamma t/2}}{1 + e^{-\Gamma\tau_p/2}} \quad (\text{II.3.40})$$

The function

$$g(t) = \frac{e^{-\Gamma t/2} + e^{-\Gamma\tau_p/2} e^{\Gamma t/2}}{1 + e^{-\Gamma\tau_p/2}}, \quad 0 < t < \tau_p, \quad (\text{II.3.41})$$

is plotted in Fig. 1.* It has a minimum value of

$$g(\frac{\tau_p}{2}) = \frac{2e^{-\Gamma\tau_p/4}}{1 + e^{-\Gamma\tau_p/2}} \quad (\text{II.3.42})$$

at one-half the Poincaré period. For earlier times, $t < \pi\hbar/d$, the amplitude $a_c(t)$ exhibits an approximately exponential decay. The Poincaré period, $2\pi\hbar/d$, increases with decreasing spacing constant, d ; hence, when $d \rightarrow 0$ (continuous S_A) the Poincaré period becomes infinite, and the exponential decay prevails for all values of the time, as in (II.3.34). The system is thus dissipative for the time interval $0 < t < \tau_p/2$.

(2) For S_A sparse $((\pi\kappa)^2 \ll d^2$ or $\Gamma\tau_p \ll 4$), Eq. (II.3.38) yields:

$$a_c(t) \approx e^{-i\omega_0 t} \frac{\cosh[\frac{1}{2} \sqrt{\Gamma\tau_p} - \frac{t}{(\hbar/\kappa)}]}{\cosh(\frac{1}{2} \sqrt{\Gamma\tau_p})} \approx e^{-i\omega_0 t} \quad \text{for } 0 < t < \tau_p, \quad (\text{II.3.43})$$

since

* See page 167 for figure.

$$\cosh(\frac{1}{2} \sqrt{\Gamma\tau_p}) = \frac{e^{(1/2)\sqrt{\Gamma\tau_p}} + e^{-(1/2)\sqrt{\Gamma\tau_p}}}{2} = \frac{(1 + \frac{1}{2} \sqrt{\Gamma\tau_p} + \dots) + (1 - \frac{1}{2} \sqrt{\Gamma\tau_p} + \dots)}{2} \approx 1, \quad (\text{II.3.44})$$

and the argument of the cosh in the numerator of (II.3.43) is restricted between the values 0 and $\frac{1}{2} \sqrt{\Gamma\tau_p}$, that is

$$0 \leq \left| \frac{1}{2} \sqrt{\Gamma\tau_p} - \frac{t}{(\hbar/\kappa)} \right| \leq \frac{1}{2} \sqrt{\Gamma\tau_p}, \quad (\text{II.3.45})$$

for all times $0 \leq t \leq \tau_p$.

In the case of a sparse additional subspace, S_A , the system is never dissipative.

D. Continuous Additional Subspace, S_A

In the case of a finite-A nucleus physically meaningful additional subspaces, S_A , may be constructed with the help of the continuum of the unbound single-particle states. In this section we show how the formulae and relations derived in the previous sections for discrete spectra can be modified to apply for continuous additional spectra as well. This modification is done through the use of the formula^{53,57}

$$\frac{1}{x-\omega} = P \frac{1}{x-\omega} - \zeta(\omega) \delta(x-\omega), \quad (\text{II.3.46})$$

which treats properly under the integrations the singularities arising in the transition from a discrete to a continuous spectrum. The constant $\zeta(\omega)$ here is not equal to the value $i\pi$, common in scattering theory, but is an unknown to be determined; it will turn out to be the limit of the quantity $\pi \cot(\frac{\pi\tau_v}{d_v})$ of Eq. (II.3.25a) when $d_v \rightarrow 0$.

The eigenstates $|v\rangle$ will now be labelled by a continuum variable $|\omega\rangle$. The amplitudes $Y(\vec{m}, \vec{l}; \omega)$, which correspond to the $Y_{\vec{m}\vec{l}}(\omega_v)$ of

Eq. (II.3.18) are:

$$Y(\tilde{m}, \tilde{l}; \omega) = f_c(\omega) K(\tilde{m}, \tilde{l}; \omega_c) \left(P \frac{1}{\hbar\omega - \epsilon(\tilde{m}, \tilde{l})} + \zeta(\omega) \delta(\hbar\omega - \epsilon(\tilde{m}, \tilde{l})) \right). \quad (\text{II.3.47})$$

The dispersion relation (II.3.25a) is modified to

$$\hbar\omega - \hbar\omega_0 = \zeta(\omega) \frac{\hbar\Gamma(\omega)}{2\pi}, \quad (\text{II.3.48a})$$

where

$$\hbar\omega_0 = \hbar\omega_c + F - G \quad (\text{II.3.48b})$$

and

$$\left\{ F = P \iint d\tilde{m} d\tilde{l} \frac{|K(\tilde{m}, \tilde{l}; \omega_c)|^2}{\hbar\omega - \epsilon(\tilde{m}, \tilde{l})}; \quad G = \iint d\tilde{m} d\tilde{l} \frac{|\Lambda(\tilde{m}, \tilde{l}; \omega_c)|^2}{\hbar\omega + \epsilon(\tilde{m}, \tilde{l})} \right\}. \quad (\text{II.3.49})$$

The width $\Gamma(\omega)$ is

$$\Gamma(\omega) = \frac{2\pi}{\hbar} \iint d\tilde{m} d\tilde{l} |K(\tilde{m}, \tilde{l}; \omega_c)|^2 \delta(\hbar\omega - \epsilon(\tilde{m}, \tilde{l})). \quad (\text{II.3.50})$$

We use the orthonormality condition, $\langle \omega' | \omega \rangle = \delta(\hbar\omega' - \hbar\omega)$, and the value of the constant $\zeta(\omega)$, calculated through Eq. (II.3.48a) to determine the amplitudes $f_c(\omega)$. Following the same steps as in Ref. 53 one finds

$$|f_c(\omega)|^2 = \frac{1}{2\pi\hbar} \frac{\Gamma(\omega)}{(\omega - \omega_0)^2 + (\frac{\Gamma(\omega)}{2})^2}. \quad (\text{II.3.51})$$

When the shift $\omega_0 - \omega_c$ in the collective frequency, ω_c (Eq. II.3.48b)), is small, and the spreading of the strength of the collective state is limited, the coefficients $f_c(\omega)$ (Eq. (II.3.51)) are appreciably different than zero only inside a narrow frequency band around ω_c . Then, because of the slow variation of the decay width, $\Gamma(\omega)$ (Eq. (II.3.50)), implied by condition (II.3.17a), the coefficients $|f_c(\omega)|^2$ are approximately equal to

*When the hole index, $\tilde{l}$, is discrete the corresponding integral over $\tilde{l}$ is to be understood as a sum.

$$|f_c(\omega)|^2 \approx \frac{1}{2\pi\hbar} \frac{\Gamma(\omega_c)}{(\omega - \omega_0)^2 + [\Gamma(\omega_c)/2]^2}, \quad (\text{II.3.52})$$

where both quantities ω_0 and $\Gamma(\omega_c)$ are evaluated at the point $\omega = \omega_c$.

Finally, the probability amplitude, $a_c(t)$, for finding the system in the collective state at any time t (Eq. (II.3.33)) in the case of continuous S_A is the following integral:

$$a_c(t) = \int_{-\infty}^{+\infty} |f_c(\omega)|^2 e^{-i\omega t} d(\hbar\omega). \quad (\text{II.3.53})$$

Substitution of Eq. (II.3.52) into (II.3.53) yields

$$a_c(t) = \frac{\Gamma(\omega_c)}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t} d\omega}{(\omega - \omega_0)^2 + [\Gamma(\omega_c)/2]^2} = e^{-i\omega_0 t} e^{-[\Gamma(\omega_c)t]/2}. \quad (\text{II.3.54})$$

E. The Energy Dissipation Rate

As discussed in Sec. II.2 (Eq. (II.2.12)), the collective energy $E_c(t)$ of the system at any time t is the expectation value of the collective Hamiltonian $H_c = \hbar\omega_c Q_c^\dagger Q_c$ with the wave packet $|\phi(t)\rangle$ which describes the time evolution of the system.

Using the expansion (II.3.32) for the wave packet $|\phi(t)\rangle$ and the expansion (II.3.30) for the collective state operator, $Q_c^\dagger$, one finds in RPA

$$\begin{aligned} E_c(t) &= \hbar\omega_c \langle \phi(t) | Q_c^\dagger Q_c | \phi(t) \rangle \\ &= \hbar\omega_c \sum_{\mu\nu} f_{\mu c} f_{\nu c}^* e^{i(\omega_\mu - \omega_\nu)t} \langle \mu | Q_c^\dagger Q_c | \nu \rangle \\ &= \hbar\omega_c \sum_{\mu\nu\kappa\lambda} f_{\mu c} f_{\nu c}^* f_{\lambda c}^* f_{\kappa c} e^{i(\omega_\mu - \omega_\nu)t} \langle 0 | \hat{Q}_\mu^\dagger \hat{Q}_\lambda^\dagger \hat{Q}_\kappa \hat{Q}_\nu | 0 \rangle_S \\ &= \hbar\omega_c \sum_{\mu\nu} |f_{\mu c}|^2 |f_{\nu c}|^2 e^{i(\omega_\mu - \omega_\nu)t} \\ &= \hbar\omega_c |a_c(t)|^2. \end{aligned} \quad (\text{II.3.55})$$

Here, $a_c(t)$ is the probability amplitude given by Eq. (II.3.33).

With the result (II.3.40) or (II.3.54) for the probability amplitude $a_c(t)$, one can interpret the rate of decrease in the collective energy, $E_c(t)$, as the energy dissipation rate, $\dot{E}_{diss}(t)$, from the collective degree of freedom to the intrinsic individual degrees of freedom represented by the states of the additional subspace, S_A :

$$\dot{E}_{diss}(t) = \frac{d}{dt} E_c(t) = -\Gamma E_c(t) = -\Gamma \hbar \omega_c e^{-\Gamma t}, \quad 0 < t < \frac{\tau_D}{2} - \frac{2}{\Gamma}. \quad (II.3.56)$$

The energy dissipation rate (II.3.56) is proportional to the collective energy itself which, therefore, exhibits a simple exponential decay as a function of time. This behavior differs from that of the classical damped harmonic oscillator whose energy dissipation rate is proportional to the kinetic energy and, therefore, oscillates while it decays. In Ch. IV we will examine this behavior in detail, and we will show how it goes over to that of the classical damped oscillator for certain multi-boson packets which describe classical vibrations of the mean field.

The result (II.3.55) for the collective energy at any time during the damping of the one-phonon state has the following simple structure:

(collective energy at any time)

$$= (\text{energy of the initial state}) \times \left(\text{probability for finding the system in the initial state} \right). \quad (II.3.57)$$

Because of its simplicity, relation (II.3.57) looks appealing from an intuitive viewpoint, so that one could have been led to use (II.3.57)--instead of deriving it by first specifying a collective Hamiltonian--as the general definition for the collective energy. This, however, would lead to errors since the relation (II.3.57) is true only in the case of a one-phonon initial state, but fails in the case of more complex (e.g., n-phonon) initial states, as it will be shown in Chs. III and IV.

II.4. The Damping of the One-Phonon Collective State in the Time-Dependent Second RPA

In Sec. II.3, we included additional states in S_A of the lp-lh structure only. Therefore, the usual RPA equations were sufficient in determining the approximate "eigenstates" $|v\rangle$ in the larger subspace, $S = S_R + S_A$. However, such a drastic simplification in the structure of the additional subspace has physically the disadvantage of associating the energy dissipation rate, in the case of a finite-A nucleus, entirely with the escape width, $\Gamma^\dagger$, for direct particle emission to the continuum.

In this Section, we relax this restriction and show how the energy dissipation rate can be associated not only with the escape width $\Gamma^\dagger$, but also in part with the spreading width $\Gamma^\ddagger$, for transitions from the collective state to more complicated nuclear compound states (and, in particular, in the first stage, to their 2p-2h component). We accomplish this goal by utilizing the so-called second RPA⁴²⁻⁴⁴ which allows the incorporation of 2p-2h, as well as lp-lh states, in S_A .

The creation operators $\hat{Q}_v^\dagger$ of Eq. (II.2.4) in the second RPA are approximated by

$$\begin{aligned} \hat{Q}_{v,RPAII}^\dagger = & \sum_{m1} \{ Y_{m1}(\omega_v) a_{m1}^\dagger a_1 - Z_{m1}(\omega_v) a_{1m}^\dagger a_m \} \\ & + \sum_{\tilde{m}\tilde{1}} \{ Y_{\tilde{m}\tilde{1}}(\omega_v) a_{\tilde{m}\tilde{1}}^\dagger a_{\tilde{1}} - Z_{\tilde{m}\tilde{1}}(\omega_v) a_{\tilde{1}\tilde{m}}^\dagger a_{\tilde{m}} \} \\ & + \sum_{m<n} \sum_{1<j} \{ Y_{m1j}(\omega_v) a_{m n j}^\dagger a_{1j} - Z_{m1j}(\omega_v) a_{1j}^\dagger a_{n m} \}. \end{aligned} \quad (II.4.1a)$$

As in Sec. II.3, the tilded indices here denote lp-lh additional states in S_A . The $\hat{Q}_{v,RPAII}^\dagger$ operators of Eq. (II.4.1) satisfy further the relations

$$|v\rangle_{\text{RPAII}} = \hat{Q}_{v,\text{RPAII}}^\dagger |0\rangle_S \quad \text{and} \quad \hat{Q}_{v,\text{RPAII}} |0\rangle_S = 0, \quad (\text{II.4.1b})$$

$$[H, \hat{Q}_{v,\text{RPAII}}^\dagger] \approx \mathcal{H}_{v,\text{RPAII}} \hat{Q}_{v,\text{RPAII}}^\dagger, \quad (\text{II.4.2})$$

in analogy with Eqs. (II.2.4) and (II.2.5) of Sec. II.2 and with Eqs. (II.3.1) and (II.3.2) of Sec. II.3. H in Eq. (II.4.2) is the complete exact Hamiltonian which includes all the residual interaction terms providing finite coupling between the restricted subspace S_R and the additional subspace S_A .

Following the same steps as in Subsec. II.3.A, one shows that the time-dependent second RPA yields the following approximation

$$|\phi(t)\rangle \approx e^{-iE_0 t/\hbar} \sum_v b_{v,\text{RPAII}} e^{-i\omega_{v,\text{RPAII}} t} |v\rangle_{\text{RPAII}} \quad (\text{II.4.3})$$

for the wave packet $|\phi(t)\rangle$ which describes the damping of the one-phonon collective state. The packet (II.4.3) solves the initial value problem posed in Eq. (II.2.6) of Sec. II.2.

Since the remainder of the discussion in this section will be based entirely upon this second RPA, we omit henceforth the subscript RPAII.

The amplitudes $Y_{mi}(\omega_v)$, $Z_{mi}(\omega_v)$, $Y_{\tilde{m}\tilde{i}}(\omega_v)$, $Z_{\tilde{m}\tilde{i}}(\omega_v)$, $Y_{m\tilde{m}ij}(\omega_v)$, $Z_{m\tilde{m}ij}(\omega_v)$, and the frequencies ω_v determining the states $|v\rangle$ in Eq. (II.4.1) obey the following second RPA equations (see Appendix A for more details):

$$\begin{bmatrix} A_{11} & B_{11} \\ -B_{11}^* & -A_{11}^* \end{bmatrix} \begin{bmatrix} Y_1(\omega_v) \\ Z_1(\omega_v) \end{bmatrix} + \begin{bmatrix} A_{1\tilde{1}} & B_{1\tilde{1}} \\ -B_{1\tilde{1}}^* & -A_{1\tilde{1}}^* \end{bmatrix} \begin{bmatrix} Y_{\tilde{1}}(\omega_v) \\ Z_{\tilde{1}}(\omega_v) \end{bmatrix} + \begin{bmatrix} A_{12} & B_{12} \\ -B_{12}^* & -A_{12}^* \end{bmatrix} \begin{bmatrix} Y_2(\omega_v) \\ Z_2(\omega_v) \end{bmatrix} = \mathcal{H}_{1v} \begin{bmatrix} Y_1(\omega_v) \\ Z_1(\omega_v) \end{bmatrix} \quad (\text{II.4.4})$$

$$\begin{bmatrix} A_{\tilde{1}\tilde{1}} & B_{\tilde{1}\tilde{1}} \\ -B_{\tilde{1}\tilde{1}}^* & -A_{\tilde{1}\tilde{1}}^* \end{bmatrix} \begin{bmatrix} Y_{\tilde{1}}(\omega_v) \\ Z_{\tilde{1}}(\omega_v) \end{bmatrix} + \begin{bmatrix} A_{\tilde{1}1} & B_{\tilde{1}1} \\ -B_{\tilde{1}1}^* & -A_{\tilde{1}1}^* \end{bmatrix} \begin{bmatrix} Y_1(\omega_v) \\ Z_1(\omega_v) \end{bmatrix} + \begin{bmatrix} A_{\tilde{1}2} & B_{\tilde{1}2} \\ -B_{\tilde{1}2}^* & -A_{\tilde{1}2}^* \end{bmatrix} \begin{bmatrix} Y_2(\omega_v) \\ Z_2(\omega_v) \end{bmatrix} = \mathcal{H}_{\tilde{1}v} \begin{bmatrix} Y_{\tilde{1}}(\omega_v) \\ Z_{\tilde{1}}(\omega_v) \end{bmatrix}$$

$$\begin{bmatrix} A_{21} & B_{21} \\ -B_{21}^* & -A_{21}^* \end{bmatrix} \begin{bmatrix} Y_1(\omega_v) \\ Z_1(\omega_v) \end{bmatrix} + \begin{bmatrix} A_{2\tilde{1}} & B_{2\tilde{1}} \\ -B_{2\tilde{1}}^* & -A_{2\tilde{1}}^* \end{bmatrix} \begin{bmatrix} Y_{\tilde{1}}(\omega_v) \\ Z_{\tilde{1}}(\omega_v) \end{bmatrix} + \begin{bmatrix} A_{22} & B_{22} \\ -B_{22}^* & -A_{22}^* \end{bmatrix} \begin{bmatrix} Y_2(\omega_v) \\ Z_2(\omega_v) \end{bmatrix} = \mathcal{H}_{2v} \begin{bmatrix} Y_2(\omega_v) \\ Z_2(\omega_v) \end{bmatrix}$$

The index l here stands for a set of $1p-1h$ indices (m, i) , and the index $\tilde{l}$ for a set of $2p-2h$ indices (m, n, i, j) . The submatrices A_{11} , B_{11} describe the couplings inside the restricted subspace S_R and have the form

$$A_{mi,nj} = \delta_{mn} \delta_{ij} (\epsilon_m - \epsilon_i) + V_{mj} \delta_{in}; \quad B_{mi,nj} = V_{mi} \delta_{jn}. \quad (\text{II.4.5a})$$

The submatrices $A_{1\tilde{1}}$, $B_{1\tilde{1}}$, $A_{\tilde{1}2}$, $B_{\tilde{1}2}$ describe the couplings between the subspaces S_R and S_A . They have the following form

$$A_{mi,pk} = V_{mk} \delta_{ip}; \quad B_{mi,pk} = V_{mp} \delta_{ik}; \quad (\text{II.4.5b})$$

$$A_{mi,pqkl} = V_{mp} \delta_{ik} + V_{kpq} \delta_{il} + V_{lkip} \delta_{mq} + V_{klqi} \delta_{mp}; \quad (\text{II.4.5c})$$

$$B_{mi,pqkl} = 0. \quad (\text{II.4.5d})$$

In writing (II.4.5) we assumed that the couplings among the states of the additional subspace S_A can be neglected. Then

$$E_{\tilde{m}\tilde{i},p\tilde{k}} = (\epsilon_{\tilde{m}} - \epsilon_{\tilde{i}}) \delta_{\tilde{m}\tilde{i}} \delta_{p\tilde{k}} \quad (\text{II.4.5e})$$

and

$$E_{m\tilde{m}ij,pqkl} = (\epsilon_m + \epsilon_n - \epsilon_i - \epsilon_j) \mathcal{U}(pq) \mathcal{U}(kl) \delta_{mp} \delta_{nq} \delta_{ik} \delta_{jl}, \quad (\text{II.4.5f})$$

where $\mathcal{U}(rs)$ is the antisymmetrizer for indices r, s .

We address two simplified cases specified by the assumed characteristics of the states in the additional subspace S_A .

- (1) S_A contains $1p-1h$ states only ($A_{12} = 0$). This case was examined in Sec. II.3.
- (2) S_A contains $2p-2h$ states only ($A_{1\tilde{1}} = B_{1\tilde{1}} = 0$). Then the energy dissipation is associated entirely with the spreading width, $\Gamma^\dagger$, for transitions from the collective state to more complicated compound states. This latter case will be examined in the rest of this section.

Note, however, that some intermediate combination of these two extremes will generally apply to realistic situations.

When S_A contains 2p-2h states only, Eq. (II.4.4) reduces to the following matrix equation:

$$\begin{bmatrix} A_{11} & B_{11} & A_{12} & B_{12} \\ -B_{11}^* & -A_{11}^* & -B_{12}^* & -A_{12}^* \\ A_{21} & B_{21} & E_{22} & 0 \\ -B_{21}^* & -A_{21}^* & 0 & -E_{22} \end{bmatrix} \begin{bmatrix} Y_1(\omega_v) \\ Z_1(\omega_v) \\ Y_2(\omega_v) \\ Z_2(\omega_v) \end{bmatrix} = \kappa \omega_v \begin{bmatrix} Y_1(\omega_v) \\ Z_1(\omega_v) \\ Y_2(\omega_v) \\ Z_2(\omega_v) \end{bmatrix} \quad (\text{II.4.6})$$

When the submatrices A_{12} are set equal to zero (the matrices B_{12} are already zero), the subspaces S_R and S_A decouple and Eq. (II.4.6) has simple eigenvector solutions of the form

$$\begin{pmatrix} 0 \\ 0 \\ \delta_{mp} \delta_{nq} \delta_{ik} \delta_{lj} \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_{mp} \delta_{nq} \delta_{ik} \delta_{jl} \end{pmatrix} \text{ and/or } \begin{pmatrix} y_1(\omega_a) \\ z_1(\omega_a) \\ 0 \\ 0 \end{pmatrix} \quad (\text{II.4.7})$$

where $m < n$ and $i < j$ and the index a runs over the dimensions of the initial subspace S_R . The vectors $\begin{pmatrix} y_1(\omega_a) \\ z_1(\omega_a) \end{pmatrix}$ are eigensolutions of the simple RPA equations (II.3.10) in the restricted subspace S_R .

Following the same steps as in Subsec. II.3.B, we expand the eigenvectors-solutions of Eq. (II.4.6) in the complete base set of (II.4.7):

$$\begin{pmatrix} Y_1(\omega_v) \\ Z_1(\omega_v) \\ Y_2(\omega_v) \\ Z_2(\omega_v) \end{pmatrix} = \sum_a f_{va} \begin{pmatrix} y_1(\omega_a) \\ z_1(\omega_a) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ Y_2(\omega_v) \\ Z_2(\omega_v) \end{pmatrix} \quad (\text{II.4.8})$$

Substituting the expansion (II.4.8) into Eq. (II.4.6), using Eq. (II.3.10) and the second RPA orthonormality properties (A.27) of the base vectors (II.4.7), one gets

$$(\kappa \omega_v - \kappa \omega_b) f_{vb} = \sum_{\substack{p < q \\ k < l}} \{ K_{pqkl}^*(\omega_b) Y_{pqkl}(\omega_v) + \Lambda_{pqkl}^*(\omega_b) Z_{pqkl}(\omega_v) \} \quad (\text{II.4.9a})$$

$$-(\kappa \omega_v + \kappa \omega_b) f_{v\bar{b}} = \sum_{\substack{p < q \\ k < l}} \{ \Lambda_{pqkl}(\omega_b) Y_{pqkl}(\omega_v) + K_{pqkl}(\omega_b) Z_{pqkl}(\omega_v) \} \quad (\text{II.4.9b})$$

$$(\kappa \omega_v - \epsilon_{m1} - \epsilon_{nj}) Y_{mnij}(\omega_v) = \sum_a \{ f_{va} K_{mnij}(\omega_a) + f_{v\bar{a}} \Lambda_{mnij}^*(\omega_a) \} \quad (\text{II.4.9c})$$

$$-(\kappa \omega_v + \epsilon_{m1} + \epsilon_{nj}) Z_{mnij}(\omega_v) = \sum_a \{ f_{va} \Lambda_{mnij}(\omega_a) + f_{v\bar{a}} K_{mnij}^*(\omega_a) \} \quad (\text{II.4.9d})$$

where

$$K_{pqkl}(\omega_b) = \sum_m \{ v_{pqm} y_{mk}(\omega_b) + v_{pqkm} y_{ml}(\omega_b) \} + \sum_i \{ v_{piik} y_{qi}(\omega_b) + v_{qikl} y_{pi}(\omega_b) \}, \quad (\text{II.4.10a})$$

$$\Lambda_{pqkl}(\omega_b) = \sum_m \{ v_{m\bar{p}q} z_{mk}(\omega_b) + v_{kmpq} z_{ml}(\omega_b) \} + \sum_i \{ v_{ikpi} z_{qi}(\omega_b) + v_{klqi} z_{pi}(\omega_b) \}. \quad (\text{II.4.10b})$$

The indices $\bar{a}, \bar{b}$ with a bar refer to the unphysical RPA eigenvectors with negative energies as discussed in Subsec. II.3.B.

Equation (II.4.9) has the same structure as Eq. (II.3.15) of Sec. II.3. However, the matrix elements $K_{pqkl}(\omega_b)$ and $\Lambda_{pqkl}(\omega_b)$, which describe the coupling of the RPA eigenstates in the restricted subspace S_R to the additional states of S_A , have now a dependence on the matrix elements of the two-body force and the RPA amplitudes $y_1(\omega_b)$ and $z_1(\omega_b)$ different from the one in Sec. II.3 (Eq. (II.3.16)) where the additional subspace S_A contained 1p-1h instead of 2p-2h states. Specifically, the two-body

matrix elements in Eq. (II.4.10) have 3p-1h or 1p-3h indices, unlike those in Eq. (II.3.16) which are labelled by a set of 2p-2h indices. Observe further that the coupling matrix elements $K_{pqkl}(\omega_b)$ depend only on the RPA amplitudes $y_1(\omega_b)$ and not on the amplitudes $z_1(\omega_b)$, in contrast with the $K_{n_j}(\omega_b)$ in Eq. (II.3.16a). An analogous remark is also valid for the matrix elements $\Lambda_{pqkl}(\omega_b)$ and $\Lambda_{n_j}(\omega_b)$ of Eq. (II.4.10b) and (3.16b) respectively.

We proceed to find an approximate solution for Eq. (II.4.9) in the same way as we did for Eq. (II.3.15) of Subsec. II.3.B.

Closed form approximate solutions can be obtained under the "slow variation" condition, that the coupling matrix elements $|K_{pqkl}(\omega_b)|$ and $|\Lambda_{pqkl}(\omega_b)|$ and the spacing parameter d_{pqkl} vary slowly with the energies $\epsilon_{pqkl} = \epsilon_{pk} + \epsilon_{ql}$ of the spectrum of the additional subspace S_A .

The most general solution under the "slow variation" condition is examined in Appendix C. However, here as in Subsec. II.3.B, we restrict ourselves to the physically simple case when the S_R -to- S_A coupling is "weak" in the sense of conditions (II.3.17b). The approximate solutions which correspond to this "weak" coupling have the form,

$$\left. \begin{aligned} f_{vb} &= f_{vc} \delta_{bc} ; & f_{vb} &= 0 ; \\ y_{pqkl}(\omega_v) &= f_{vc} \frac{K_{pqkl}(\omega_b)}{\hbar\omega_v - \epsilon_{pqkl}} ; & z_{pqkl}(\omega_v) &= -f_{vc} \frac{\Lambda_{pqkl}(\omega_b)}{\hbar\omega_v + \epsilon_{pqkl}} \end{aligned} \right\} \quad (II.4.11)$$

Repeating the same steps as in Subsec. II.3.B (from Eq. (II.3.18) to Eq. (II.3.29)) one finds:

$$|f_{vc}|^2 \approx \left\{ 1 + \left(\frac{\pi K}{D} \right)^2 + \frac{(\hbar\omega_v - \hbar\omega_0)^2}{K^2} \right\}^{-1} \quad (II.4.12)$$

where

$$K = |K_{pqkl}(\omega_c)| \left| \epsilon_{pqkl} \hbar\omega_c \right| \quad (II.4.13)$$

and

$$D = d_{pqkl} \left| \epsilon_{pqkl} \hbar\omega_c \right| \quad (II.4.14)$$

and ω_0 is the shifted collective frequency.

In the case of discrete S_A , the probability amplitude, $a_c(t)$, for finding the system in the one-phonon collective state is calculated exactly as in Subsec. II.3.C. The Poincaré recurrence period depends now on the 2p-2h level spacing D ,

$$\tau_p = \frac{2\pi K}{D} \quad (II.4.15)$$

As in Subsec. II.3.C, the system is dissipative, for a dense additional subspace S_A ($(\pi K)^2 \gg D^2$ or $\Gamma\tau_p \gg 4$), during the time interval $0 < t < \tau_p/2$. Then

$$a_c(t) \approx e^{-i\omega_0 t} \frac{e^{-\Gamma t/2} + e^{-\Gamma\tau_p/2} e^{\Gamma t/2}}{1 + e^{-\Gamma\tau_p/2}}, \quad 0 < t < \tau_p \quad (II.4.16)$$

as in Subsec. II.3.C (Eq. (II.3.40)). However, the decay width Γ is here entirely associated with the spreading width, $\Gamma^\dagger$, and given by

$$\Gamma = \Gamma^\dagger = \frac{2\pi}{K} \frac{K^2}{D} \quad (II.4.17)$$

For heavy- and medium-weight finite-A nuclei the 2p-2h additional spectrum is generally discrete but dense, so that Eqs. (II.4.15), (II.4.16) and (II.4.17) are expected to apply, unlike the case of the 1p-1h additional spectrum where the continuous S_A limit is physically relevant. However, for the sake of mathematical completeness one can repeat the steps of Subsec. II.3.D and derive the following results for the case of a continuous additional 2p-2h subspace S_A :

$$a_c(t) = e^{-i\omega_0 t} e^{-(\Gamma(\omega_c)t)/2}, \quad (\text{II.4.18})$$

where

$$\Gamma(\omega_c) = \Gamma^+(\omega_c) = \frac{2\pi}{\mathcal{M}} \iiint dp dq dk dz |K_{pqkl}(\omega_c)|^2 \delta(\mathcal{M}\omega_c - \epsilon(p,k) - \epsilon(q,l)). \quad (\text{II.4.19})$$

Finally, for the energy dissipation rate one finds following the same steps as in Sec. II.3.E,

$$\dot{E}_{\text{diss}}(t) = \frac{d}{dt} E_c(t) = -\Gamma E_c(t) = -\mathcal{M}\omega_c e^{-\Gamma t}, \quad 0 < t < \frac{\tau_p}{2} - \frac{2}{\Gamma}. \quad (\text{II.4.20})$$

However, the decay constant Γ is now entirely associated with the spreading width, Γ^+ , given either by Eq. (II.4.17) or by Eq. (II.4.20).

The approach utilized here for determining the spreading width contrasts with previous works^{59,60} on spreading widths which extend the simple RPA by adding an imaginary part to the self-energies of individual particles and holes. In particular, these complex self-energies result⁵⁹ from single-particle strength functions that are concentrated around the original single-particle energies--unlike the collective state strength function, here (Eq. (II.4.12)), which is concentrated around the one-phonon energy, $\mathcal{M}\omega_0$. In the time-dependent picture, these complex single-particle energies lead to unstable single-particle states with a finite lifetime, and, thus, with a finite mean free path^{36,37}--unlike the present second RPA model where the single-particle states are stable and, thus, the corresponding mean free path is infinite.

Rather, with respect to the result (II.4.17) for the spreading width, the present model is a generalization to the RPA of the schematic model utilized by Mahalia et al.⁶¹ to calculate the damping widths of Giant resonances in heavy and medium-weight nuclei. In contrast to the present work, however, these authors described the one-phonon collective state as an eigensolution of the TDA matrix and, subsequently, used a first-order time-dependent perturbation.

III. THE DAMPING OF THE STATIONARY SHARP n-PHONON COLLECTIVE STATES

III.1. Introduction

In Chap. II we developed a model describing the damping of the one-phonon collective state, $|c\rangle$, either in the usual RPA or in the second RPA. In this chapter we examine how the sharp n-phonon collective states, $|c^{(n)}\rangle$, decay, and we show that the collective energy here, in contrast with the one-phonon case, is not proportional to the probability for finding the system in the initial collective state. In particular, the collective energy here is proportional to the average number of phonons in the system instead of to the probability, $|a_c^{(n)}(t)|^2$, for finding the system in the initial n-phonon collective state. However, the energy dissipation rate here is again proportional to the total collective energy as in the one-phonon case, and, thus, it exhibits a simple exponential decay.

III 2. The Probability Amplitude $a_c^{(n)}(t)$ for Finding the System In An Initial n-Phonon Collective State $|c^{(n)}\rangle$

These stationary n-phonon collective states are created by repeated action of the RPA creation operator $Q_c^\dagger$, defined by (II.3.11b) when $a=c$;

$$|c^{(n)}\rangle = \frac{(Q_c^\dagger)^n |0\rangle_{S_R}}{\sqrt{n!}}. \quad (\text{III.2.1})$$

When the S_R -to- S_A coupling is neglected, the states $|c^{(n)}\rangle$ are approximate eigenstates of the many-body Hamiltonian, H :

$$[H, (Q_c^\dagger)^n] \approx n \hbar \omega_c (Q_c^\dagger)^n. \quad (\text{III.2.2})$$

Equation (III.2.2) is derived by using the operator identity (II.3.4)

and the RPA eigenvalue equation

$$[H, Q_c^\dagger] \approx \hbar \omega_c Q_c^\dagger, \quad (\text{III.2.3})$$

valid for the one-phonon state within the restricted subspace, S_R . The factor $\sqrt{n!}$ in the denominator of (III.2.1) guarantees proper normalization;

$$\langle c^{(n)} | c^{(n)} \rangle = 1, \quad (\text{III.2.4})$$

since, for a product of several boson operators, one has

$$S_R \langle 0 | Q_{\kappa_1} \dots Q_{\kappa_n} Q_{\lambda_1}^\dagger \dots Q_{\lambda_n}^\dagger | 0 \rangle_{S_R} = \mathcal{P}_{\lambda_1, \dots, \lambda_n} \delta_{\kappa_1, \lambda_1} \delta_{\kappa_2, \lambda_2} \dots \delta_{\kappa_n, \lambda_n}, \quad (\text{III.2.5})$$

where $\mathcal{P}_{\lambda_1, \dots, \lambda_n}$ stands for the permutations of the n elements $\lambda_1, \dots, \lambda_n$. The total number of terms in the right-hand side of Eq. (III.2.5) is $n!$.

In analogy with the damping of the one-phonon collective state (see Sec. II.2), we describe the damping of the n-phonon collective state by

propagating it in time according to the complete Hamiltonian, H . The complete Hamiltonian H includes all the interaction terms providing finite coupling between the restricted subspace S_R and the additional subspace S_A . Then the wave packet $|\phi^{(n)}(t)\rangle$ describing the damping is

$$|\phi^{(n)}(t)\rangle = e^{-iHt/\hbar} |c^{(n)}\rangle. \quad (\text{III.2.6})$$

To find, either in the RPA or the second RPA, an approximate solution for the wave packet (III.2.6), we use the expansion of the collective one-phonon state over the set of the "eigenstates" $|v\rangle$ in the extended subspace S ;

$$|c\rangle \approx \sum_v f_{vc}^* |v\rangle \quad (\text{III.2.7a})$$

or

$$Q_c^\dagger \approx \sum_v f_{vc}^* Q_v^\dagger \quad (\text{III.2.7b})$$

where the coefficients f_{vc} are given in the case of the simple RPA by Eq. (II.3.29) and in the case of the second RPA by Eq. (II.4.12). Since, for both the simple and Second RPA, the commutation

$$[H, Q_v^\dagger] \approx \hbar \omega_v Q_v^\dagger \quad (\text{III.2.8})$$

is valid in the extended subspace S and, since

$$\begin{aligned} e^{-iHt/\hbar} Q_v^\dagger e^{iHt/\hbar} &= Q_v^\dagger - \frac{it}{\hbar} [H, Q_v^\dagger] + \frac{1}{2!} \left(\frac{it}{\hbar}\right)^2 [H, [H, Q_v^\dagger]] - \dots \\ &\approx Q_v^\dagger \left(1 - i\omega_v t + \frac{(i\omega_v t)^2}{2!} - \dots\right) \\ &= Q_v^\dagger e^{-i\omega_v t}, \end{aligned} \quad (\text{III.2.9})$$

one finds

$$\begin{aligned} |\phi^{(n)}(t)\rangle &= e^{-iHt/\hbar} |c^{(n)}\rangle \\ &= e^{-iHt/\hbar} \frac{(Q_c^\dagger)^n |0\rangle_{S_R}}{\sqrt{n!}} \\ &\approx \frac{e^{-iHt/\hbar}}{\sqrt{n!}} \sum_{v_1, v_2, \dots, v_n} f_{v_1, c}^* f_{v_2, c}^* \dots f_{v_n, c}^* Q_{v_1}^\dagger Q_{v_2}^\dagger \dots Q_{v_n}^\dagger |0\rangle_S \\ &\approx \frac{e^{-iE_0, S t/\hbar}}{\sqrt{n!}} \sum_{v_1, v_2, \dots, v_n} f_{v_1, c}^* f_{v_2, c}^* \dots f_{v_n, c}^* \\ &\quad \times e^{-i(\omega_{v_1} + \omega_{v_2} + \dots + \omega_{v_n})t} Q_{v_1}^\dagger Q_{v_2}^\dagger \dots Q_{v_n}^\dagger |0\rangle_S. \end{aligned} \quad (\text{III.2.10})$$

Then the probability amplitude, $a_c^{(n)}(t)$, for finding, at a later time, the system in the initial n -phonon collective state is

$$\begin{aligned} a_c^{(n)}(t) &= e^{iE_0, S t/\hbar} \langle c^{(n)} | \phi^{(n)}(t) \rangle \\ &\approx \frac{1}{n!} \sum_{\substack{\mu_1, \dots, \mu_n \\ v_1, \dots, v_n}} f_{\mu_n, c} \dots f_{\mu_1, c} f_{v_1, c}^* \dots f_{v_n, c}^* \\ &\quad \times e^{-i(\omega_{v_1} + \dots + \omega_{v_n})t} \langle 0 | Q_{\mu_n} \dots Q_{\mu_1} Q_{v_1}^\dagger \dots Q_{v_n}^\dagger | 0 \rangle_S \\ &= \left(\sum_v |f_{vc}|^2 e^{-i\omega_v t} \right)^n = [F(t)]^n, \end{aligned} \quad (\text{III.2.11})$$

where we utilized Eq. (III.2.5). The expression

$$F(t) = \sum_v |f_{vc}|^2 e^{-i\omega_v t} \quad (\text{III.2.12})$$

equals the probability amplitude $a_c(t)$ for the one-phonon collective state. $F(t)$ exhibits, therefore, the same time behavior as the one discussed in Subsec. II.3.C, i.e., it exhibits a damped time dependence for the first half of the Poincaré recurrence period.

III.3. The Energy Dissipation Rate $\dot{E}_{\text{diss}}^{(n)}(t)$

In the case of a decaying initial n -phonon state, the collective energy $E_c^{(n)}(t)$ is defined as the expectation value of the collective Hamiltonian $H_c = \sum_c \omega_c Q_c^\dagger Q_c$ (Eq. (II.2.11)) with the wave packet (III.2.10).

One has

$$E_c^{(n)}(t) = \sum_c \omega_c \langle \phi^{(n)}(t) | Q_c^\dagger Q_c | \phi^{(n)}(t) \rangle$$

$$= \frac{\sum_c \omega_c}{n!} \sum_{\substack{\mu_1, \dots, \mu_n \\ \kappa, \lambda \\ \nu_1, \dots, \nu_n}} f_{\mu_n, c} \dots f_{\mu_1, c} \frac{1}{\kappa c} f_{\lambda c} f_{\nu_1, c} \dots f_{\nu_n, c}$$

$$\times e^{i(\omega_{\mu_n} + \dots + \omega_{\mu_1} - \omega_{\nu_1} - \dots - \omega_{\nu_n})t}$$

$$\times {}_S \langle 0 | \hat{Q}_{\mu_n} \dots \hat{Q}_{\mu_1} \hat{Q}_\kappa^\dagger \hat{Q}_\lambda \hat{Q}_{\nu_1}^\dagger \dots \hat{Q}_{\nu_n}^\dagger | 0 \rangle_S. \quad (\text{III.3.1})$$

Observe that in Eq. (III.3.1) the creation operator $\hat{Q}_\kappa^\dagger$ lies to the left of the annihilation operator $\hat{Q}_\lambda$, unlike the case of Eq. (III.2.5) where all the creation operators were to the right of every and all the annihilation operators. However, it is straightforward, using the boson commutation relations (A.29e), to calculate the result:

$${}_S \langle 0 | \hat{Q}_{\mu_n} \dots \hat{Q}_{\mu_1} \hat{Q}_\kappa^\dagger \hat{Q}_\lambda \hat{Q}_{\nu_1}^\dagger \dots \hat{Q}_{\nu_n}^\dagger | 0 \rangle_S$$

$$= \delta_{\kappa, \mu_1} \delta_{\lambda, \mu_2} \dots \delta_{\nu_1, \lambda} \delta_{\nu_2, \mu_2} \dots \delta_{\nu_n, \mu_n}$$

$$+ \sum_{s=2}^{n-1} \delta_{\kappa, \mu_s} \delta_{\mu_1, \dots, \mu_{(s-1)}, \lambda, \mu_{(s+1)}, \dots, \mu_n} \delta_{\nu_1, \mu_1} \dots \delta_{\nu_{(s-1)}, \mu_{(s-1)}}$$

$$\times \delta_{\nu_s, \lambda} \delta_{\nu_{(s+1)}, \mu_{(s+1)}} \dots \delta_{\nu_n, \mu_n}$$

$$+ \delta_{\kappa, \mu_n} \delta_{\mu_1, \dots, \mu_{(n-1)}, \lambda} \delta_{\nu_1, \mu_1} \dots \delta_{\nu_{(n-1)}, \mu_{(n-1)}} \delta_{\nu_n, \lambda}. \quad (\text{III.3.2})$$

The total number of terms in the right-hand side of Eq. (III.3.2) is $n(n!)$.

With the help of Eq. (III.3.2), one gets from Eq. (III.3.1):

$$E_c^{(n)}(t) = n \sum_c \omega_c \left(\sum_\mu |f_{\mu c}|^2 \right)^{n-1} \left(\sum_\kappa |f_{\kappa c}|^2 e^{i\omega_\kappa t} \right) \left(\sum_\lambda |f_{\lambda c}|^2 e^{-i\omega_\lambda t} \right)$$

$$= n \sum_c \omega_c |F(t)|^2. \quad (\text{III.3.3})$$

In deriving Eq. (III.3.3), we have utilized the property

$$\sum_c |f_{\mu c}|^2 = \langle c | c \rangle = 1. \quad (\text{III.3.4})$$

We observe that the collective energy in (III.3.3), in contrast with Eq. (II.3.55), is not proportional to the probability for finding the system in the initial collective state; i.e.,

$$E_c^{(n)}(t) \neq E_c^{(n)}(t=0) |a_c^{(n)}(t)|^2 \quad \text{for } n \geq 2. \quad (\text{III.3.5})$$

Rather, the collective energy $E_c^{(n)}(t)$ is proportional to the average number of phonons in the system, that is, proportional to

$$\langle \phi^{(n)}(t) | \hat{Q}_c^\dagger \hat{Q}_c | \phi^{(n)}(t) \rangle = n |F(t)|^2. \quad (\text{III.3.6})$$

As in the one-phonon case for a dense additional subspace S_A ($\Gamma_P \gg 4$), the system is dissipative in the time interval $0 < t < \frac{\tau_P}{2}$, where τ_P is the Poincaré recurrence period (Eq. (II.3.39) and (II.4.15)). During the time interval the energy dissipation rate $\dot{E}_{\text{diss}}^{(n)}(t)$ is defined as the time derivative of the collective energy. One has

$$\dot{E}_{\text{diss}}^{(n)}(t) = \frac{d}{dt} E_c^{(n)}(t) = -\Gamma E_c^{(n)}(t) = -n \Gamma \sum_c \omega_c e^{-\Gamma t}, \quad 0 < t < \frac{\tau_P}{2} - \frac{2}{\Gamma}. \quad (\text{III.3.7})$$

The energy dissipation rate (III.3.7) is again, as in the one-phonon case, proportional to the collective energy itself and, therefore, exhibits a simple exponential decay as a function of time. This behavior differs from that of a classical damped harmonic oscillator, whose energy dissipation rate is proportional to the kinetic energy and, therefore, oscillates while it decays. In Ch. IV we will show how this behavior goes over to that of the classical damped oscillator for certain coherent multi-boson packets, which describe classical time-varying vibrations of the mean field.

IV. DAMPING OF THE TIME-VARYING OSCILLATIONS OF THE MEAN FIELD (COHERENT MULTI-PHONON INITIAL WAVE PACKETS)

IV.1 Introduction

In Ch. IV we describe the damping of coherent time-varying mean field oscillations and construct a nonlinear frictional Hamiltonian.

In Sec. IV.2, we show how the collective one-phonon RPA state (or more generally, any n -phonon stationary state) is unable to describe time-varying *undamped* oscillations of the mean field since such oscillations require wave packets formed by linear superposition of the RPA n -phonon eigenstates. We exploit the analogy between the RPA approximation within the restricted subspace and the *undamped* quantal harmonic oscillator to introduce canonical position and momentum operators for the distortion of the mean field--here the collective degree of freedom--within the restricted subspace, S_R .

In Sec. IV.3, we consider the problem in the full set of extended subspaces. The time-dependent oscillations of the mean field become then damped, and are described by the time evolution of a wave packet either according to the RPA or the Second RPA of the complete Schrödinger equation when initialized with the coherent wave packet describing undamped mean field oscillations.

In Sec. IV.4 we proceed to introduce explicitly position and momentum canonical operators for the collective degree of freedom within the extended subspace, $S = S_R + S_A$. In contrast with Chs. II and III--where the collective Hamiltonian was defined to be completely within the restricted subspace, S_R --we here define a generalized collective Hamiltonian by partitioning the total Hamiltonian into a collective part, an interaction part, and an internal part through the construction of a nonlinear frictional Hamiltonian, which depends only upon the canonical collective position and momentum coordinates and upon the wave packet describing the

damping. This frictional Hamiltonian is constructed to prescribe precisely the same time behavior for these collective coordinates as does the total Hamiltonian in the RPA and the Second RPA. We calculate the collective energy of the system as the sum of the collective kinetic and the collective potential energy parts of the nonlinear frictional Hamiltonian, and extract its time decay. The energy dissipation rate has the property of being proportional to the square of the collective velocity. It is, therefore, proportional to the kinetic part of the collective energy, and oscillates while decaying. This is also true for the classical damped harmonic oscillator, but contrasts with the dissipation of the sharp π -phonon collective states (discussed in Chs. II and III where the energy dissipation rate was shown to be proportional to the total collective energy, and thus, exhibited a simple exponential decay).

IV.2. Undamped Time-Varying Oscillations of the Mean Field

In this section we consider the undamped oscillations of the mean field. In our model, this undamped limit is obtained when the S_R to S_A coupling is set equal to zero.

A. The Mean Field Operator

The single-particle mean field is defined^{40a} in the ph representation by

$$u_{\alpha\alpha'}(t) = \sum_{\beta\beta'} V_{\alpha\beta',\alpha'\beta} \rho_{\beta\beta'}(t), \quad (\text{IV.2.1a})$$

where the single-particle density matrix is

$$\rho_{\beta\beta'}(t) = \langle \chi(t) | a_{\beta'}^\dagger a_{\beta} | \chi(t) \rangle, \quad (\text{IV.2.1b})$$

and $|\chi(t)\rangle$ is the many-body wave function of the system under consideration. We define the operator for the mean field as follows:

$$U_{\alpha\alpha'} = \sum_{\beta\beta'} V_{\alpha\beta',\alpha'\beta} a_{\beta'}^\dagger a_{\beta}, \quad (\text{IV.2.2a})$$

and write it as a sum

$$U_{\alpha\alpha'} = U_{\alpha\alpha'}^{pp, hh} + U_{\alpha\alpha'}^{ph}, \quad (\text{IV.2.2b})$$

where the particle-particle and hole-hole part is

$$U_{\alpha\alpha'}^{pp, hh} = \sum_{mn} V_{\alpha mn, \alpha' n} a_m^\dagger a_n + \sum_{ij} V_{\alpha i \alpha' j} a_i^\dagger a_j, \quad (\text{IV.2.2c})$$

and the particle-hole part is

$$U_{\alpha\alpha'}^{ph} = \sum_n \{ V_{\alpha n \alpha' j} a_n^\dagger a_j + V_{\alpha j \alpha' n} a_j^\dagger a_n \}. \quad (\text{IV.2.2d})$$

Here, the ph indices (n, j) denote a state in the restricted subspace S_R .

As explained in Chap. II the RPA in S_R yields a set of "eigenstates"

(Eq. (II.3.11))

$$|a\rangle = Q_a^\dagger |0\rangle_{S_R}, \quad (\text{IV.2.3a})$$

where the creation operators $Q_a^\dagger$ are linear combinations of the ph pairs,

$$Q_a^\dagger = \sum_{nj} \{ y_{nj}(\omega_a) a_n^\dagger a_j + z_{nj}(\omega_a) a_j^\dagger a_n \}. \quad (\text{IV.2.3b})$$

Using the boson approximation (see Appendix A, Eq. (A.25a))

$$[a_{j,a}^\dagger, a_{m,a}^\dagger] \approx \delta_{mj} \delta_{1j} \quad (IV.2.4)$$

one can invert the relation (IV.2.3b)

$$a_{nj}^\dagger = \sum_a \{y_{nj}^*(\omega_a) Q_a^\dagger + z_{nj}(\omega_a) Q_a\} \quad (IV.2.5)$$

The index a runs over the dimensions of the restricted subspace S_R .

Substitution of Eq. (IV.2.5) into Eq. (IV.2.2d) yields the ph portion

of the mean field operator expressed in terms of the $Q_a^\dagger$'s and Q_a 's:

$$U_{\beta\beta'}^{ph} = \sum_a \{b_{\beta\beta',a}^* Q_a^\dagger + b_{\beta\beta',a} Q_a\} \quad (IV.2.6a)$$

where

$$b_{\beta\beta',a} = \sum_j \{V_{\beta n \beta' j} z_{nj}(\omega_a) + V_{\beta j \beta' n} y_{nj}(\omega_a)\} \quad (IV.2.6b)$$

B. Lack of Correspondence between the Stationary n-phonon RPA Collective

States, $e^{-i\omega_c t} |c^{(n)}\rangle$, and the Time-Varying Oscillations of the Mean Field

We now show that the stationary RPA collective eigenstates do not describe a time-varying mean field. In fact, if one uses the wave function $e^{-i\omega_c t} |c^{(n)}\rangle$ in Eq. (IV.2.1), one finds for the mean field:

$$U_{\alpha\alpha'}^{pp, hh} = \langle c^{(n)} | U_{\alpha\alpha'}^{pp, hh} | c^{(n)} \rangle = \sum_j V_{\alpha j \alpha' j} \quad (IV.2.7a)$$

and

$$U_{\alpha\alpha'}^{ph}(t) = \langle c^{(n)} | U_{\alpha\alpha'}^{ph} | c^{(n)} \rangle = 0 \quad (IV.2.7b)$$

since, in the RPA description (as discussed in Appendix E), the single-particle density matrix is diagonal for an RPA eigenstate, i.e.,

$$\langle c^{(n)} | a_{\alpha'}^\dagger a_{\alpha} | c^{(n)} \rangle = S_R \langle 0 | a_{\alpha'}^\dagger a_{\alpha} | 0 \rangle_{S_R} = \begin{cases} \delta_{\alpha\alpha'}, & \text{when } c=h \\ 0, & \text{otherwise} \end{cases} \quad (IV.2.8)$$

The mean field presented by Eq. (IV.2.7) is identical with the static Hartree-Fock mean field, and in particular, is time independent. The fact that the mean field corresponding to an RPA-approximate eigenstate is time independent should not be a surprising result since an eigenstate describes a state all of whose physical properties are constant in time.

C. Wave Packet Description of Undamped Oscillations of the Mean Field

Phenomenological descriptions of nuclear collective motion are often^{40a} based upon the small amplitude oscillations of the mean field around its static Hartree-Fock equilibrium shape. Frequently, also the mean field is approximated by a phenomenological shell model potential. In the present work, we describe such undamped oscillatory behavior of the mean field by the following multi-phonon coherent wave packet*

$$\begin{aligned} |\psi(t)\rangle &= \exp \left\{ -\frac{\epsilon^2}{2} + \epsilon e^{-i(\omega_c t + \phi)} Q_c^\dagger \right\} |0\rangle_{S_R} \\ &= e^{-\epsilon^2/2} \sum_{n=0}^{\infty} \frac{\epsilon^n e^{-i\ln(\omega_c t + \phi)}}{\sqrt{n!}} |c^{(n)}\rangle \end{aligned} \quad (IV.2.9)$$

where the states $|c^{(n)}\rangle$ are created through repeated action of the RPA collective operator, $Q_c^\dagger$, on the ground state, i.e.,

$$|c^{(n)}\rangle = \frac{(Q_c^\dagger)^n |0\rangle_{S_R}}{\sqrt{n!}}, \quad n=0,1,2,\dots,\infty \quad (IV.2.10)$$

*The wave packet (IV.2.9) has the same form as the Glauber packets used in the quantum theory of coherent light.^{38,62,63} We thus expect the model developed in Sec. IV.3 for the damping of the coherent packet (IV.2.9) to have possible applications in quantum optics as well.

and ϕ is an arbitrary initial phase. In Eq. (IV.2.9),

ϵ is a parameter which specifies the amplitude of the oscillations, and the factor $e^{-\epsilon^2/2}$ provides proper normalization.

For the wave function (IV.2.9), one finds that the single-particle density matrix of Eq. (IV.2.1b) (see Appendix E) has the following pp and hh matrix elements:

$$\rho_{mn}(t) = 0; \rho_{ij}(t) = \delta_{ij}. \quad (IV.2.11)$$

It follows that the corresponding pp and hh part of the mean field is again given by the Hartree-Fock static field as specified in Eq. (IV.2.7a). Now, however, in contrast with the situation for collective eigensolutions discussed in subsec. IV.2.B, the expectation value of the operator $U_{\alpha\alpha}^{ph}$ of Eq. (IV.2.6a) is non-zero and time dependent; there results a time-dependent part of the mean field given by the expression:

$$u_{\beta\beta'}^{ph}(t) = \sum_a \{ b_{\beta',a}^* \langle \psi(t) | Q_a^\dagger | \psi(t) \rangle + b_{\beta\beta',a} \langle \psi(t) | Q_a | \psi(t) \rangle \}. \quad (IV.2.12)$$

In Appendix E we show that the matrix elements simplify to δ -functions,

$$\langle \psi(t) | Q_a^\dagger | \psi(t) \rangle = \langle \psi(t) | Q_a | \psi(t) \rangle^* = \epsilon e^{i(\omega_c t + \phi)} \delta_{ac}, \quad (IV.2.13)$$

so that the time-dependent mean field becomes

$$u_{\beta\beta'}^{ph}(t) = \epsilon \{ b_{\beta',c}^* e^{i(\omega_c t + \phi)} + b_{\beta\beta',c} e^{-i(\omega_c t + \phi)} \}. \quad (IV.2.14)$$

Thus, the time-dependent part of the mean field oscillates sinusoidally with frequency ω_c and with an amplitude proportional to the parameter ϵ . The time-dependent mean field (IV.2.14) calculated with the coherent RPA wave packet (IV.2.9) is equal to the time-dependent mean field calculated in the theory of small amplitude TDHF.^{40a} This equality results from the fact that both the RPA and the small amplitude TDHF yield the same single-particle density matrix elements, despite the fact that the RPA coherent wave packet (IV.2.9) is a many-body wave function different from the TDHF determinants.

D. Collective Energy For the Multi-Phonon Coherent Packet (IV.2.9)

In Sec. II.2 we defined the collective Hamiltonian, H_c , by decoupling the subspaces S_R and S_A . We found

$$H_c = \hbar\omega_c Q_c^\dagger Q_c. \quad (IV.2.15)$$

Since in this section we have neglected the S_R -to- S_A coupling, the definition (IV.2.15) for the collective Hamiltonian is also appropriate here. For the system described by the wave function (IV.2.9), it leads to the collective energy:

$$E_c(t) = \hbar\omega_c \langle \psi(t) | Q_c^\dagger Q_c | \psi(t) \rangle \\ = \hbar\omega_c e^{-\epsilon^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\epsilon^{m+n} e^{i(m-n)(\omega_c t + \phi)}}{m!n!} S_R \langle 0 | Q_c^m Q_c^\dagger Q_c (Q_c^\dagger)^n | 0 \rangle_{S_R}. \quad (IV.2.16a)$$

The expectation value in (IV.2.16a) of the product of the boson operators $Q_c^\dagger$ and Q_c over their own vacuum yields a non-zero value only when $m=n$.

Using Eq. (III.3.2) one finds

$$S_R \langle 0 | Q_c^{m\dagger} Q_c (Q_c^\dagger)^n | 0 \rangle_{S_R} = n(n!) \delta_{nm}. \quad (IV.2.16b)$$

It follows then that

$$E_c(t) = \hbar\omega_c e^{-\epsilon^2} \sum_{n=0}^{\infty} \frac{\epsilon^{2n} 2n}{n!} \\ = \epsilon^2 \hbar\omega_c e^{-\epsilon^2} \sum_{n=1}^{\infty} \frac{2(n-1)}{(n-1)!} \\ = \epsilon^2 \hbar\omega_c. \quad (IV.2.17)$$

The result (IV.2.17) for the collective energy is not surprising since the operators $Q_c^\dagger$ and Q_c behave like the raising and lowering operators of a simple harmonic oscillator. Unlike the energy $n\hbar\omega_c$ of the n -phonon eigenstates considered in Chaps. II and III, the energy (IV.2.17) of the coherent wave packet (IV.2.9) is not quantized; on the contrary, it depends on the continuous variable ϵ .

E. Generalized Position and Canonical Momentum Operators for the Collective Degree of Freedom

Within the restricted subspace S_R the RPA collective operators $Q_c^\dagger$ and Q_c behave like the raising and lowering operators of an undamped quantum harmonic oscillator of frequency ω_c . Thus, when the S_R -to- S_A coupling is neglected, the RPA is analogous to an undamped simple harmonic oscillator whose Hamiltonian, H_{SHO} , is

$$H_{SHO} = \frac{1}{2} m_c \omega_c^2 (Q_c^\dagger + Q_c)^2. \quad (IV.2.18)$$

This analogy enables one to introduce a generalized position operator for the collective degree of freedom as follows:⁶⁴

$$\sigma_c = Q_c^\dagger + Q_c. \quad (IV.2.19)$$

The corresponding momentum operator, p_c , being canonical to σ_c , satisfies the commutation relation

$$[\sigma_c, p_c] = i\hbar. \quad (IV.2.20)$$

To determine the momentum operator, p_c , we make use of the relation

$$p_c = m_c \dot{\sigma}_c, \quad (IV.2.21)$$

where $\dot{\sigma}_c$ is the velocity operator and m_c the inertial parameter for the collective degree of freedom. The velocity operator $\dot{\sigma}_c$ is determined by the general definition⁶⁵ in the Schrödinger representation of the time-derivative, $\hat{A}$, of any operator $\hat{A}$:

$$\hat{A} = \frac{1}{i\hbar} [H, \hat{A}] + \frac{\partial \hat{A}}{\partial t}. \quad (IV.2.22)$$

Here, H is the complete, exact, many-body Hamiltonian. Calculating the commutator $[H, \sigma_c]$ in RPA within the restricted subspace S_R , one finds:

$$\dot{\sigma}_c = i\omega_c (Q_c^\dagger - Q_c). \quad (IV.2.23)$$

The inertial parameter m_c is determined by substituting Eqs. (IV.2.21) and (IV.2.23) in Eq. (IV.2.20). Using the boson commutation relations one finds:

$$\begin{aligned} i\hbar &= m_c [\sigma_c, \dot{\sigma}_c] = i m_c \omega_c [Q_c^\dagger + Q_c, Q_c^\dagger - Q_c] \\ &= 2i m_c \omega_c. \end{aligned} \quad (IV.2.24)$$

Thus

$$m_c = \frac{\hbar}{2\omega_c}. \quad (IV.2.25)$$

Equation (IV.2.21) then yields:

$$p_c = \frac{i\hbar}{2} (Q_c^\dagger - Q_c). \quad (IV.2.26)$$

In relation to the definition (IV.2.22) of a velocity operator, it is worth remembering that Eq. (IV.2.22) is the quantum analog of the general formula of classical mechanics for the total time-derivative of a function $A(q, p, t)$, where q and p are position and momentum canonical variables. Indeed one has in classical mechanics:

$$\frac{dA}{dt} = [A, H]_{PB} + \frac{\partial A}{\partial t}, \quad (IV.2.27)$$

where $[A, H]_{PB}$ is a Poisson bracket. Equation (IV.2.22) can then be viewed as resulting^{65,66} from (IV.2.27) through the substitution of the Poisson bracket with $1/i\hbar$ times the commutator of the corresponding quantal operators $\hat{A}$ and H . Notice further that Hamilton's canonical equations of motion are a special case of (IV.2.27) when the function $A(q, p, t)$ is simply either q or p .

Calculating the expectation value for the position operator with the wave packet (IV.2.9), one gets using the boson commutation relations:

$$\begin{aligned}
\langle \psi(t) | \sigma_c | \psi(t) \rangle &= e^{-\epsilon^2} \sum_{m=0, n=0}^{\infty} \frac{\epsilon^{m+n} e^{i(m-n)(\omega_c t + \phi)}}{m!n!} \\
&\quad \times \langle 0 | Q_c^m (Q_c^\dagger + Q_c) (Q_c^\dagger)^n | 0 \rangle_{S_R} \\
&= e^{-\epsilon^2} \sum_{m=0, n=0}^{\infty} \frac{\epsilon^{m+n} e^{i(m-n)(\omega_c t + \phi)}}{m!n!} (\delta_{m,n+1} (m!) + \delta_{m+1,n} (n!)) \\
&= e^{-\epsilon^2} \left(\sum_{m=1}^{\infty} \frac{\epsilon^{2(m-1)}}{(m-1)!} \right) \left(e^{i(\omega_c t + \phi)} + e^{-i(\omega_c t + \phi)} \right) \\
&= 2\epsilon \cos(\omega_c t + \phi). \quad (IV.2.28)
\end{aligned}$$

The result (IV.2.28) is well known from the theory⁶⁴ of the simple undamped harmonic oscillator. It shows that the expectation value of the position operator over the multi-phonon coherent wave packet (IV.2.9) behaves like the coordinate of a *classical* undamped oscillator.

F. The Collective Hamiltonian, H_c , as a Sum of Kinetic and Potential

Energy Terms

The introduction of position and momentum canonical collective coordinates allows the expression of the collective Hamiltonian, H_c , as a sum of kinetic and potential energy terms minus the zero point, $\hbar\omega_c/2$, energy. In fact, starting with Eq. (IV.2.19) for the collective coordinate, Eq. (IV.2.26) for the collective momentum, and Eq. (IV.2.25) for the inertial parameter, one can repeat the same steps as in Ref. 64, and derive the following operator relation:

$$H_c = \hbar\omega_c Q_c^\dagger Q_c = \frac{p_c^2}{2m_c} + \frac{1}{2} m_c \omega_c^2 Q_c^2 - \frac{\hbar\omega_c}{2}. \quad (IV.2.29)$$

G. Hilbert Space Versus Phase Space

With the use of the coherent wave packet (IV.2.9) and the introduction of the collective coordinates, Q_c and p_c , one is in a position to describe the collective motion in two different ways. The first description utilizes the Hilbert subspace spanned by any or all the n -phonon collective states $|c^{(n)}\rangle$. The second description utilizes the phase space spanned by the position, Q_c , and momentum, p_c , collective coordinates. These two descriptions are equivalent in all aspects when the *undamped* collective motion is considered. However, as it will be discussed in Ch. V, they differ when the *damped* collective motion is considered. In this case, from the Hilbert subspace view, there is a loss of both probability and energy. On the contrary, the phase space view leads to nonlinear frictional effective Hamiltonians which *do* conserve the probability while dissipating energy.

IV.3. Damped Time-Varying Oscillations of the Mean Field

A. Wave Packet Description of Damped Oscillations of the Mean Field

In this section we generalize the treatment of the multi-phonon coherent wave packet, so that it is able to describe damped oscillations of the mean field as a result of a finite coupling between the S_R and S_A subspaces. In subsec. IV.2.1 we found that time-varying *undamped* oscillations of the mean field cannot be described by the one-phonon collective RPA eigenstate, nor by any other n-phonon RPA eigenstate. They can, however, be described by a multi-phonon minimum uncertainty coherent wave packet (IV.2.9) formed by a superposition of n-phonon eigenstates. We here generalize this description to include the damping of the mean field oscillations via a straightforward extension of the general microscopic process formulated in Sec. II.2. In particular, we treat the time evolution of the following wave packet:

$$|\psi(t)\rangle = e^{-iHt/\hbar} \exp\left[-\frac{\epsilon^2}{2} + \epsilon e^{-i\phi} Q_c^\dagger\right] |0\rangle_{S_R}, \quad (IV.3.1)$$

where H is the complete Hamiltonian, including now all interaction terms which provide finite coupling between the subspaces S_R and S_A . The wave packet (IV.3.1) implies the initial condition

$$|\psi(t=0)\rangle = \exp\left[-\frac{\epsilon^2}{2} + \epsilon e^{-i\phi} Q_c^\dagger\right] |0\rangle_{S_R} \quad (IV.3.2)$$

instead of the single eigenfunction initialization

$$|\phi(t=0)\rangle = |c\rangle, \quad (IV.3.3)$$

utilized in Sec. II.2.

We calculate the wave packet (IV.3.1) either in the RPA or in the second RPA approximation while taking into account the finite coupling between the subspaces S_R and S_A . To carry out this calculation we utilize either the simplified solution (II.3.18) for the RPA eigenstates $|v\rangle$ in S , or the simplified solution (II.4.12) for the second RPA eigenstates

$|v\rangle$ in S . This simplified solution allows the expansion (see Sec. II.3, in particular Eqs. (II.3.30) and (II.3.31))

$$|c\rangle \approx \sum_v f_{vc}^* |v\rangle \quad (IV.3.4a)$$

and

$$Q_c^\dagger \approx \sum_v f_{vc}^* Q_v^\dagger. \quad (IV.3.4b)$$

One then finds

$$|\psi(t=0)\rangle \approx e^{-\epsilon^2/2} \exp\left\{\epsilon e^{-i\phi} \sum_v f_{vc}^* Q_v^\dagger\right\} |0\rangle_S. \quad (IV.3.5)$$

The wave packet (IV.3.1) then becomes

$$\begin{aligned} |\psi(t)\rangle &\approx e^{-iHt/\hbar} \exp\left[-\frac{\epsilon^2}{2} + \epsilon e^{-i\phi} \sum_v f_{vc}^* Q_v^\dagger\right] |0\rangle_S \\ &= e^{-\epsilon^2/2} e^{-iHt/\hbar} \left(\sum_{m=0}^{\infty} \frac{\epsilon^m e^{-im\phi}}{m!} \sum_{v_1, \dots, v_m} f_{v_1, c}^* \dots f_{v_m, c}^* Q_{v_1}^\dagger \dots Q_{v_m}^\dagger\right) |0\rangle_S \end{aligned} \quad (IV.3.6)$$

Using the relation (Eq. (III.2.9))

$$e^{-iHt/\hbar} Q_v^\dagger e^{iHt/\hbar} \approx Q_v^\dagger e^{-i\omega_v t}, \quad (IV.3.7)$$

the final result for the wave packet $|\psi(t)\rangle$ is:

$$|\psi(t)\rangle \approx e^{-iE_0 S t/\hbar} \exp\left[-\frac{\epsilon^2}{2} + \epsilon \sum_v f_{vc}^* e^{-i(\omega_v t + \phi)} Q_v^\dagger\right] |0\rangle_S. \quad (IV.3.8)$$

The probability amplitude, $A_c(t)$, for finding the system in the initial state at a later time t is:

$$\begin{aligned} A_c(t) &= e^{iE_0 S t/\hbar} \langle \psi(t=0) | \psi(t) \rangle \\ &= e^{-\epsilon^2} \sum_{m=0, n=0}^{\infty} \frac{\epsilon^{m+n} e^{i(m-n)\phi}}{m!n!} \sum_{\substack{\mu_1, \dots, \mu_m \\ \nu_1, \dots, \nu_n}} f_{\mu_1, c} \dots f_{\mu_m, c} f_{\nu_n, c}^* \dots f_{\nu_1, c}^* \\ &\quad \times e^{-i(\omega_{\nu_n} + \dots + \omega_{\nu_1})t} \langle 0 | Q_{\mu_1} \dots Q_{\mu_m} Q_{\nu_n}^\dagger \dots Q_{\nu_1}^\dagger | 0 \rangle_S. \end{aligned} \quad (IV.3.9)$$

The expectation value in Eq. (IV.3.9) of the product of the boson operators is different than zero only when $m=n$. Then one has

$${}_S \langle 0 | q_{\mu 1} \dots q_{\mu n} \hat{Q}_{\nu n}^\dagger \dots \hat{Q}_{\nu 1}^\dagger | 0 \rangle_S = \mathcal{P}_{\nu 1, \dots, \nu n} \delta_{\mu 1, \nu 1} \delta_{\mu 2, \nu 2} \dots \delta_{\mu n, \nu n} \quad (\text{IV.3.10})$$

where $\mathcal{P}_{\nu 1, \dots, \nu n}$ stands for the permutations of the n elements $\nu 1, \dots, \nu n$.

With the help of (IV.3.10) the probability amplitude (IV.3.9) becomes

$$A_c(t) = e^{-\epsilon^2} \sum_{n=0}^{\infty} \frac{\epsilon^{2n}}{n!} \left(\sum_{\nu} |f_{\nu c}|^2 e^{-i\omega_{\nu} t} \right)^n \\ = \exp \{ \epsilon^2 F(t) - \epsilon^2 \}, \quad (\text{IV.3.11})$$

where

$$F(t) = \sum_{\nu} |f_{\nu c}|^2 e^{-i\omega_{\nu} t}. \quad (\text{IV.3.12})$$

The quantity $F(t)$ here--as in Chap. III (Eq. (III.2.12)), where the damping of a sharp n -phonon collective state was considered--is equal to the prob-

ability amplitude $a_c(t)$ (Eq. (II.3.33)) for the decay of the one-phonon

state. $F(t)$ exhibits, therefore, the same time behavior as the

one discussed in Subsec. II.3. In particular, in the case of

a dense additional subspace S_A ($\Gamma \tau_p \gg 4$) the system is dissipative

during the first half of the Poincaré period, τ_p , and

$$F(t) \approx e^{-i\omega_0 t - (\Gamma t/2)}; \quad 0 < t < \frac{\tau_p}{2} - \frac{\Gamma}{2}. \quad (\text{IV.3.13})$$

Equation (IV.3.13) is valid for both cases when S_A includes either $1p-1h$

or $2p-2h$ states. However, in these two cases the shifted frequency,

ω_0 , and the decay width, Γ , have a different dependence upon the two-body

force and upon the RPA amplitudes determining the one-phonon collective state.

Specifically, in the former case, the results of Sec. II.3 for the escape

width apply, while in the latter case one should use the results of

Sec. II.4 for the spreading width.

B. Explicit Demonstration of the Damping of the Oscillations of the

Mean Field

(i) The additional subspace S_A has states of the $2p-2h$ structure only.

Since the mean field does not depend on the two-body, but only on the one-body density matrix, the $2p-2h$ additional states do not contribute any terms in the definition of the mean field operator. Working as in Subsec. IV.2.A we find that Eq. (IV.2.2d) for the particle-hole part, $\hat{U}_{\alpha\alpha'}^{ph}$, of the mean field operator is still valid when the S_R -to- S_A coupling is considered,

$$\hat{U}_{\alpha\alpha'}^{ph} = \sum_{nj} \{ \bar{v}_{\alpha n \alpha' j} a_n^\dagger a_j + v_{\alpha j \alpha' n} a_j^\dagger a_n \}. \quad (\text{IV.3.14})$$

The ph indices (n, j) denote a state in the restricted subspace S_R .

As explained in Sec. II.4 the second RPA in the extended subspace

S yields a set of "eigenstates", $|v\rangle$ (Eq. (II.4.1)), such that

$$|v\rangle = \hat{Q}_v^\dagger |0\rangle_S, \quad (\text{IV.3.15})$$

where the creation operators $\hat{Q}_v^\dagger$ are

$$\hat{Q}_v^\dagger = \sum_{mi} \{ Y_{mi}(\omega_v) a_m^\dagger a_i - Z_{mi}(\omega_v) a_i^\dagger a_m \} \\ + \sum_{\substack{m < n \\ i < j}} \{ Y_{mni j}(\omega_v) a_m^\dagger a_n^\dagger a_j a_i - Z_{mni j}(\omega_v) a_i^\dagger a_j^\dagger a_n a_m \}. \quad (\text{IV.3.16})$$

Inverting Eq. (IV.3.16) we find

$$a_n^\dagger a_j = \sum_{\nu} \{ [\hat{Q}_\nu^\dagger a_n^\dagger a_j] \hat{Q}_\nu^\dagger - [\hat{Q}_\nu^\dagger a_n^\dagger a_j] \hat{Q}_\nu \}. \quad (\text{IV.3.17})$$

Using the boson approximation to calculate the commutators in (IV.3.17), one gets:

$$a_n^\dagger a_j = \sum_{\nu} \{ Y_{nj}(\omega_\nu) \hat{Q}_\nu^\dagger + Z_{nj}(\omega_\nu) \hat{Q}_\nu \}. \quad (\text{IV.3.18})$$

With the help of Eq. (IV.3.18), Eq. (IV.3.14) for the particle-

hole part of the mean field operator yields

$$\hat{U}_{\alpha\alpha'}^{ph} = \sum_{\nu} \{ B_{\alpha\alpha', \nu}^{(II)} \hat{Q}_\nu^\dagger + B_{\alpha\alpha', \nu}^{(II)} \hat{Q}_\nu \}. \quad (\text{IV.3.19a})$$

where

$$B_{aa',v}^{(II)} = \sum_{nj} \{V_{ana'j} Z_{nj}(\omega_v) + V_{aja'n} Y_{nj}(\omega_v)\}. \quad (IV.3.19b)$$

Equations (IV.3.19) are a straightforward extension of Eqs. (IV.2.6).

When the approximate solution (II.4.12) of Sec. II.4 for the second RPA eigenstates $|v\rangle$ in the extended subspace S is used, one finds:

$$B_{\beta\beta',v}^{(II)} \approx f_{vc} b_{\beta\beta',c} \quad (IV.3.20)$$

where the $b_{\beta\beta',c}$ are given by Eq. (IV.2.6b) when $a=c$. Equation (IV.3.19a) then yields

$$\hat{u}_{aa'}^{ph} = (\sum_v f_{vc}^* \hat{Q}_v^\dagger) b_{aa',c}^* + (\sum_v f_{vc} \hat{Q}_v) b_{aa',c} \quad (IV.3.21)$$

In Appendix F we have calculated

$$\begin{aligned} \langle \Psi(t) | \sum_v f_{vc}^* \hat{Q}_v^\dagger | \Psi(t) \rangle &= \langle \Psi(t) | \sum_v f_{vc} \hat{Q}_v | \Psi(t) \rangle^* \\ &= e^{i\phi} F^*(t). \end{aligned} \quad (IV.3.22)$$

Using the result (IV.3.13) for the quantity $F(t)$ (valid in the case of a dense additional subspace), one finds for the time-varying part of the mean field:

$$\begin{aligned} u_{aa'}^{ph}(t) &= \langle \Psi(t) | \hat{u}_{aa'}^{ph} | \Psi(t) \rangle \\ &\approx e^{-\Gamma t/2} [b_{aa',c}^* e^{i(\omega_0 t + \phi)} + b_{aa',c} e^{-i(\omega_0 t + \phi)}], \quad 0 < t < \frac{\tau_p}{2} - \frac{2}{\Gamma}. \end{aligned} \quad (IV.3.23)$$

The time-dependent part of the mean fields performs damped harmonic oscillations during the first half of the Poincaré recurrence period.

(ii) The additional subspace S_A has states of the lp-lh structure only.

When the additional subspace includes lp-lh states, the particle-hole part of the mean field operator becomes

$$\begin{aligned} \hat{u}_{aa'}^{ph} &= \sum_{nj} \{V_{ana'j} a_n^\dagger a_j + V_{aja'n} a_j^\dagger a_n\} \\ &+ \sum_{\tilde{n}\tilde{j}} \{V_{a\tilde{n}a'\tilde{j}} a_{\tilde{n}}^\dagger a_{\tilde{j}} + V_{a\tilde{j}a'\tilde{n}} a_{\tilde{j}}^\dagger a_{\tilde{n}}\}, \end{aligned} \quad (IV.3.24)$$

where indices (n,j) denote lp-lh states in the restricted subspace, S_R , and indices $(\tilde{n},\tilde{j})$ denote lp-lh states in the additional subspace, S_A . As explained in Sec. II.3, the simple RPA in the extended subspace, S , yields a set of "eigenstates" $|v\rangle$ (Eq. (II.3.1)) such that

$$|v\rangle = \hat{Q}_v^\dagger |0\rangle_S, \quad (IV.3.25)$$

where the creation operators $\hat{Q}_v^\dagger$ —in contrast with Eq. (IV.3.16)—are now given by

$$\begin{aligned} \hat{Q}_v^\dagger &= \sum_{m1} \{Y_{m1}(\omega_v) a_m^\dagger a_1 - Z_{m1}(\omega_v) a_1^\dagger a_m\} \\ &+ \sum_{\tilde{m}\tilde{1}} \{Y_{\tilde{m}\tilde{1}}(\omega_v) a_{\tilde{m}}^\dagger a_{\tilde{1}} - Z_{\tilde{m}\tilde{1}}(\omega_v) a_{\tilde{1}}^\dagger a_{\tilde{m}}\}. \end{aligned} \quad (IV.3.26)$$

Inverting Eq. (IV.3.26) and substituting in Eq. (IV.3.24), one finds

$$\hat{u}_{aa'}^{ph} = \sum_v \{B_{aa',v}^{(I)*} \hat{Q}_v^\dagger + B_{aa',v}^{(I)} \hat{Q}_v\}, \quad (IV.3.27a)$$

where

$$\begin{aligned} B_{aa',v}^{(I)} &= \sum_{nj} \{V_{ana'j} Z_{nj}(\omega_v) + V_{aja'n} Y_{nj}(\omega_v)\} \\ &+ \sum_{\tilde{n}\tilde{j}} \{V_{a\tilde{n}a'\tilde{j}} Z_{\tilde{n}\tilde{j}}(\omega_v) + V_{a\tilde{j}a'\tilde{n}} Y_{\tilde{n}\tilde{j}}(\omega_v)\}. \end{aligned} \quad (IV.3.27b)$$

When the approximate solution (II.3.18) of Sec. II.3 for the simple RPA eigenstates $|v\rangle$ in the extended subspace, S , is used, one finds:

$$B_{aa',v}^{(I)} = f_{vc} b_{aa',c} + f_{vc} \sum_{\tilde{n}\tilde{j}} \left\{ \frac{V_{a\tilde{n}a'\tilde{n}} K_{\tilde{n}\tilde{j}}(\omega_c)}{\hbar\omega_v - \epsilon_{\tilde{n}\tilde{j}}} - \frac{V_{a\tilde{n}a'\tilde{j}} K_{\tilde{n}\tilde{j}}(\omega_c)}{\hbar\omega_v + \epsilon_{\tilde{n}\tilde{j}}} \right\}. \quad (IV.3.28)$$

Taking into consideration the slow variation condition (II.3.17a) and repeating the same steps from Eq. (II.3.20) until Eq. (II.3.24), one finds:

$$\sum_{\tilde{n}\tilde{j}} \frac{V_{a\tilde{n}a'\tilde{n}} K_{\tilde{n}\tilde{j}}(\omega_c)}{\hbar\omega_v - \epsilon_{\tilde{n}\tilde{j}}} \approx v K \frac{\pi}{d_v} \cot\left(\frac{\pi\tau_v}{d_v}\right) + F(V_{a\tilde{j}a'\tilde{n}} K_{\tilde{n}\tilde{j}}(\omega_c); \omega_v), \quad (IV.3.29)$$

where

$$V = V_{aj'a'n} \left| \begin{matrix} \epsilon_{nj} \hbar \omega_c \\ \epsilon_{nj} \hbar \omega_c \end{matrix} \right| ; K = K_{nj}(\omega_c) \left| \begin{matrix} \epsilon_{nj} \hbar \omega_c \\ \epsilon_{nj} \hbar \omega_c \end{matrix} \right| , \quad (IV.3.30)$$

and

$$F(V_{aj'a'n}, K_{nj}(\omega_c); \omega_v) = \sum_{nj} \frac{V_{aj'a'n} K_{nj}(\omega_c)}{\epsilon_{pk} + \frac{1}{2} - \epsilon_{nj}} . \quad (IV.3.31)$$

Setting

$$G(V_{aj'a'n}, \Lambda_{nj}(\omega_c); \omega_v) = \sum_{nj} \frac{V_{aj'a'n} \Lambda_{nj}(\omega_c)}{\hbar \omega_v + \epsilon_{nj}} , \quad (IV.3.32)$$

and using Eq. (IV.3.29) and the dispersion relation (II.3.25a), one gets from Eq. (IV.3.28):

$$B_{aa',v}^{(I)} \approx f_{vc} b_{aa',c} + f_{vc} \left[\frac{V}{K^*} (\hbar \omega_v - \hbar \omega_0) + F(V_{aj'a'n}, K_{nj}(\omega_c); \omega_v) - G(V_{aj'a'n}, \Lambda_{nj}(\omega_c); \omega_v) \right] . \quad (IV.3.33)$$

The term F-G in (IV.3.33) is part of the shift in the collective frequency. Because of the weak coupling condition (II.3.17b)--that this shift is small--the term F-G can be neglected. Thus,

$$B_{aa',v}^{(I)} \approx f_{vc} b_{aa',c} + f_{vc} \frac{V}{K^*} (\hbar \omega_v - \hbar \omega_0) . \quad (IV.3.34)$$

Equation (IV.3.27a) then yields

$$\begin{aligned} \hat{U}_{aa'}^{ph} &= \left(\sum_v f_{vc}^* \hat{Q}_v^\dagger \right) b_{a',c}^* + \left(\sum_v f_{vc} \hat{Q}_v \right) b_{aa',c} \\ &+ \frac{V}{K} \sum_v f_{vc}^* (\hbar \omega_v - \hbar \omega_0) \hat{Q}_v^\dagger + \frac{V}{K^*} \sum_v f_{vc} (\hbar \omega_v - \hbar \omega_0) \hat{Q}_v . \end{aligned} \quad (IV.3.35)$$

For a dense additional subspace, the coefficients f_{vc}^* (Eq. (II.3.19)) are centered around the energy, $\hbar \omega_0$. Because of this and the small spreading of the f_{vc}^* 's around $\hbar \omega_0$, the last two terms in Eq. (IV.3.35) yield negligible contributions to the expectation value of $\hat{U}_{aa'}^{ph}$. One thus has:

$$u_{ac}^{ph}(t) = \langle \Psi(t) | \hat{U}_{aa'}^{ph} | \Psi(t) \rangle$$

$$\approx b_{a',c}^* \langle \Psi(t) | \left(\sum_v f_{vc}^* \hat{Q}_v^\dagger \right) | \Psi(t) \rangle + b_{aa',c} \langle \Psi(t) | \left(\sum_v f_{vc} \hat{Q}_v \right) | \Psi(t) \rangle . \quad (IV.3.36)$$

Using Eq. (IV.3.22) and (IV.3.13), one finds again for the time-varying part of the mean field:

$$u_{aa'}^{ph}(t) \approx e^{-\Gamma t/2} \left[b_{c',a,c}^* e^{i(\omega_0 t + \phi)} + b_{aa',c} e^{-i(\omega_0 t + \phi)} \right] \quad (IV.3.37)$$

$$0 < t < \frac{\tau_p}{2} - \frac{2}{\Gamma} .$$

The time-dependent part of the mean field performs again damped harmonic oscillations during the first half of the Poincaré recurrence period.

IV.4. The Frictional Hamiltonian and the Energy Dissipation Rate

A. Position and Momentum Canonical Operators for the Collective Degree of Freedom Within the Extended Subspace, $S = S_R + S_A$. The Time Decay of Their Expectation Values

By analogy with the oscillator relationship (IV.2.19), the generalized position operator, $\hat{\sigma}_c$, for the collective degree of freedom in the extended subspace, $S = S_R + S_A$, is defined as

$$\begin{aligned}\hat{\sigma}_c &= \hat{Q}_c^\dagger + \hat{Q}_c \\ &= \sum_{\nu} \{f_{\nu c}^* \hat{Q}_{\nu}^\dagger + f_{\nu c} \hat{Q}_{\nu}\}.\end{aligned}\quad (\text{IV.4.1})$$

The corresponding momentum operator, $\hat{p}_c$, being canonical to $\hat{\sigma}_c$, satisfies the commutation relation

$$[\hat{\sigma}_c, \hat{p}_c] = i\hbar. \quad (\text{IV.4.2})$$

The momentum operator, $\hat{p}_c$, is determined by the relation

$$\hat{p}_c = M_c \hat{\dot{\sigma}}_c, \quad (\text{IV.4.3})$$

where $\hat{\dot{\sigma}}_c$ is the generalized velocity operator, and M_c the collective inertial parameter. The velocity operator is prescribed by the general definition (IV.2.22) of a time derivative. One has

$$\hat{\dot{\sigma}}_c = \frac{1}{i\hbar} [H, \hat{\sigma}_c]. \quad (\text{IV.4.4})$$

Here, H is the complete many-body Hamiltonian including all finite coupling terms between the S_R and S_A subspaces. The commutator in (IV.4.4) is calculated either in the simple or the second RPA within the extended subspace S . In both cases one finds:

$$\hat{\dot{\sigma}}_c = i \sum_{\nu} \omega_{\nu} \{f_{\nu c}^* \hat{Q}_{\nu}^\dagger - f_{\nu c} \hat{Q}_{\nu}\}, \quad (\text{IV.4.5})$$

since within the extended subspace S the following relation holds:

$$[H, \hat{Q}_{\nu}^\dagger] \approx \hbar \omega_{\nu} \hat{Q}_{\nu}^\dagger \quad (\text{IV.4.6})$$

for both the simple and second RPA. Observe that when the S_R -to- S_A coupling is considered, the velocity operator $\hat{\dot{\sigma}}_c$ --in contrast with

Eq. (IV.2.23)--is different than the expression $i\omega_0(\hat{Q}_c^\dagger - \hat{Q}_c)$ familiar from the theory of the simple *undamped* harmonic oscillator. The inertial parameter M_c is determined by substituting Eqs. (IV.4.3) and (IV.4.5) in Eq. (IV.4.2). Using the boson commutation relations one gets

$$\begin{aligned}i\hbar &= M_c [\hat{\dot{\sigma}}_c, \hat{\sigma}_c] = iM_c \left[\sum_{\nu} (f_{\nu c}^* \hat{Q}_{\nu}^\dagger + f_{\nu c} \hat{Q}_{\nu}), \sum_{\mu} \omega_{\mu} (f_{\mu c}^* \hat{Q}_{\mu}^\dagger - f_{\mu c} \hat{Q}_{\mu}) \right] \\ &= 2iM_c \sum_{\nu} \omega_{\nu} |f_{\nu c}|^2 \\ &= 2iM_c \omega_0.\end{aligned}\quad (\text{IV.4.7})$$

Thus,

$$M_c = \frac{\hbar}{2\omega_0}. \quad (\text{IV.4.8})$$

We calculate now the expectation value, $\langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle$, of the collective position operator. Using Eq. (IV.3.22) one finds

$$\begin{aligned}\langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle &= \langle \Psi(t) | \left(\sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^\dagger \right) | \Psi(t) \rangle + \langle \Psi(t) | \left(\sum_{\nu} f_{\nu c} \hat{Q}_{\nu} \right) | \Psi(t) \rangle \\ &= e^{i\phi} F^*(t) + e^{-i\phi} F(t) \\ &\approx 2e^{-\Gamma t/2} \cos(\omega_0 t + \phi), \quad 0 < t < \frac{T_p}{2} - \frac{2}{\Gamma}.\end{aligned}\quad (\text{IV.4.9})$$

The quantity $F(t)$ is given by Eq. (IV.3.12) and is approximated by Eq. (IV.3.13) in the case of a dense additional subspace S_A .

For the expectation value, $\langle \Psi(t) | \hat{p}_c | \Psi(t) \rangle$, of the collective momentum operator, one has

$$\begin{aligned}\langle \Psi(t) | \hat{p}_c | \Psi(t) \rangle &= iM_c \langle \Psi(t) | \left(\sum_{\nu} \omega_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^\dagger \right) | \Psi(t) \rangle \\ &\quad - iM_c \langle \Psi(t) | \left(\sum_{\nu} \omega_{\nu} f_{\nu c} \hat{Q}_{\nu} \right) | \Psi(t) \rangle.\end{aligned}\quad (\text{IV.4.10})$$

In Appendix F we have calculated that

$$\begin{aligned}\langle \Psi(t) | \left(\sum_{\nu} \omega_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^\dagger \right) | \Psi(t) \rangle &= \langle \Psi(t) | \left(-i \sum_{\nu} \omega_{\nu} f_{\nu c} \hat{Q}_{\nu} \right) | \Psi(t) \rangle^* \\ &= e^{i\phi} \frac{dF^*(t)}{dt}.\end{aligned}\quad (\text{IV.4.11})$$

Taking into account Eq. (IV.4.9), one gets

$$\langle \psi(t) | \hat{p}_c | \psi(t) \rangle = M_c \frac{d}{dt} \langle \psi(t) | \hat{q}_c | \psi(t) \rangle. \quad (\text{IV.4.12})$$

In the first half of the Poincaré recurrence period, the expectation value of the position and momentum operators follow a damped harmonic oscillation.

B. A Classical Equation of Motion For the Expectation Values of the Collective Coordinates

As shown in Subsec. IV.4.A the expectation values of the position and momentum operators over the wave packet (IV.3.8) follow a damped harmonic oscillation for the first half of the Poincaré recurrence period (the "slow variation", "weak" coupling and dense additional subspace conditions are implied). Using Eqs. (IV.4.9) and (IV.4.12) and carrying out the time derivations, one obtains the relation

$$\langle \psi(t) | \frac{d\hat{p}_c}{dt} | \psi(t) \rangle = -\Gamma \langle \psi(t) | \hat{p}_c | \psi(t) \rangle - F_c \left(\omega_0^2 + \frac{\Gamma^2}{4} \right) \langle \psi(t) | \hat{q}_c | \psi(t) \rangle, \quad (IV.4.13)$$

$$0 < t < \frac{\tau_p}{2} - \frac{2}{\Gamma}.$$

This relation (IV.4.13) for the expectation values of $\hat{q}_c$ and $\hat{p}_c$ has the same form as Newton's equation of motion for a classical damped harmonic oscillator. We can thus interpret (IV.4.13) as describing two kinds of collective forces: a conservative force proportional to the collective coordinate and a frictional force proportional to the corresponding momentum.

Equation (IV.4.13) is a remarkable result from a quantum mechanical standpoint, since for its derivation we did not invoke the limit of large quantum numbers--specifically, the limit $\epsilon \gg 1$ for the wave function (IV.3.8)--as required by Bohr's correspondence principle in order to obtain classical results. On the contrary, the linearized equations of motion used here are a good approximation to the exact many-body problem for values $\epsilon \ll 1$ (small amplitude oscillations around the ground state).

C. Construction of the Frictional Hamiltonian, $\mathcal{H}_F$. Definition of a Generalized Collective Hamiltonian, $\mathcal{H}_c$.

In this section--in contrast to Sec. IV.2--the effect of the S_R -to- S_A coupling is being considered. Then the total Hamiltonian, H , may be split into three terms within the extended subspace S according to the scheme:

$$\left(\begin{array}{c} \text{total} \\ \text{Hamiltonian} \end{array} \right) = \left(\begin{array}{c} \text{collective} \\ \text{Hamiltonian} \end{array} \right) + \left(\begin{array}{c} \text{interaction} \\ \text{Hamiltonian} \end{array} \right) + \left(\begin{array}{c} \text{internal} \\ \text{Hamiltonian} \end{array} \right) \quad (\text{IV.4.14})$$

This partition is not unique since, in general, the interaction Hamiltonian contributes to the total energy a non-zero term which can be reallocated in more than one way between the collective and internal parts, while at the same time the following limiting condition holds:

$$\left(\begin{array}{c} \text{collective} \\ \text{Hamiltonian} \end{array} \right) \xrightarrow[\text{coupling}]{S_R \text{ to } S_A} \mathcal{H}_c Q_c^\dagger Q_c; \quad (IV.4.15)$$

that is, when the subspaces S_R and S_A uncouple, the collective Hamiltonian part reduces to $H_c = \mathcal{H}_c Q_c^\dagger Q_c$ (Eq. (IV.2.15)), and is simply proportional to the number operator for the collective phonons in the system.

However, the introduction in Subsec. IV.4.A inside the extended subspace S of the proper canonical collective coordinates allows a physically plausible definition of a generalized collective Hamiltonian, $\mathcal{H}_c$, through the construction of a frictional Hamiltonian, $\mathcal{H}_F$. This frictional Hamiltonian is constructed to prescribe--during the first half of the Poincaré recurrence period--precisely the same time behavior for the collective coordinates as the exact Hamiltonian, H , produces in either the simple or the second EPA within S . Accordingly, we define $\mathcal{H}_F$ by requiring that its substitution for H in Eqs. (IV.4.4) and (IV.4.13) leaves those equations unaltered, that is, we require:

$$\frac{1}{\hbar}[\mathcal{H}_F, \hat{\sigma}_c] = \frac{\hat{p}_c}{M_c} \quad (\text{IV.4.16a})$$

and

$$\frac{1}{\hbar}\langle\psi(t)|[\mathcal{H}_F, \hat{p}_c]|\psi(t)\rangle = -\Gamma\langle\psi(t)|\hat{p}_c|\psi(t)\rangle - M_c(\omega_0^2 + \frac{\Gamma^2}{4})\langle\psi(t)|\hat{\sigma}_c|\psi(t)\rangle. \quad (\text{IV.4.16b})$$

Such a frictional Hamiltonian has the following most general dependence

on the collective coordinates:

$$\mathcal{H}_F = \sum_{m=0} \sum_{n=0} C_{m,n}(t) \hat{\sigma}_c^m \hat{p}_c^n, \quad (\text{IV.4.17})$$

where $C_{m,n}(t)$ are time-dependent constants yet to be determined. Substituting Eq. (IV.4.17) into Eq. (IV.4.16) and using the commutation relation (IV.4.2) for the canonical coordinates $\hat{\sigma}_c$ and $\hat{p}_c$, one gets

$$C_{n,m}(t) = \begin{cases} \frac{1}{2M_c}, & \text{when } m=0, n=2 \\ \frac{M_c}{2}(\omega_0^2 + \frac{\Gamma^2}{4}), & \text{when } m=2, n=0 \\ \Gamma\langle\psi(t)|\hat{p}_c|\psi(t)\rangle, & \text{when } m=1, n=0 \\ \text{arbitrary,} & \text{when } m=0, n=0 \\ 0, & \text{in all other cases.} \end{cases} \quad (\text{IV.4.18})$$

Hence, the frictional Hamiltonian is

$$\mathcal{H}_F = \frac{\hat{p}_c^2}{2M_c} + \frac{M_c}{2}(\omega_0^2 + \frac{\Gamma^2}{4})\hat{\sigma}_c^2 + \Gamma\hat{\sigma}_c\langle\psi(t)|\hat{p}_c|\psi(t)\rangle + C_{c,0}(t). \quad (\text{IV.4.19})$$

The first term in (IV.4.19) is interpreted as the collective kinetic energy operator, and the second term as the collective potential energy operator. The third term, interpreted as the interaction Hamiltonian, is time-irreversible and thus dissipative describing the transfer of energy from the collective degree of freedom to the individual intrinsic degrees of freedom. The generalized conservative collective Hamiltonian, $\mathcal{H}_c$, is defined as the sum of the collective kinetic and collective potential energy parts of the frictional Hamiltonian, i.e.:

$$\mathcal{H}_c = \frac{\hat{p}_c^2}{2M_c} + \frac{M_c}{2}(\omega_0^2 + \frac{\Gamma^2}{4})\hat{\sigma}_c^2. \quad (\text{IV.4.20})$$

Observe that definition (IV.4.20), despite the structural resemblance, is substantially different from the definition (Eq. (IV.2.29))

$$H_c = \hbar\omega_c Q_c^\dagger Q_c = \frac{\hat{p}_c^2}{2m_c} + \frac{1}{2}m_c\omega_c^2\hat{\sigma}_c^2 - \frac{\hbar\omega_c}{2} \quad (\text{IV.4.21})$$

of the collective Hamiltonian specified when the S_R -to- S_A coupling is neglected. This difference results not only from the term $M_c\Gamma^2\hat{\sigma}_c^2/8$ in (IV.4.20) but also from the fact that the momentum operator $\hat{p}_c$ within S (Eq. (IV.4.5)) has a finite projection upon the additional subspace S_A , unlike the momentum operator p_c in Eq. (IV.2.26) which lies entirely inside the restricted subspace S_R .

The time-dependent constant, $C_{0,0}(t)$, in the frictional Hamiltonian (IV.4.19) is specified by requiring that the contribution of the interaction Hamiltonian to the total energy is zero. One finds

$$\mathcal{H}_{\text{interaction}} = \Gamma(\hat{\sigma}_c - \langle\psi(t)|\hat{\sigma}_c|\psi(t)\rangle)\langle\psi(t)|\hat{p}_c|\psi(t)\rangle. \quad (\text{IV.4.22})$$

The generalized internal Hamiltonian is simply:

$$\mathcal{H}_{\text{internal}} = H - \mathcal{H}_F. \quad (\text{IV.4.23})$$

D. The Energy Dissipation Rate, $\dot{\mathcal{E}}_{\text{diss}}(t)$, for the case of Simple Harmonic Oscillations of the Mean Field.

The collective excitation energy, $\mathcal{E}_c(t)$, is defined here as the expectation value of the generalized collective Hamiltonian, $\mathcal{H}_c$ (Eq. (IV.4.20)), with the wave packet, $|\psi(t)\rangle$ (Eq. (IV.3.8)), describing the damping, less the zero point energy of the collective degree of freedom, i.e.,

$$\mathcal{E}_c(t) = \langle\psi(t)|\mathcal{H}_c|\psi(t)\rangle - S\langle 0|\mathcal{H}_c|0\rangle_S. \quad (\text{IV.4.24})$$

A straightforward algebraic calculation carried out in Appendix F yields the remarkable result that the collective excitation energy depends only

upon the squares of the expectation values, rather than upon the expectation values of the squares of $\hat{q}_c$ and $\hat{p}_c$:

$$\mathcal{E}_c(t) = \frac{\langle \Psi(t) | \hat{p}_c^2 | \Psi(t) \rangle^2}{2M_c} + \frac{M_c}{2} \left(\omega_0^2 + \frac{\Gamma^2}{4} \right) \langle \Psi(t) | \hat{q}_c^2 | \Psi(t) \rangle^2. \quad (\text{IV.4.25})$$

The rate of dissipation, $\dot{\mathcal{E}}_{\text{diss}}(t)$ (during the first half of the Poincaré recurrence period) of the collective energy is defined as the time derivative of the collective energy (IV.4.25), and calculated using (IV.4.13) to be:

$$\dot{\mathcal{E}}_{\text{diss}}(t) = -2\Gamma \frac{\langle \Psi(t) | \hat{p}_c^2 | \Psi(t) \rangle^2}{2M_c} \approx -2\epsilon^2 \Gamma \omega_0 e^{-\Gamma t} \sin^2(\omega_0 t + \phi), \quad (IV.4.26)$$

$$0 < t < \frac{T_P}{2} - \frac{2}{\Gamma}.$$

The last step in (IV.4.26) follows because the damping is weak, so that $\Gamma/2 \ll \omega_0$. The energy dissipation rate (IV.4.26) is proportional to the kinetic collective energy as for a classical damped harmonic oscillator, but in contrast with the decay of a sharp n-phonon collective eigenstate (Chaps. II and III).^{*} However, one should remember that the simple classical result (IV.4.26) has been derived from the present RPA microscopic model for nuclear dissipation only under the "slow variation", "weak" coupling, and dense additional subspace conditions (see Chap. II). Any relaxation of these conditions is expected to lead to more general results away from this correspondence with classical mechanics.

^{*}The energy dissipation rate for a sharp n-phonon eigenstate remains proportional to the total collective energy even when the generalized collective Hamiltonian $\mathcal{H}_c$ (Eq. (IV.4.20))--instead of the Hamiltonian H_c (Eq. (IV.2.15))--is used to calculate the collective energy of the system.

E. The Physical Meaning of the Decoupled-Subspaces Collective

$$\text{Hamiltonian, } H_c = \hbar \omega_c Q_c^\dagger Q_c.$$

As has already been discussed, the definition $H_c = \hbar \omega_c Q_c^\dagger Q_c$ for the collective Hamiltonian--at which one arrives by decoupling the subspaces S_R and S_A --is not unique. Taking into account the S_R -to- S_A coupling, we specified a generalized collective Hamiltonian, $\mathcal{H}_c$, given by Eq. (IV.4.20). Yet, the definition $H_c = \hbar \omega_c Q_c^\dagger Q_c$ has the following physical interpretation.

Observe that the operator $Q_c^\dagger Q_c$ counts the number of collective phonons in the system. Its expectation with the coherent wave packet (IV.3.8) describing the damping of mean field oscillations is (see Appendix F):

$$\langle \Psi(t) | Q_c^\dagger Q_c | \Psi(t) \rangle = \epsilon^2 e^{-\Gamma t}, \quad 0 < t < \frac{T_P}{2} - \frac{2}{\Gamma}. \quad (IV.4.27)$$

For the first half of the Poincaré recurrence period, this expectation value of the number operator exhibits a simple exponential time decay. Through comparison with the time dependence of $\langle \Psi(t) | \hat{q}_c^2 | \Psi(t) \rangle$ (Eq. (IV.4.9)), one can interpret (IV.4.27) as describing the decay of the amplitude of the oscillatory motion performed by the collective variables.

The expectation value $\langle \Psi(t) | H_c | \Psi(t) \rangle$ yields the average energy carried by the phonons in the system, which is proportional to the amplitude of the oscillation.

V. CONNECTION WITH THE QUANTUM MECHANICS OF THE DAMPED HARMONIC
OSCILLATOR AND THE PHENOMENOLOGICAL THEORIES
OF NONLINEAR QUANTAL FRICTION

V.1. Introduction

The quantum mechanics of the damped harmonic oscillator, and of friction in general, constitutes a fundamental--yet still open--theoretical problem with applications to a variety of fields, including nuclear physics, quantum optics, solid state, etc., and considerable effort has been devoted to it.^{32-35,38,39}

As has early been recognized,^{38,39} dissipation should arise from interactions between the actually observed degree of freedom (in our case the collective mode) and others (sometimes labelled the reservoir) into which energy flows in an "irreversible" way. For a dissipation description to apply, it is necessary that the details of the reservoir structure be suppressed and that the description be made complete within the subspace of the observed degree of freedom. However, as mentioned by Dekker,³⁸ "...the calculations from first principles" have been "little transparent as a rule, and the transition from reversibility to irreversibility remains a difficult problem...".

The present work makes a contribution to the subject. Indeed, starting from first principles at the microscopic level, we have extracted from the full problem (observed degree of freedom plus reservoir) approximate reversible solutions which, under appropriate conditions, describe the damping of the one-phonon (Ch. II), n-phonon (Ch. III), and multi-phonon (Ch. IV) initial collective states either in the time-dependent simple RPA or the second RPA. Both these RPA equations

of motion and their solutions remain *reversible* during the total span of the Poincaré recurrence period. The solutions, however, approach those of an *irreversible* process during the early portion of the Poincaré period (Subsec. II.3.C).

In Subsec. IV.4.C, for the special case of a decaying coherent multi-phonon packet, we have constructed an *irreversible* frictional Hamiltonian which describes the time decay of the collective coordinates in the same way as does the complete *reversible* Hamiltonian during the early portion of the Poincaré period. However, throughout Ch. IV we used, as wave functions, the packet $|\Psi(t)\rangle$ (Eq. (IV.3.8)) which describes the evolution of the system in the extended space (observed degree of freedom plus reservoir) and, hence, carries superfluous, from the collective point of view, information about the intrinsic degrees of freedom. There remains a final step in providing a description of dissipation complete within the collective coordinate subspace, that is the specification of a wave function $\mathcal{G}(\sigma, t)$ which depends exclusively on the collective coordinate and, yet, from the collective viewpoint, describes dissipation in the same way as does the full packet $|\Psi(t)\rangle$ during the early portion of the Poincaré recurrence period. This step is carried out in the next section.

V.2. Complete Description of Dissipation Solely Within the Collective Coordinate Subspace

We provide a description of dissipation complete within the collective coordinate subspace by seeking a wave function, $\mathcal{G}(\sigma, t)$, which depends solely on the collective coordinate, and obeys the following Schrödinger equation for a forced harmonic oscillator:

$$\mathcal{H}_F \mathcal{G}(\sigma, t) = i\hbar \frac{d}{dt} \mathcal{G}(\sigma, t). \quad (V.2.1a)$$

The frictional Hamiltonian $\mathcal{H}_F$ is the one constructed in Subsec. IV.4.C (Eq. (IV.4.19) combined with Eq. (IV.4.22)), i.e.,

$$\mathcal{H}_F = \frac{\hat{p}_c^2}{2M_c} + \frac{M_c}{2} \left(\omega_0^2 + \frac{\Gamma^2}{4} \right) \hat{\sigma}_c^2 + \Gamma M_c (\hat{\sigma}_c - f(t)) \frac{df(t)}{dt}, \quad (V.2.1b)$$

where, during the first half of the Poincaré period, the explicitly time-dependent function $f(t)$ equals the expectation value of the collective coordinate, $\hat{\sigma}_c$, with the packet $|\Psi(t)\rangle$ (Eq. (IV.4.9)):

$$f(t) = \langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle = 2\epsilon e^{-\Gamma t/2} \cos(\omega_0 t + \phi). \quad (V.2.1c)$$

Being canonical, the collective coordinates $\hat{\sigma}_c$ and $\hat{p}_c$ have the following σ -representation:

$$\hat{\sigma}_c \rightarrow \sigma \quad \text{and} \quad \hat{p}_c \rightarrow p = \frac{\hbar}{i} \frac{d}{d\sigma}. \quad (V.2.1d)$$

Solution for Expectation Values Using (V.2.1)

The expectation values of σ and p with the wave function $\mathcal{G}(\sigma, t)$ are denoted with a bar, i.e.,

$$\bar{\sigma}(t) = \int_{-\infty}^{\infty} \mathcal{G}^*(\sigma, t) \sigma \mathcal{G}(\sigma, t) d\sigma. \quad (V.2.2)$$

We now calculate the second time derivative of the expectation value $\bar{\sigma}(t) = M_c \bar{\sigma}(t)$ using the Schrödinger equation (V.2.1) for a forced harmonic oscillator.

One has:

$$\ddot{\bar{\varphi}}(t) = \frac{1}{\hbar} \int_{-\infty}^{+\infty} \varphi^*(\sigma, t) [\mathcal{H}_F, \hat{p}] \varphi(\sigma, t) d\sigma, \quad (V.2.3)$$

from which it follows:

$$\ddot{\bar{\sigma}}(t) + (\omega_0^2 + \frac{\Gamma^2}{4}) \bar{\sigma}(t) = -\Gamma \frac{d}{dt} [2\epsilon e^{-\Gamma t/2} \cos(\omega_0 t + \phi)]. \quad (V.2.4)$$

The second-order differential equation (V.2.4) is linear and identical with the equation of motion of a classical forced harmonic oscillator. Direct substitution shows that the following expression is a special solution to the differential equation (V.2.4):

$$\bar{\sigma}(t) = 2\epsilon e^{-\Gamma t/2} \cos(\omega_0 t + \phi). \quad (V.2.5)$$

The special solution (V.2.5) is equal to the expectation value $\langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle$ (Eq. (V.2.1c)). However, the special solution (V.2.5) is not the most general solution of the forced harmonic oscillator equation (V.2.4). When one takes into account the free vibrations, this most general solution of (V.2.4) has the form:

$$\bar{\sigma}(t) = A \sin(t\sqrt{\omega_0^2 + \frac{\Gamma^2}{4}}) + B \cos(t\sqrt{\omega_0^2 + \frac{\Gamma^2}{4}}) + 2\epsilon e^{-\Gamma t/2} \cos(\omega_0 t + \phi). \quad (V.2.6)$$

To determine the constants A and B , one needs initial conditions for the position $\bar{\sigma}_0$ and velocity $\dot{\bar{\sigma}}_0$ variables. Since we want the wave function $\varphi(\sigma, t)$ to describe the same dissipation process as does the wave function $|\Psi(t)\rangle$ in the full space, it is natural to require that initially the corresponding expectation values are equal. With the help of (V.2.1c), one then has the following initial conditions:

$$\bar{\sigma}_0 = \langle \Psi(t=0) | \hat{\sigma}_c | \Psi(t=0) \rangle = 2\epsilon \cos \phi \quad (V.2.7a)$$

and

$$\dot{\bar{\sigma}}_0 = \langle \Psi(t=0) | \frac{\hat{p}_c}{M_c} | \Psi(t=0) \rangle = -2\epsilon(\omega_0 \sin \phi + \frac{\Gamma}{2} \cos \phi). \quad (V.2.7b)$$

For the constants A and B , conditions (V.2.7) yield:

$$A = B = 0. \quad (V.2.8)$$

The effect of the initial conditions (V.2.7) is to eliminate the free vibrations from the solution of the forced harmonic oscillator, and to restrict the solution of (V.2.4) to the special term (V.2.5). Observe that this special solution is equal at all times to the expectation value of $\hat{\sigma}_c$ with the full packet $|\Psi(t)\rangle$, that is the initial values (V.2.7) guarantee for all times ($0 < t < \frac{\tau_p}{2} - \frac{2}{\Gamma}$):

$$\bar{\sigma}(t) \stackrel{\text{def.}}{=} \int_{-\infty}^{+\infty} \varphi^*(\sigma, t) \sigma \varphi(\sigma, t) d\sigma = \langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle \stackrel{\text{def.}}{=} f(t), \quad (V.2.9a)$$

and

$$\bar{\mathcal{P}}(t) \stackrel{\text{def.}}{=} \int_{-\infty}^{+\infty} \varphi^*(\sigma, t) (\frac{\hbar}{i} \frac{d}{d\sigma}) \varphi(\sigma, t) d\sigma = \langle \Psi(t) | \hat{p}_c | \Psi(t) \rangle \stackrel{\text{def.}}{=} M_c \frac{df(t)}{dt}. \quad (V.2.9b)$$

The special solution (V.2.5) is also a solution of a damped harmonic oscillator and, thus, it can be obtained from the following differential equation:

$$\ddot{\bar{\sigma}}(t) + \Gamma \dot{\bar{\sigma}}(t) + (\omega_0^2 + \frac{\Gamma^2}{4}) \bar{\sigma}(t) = 0. \quad (V.2.10)$$

This differential equation (V.2.10) for the expectation value

$\bar{\sigma}(t) = \int_{-\infty}^{+\infty} \varphi^*(\sigma, t) \sigma \varphi(\sigma, t) d\sigma$ results from the following nonlinear frictional Schrödinger equation:

$$\left\{ -\frac{\hbar^2}{2M_c} \frac{d^2}{d\sigma^2} + M_c (\omega_0^2 + \frac{\Gamma^2}{4}) \sigma^2 + \Gamma(\sigma - \bar{\sigma}(t)) \bar{\mathcal{P}}(t) \right\} \varphi(\sigma, t) = \hbar \frac{d}{dt} \varphi(\sigma, t). \quad (V.2.11)$$

where the collective variables $\bar{\sigma}(t)$ and $\bar{\mathcal{P}}(t)$ have replaced the explicitly time-dependent functions $f(t)$ and $M_c \frac{df(t)}{dt}$.

The nonlinear frictional Schrödinger equation (V.2.11) is equivalent to the Schrödinger equation (V.2.1) for the linear forced harmonic oscillator, when the initial conditions (V.2.7) are taken into account.

This nonlinear frictional Schrödinger equation (V.2.11) provides a description of dissipation complete within the collective coordinate subspace. Even more, this nonlinear Schrödinger equation is time irreversible.

This *irreversible* description within the collective coordinate subspace parallels--during the early part of the Poincaré period, as far as the expectation values and, hence, the energy loss rate are concerned--the *reversible* evolution of the entire system in the full space. Therefore, the system is dissipative, properly so called in the precise sense of Fano's⁶⁷ definition of irreversible processes, according to which a system is dissipative only when its *reversible* dynamical equations within the full space can be replaced within the collective coordinate subspace by an approximate *irreversible* dynamical equation. The fact that Fano utilizes density matrices instead of wave functions is a purely technical distinction of no physical consequence.

V.3. Comparison with the Kostin-Albrecht and Kostin-Kan

Friction Operators

Nonlinear frictional Hamiltonians, $\tilde{\mathcal{H}}_F(\sigma, \bar{\sigma}, \bar{\rho})$, depending on the collective variables $\bar{\sigma}(t)$ and $\bar{\rho}(t)$ have already been proposed in the framework of phenomenological quantum friction theories.^{34,35,39b} In particular, here the microscopically derived friction term,

$$W^{F-A} = (\sigma - \bar{\sigma}(t)) \bar{\rho}(t), \quad (V.3.1)$$

in the Schrödinger equation (V.2.11) has the same form with one proposed by Kostin^{39b} (1975) and Albrecht³⁴ (1975). The nonlinear Schrödinger equation (V.2.11) has already been solved by Hasse.³⁵ His solution has the form of a minimum uncertainty packet and is:

$$\varphi(\sigma, t) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp \left\{ -\frac{\alpha}{2} (\sigma - \bar{\sigma}(t))^2 + \frac{i}{\hbar} \left[\bar{\rho}(t) (\sigma - \bar{\sigma}(t)) + \int_0^t L(t') dt' - \frac{\hbar^2 \alpha t}{2M_c} \right] \right\}, \quad (V.3.2a)$$

where

$$\alpha = \frac{M_c (\omega_0^2 + \frac{\Gamma^2}{4})^{1/2}}{\hbar} \quad (V.3.2b)$$

and

$$L(t) = \frac{M_c}{2} \left\{ \dot{\bar{\sigma}}^2(t) - (\omega_0^2 + \frac{\Gamma^2}{4}) \bar{\sigma}^2(t) \right\}. \quad (V.3.2c)$$

Earlier, Kostin^{39a} (1972) and Kan and Griffin^{32,33} (1974) had proposed a formally different friction term. This Kostin-Kan friction term is the following:

$$W^{K-K} = \frac{\hbar}{2i} \left[\ln \frac{\varphi(\sigma, t)}{\varphi^*(\sigma, t)} - \ln \frac{\varphi^*(\sigma, t)}{\varphi(\sigma, t)} \right]. \quad (V.3.3)$$

However, for the damped SHO packet (V.3.2), a straightforward calculation shows that the Kostin-Kan term (V.3.3) and the Kostin-Albrecht term (V.3.1) are equal:

$$W^{K-K}(\varphi) = W^{K-A}(\varphi). \quad (V.3.4)$$

Therefore, it is not surprising that both the Kostin-Kan and the Kostin-Albrecht nonlinear SHO Schrödinger equations have the same packet (V.3.2) as solution.³⁵

Besides the friction term (V.3.1), Albrecht³⁴ and Hasse³⁵ have proposed a whole class of Schrödinger friction operators, $\mathcal{W}^{A-H}$:

$$\mathcal{W}^{A-H} = (\sigma - \bar{\sigma})[C\hat{p} + (1-C)\bar{\hat{p}}] - \frac{iM_C}{2}, \quad (V.3.5)$$

where C is an arbitrary real constant. The Kostin-Albrecht term (V.3.1) results from (V.3.5) when $C = 0$. In the framework of the present microscopic model, the class of friction operators (V.3.5) results when one relaxes the first condition (IV.4.16a) for constructing the frictional Hamiltonian, $\mathcal{H}_F$. In particular, the class of friction operators (V.3.5) results when one requires the following condition:

$$\frac{i}{\hbar} \langle \psi(t) | [\mathcal{H}_F, \hat{\sigma}_C] | \psi(t) \rangle = - \langle \psi(t) | \frac{\hat{p}_C}{M_C} | \psi(t) \rangle \quad (V.3.6)$$

instead of

$$\frac{i}{\hbar} [\mathcal{H}_F, \hat{\sigma}_C] = \frac{\hat{p}_C}{M_C}, \quad (V.3.7)$$

that is when the first of Hamilton's equations of motion corresponding to the frictional Hamiltonian, $\mathcal{H}_F$, is satisfied over expectation values instead of by the operators themselves.

The reversible description of the time evolution of the system within the full space is unique. However, the suppression of the intrinsic degrees of freedom allows more than one Schrödinger friction operator, all of which provide equivalent descriptions from the collective coordinate viewpoint.

V.4. Phase Space Division Versus Hilbert Space Division

In the previous section V.3 we provided a description of dissipation complete within the collective coordinate subspace while suppressing the details of the intrinsic coordinates associated with the rest of the phase space. This division of the total phase space between collective coordinate subspace and reservoir led to nonlinear frictional Hamiltonians, $\mathcal{H}_F$, which depended only on the collective coordinates σ and $\hat{p}$ and their expectation values. Since the collective coordinates are hermitean operators, the frictional Hamiltonians themselves are also hermitean. As a result, these nonlinear frictional Hamiltonians conserve the norm of the wave function $\mathcal{J}(\sigma, t)$, i.e.:

$$\frac{d}{dt} \int_{-\infty}^{+\infty} |\mathcal{J}(\sigma, t)|^2 d\sigma = 0. \quad (V.4.1)$$

Equation (V.4.1) shows that only the energy flows irreversibly out from the collective coordinate subspace; the probability does not.

This division of the total phase space between collective coordinate subspace and reservoir contrasts with another natural division of the total Hilbert space between a restricted Hilbert subspace and an additional Hilbert subspace. This Hilbert space division was adopted naturally from the outset in the development of the present model. However, in contrast to the collective coordinate viewpoint, from the restricted Hilbert subspace viewpoint, there is a loss of both probability and energy.

V.5. The Question of Stationary Solutions of the

Nonlinear Schrödinger Eq. (V.2.11)

In the literature, it has been noticed^{34,35} that--besides the damped packet solutions (V.3.2)--the nonlinear Schrödinger equation (V.2.11) has also stationary solutions $\varphi_n(\sigma)$ which describe no damping whatsoever. These stationary solutions are the eigenstates of the undamped SHO resulting from (V.2.11) when the Kostin-Albrecht friction operator, $\hat{W}^{K-A} = (\sigma - \bar{\sigma}(t))\bar{\varphi}(t)$, is set equal to zero. These stationary solutions $\varphi_n(\sigma)$ yield indeed a zero value for the momentum expectation value $\bar{p}(t)$ and thus a zero value for the friction operator $\hat{W}^{K-A}$.

The wave functions $\varphi_n(\sigma)$ are defined within the collective coordinate subspace. Let us now consider the wave packets $|\phi^{(n)}(t)\rangle$ (Eq. (III.2.5)) describing the damping of sharp n-phonon initial states within the extended Hilbert subspace. These packets, $|\phi^{(n)}(t)\rangle$, have also the property that the momentum expectation value, $\langle \phi^{(n)}(t) | \hat{p}_c | \phi^{(n)}(t) \rangle$, is zero at all times. Therefore, these damped packets, $|\phi^{(n)}(t)\rangle$, in the extended subspace relate to the stationary undamped solutions, $\varphi_n(\sigma)$, in the collective coordinate subspace.

We conclude that the nonlinear frictional Schrödinger equation (V.2.11) does not describe dissipation in extreme quantal cases (i.e., sharp n-phonon initial collective states). However, as discussed in Sec. V.2, Eq. (V.2.11) does describe dissipation in cases with a classical analog (i.e., minimum uncertainty initial packets).

VI. CONNECTION WITH THE VIBRATING POTENTIAL MODEL FOR NUCLEAR

COLLECTIVE MOTION AND THE SWIATECKI WALL FORMULA

FOR NUCLEAR DISSIPATION

VI.1. Introduction

In Ch. VI we focus on the connection of the present microscopic model with the phenomenological Vibrating Potential Model^{40a,45} for the nuclear collective motion and, thereby, with the Swiatecki wall formula.^{21,22,26}

With the simple two-body schematic interaction which leads to the Vibrating Potential Model^{40a,45} for surface vibrations, the mean field depends only upon the collective position variable, which specifies the deformation parameter of the average nuclear surface. For this schematic interaction, the value of the proportionality coefficient in the energy dissipation rate (Eq. (IV.4.26)) disagrees in general with the corresponding constant of the Swiatecki wall formula. Indeed, even in the special limit of an infinite square well shell model potential of very large nuclear radius, which corresponds to Swiatecki's model of a semi-infinite Fermi gas, this proportionality coefficient is shown to have a specific dependence upon the ratio, η , of the energy of the one-phonon collective state versus the Fermi energy. However, in this special limit, the low frequency value of this proportionality coefficient agrees with the Swiatecki constant.

VI.2. The Swiatecki Wall Formula as a Special Limit of the SHO Energy Dissipation Rate (IV.4.26)

In Chapt. IV we showed how the damping of initial minimum uncertainty packets leads to a classical energy dissipation rate (IV.4.26); that is, to an energy dissipation rate proportional to the kinetic energy (or, equivalently, proportional to the square of the collective velocity):

$$\dot{\mathcal{E}}_{\text{diss}}(t) = -M_c \Gamma \langle \dot{\psi}(t) | \hat{p}_c | \dot{\psi}(t) \rangle^2. \quad (\text{VI.2.1})$$

The proportionality coefficient

$$\gamma' = M_c \Gamma = \frac{\hbar \Gamma}{2\omega_0} \quad (\text{VI.2.2})$$

has been explicitly calculated both in the simple RPA and in the second RPA approximations. Such a classical energy loss rate, as well as the resulting classical damped SHO equation of motion, have in the past provided the starting point for the phenomenological theories^{32-35,38,39} and applications^{9,27b} of quantum friction.

Using a classical one-body dissipation process in a semi-infinite medium, Swiatecki²¹ derived a specific constant value for this proportionality coefficient. In this section we will see how, from an appropriate—albeit unrealistic—limit of the expression (VI.2.1), when the size of the system tends to infinity, one can extract the constant which Swiatecki's classical "wall formula" would imply for the energy loss rate of a spherical droplet oscillating in a pure (sufficiently low) multipole mode.

In the Swiatecki wall model the collective degree of freedom is represented by a moving wall which is an idealization of the nuclear mean field. Since nuclei are leptodermous systems, the collective variables are then associated with the deformation parameters of the average nuclear surface. On the other hand, the general collective variable

$\langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle$ in Eq. (VI.2.1) can represent modes of oscillations of the nuclear system more complicated than surface vibrations (e.g., modes incorporating compression). However, this collective variable, $\langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle$, becomes identical with the deformation parameter of the nuclear surface when the schematic separable interaction--which accounts in the Vibrating Potential Model for the small amplitude motion of the nuclear surface--is used:

$$V_{mj}^\lambda = -\xi D_{m1}^\lambda D_{nj}^\lambda \quad (\text{VI.2.3a})$$

where

$$D_{\beta\beta}^\lambda = -\langle \beta | r \frac{du^{(0)}(r)}{dr} Y_{\lambda 0}(\theta) | \beta' \rangle. \quad (\text{VI.2.3b})$$

Here the $Y_{\lambda 0}(\theta)$ are the spherical harmonics of multipolarity $(\lambda, 0)$ and $u^{(0)}(r)$ describes the shell model potential. We assume spherical symmetry and, since the nucleus is a leptodermous system, we approximate the shell model potential with a square well:

$$u^{(0)}(r) \approx u_0 \theta(r-R). \quad (\text{VI.2.3c})$$

$\theta(r-R)$ is the unit step function and R the nuclear radius.

With the schematic interaction (VI.2.3) and definition (IV.4.1) for the collective position operator $\hat{\sigma}_c$, Eq. (IV.3.36) for the matrix elements, $u_{\beta\beta}^{\text{ph}}(t)$, of the mean field yields:

$$u_{\beta\beta}^{\text{ph}}(t) = n(\omega_c) D_{\beta\beta}^\lambda \langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle, \quad (\text{VI.2.4})$$

where we used the relation,

$$b_{\beta\beta',c} = -\xi D_{\beta\beta'}^\lambda \sum_{nj} \{ D_{nj}^\lambda Y_{nj}(\omega_c) + D_{nj}^\lambda Z_{nj}(\omega_c) \} = D_{\beta\beta'}^\lambda n(\omega_c). \quad (\text{VI.2.5})$$

In the coordinate representation, the time-varying part of the mean field is then

$$u^{\text{ph}}(\vec{r}, \vec{r}', t) = n(\omega_c) \langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle \sum_{\beta\beta'} \langle \vec{r} | \beta \rangle D_{\beta\beta'}^\lambda \langle \beta' | \vec{r}' \rangle \\ = -n(\omega_c) \langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle r \frac{du^{(0)}(r)}{dr} Y_{\lambda 0}(\theta) \delta(\vec{r}-\vec{r}'). \quad (\text{VI.2.6})$$

With the assumption $\epsilon \ll 1$ (ϵ specifies the amplitude of oscillations, see Eq. (IV.4.9)), and the identification,

$$n(\omega_c) \langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle = \alpha_{\lambda 0}(t), \quad (\text{VI.2.7})$$

where $\alpha_{\lambda 0}(t)$ are the deformation parameters of the average nuclear surface, Eq. (VI.2.6) does indeed describe small amplitude surface vibrations of the mean field of a leptodermous system.^{40a} With Eq. (VI.2.7), Eq. (VI.2.1) for the energy dissipation rate yields:

$$\dot{\mathcal{E}}_{\text{diss}}(t) = -\frac{M_c \Gamma}{n^2(\omega_c)} \dot{\alpha}_{\lambda 0}^2(t). \quad (\text{VI.2.8})$$

We want now to compare Eq. (VI.2.8) with the energy dissipation rate calculated with Swiatecki's wall formula for the same nuclear (spherical) boundary. In Ref. 21 it was found

$$\dot{\mathcal{E}}_{\text{diss,SWF}}(t) = -\rho \bar{v} R^4 \dot{\alpha}_{\lambda 0}^2(t), \quad (\text{VI.2.9})$$

where $\rho = \mu(k_F^3/6\pi^2)$ is the mass density of nuclear matter and $\bar{v} = 3\hbar k_F/4\mu$ is the average velocity of a Fermi gas distribution at zero temperature; μ is the nucleon mass

The proportionality coefficient,

$$\gamma = \frac{M_c \Gamma}{n^2(\omega_c)} = \frac{\hbar}{2\omega_c} \frac{\Gamma}{n^2(\omega_c)}, \quad (\text{VI.2.10})$$

in Eq. (VI.2.8) is in general different from the Swiatecki proportionality constant,

$$\gamma_{\text{SWF}} = \rho \bar{v} R^4, \quad (\text{VI.2.11})$$

since the Swiatecki wall model is a semi-infinite system, while the energy dissipation rate (VI.2.8) was derived for a finite-A nucleus. In

particular, in the Swiatecki wall model the collective mode (the moving rigid wall) loses energy to the particles of a Fermi gas. Thus, in Swiatecki's model, in contrast to finite-A nuclei, the bound particles have a continuous range of kinetic energies, from zero up to the Fermi energy ϵ_F ; an analogous condition would prevail in our model if we replace the discrete single-particle bound spectrum of a finite system by a continuous spectrum. For the finite-A nuclei occurring in the periodic table this substitution is not realistic, since it would lead to the decay of even the low-lying isoscalar collective modes in contrast to the experimental results. However, as a model problem one can imagine a fictitious nucleus with sufficiently large radius such that this substitution could be valid. To simplify the calculations we use an infinitely deep square well as the shell model potential.

Moreover, the phenomenological Swiatecki model is based on the assumption of a gas of *independent* nucleons, not only in providing for the collective motion, but also in describing the nucleon-surface interactions. Thus, it contains *solely* one-body forces. In the framework of the present microscopic model, such a case effectively arises when the additional subspace S_A contains only 1p-1h states. Thus, in this large radius/infinite square well limit, we calculate only the value of the escape width $\Gamma^\dagger$. Then the decay width Γ in Eq. (VI.2.10) is given by (see Eq. (II.3.50))

$$\Gamma = \Gamma^\dagger = \frac{2\pi}{\hbar} \iint d\tilde{m} d\tilde{l} |K(\tilde{n}, \tilde{l}; \omega_c)|^2 \delta(\hbar\omega_c - \epsilon(\tilde{n}, \tilde{l})). \quad (\text{VI.2.12})$$

We first observe that

$$K(\tilde{n}, \tilde{l}; \omega_c) = D^\lambda(\tilde{n}, \tilde{l}) n(\omega_c), \quad (\text{VI.2.13})$$

when the schematic interaction (VI.2.3) is used. Equation (VI.2.12) then yields:

$$\Gamma = \frac{\hbar\Gamma^\dagger}{2\omega_0 n^2(\omega_c)} = \frac{\pi}{\omega_c} \iint d\tilde{m} d\tilde{l} D^2(\tilde{m}, \tilde{l}) \delta(\hbar\omega_c - \epsilon(\tilde{m}, \tilde{l})). \quad (\text{VI.2.14})$$

The calculation of the integral (VI.2.14) in the large radius/infinite square well potential limit is carried out in Appendix G. Under the assumption that

$$\lambda \ll k_F R, \quad (\text{VI.2.15})$$

the result is:

$$\gamma_{\text{limit}} = \frac{\hbar(k_F R)^4}{8\pi^2} \frac{1}{n} \left\{ \frac{(1+\eta)^{3/2} - (1-\eta)^{3/2}}{3} + \frac{\eta^2}{4} \int_{-1}^{-\eta} dx \ln \left[\frac{2\sqrt{x^2+\eta x} - 2x-\eta}{2\sqrt{x^2-\eta x} - 2x+\eta} \right] + \frac{\eta^2}{4} \int_{-\eta}^0 dx \ln \left[\frac{\eta}{2\sqrt{x^2-\eta x} - 2x+\eta} \right] \right\}; \quad \eta \leq 1, \quad (\text{VI.2.16})$$

where the ratio η is:

$$\eta = \frac{\hbar\omega_c}{\epsilon_F}. \quad (\text{VI.2.17})$$

Unlike the Swiatecki constant, γ_{limit} depends on the collective frequency ω_c . For the low frequency limit, however, one finds:

$$\gamma_{\text{limit}} = \frac{\hbar(k_F R)^4}{8\pi^2} \quad \eta \rightarrow 0. \quad (\text{VI.2.18})$$

The low frequency limit of expression (VI.2.16) agrees exactly with the classical Swiatecki constant (VI.2.11).

Swiatecki's phenomenology for nuclear dissipation is based on the picture of a classical gas of *independent* particles, which interact with the walls of the container. Since the particles are viewed as independent, this theory contains *solely* one-body forces. The present microscopic model starts with two-body forces. These two-body forces are averaged into effective one-body forces when the simple RPA is used. A model containing *solely* one-body forces results from the present microscopic theory only when the dynamical equations of motion in the extended subspaces remain within the simple RPA framework, i.e., when the couplings to 2p-2h states are neglected and S_A has additional states of the 1p-1h structure only. For a finite-A nucleus, however, this has the disadvantage of associating the energy dissipation entirely with the escape width for direct

particle emission to the continuum. The escape width for direct particle emission has been estimated by a variety of approaches.⁶⁸⁻⁷¹ While it seems to account for most of the experimental width of the Giant dipole vibration in light nuclei,^{68,69,71} it is found much smaller than the experimental width of heavy and medium-weight nuclei.^{70,71} The large values of dissipation, which Swiatecki's wall formula yields, are due to the semi-infinite Fermi gas limit. This suggests that, in the case of finite-A nuclei, in addition to the one-body forces responsible for the collective motion, a theory for nuclear dissipation needs to consider the *residual two-body* interactions responsible for the spreading width to compound nuclear states.

As shown in Sec. II.4, the present model can accommodate such an extension via the second RPA. In this case, formula (VI.2.10) for the proportionality coefficient γ is still valid, when the schematic interaction (VI.2.3) is used. However, the width Γ will now be the sum of escape plus spreading width and can be either calculated microscopically with the help of Eqs. (II.3.50) and (II.4.17), or determined directly from experimental values. The constant $\alpha^2(\omega_c)$ appearing in (VI.2.10) is related to the inertial parameter, B_λ , of the Vibrating Potential Model as follows:^{40a}

$$\alpha^2(\omega_c) = \frac{\omega_0 B_\lambda}{2\hbar} \quad (\text{VI.2.19})$$

In light of the present analysis, the fact that Swiatecki's wall formula for medium-weight and heavy nuclei gives widths of the same order of magnitude with the experimental values⁸ seems purely accidental. Even more crucial is the $A^{-1/2}$ qualitative dependence⁸ it provides on the mass number A. That is, according to Swiatecki's formula, the widths increase with decreasing A. This, however, runs contrary to the experimental results (for ¹⁶O the experiment yields widths ≈ 15 MeV, while for ²⁰⁸Pb, it yields widths ≈ 4 MeV⁵⁹).

VI.3 Comparison with Other Different Physical Approaches

to the Wall Formula

A. Absence of Randomization Assumption in the Microscopic RPA Model in Contrast to the Classical Wall Model

In the previous Section VI.2, we derived the constant coefficient which Swiatecki's classical wall formula (Eq. (VI.2.9)) would imply for the energy loss rate of a spherical droplet oscillating in a pure (sufficiently low) multipole mode. In particular, we showed that this Swiatecki constant agrees precisely with the low frequency value of an appropriate limit of the general expression (II.4.26) for the energy dissipation rate provided by the present microscopic RPA model. Specifically, this appropriate limit involved

- (i) treating additional states of the $1p-1h$ structure only (the treatment of dissipation is thereby restricted entirely within the usual RPA framework and thus is equivalent to treating independent nucleons);
- (ii) assuming a fictitious nucleus of sufficiently large radius so that the discrete particle spectrum can be approximated by a continuum (then the nucleons effectively comprise a Fermi gas);
- (iii) assuming small amplitude surface vibrations ($\epsilon \ll 1$) and using the schematic Vibrating Potential Model interaction (VI.2.3) (this small amplitude limit guarantees small surface velocities, since the amplitude of the surface velocity is proportional to $\epsilon\omega_0$. The collective frequency, ω_0 , however, is not restricted to small values); and
- (iv) utilizing an infinitely deep square well as the shell model potential.

Yet, the fact that the wall formula was derived from the present microscopic RPA model in an appropriate limit does not mean that this limit reproduces all the physical assumptions entering in the classical wall model for one-body nuclear dissipation. In particular, the classical Swiatecki wall formula for a gas of independent particles with mass density ρ and average velocity $\bar{v}$, in a container of constant volume, whose walls deform with normal velocities $\dot{z}$, has been calculated to imply:

$$\dot{E}_{\text{diss,SWF}}(t) = \rho \bar{v} \oint \dot{z}^2 d\sigma, \quad (\text{VI.3.1})$$

where the integral is over the surface of the container. However, in the framework of the classical wall model, formula (VI.3.1) can be derived without any randomization assumptions only in the case of a simple plane boundary. Indeed, in the case of a plane boundary the particles reflected from the boundary cannot return to interact again with the wall and, therefore, the collective degree loses its energy irreversibly. However, when the gas is contained in a finite container, the particles collide repeatedly with the boundary. Then, in order to apply the classical wall model to the individual surface elements of a finite container, a randomization assumption is necessary, i.e., that each element of the surface continues to be bombarded by particles as if these originated in a randomized distribution.

On the other hand, the present microscopic RPA model for nuclear dissipation contrasts with the classical wall model in that it does not require any such randomization assumptions. Specifically, for a finite-A system, the collective degree of freedom loses energy during the early portion of the Poincaré recurrence period and gains it back during the late portion of the Poincaré recurrence period (see Sec. II.3.C). This Poincaré

recurrence period is inversely proportional to the level spacing of the additional quantum states. Since the microscopic theory of the finite-A nucleus allows the inclusion of compound nuclear states (which have a small level spacing) in the additional subspace, S_A , the Poincaré period can become much larger than the time during which the collective degree of freedom is observed, and, therefore, only the loss of collective energy can be registered.

In the framework of the classical wall model, two distinct mechanisms have been proposed as the randomizing agent: (i) the successive collisions with differently moving surface elements, and (ii) the two-body stochastic collisions between the gas particles.

(i) The Successive Collisions with Different Surface Elements

This mechanism was investigated by Koonin and Randrup.^{22b} These authors used a perturbative linear response treatment for a classical gas in a spherical container whose surface vibrates in a definite multipole mode. Their results indicated very much reduced dissipation rates for the low multipole modes ($\lambda=0$ and $\lambda=2$). However, as recognized by the same authors, in the case of a spherical surface the validity of the perturbative treatment is doubtful because the response function does not have a short relaxation time. Nevertheless, the work of Koonin and Randrup suggested the possibility for the energy dissipation rate to have a dependence on the multipolarity λ of the surface vibration, and later a phenomenological formula with an explicit dependence on λ was proposed by Soroushian.²⁴ However, the present microscopic RPA model in the large radius limit* yields no such λ dependence. In particular, the proportionality coefficient γ_{limit} (Eq. (VI.2.16)) is independent of λ for all multipoles with $\lambda \ll k_F R$.

*Observe that this large radius (or large nucleus) limit is precisely the same limit considered by Koonin and Randrup^{22b} (see especially Sec. 3 of Ref. 22b).

(11) The Two-Body Stochastic Collisions

The possibility for this mechanism of being the randomizing agent in the classical wall model has been investigated by Hasse.²⁵ This author used a collisional Boltzmann equation to describe the bulk of a gas with a plane boundary. The vibrational motion of the wall was taken into account through boundary conditions on the collisional Boltzmann equation. Hasse's result was that a large but finite mean free path had no effect on the Swiatecki wall formula.

However, the present work shows that models for nuclear dissipation can be formulated without considering any randomizing processes (i.e., two-body binary stochastic collisions). The following point can help to illustrate the contrast between the present work and Hasse's approach. The finite system time-dependent equations of the usual RPA--utilized here to derive the expression (VI.2.14) for the proportionality coefficient γ --go over in the infinite medium/long wavelength limit to the collisionless Boltzmann (or Vlasov) equation.⁷² This Vlasov equation does not have a collision term.

B. Absence of Any Effect on the Wall Formula Due to the Explicit Self-consistency of the Microscopic RPA Treatment

In the classical Swiatecki model the self-consistent nuclear mean field has been replaced by an externally driven moving wall. This replacement of the mean field by an externally driven wall (or any other form of an externally driven one-body potential) leaves open the question whether

self-consistency is violated. Self-consistency expresses the physical fact that the source of the time dependence of the nuclear mean field is the motion of the individual nucleons themselves. Sierk, Koonin and Nix²³ have suggested that the classical wall model does indeed violate self-consistency. They proposed a modification, which attempted to incorporate self-consistency. In particular, they proposed that the normal velocity of the moving nuclear surface in the Swiatecki formula be replaced by the normal derivative of this velocity. The present microscopic RPA model is explicitly self-consistent in the sense that the nuclear mean field was calculated from microscopic wave functions describing the motion of the individual nucleons (see Sec. IV.3.8). However, in contrast with the Sierk, Koonin and Nix²³ proposal the microscopic energy dissipation formula (VI.2.1) does not depend on the derivatives of the collective velocity, but on the collective velocity itself.

C. Contrast with the Linear Response Derivation of Koonin et al.

Koonin, Hatch and Rasmussen^{22a} gave a quantal derivation of a wall formula by applying the linear response formalism to the special case of a Fermi gas of particles bounded by a flat infinitely hard potential wall. However, their linear response approach is a time-perturbative treatment, and does not yield the classical dissipation result--i.e., energy dissipation rates proportional to the square of the collective velocity--unless the assumption of slowly changing (adiabatic/small frequency) collective motion is invoked.* In contrast, the present nonperturbative RPA treatment yields

* This assumption of slow variation for the collective motion is also essential in the linear response theory of Hofmann and Siemens (see Ch. VII for details).

the classical dissipation result (Eq. (VI.2.1)) for all values of the collective frequency. Because of this restriction to the adiabatic regime, it is not surprising that the proportionality coefficient for the energy loss rate calculated by Koonin et al. is frequency independent and, thus, agrees with the corresponding Swiatecki constant for a flat boundary.

VII. CONTRAST WITH THE LINEAR RESPONSE APPROACH TO NUCLEAR DISSIPATION

VII.1 Contrast with the Linear Response Within the 1p-1h Subspace

Hofmann and Siemens²⁸ have proposed a linear response approach to nuclear dissipation. This linear response approach is a one-body dissipation theory, since the damping of collective motion is viewed as resulting from the interaction of independent nucleons with the time-varying part of the nuclear mean field. However, this nuclear mean field is not calculated self-consistently from the two-body forces between the individual nucleons, but is replaced by an externally driven one-body potential suggested from classical phenomenological models for the collective motion, in particular, the cranking model.²⁸ Thus, this linear response approach—as well as the Swiatecki wall model²¹ and the theory of Gross and co-workers²⁷—is a semi-classical theory, in the specific sense that only the particles are described quantum mechanically, while the motion of the mean field is not quantized. Consequently, the nuclear mean field (and the corresponding collective coordinates) lack any internal structure, in contrast with the present microscopic RPA theory, where the nuclear mean field is described by an ensemble of collective phonons. This lack of internal structure restricts nuclear dissipation to cases which have a classical analog, whereas the microscopic RPA theory is able to describe the damping in extreme quantal cases, i.e., sharp n-phonon initial collective states.

The linear response approaches utilize a time-dependent perturbative treatment, unlike the present nonperturbative RPA theory. As a result, the linear response approach describes classical dissipation—energy loss rates proportional to kinetic energy—only in the case of slow charging (adiabatic) motion. Accordingly, the inertial parameters for the collective degree of freedom derived in the linear response theory are those obtained²⁸ in the nuclear cranking model. These cranking model parameters contrast

with the Vibrating Potential Model parameters^{40a} derived in the present RPA mode. (The cranking model is the adiabatic limit of the Vibrating Potential Model.^{40a})

Because of this restriction to the adiabatic motion, it is not surprising that the dissipation coefficients calculated by the linear response theory are the low frequency limit of the corresponding friction coefficients derived in the present RPA model. In particular, the friction coefficient, γ_{LRT} , for the damping of small amplitude vibrations of a spherical droplet was found^{28b} in the linear response theory at zero temperature to be

$$\gamma_{LRT} = \pi \sum_{m1} |\langle m | \hat{P} | 1 \rangle|^2 \left[\frac{\partial}{\partial \omega} [\delta(\omega - \epsilon_{m1}) - \delta(\omega + \epsilon_{m1})] \right]_{\omega=0}. \quad (\text{VII.1.1})$$

Expression (VII.1.1) is frequency independent. It is also the low frequency limit of expression (VI.2.14) for the friction coefficient resulting from the RPA model. In later papers, motivated by the derivation of a quantal Fokker-Planck equation, Hofmann et al.^{28d} and Werner et al.³⁰ suggested a finite frequency friction coefficient identical to expression (VI.2.14) of the RPA model. However, this Fokker-Planck approach yielded^{28d} an energy dissipation rate different than classical dissipation.

Since the linear response theories construct the interaction Hamiltonian between the collective degree of freedom and the individual nucleons by utilizing the phenomenological cranking model, they correspond to utilizing a schematic two-body interaction in the microscopic RPA model. The microscopic RPA model, however, is able to treat more general two-body interactions than these separable interactions.

VII.2 Contrast with Extensions of the Linear Response Theories to

Include Couplings to 2p-2h States

The linear response theory of Hofmann and Siemens²⁸ was first formulated under the assumption that the nucleons constitute a Fermi gas. The continuous single-particle spectrum of a Fermi gas provided for dense additional 1p-1h subspaces and thus led to irreversible energy losses. In a later paper³⁰ Werner et al. recognized that for a finite-A system a problem of principle arises within the framework of the linear response approach, when restricted inside the 1p-1h subspace. How does one incorporate irreversibility when calculating friction coefficients for a finite system with a discrete 1p-1h subspace? Even more, due to the low frequency limit, the friction coefficients calculated from the linear response approach in the case of a 1p-1h spectrum possessing a gap turned out, at zero temperatures, to be always identically zero.*

Within the framework of linear response, Jensen et al.²⁹ and Werner et al.³⁰ proposed to attribute to each individual single-particle and hole energy (ϵ_m and ϵ_1 , respectively) some finite uncertainty for decaying to 2p-1h states. That is, they utilized a class of residual two-body interactions

*In the RPA dissipation model the damping of collective motion for a finite-A nucleus within the 1p-1h space is associated with the escape width for direct particle emission to the continuum. For low frequency vibrations this escape width is zero, as expected.

to dress the self-energies of individual particles and holes with an imaginary part. This approach parallels the Fermi liquid theories. This dressing of the individual single-particle energies, however, is associated^{29,30} with the optical potential for one-particle processes and, in a time-dependent picture, is equivalent to unstable single-particle states, which decay due to real transitions to 2p-1h nuclear states degenerate with the original single-particle state. Since a single-particle's lifetime is proportional to the mean free path for binary collisions between individual particles, this extension of the linear response theory is expected to be valid for cases when the nuclear mean free path is short compared to the nuclear dimensions. Such an assumption, however, is contrary to the long mean free path property for low temperatures in finite-A nuclei. Indeed, Werner et al.³⁰ find an approximately linear dependence on temperature of the friction coefficients. Accordingly, their friction coefficients are zero at zero temperature.

In the present microscopic model an extension to include a certain set of residual two-body couplings to 2p-2h states has been formulated by utilizing the Second RPA (see Sec.II.4). However, this extension is different from dressing the single-particle energies with an imaginary part. Indeed, the single-particle energies, ϵ_m and ϵ_1 , in the Second RPA equations (II.4.4) are equal to the stationary Hartree-Fock energies, and the single-particle states are stable having an infinite lifetime. Thus the mean free path is infinite. Yet, the collective state decays due to the presence of a dense 2p-2h spectrum around the one-phonon collective energy.

$\hbar\omega_c$. In the case of heavy and medium-weight nuclei, at the relatively high energy of the giant vibrations, there always exists such a dense 2p-2h spectrum allowing transitions from the collective state to more complicated nuclear compound states. This dissipative mechanism depends only on the presence of a dense 2p-2h spectrum and is operating at zero temperature, unlike the extension of Jensen et al. and Werner et al.

VIII. SUMMARY

In Ch. II. we have formulated a model of a general microscopic dissipative process, which takes into account the long mean free path of nucleons with respect to nuclear dimensions. In particular, this model describes nuclear dissipation in the limit of infinite mean free path. This process does not require any assumptions of a statistical nature, such as heat baths, temperature or randomization. It attributes the damping of the nuclear collective motion to the coherent coupling of the collective state—as opposed to incoherent individual couplings of the constituent parts—to more complicated nuclear states, degenerate with the collective state, via residual interactions. The dissipation of the collective motion is described by the time-evolution of wave packets which, initialized with the collective state, solve an approximate Schroedinger equation within extended subspaces larger than the restricted original subspace of the undamped collective motion.

This model, applied within the simple RPA framework (Sec. II.3), associates the energy dissipation rate with the escape width, Γ^+ , for direct particle emission to the continuum. Applied within the Second RPA framework (Sec. II.4), it associates the energy dissipation rate—besides Γ^+ —with the spreading width, Γ^+ , to nuclear compound states.

Chapter II described the damping of a one-phonon initial state, while Ch. III described the damping of any sharp n -phonon initial state. For both cases the energy dissipation rate was found proportional to the average number of collective phonons in the system, rather than proportional to the probability of finding the system in the initial collective state. Both cases of a discrete and continuous additional subspace have been examined. For a discrete and dense additional subspace, the system

is dissipative during the early portion of the Poincaré recurrence period. For a continuous additional subspace, the Poincaré recurrence period is infinite. Discrete, but sparse, additional subspaces do not lead to dissipation.

Chapter IV described the damping of the time-varying oscillations of the nuclear mean field. The description of such oscillations required initial packets formed by linear superposition of sharp n -phonon states. Specifically, minimum uncertainty coherent initial packets have been used. In this chapter, explicit position and momentum canonical coordinates have been introduced by analogy with the SHO. We proceeded to define a generalized collective Hamiltonian by partitioning the total Hamiltonian into a collective part, an interaction part, and an internal part through the construction of a nonlinear frictional Hamiltonian, which depends only upon the collective coordinates and upon the wave packet describing the damping. This frictional Hamiltonian was constructed to prescribe precisely the same time behavior for these collective coordinates as does the total Hamiltonian in the RPA and the Second RPA. We calculated the collective energy of the system as the sum of the collective kinetic and collective potential energy parts of the nonlinear frictional Hamiltonian. The energy dissipation rate was found proportional to the kinetic part of the collective energy, which thus oscillates while decaying. This is also true for classical dissipation, unlike the case of sharp n -phonon states, where the energy loss rate was proportional to the total collective energy, which thus exhibited a simple exponential decay.

Chapter V provided a description of dissipation complete, from the collective viewpoint, within the collective coordinate subspace. This enabled the connection with phenomenological nonlinear quantal friction.

The Kostin-Albrecht^{39b,34} and Kostin-Kan^{39a,32} Schrödinger friction operators have been rederived. This description within the collective coordinate subspace is *irreversible*, unlike the *reversible* description of damping within the extended Hilbert subspaces. Moreover, this description within the collective coordinate subspace conserves the probability, unlike the description which utilizes a division of the Hilbert space into a restricted and an additional subspace.

Chapter VI made the connection with the Vibrating Potential Model for nuclear collective motion, and, thereby, with the classical Swiatecki wall formula. The Swiatecki constant value for the proportionality coefficient in the energy dissipation rate was derived—for low frequencies—by using the microscopic formula for escape widths and going to the limit of an infinitely deep square well of very large radius. This limit, however, is unrealistic for nuclei in the periodic table, since it substitutes the sparse, discrete 1p-1h spectrum of finite-A nuclei by a dense one and would lead to the damping of even the low-lying isoscalar collective modes. For a finite-A nucleus, a dissipation mechanism based solely on one-body forces accounts only for the escape width to the continuum. Since for medium-weight and heavy nuclei the escape width is much smaller than the total experimental width, a complete theory for nuclear dissipation must incorporate a certain class of two-body residual interactions, besides the one-body interactions responsible for the collective motion. The present microscopic model allowed such an extension through the Second RPA (Sec. II.4).

Chapter VII contrasted the present model to the linear response theories.²⁸ The linear response theories utilize a time-dependent perturbative approach, unlike the present RPA model. As a result, the linear response theories relate to the cranking model which is the adiabatic limit

of the Vibrating Potential Model. The present model also contrasts with the linear response theories in the way the residual two-body forces are taken into consideration.

Observe that the residual two-body interactions, as utilized by the present model, do not generate binary collisions between individual particles in the sense of a collision term, that is, real transitions from single-particle initial excitations to 2p-1h states, degenerate with the original single-particle state. The lifetime of the single-particle excitations entering in the Second RPA extensions is infinite, and, correspondingly, the mean free path remains infinite. This infinite single-particle lifetime is reflected in the fact that the diagonal elements of the one-body density matrix (the occupation probabilities of the single-particle levels) are, here, time independent. To describe the damping of the mean field oscillations, the present dissipative process involves exclusively the off-diagonal density matrix elements (Ch. IV), unlike the case with the transport theories¹²⁻¹⁸ where the diagonal elements solely contribute to the dissipation process.

Observe also that the starting point of the present microscopic model are the two-body forces between nucleons. The two-body forces are averaged into effective one-body interactions, when the simple RPA is used. However, when the additional subspace, S_A , contains 2p-2h states, the dynamical equations of motion in the extended subspace are described by extensions of the simple RPA (here referred to as Second RPA). The couplings between 1p-1h and 2p-2h states are due to *residual two-body* interactions persisting beyond the averaging of the simple RPA. These residual two-body interactions, as utilized by the present model, account for couplings between nucleons and the collective phonons. In the case of a leptodermous and incompressible system, a phenomenological viewpoint

would consider the present treatment of the two-body residual forces as describing interactions of nucleons with a moving surface of the system.* Thus, even when applied in the framework of the Second RPA, the present microscopic model relates to the "one-body/long mean free path" phenomenological theories, instead of to the "two-body/short mean free path" theories where dissipation is always a bulk phenomenon. However, because of the residual two-body forces in the Second RPA, the use of the term "one-body" to characterize all long mean free path dissipative phenomena in finite nuclei is misleading. A better term is particle-phonon** dissipation. This term contrasts with the particle-particle dissipation of ordinary viscosity.

We further note that the dissipation mechanism presented in this dissertation does not require any assumptions of a statistical nature such as heat baths, temperature, or randomization. The decay it describes is the behavior of an approximate solution of truly reversible dynamical equations of motion. In particular, the dissipative process here, as formulated in the extended subspaces, is a reversible, rather than irreversible, process, properly so called in that it is described by an approximate Schrödinger equation which honors time reversibility, rather than by a coarse grained master equation which violates it.⁴⁶ This approximate Schrödinger equation is explicitly derived in Appendix B and is referred to as the time-dependent RPA or Second RPA. Thus, in

*The nucleons interacting with the moving surface, however, are not longer independent.

**In the case of a leptodermous and incompressible system, another term can be particle-surface dissipation.

this respect also, the present theory contrasts with the transport theories,¹²⁻¹⁸ which extend the TDHF equations by the addition of a time irreversible Boltzmann "collision" term.

The present theory is rather analogous to the reversible Landau damping of plasma oscillations as described by the many-body quantum theory of ionized gases.^{47,48} Besides reversibility, the analogy consists in that both the collective plasmon mode and its decay by Landau damping are described by using the RPA approximation^{47,48} and extensions of it.⁴⁸ However, the present model describes a finite system initialized at zero temperature, whereas Landau damping theory describes an infinite medium at finite temperature. Moreover, the finite system time-dependent equations of the simple RPA go over in the infinite medium/long wavelength limit to the linearized collisionless (or Vlasov) equation⁷² which Landau first used to demonstrate the Landau damping⁴⁹ of the plasmon mode in an electron gas. Later, Van Kampen⁵⁰ and Dawson⁵¹ provided interpretations of the Landau damping closer in spirit to our model for nuclear dissipation, which ascribes the damping mechanism to the time evolution of special wave packets.

Since the present model is microscopic and quantal, it can describe dissipation in extreme quantal cases which have no classical analog, i.e., the damping of sharp n-phonon initial states. This contrasts with the phenomenological "one-body/long mean free path" approaches²¹⁻²⁸ where the phenomenology itself depends upon the existence of classical analogs in order to introduce collective variables and, thereby, the time variations of the nuclear mean field.

APPENDIX A: SUMMARY OF RPA AND SECOND RPA FORMALISM

In the usual RPA^{40,41,56} the nuclear states are superpositions of one-particle—one-hole pairs only. Extensions of the usual RPA (called higher random phase approximations) are possible when one includes np-nh components in the nuclear states. The modes specifically consisting of 1p-1h and 2p-2h nucleon excitations are treated by the second RPA.^{42-44,56} The usual RPA equations of motion have been obtained by a variety of many-body approaches.⁴⁰ The same variety of methods can be used to obtain the second RPA equations. Here, in order to obtain the second RPA, we use a specific method utilized by Rowe^{40b} to derive the usual RPA and referred to as Rowe's equations-of-motion method, to distinguish it from the linearization^{41,56} of the equations-of-motion-for-operators method. However, Rowe's approach is equivalent to the linearization approach.⁴⁰

We start with the exact eigenstates $|v\rangle$ of the complete Hamiltonian, H .

$$H|v\rangle = (E_{0,ex} + M\omega_v)|v\rangle. \quad (A.1)$$

We define operators $\hat{Q}_v^\dagger$ and $\hat{Q}_v$ such that

$$|v\rangle = \hat{Q}_v^\dagger |0\rangle_{ex} \quad \text{and} \quad \hat{Q}_v |0\rangle_{ex} = 0. \quad (A.2)$$

From the stationary Schrödinger equation (A.1), one gets the equation of motion:

$$[H, \hat{Q}_v^\dagger] |0\rangle_{ex} = M\omega_v \hat{Q}_v^\dagger |0\rangle_{ex}. \quad (A.3)$$

Multiplying from the left with a variation $\delta \hat{Q}_v$ on $|v\rangle$, one obtains

$$\langle 0 | [\delta \hat{Q}_v, [H, \hat{Q}_v^\dagger]] | 0 \rangle_{ex} = M\omega_v \langle 0 | [\delta \hat{Q}_v, \hat{Q}_v^\dagger] | 0 \rangle_{ex}. \quad (A.4)$$

In deriving (A.4), we used the fact that $\langle 0 | \hat{Q}_v^\dagger = \langle 0 | H \hat{Q}_v^\dagger = 0$.

Equation (A.4) corresponds to the complete Schrödinger equation. However, the exact problem cannot be solved in practice. By assuming that the creation operators $\hat{Q}_v^\dagger$ are linear superpositions of 1p-1h pairs, one can successively derive the TDA⁴⁰ and the usual RPA.⁴⁰ Here, we use a more general approximation for the $\hat{Q}_v^\dagger$ by considering 2p-2h states as well as 1p-1h states, that is:

$$\begin{aligned} \hat{Q}_v^\dagger = & \sum_{m1} \{ Y_{m1}(\omega_v) a_{11}^\dagger a_1 - Z_{m1}(\omega_v) a_{11}^\dagger a_m \} \\ & + \sum_{\substack{m < n \\ i < j}} \{ Y_{mnij}(\omega_v) a_m^\dagger a_n^\dagger a_j a_i - Z_{mnij}(\omega_v) a_i^\dagger a_j^\dagger a_n a_m \} \\ & + \sum_{mnij} X_{mjir}(\omega_v) a_m^\dagger a_j^\dagger a_n a_i. \end{aligned} \quad (A.5)$$

The second RPA ground state $|RPAII\rangle$ is defined by analogy with

(A.2):

$$\hat{Q}_v |RPAII\rangle = 0, \quad \text{for all } v's. \quad (A.6)$$

One can, in principle, deduce from this condition an explicit expression for the ground state. Varying $\hat{Q}_v^\dagger$ in (A.5) and substituting into Eq. (A.4), one gets five sets of equations:

$$\langle RPAII | [a_{11}^\dagger a_m, [H, \hat{Q}_v^\dagger]] | RPAII \rangle = M\omega_v \langle RPAII | [a_{11}^\dagger a_m, \hat{Q}_v^\dagger] | RPAII \rangle \quad (A.7a)$$

$$\langle RPAII | [a_{11}^\dagger a_j^\dagger a_n a_m, [H, \hat{Q}_v^\dagger]] | RPAII \rangle = M\omega_v \langle RPAII | [a_{11}^\dagger a_j^\dagger a_n a_m, \hat{Q}_v^\dagger] | RPAII \rangle \quad (A.7b)$$

$$\langle RPAII | [a_{m1}^\dagger a_1, [H, \hat{Q}_v^\dagger]] | RPAII \rangle = M\omega_v \langle RPAII | [a_{m1}^\dagger a_1, \hat{Q}_v^\dagger] | RPAII \rangle \quad (A.7c)$$

$$\langle RPAII | [a_{m1}^\dagger a_n^\dagger a_j a_1, [H, \hat{Q}_v^\dagger]] | RPAII \rangle = M\omega_v \langle RPAII | [a_{m1}^\dagger a_n^\dagger a_j a_1, \hat{Q}_v^\dagger] | RPAII \rangle \quad (A.7d)$$

$$\langle RPAII | [a_{11}^\dagger a_n^\dagger a_j a_m, [H, \hat{Q}_v^\dagger]] | RPAII \rangle = M\omega_v \langle RPAII | [a_{11}^\dagger a_n^\dagger a_j a_m, \hat{Q}_v^\dagger] | RPAII \rangle \quad (A.7e)$$

These equations contain expectation values of products of Fermion operators.

These expectation values are still very complicated to calculate

since the ground state $|RPAII\rangle$ is not the vacuum of the Fermion operators.

At this point, we use a straightforward generalization of the "quasi-boson approximation".^{40,41} We assume that the correlated ground state, $|RPAII\rangle$, does not differ very much from the Hartree-Fock ground state. We then calculate all expectation values of the commutators in Eqs. (A.7) by using the Hartree-Fock ground state, i.e.:

$$\langle \text{RPAII} | [a_{1m}^\dagger a_m, a_{\alpha\beta}^\dagger] | \text{RPAII} \rangle \approx \langle \text{HF} | [a_{1m}^\dagger a_m, a_{\alpha\beta}^\dagger] | \text{HF} \rangle = \delta_{m\alpha} \delta_{i\beta}; \quad (\text{A.8})$$

$$\langle \text{RPAII} | [a_{1j}^\dagger a_n a_m, a_{\alpha\beta}^\dagger a_{\delta\gamma}] | \text{RPAII} \rangle \approx \langle \text{HF} | [a_{1j}^\dagger a_n a_m, a_{\alpha\beta}^\dagger a_{\delta\gamma}] | \text{HF} \rangle \\ = \mathcal{U}(mn) \mathcal{U}(ij) \delta_{m\alpha} \delta_{n\beta} \delta_{j\delta} \delta_{i\gamma}, \quad (\text{A.9})$$

where $\mathcal{U}(rs)$ is the antisymmetrizer for the indices r, s . Also

$$\langle \text{RPAII} | [a_{1n}^\dagger a_j a_m, \hat{M}] | \text{RPAII} \rangle \approx \langle \text{HF} | [a_{1n}^\dagger a_j a_m, \hat{M}] | \text{HF} \rangle = 0 \quad (\text{A.10})$$

for any operator $\hat{M}$.

Substituting Eq. (A.5) into Eqs. (A.7) and using Eqs. (A.8), (A.9)

and (A.10), one finds the following second RPA equations:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & -\mathcal{A}^* \end{pmatrix} \begin{pmatrix} \mathcal{Y}(\omega_v) \\ \mathcal{Z}(\omega_v) \end{pmatrix} = \chi_{\omega_v} \begin{pmatrix} \mathcal{Y}(\omega_v) \\ \mathcal{Z}(\omega_v) \end{pmatrix} \quad (\text{A.11})$$

where $\mathcal{A}$ and $\mathcal{B}$ are 4×4 matrices;

$$\mathcal{A} = \begin{bmatrix} A_{mi,pk} & A_{mi,pqkl} \\ A_{mni,j,pk} & A_{mni,j,pqkl} \end{bmatrix} \quad (\text{A.12})$$

$$\mathcal{B} = \begin{bmatrix} B_{mi,pk} & B_{mi,pqkl} \\ B_{mni,j,pk} & B_{mni,j,pqkl} \end{bmatrix} \quad (\text{A.13})$$

and

$$\mathcal{Y}(\omega_v) = \begin{pmatrix} Y_{mi}(\omega_v) \\ Y_{mni,j}(\omega_v) \end{pmatrix}; \quad \mathcal{Z}(\omega_v) = \begin{pmatrix} Z_{mi}(\omega_v) \\ Z_{mni,j}(\omega_v) \end{pmatrix}. \quad (\text{A.14})$$

One has:

$$A_{mi,pk} = \langle \text{HF} | [a_{1m}^\dagger a_m, [H, a_p^\dagger a_k]] | \text{HF} \rangle \quad (\text{A.15a})$$

$$A_{mi,pqkl} = \langle \text{HF} | [a_{1m}^\dagger a_m, [H, a_p^\dagger a_q^\dagger a_l a_k]] | \text{HF} \rangle \quad (\text{A.15b})$$

$$A_{mni,j,pqkl} = \langle \text{HF} | [a_{1j}^\dagger a_n a_m, [H, a_p^\dagger a_q^\dagger a_l a_k]] | \text{HF} \rangle \quad (\text{A.15c})$$

$$B_{mi,pk} = -\langle \text{HF} | [a_{1m}^\dagger a_m, [H, a_k^\dagger a_p]] | \text{HF} \rangle \quad (\text{A.16a})$$

$$B_{mi,pqkl} = -\langle \text{HF} | [a_{1m}^\dagger a_m, [H, a_k^\dagger a_l^\dagger a_q a_p]] | \text{HF} \rangle \quad (\text{A.16b})$$

$$B_{mni,j,pqkl} = -\langle \text{HF} | [a_{1j}^\dagger a_n a_m, [H, a_k^\dagger a_l^\dagger a_q a_p]] | \text{HF} \rangle \quad (\text{A.16c})$$

Using the operator identity,

$$[\hat{A}[\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0, \quad (\text{A.17})$$

and the fact that

$$\langle \text{HF} | [H, [\hat{C}, \hat{A}]] | \text{HF} \rangle = E_{0,\text{HF}} \{ \langle \text{HF} | [\hat{C}, \hat{A}] | \text{HF} \rangle - \langle \text{HF} | [\hat{C}, \hat{A}] | \text{HF} \rangle \} = 0, \quad (\text{A.18})$$

one sees that

$$\mathcal{A}^\dagger = \mathcal{A} \quad (\text{hermitean}) \quad (\text{A.19a})$$

$$\tilde{\mathcal{B}} = \mathcal{B} \quad (\text{symmetric}). \quad (\text{A.19b})$$

The matrix elements, $C_{mi,p\bar{l}kq}$ and $C_{mni,j,p\bar{l}kq}$ are equal to zero:

$$C_{mi,p\bar{l}kq} = \langle \text{HF} | [a_{1m}^\dagger a_m, [H, a_p^\dagger a_{\bar{l}}^\dagger a_k a_q]] | \text{HF} \rangle = 0 \quad (\text{A.20a})$$

and

$$C_{mni,j,p\bar{l}kq} = \langle \text{HF} | [a_{1j}^\dagger a_n a_m, [H, a_p^\dagger a_{\bar{l}}^\dagger a_k a_q]] | \text{HF} \rangle = 0. \quad (\text{A.20b})$$

This can be seen by using Eqs. (A.17) and (A.18). The amplitudes $Y_{pk}(\omega_v)$, $Y_{pqkl}(\omega_v)$, $Z_{pk}(\omega_v)$ and $Z_{pqkl}(\omega_v)$ do not couple to the amplitudes $X_{p\bar{l}kq}(\omega_v)$, as long as one uses the Hartree-Fock ground state to approximately calculate the commutators in Eq. (A.7).

A straightforward calculation yields:

$$A_{mi,pk} = \epsilon_{mi} \delta_{mp} \delta_{ik} + V_{mkip}; \quad (\text{A.21a})$$

$$A_{mi,pqkl} = \mathcal{U}(pq) V_{k\bar{l}ip} \delta_{mq} - \mathcal{U}(kl) V_{mkpq} \delta_{il}; \quad (\text{A.21b})$$

$$A_{mni,j,pqkl} = (\epsilon_{mi} + \epsilon_{nj}) \mathcal{U}(pq) \mathcal{U}(kl) \delta_{mp} \delta_{nq} \delta_{jl} \delta_{ik} \\ + \mathcal{U}(kl) V_{mnpq} \delta_{ik} \delta_{jl} + \mathcal{U}(pq) V_{k\bar{l}ij} \delta_{mp} \delta_{nq} \\ - \mathcal{U}(mn) \mathcal{U}(ij) \mathcal{U}(pq) \mathcal{U}(kl) V_{m\bar{l}pj} \delta_{ik} \delta_{nq}; \quad (\text{A.21c})$$

$$B_{mi,pk} = V_{mpik} \quad (A.22a)$$

$$B_{mi,pqkl} = B_{mnij,pqkl} = 0 \quad (A.22b)$$

Here the matrix elements of the two-body force are antisymmetrized and $\epsilon_{mi} = \epsilon_m - \epsilon_i$. The single-particle energies ϵ_m and ϵ_i are the stationary Hartree-Fock ones:

$$\epsilon_m = T_{mm} + \sum_j V_{mj}m \quad (A.23a)$$

$$\epsilon_i = T_{ii} + \sum_j V_{ji}j \quad (A.23b)$$

The $X_{p2kq}(\omega_v)$ in (A.5) can be neglected since they do not couple to the rest of the Second RPA amplitudes, i.e.:

$$\begin{aligned} \hat{Q}_v^+ &= \sum_{mi} \{Y_{mi}(\omega_v) a_m^\dagger a_i - Z_{mi}(\omega_v) a_i^\dagger a_m\} \\ &+ \sum_{\substack{m < n \\ i < j}} \{Y_{mnij}(\omega_v) a_m^\dagger a_n^\dagger a_j a_i - Z_{mnij}(\omega_v) a_i^\dagger a_j^\dagger a_n a_m\}. \end{aligned} \quad (A.24)$$

For the Second RPA amplitudes, one finds when using the quasi-boson approximation (Eqs. (A.8) and (A.9)) and the definition of the correlated ground state (Eq. (A.6)):

$$Y_{mnij}(\omega_v) = \langle \text{RPAII} | a_i^\dagger a_j^\dagger a_n a_m | v \rangle \quad (A.25a)$$

$$Z_{mnij}(\omega_v) = \langle \text{RPAII} | a_m^\dagger a_n^\dagger a_j a_i | v \rangle \quad (A.25b)$$

$$Y_{mi}(\omega_v) = \langle \text{RPAII} | a_i^\dagger a_m | v \rangle \quad (A.25c)$$

$$Z_{mi}(\omega_v) = \langle \text{RPAII} | a_m^\dagger a_i | v \rangle \quad (A.25d)$$

The Second RPA orthonormality relations are:

$$\langle \mu | v \rangle = \delta_{\mu v} \frac{\omega_v}{|\omega_v|}, \quad (A.26)$$

or

$$\begin{aligned} &\sum_{mi} \{Y_{mi}^*(\omega_\mu) Y_{mi}(\omega_v) - Z_{mi}^*(\omega_\mu) Z_{mi}(\omega_v)\} \\ &+ \sum_{\substack{m < n \\ i < j}} \{Y_{mnij}^*(\omega_\mu) Y_{mnij}(\omega_v) - Z_{mnij}^*(\omega_\mu) Z_{mnij}(\omega_v)\} = \delta_{\mu v} \frac{\omega_v}{|\omega_v|}. \end{aligned} \quad (A.27)$$

The quasi-boson approximation is equivalent to calculating the contractions $\langle \text{RPAII} | a_\alpha^\dagger a_\beta | \text{RPAII} \rangle$ using the Hartree-Fock ground state instead of the correlated ground state, i.e.:

$$\langle \text{RPAII} | a_\alpha^\dagger a_\beta | \text{RPAII} \rangle \approx \langle \text{HF} | a_\alpha^\dagger a_\beta | \text{HF} \rangle. \quad (A.28)$$

The quasi-boson approximation is also equivalent to requiring that the $a_m^\dagger a_i$ pairs and the $a_m^\dagger a_n^\dagger a_j a_i$ quadruples obey the following boson commutation relations:

$$[a_j a_m, a_\alpha^\dagger a_\beta] = \delta_{m\alpha} \delta_{i\beta} \quad (A.29a)$$

$$[a_i^\dagger a_j^\dagger a_n a_m, a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma] = \mathcal{U}(mn) \mathcal{U}(ij) \delta_{m\alpha} \delta_{n\beta} \delta_{j\delta} \delta_{i\gamma} \quad (A.29b)$$

$$[a_i^\dagger a_m, a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma] = 0 \quad (A.29c)$$

$$[a_\alpha^\dagger a_\beta, a_m^\dagger a_n^\dagger a_j a_i] = 0. \quad (A.29d)$$

From the orthonormality relations (A.27) in conjunction with Eqs. (A.29a-d), one gets the boson commutation for the operators $\hat{Q}_v^+, \hat{Q}_\mu$:

$$[\hat{Q}_\mu, \hat{Q}_v^+] = \delta_{\mu v}. \quad (A.29e)$$

The simple RPA equations are derived from the Second RPA equations (A.11) when the matrices $A_{mi,pqkl}$ and $A_{mnij,pqkl}$ are set equal to zero. The simple RPA orthonormality relations are:

$$\sum_{mi} \{Y_{mi}^*(\omega_\mu) Y_{mi}(\omega_v) - Z_{mi}^*(\omega_\mu) Z_{mi}(\omega_v)\} = \delta_{\mu v} \frac{\omega_v}{|\omega_v|}. \quad (A.30)$$

Equations (II.3.7) and (II.4.4) are derived from Eq. (A.11) through a simple rearrangement of terms when one differentiates between restricted and additional subspaces.

Notice that when the S_R -to- S_A coupling is neglected, Eq. (A.29e) yields as a special case the boson commutation for the operators Q_b^+, Q_a within the restricted subspace S_R :

$$[Q_a, Q_b^+] = \delta_{ab}. \quad (A.31)$$

Finally, notice that the use of the Hartree-Fock ground state in Eqs. (A.29) is equivalent⁴⁰ to linearizing the commutators $[H, a_m^\dagger a_i]$ and $[H, a_m^\dagger a_n^\dagger a_j a_i]$.

Because of this linearization the following commutation is valid either in the simple or Second RPA (see Ref. 56, especially Ch. 8);

$$[H, \hat{Q}_\nu^\dagger] \approx \hbar\omega_\nu \hat{Q}_\nu^\dagger. \quad (\text{A.32})$$

The commutation (A.32) is a special relation stronger than the corresponding Schrödinger equation:

$$[H, \hat{Q}_\nu^\dagger] | \text{RPA II} \rangle \approx \hbar\omega_\nu \hat{Q}_\nu^\dagger | \text{RPA II} \rangle. \quad (\text{A.33})$$

APPENDIX B: ALTERNATIVE DERIVATIONS

In this appendix we (1) derive the time-dependent Second RPA equations of motion, and (2) obtain a direct solution of them which reproduces the results of Secs. II.3.D and II.4 in the case of a subspace S_A with dense spectrum.

(1) We define a wave packet (time-dependent) operator $\hat{O}^\dagger(t)$ as

$$\hat{O}^\dagger(t) = \sum_\nu b_\nu \hat{Q}_\nu^\dagger e^{-i\omega_\nu t}. \quad (\text{B.1})$$

Since the stationary state operators, $\hat{Q}_\nu^\dagger$, have the expansion (A.24), the wave packet operator, $\hat{O}^\dagger(t)$, has the following expansion in the $1p-1h/2p-2h$ base:

$$\begin{aligned} \hat{O}^\dagger(t) = & \sum_{m1} \{ \Pi_{m1}(t) a_m^\dagger a_1 - P_{m1}(t) a_1^\dagger a_m \} \\ & + \sum_{\substack{m < n \\ i < j}} \{ \Pi_{mij}(t) a_m^\dagger a_n^\dagger a_j a_i - P_{mij}(t) a_i^\dagger a_j^\dagger a_n a_m \}, \end{aligned} \quad (\text{B.2})$$

where

$$\Pi_{m1}(t) = \sum_\nu b_\nu Y_{m1}(\omega_\nu) e^{-i\omega_\nu t}, \quad (\text{B.3a})$$

$$P_{m1}(t) = \sum_\nu b_\nu Z_{m1}(\omega_\nu) e^{-i\omega_\nu t}, \quad (\text{B.3b})$$

$$\Pi_{mij}(t) = \sum_\nu b_\nu Y_{mij}(\omega_\nu) e^{-i\omega_\nu t}, \quad (\text{B.3c})$$

and

$$P_{mij}(t) = \sum_\nu b_\nu Z_{mij}(\omega_\nu) e^{-i\omega_\nu t}. \quad (\text{B.3d})$$

Using Eq. (A.6), one sees that

$$\hat{O}(t) | \text{RPA II} \rangle = 0. \quad (\text{B.4})$$

Then, it is straightforward to show that the coefficients $\Pi_{m1}(t)$, $P_{m1}(t)$,

$\Pi_{mij}(t)$, and $P_{mij}(t)$ are equal to the amplitudes of the operators $a_i^\dagger a_m$, $a_m^\dagger a_1$, $a_i^\dagger a_j^\dagger a_n a_m$, and $a_m^\dagger a_n^\dagger a_j a_i$ between the correlated ground state

$|RPAII\rangle$, and the wave packet $|\phi(t)\rangle = \hat{O}^\dagger(t)|RPAII\rangle$, i.e.,

$$\Pi_{m1}(t) = \langle RPAII | a_m^\dagger a_m \hat{O}^\dagger(t) | RPAII \rangle, \quad (B.5a)$$

$$P_{m1}(t) = \langle RPAII | a_m^\dagger a_1 \hat{O}^\dagger(t) | RPAII \rangle, \quad (B.5b)$$

$$\Pi_{m1j}(t) = \langle RPAII | a_m^\dagger a_j^\dagger a_n a_m \hat{O}^\dagger(t) | RPAII \rangle, \quad (B.5c)$$

and

$$P_{m1j}(t) = \langle RPAII | a_m^\dagger a_n^\dagger a_j a_1 \hat{O}^\dagger(t) | RPAII \rangle. \quad (B.5d)$$

Equations (B.5) are a generalization of Eqs. (A.25) for the stationary states.

As can be seen by using Eqs. (B.3) and (A.11), the amplitudes $\Pi_{m1}(t)$, $P_{m1}(t)$, $\Pi_{m1j}(t)$, and $P_{m1j}(t)$ satisfy the following time-dependent Second RPA equations:

$$\begin{pmatrix} \mathcal{A} & \mathcal{E} \\ -\mathcal{B}^* & -\mathcal{A}^* \end{pmatrix} \begin{pmatrix} \mathcal{K}(t) \\ \mathcal{P}(t) \end{pmatrix} = i\hbar \frac{d}{dt} \begin{pmatrix} \mathcal{K}(t) \\ \mathcal{P}(t) \end{pmatrix}, \quad (B.6)$$

where the 4×4 matrices $\mathcal{A}$ and $\mathcal{B}$ are given in Eqs. (A.12) and (A.13), and

$$\mathcal{K}(t) = \begin{pmatrix} \Pi_{m1}(t) \\ \Pi_{m1j}(t) \end{pmatrix}; \quad \mathcal{P}(t) = \begin{pmatrix} P_{m1}(t) \\ P_{m1j}(t) \end{pmatrix}. \quad (B.7)$$

(2)(i) S_A contains additional states of the 1p-1h structure only.

Distinguishing between indices (n,j) for the restricted subspace, S_R , and indices $(\bar{n},\bar{j})$ for the additional subspace, S_A , the time-dependent equations-of-motion (B.6) yield:

$$\begin{pmatrix} A & B & C & D \\ -B^* & -A^* & -D^* & -C^* \end{pmatrix} \begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix} = i\hbar \frac{d}{dt} \begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix}, \quad (3.8a)$$

$$\begin{pmatrix} A & B & C & D \\ -B^* & -A^* & -D^* & -C^* \end{pmatrix} \begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix} = i\hbar \frac{d}{dt} \begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix}, \quad (3.8b)$$

$$\begin{pmatrix} C & D & E & 0 \\ -D^* & -C^* & 0 & -E \end{pmatrix} \begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix} = i\hbar \frac{d}{dt} \begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix}, \quad (3.8c)$$

$$\begin{pmatrix} C & D & E & 0 \\ -D^* & -C^* & 0 & -E \end{pmatrix} \begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix} = i\hbar \frac{d}{dt} \begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix}, \quad (3.8d)$$

when one neglects the couplings among states $(\bar{m},\bar{i})$ of the additional subspace, S_A , as was assumed in Sec. II.3.B. We seek solutions of the form:

$$\begin{pmatrix} \Pi(t) \\ P(t) \\ \bar{\Pi}(t) \\ \bar{P}(t) \end{pmatrix} = \begin{pmatrix} y(\omega_c) \\ z(\omega_c) \\ \bar{\Pi}(\Omega(n), n) \\ \bar{P}(\Omega(n), n) \end{pmatrix} e^{-i\Omega(n)t + nt}, \quad (B.9)$$

where $\Omega(n)$ is a complex frequency.

Observe that a solution of the form (B.9) implies the initial condition:

$$\langle c | \phi(t=0) \rangle = 1. \quad (B.10)$$

From Eqs. (B.8c) and (B.8d), we immediately obtain:

$$\Pi_{m1}(\Omega(n), n) = \frac{K_{m1}^-(\omega_c)}{\Omega(n) - \epsilon_{m1} + i\hbar\eta} \quad (B.11a)$$

and

$$P_{m1}(\Omega(n), n) = -\frac{A_{m1}^-(\omega_c)}{\Omega(n) + \epsilon_{m1} + i\hbar\eta}, \quad (B.11b)$$

where the matrix elements, $K_{m1}^-(\omega_c)$, $A_{m1}^-(\omega_c)$, are given by Eqs. (II.3.16).

Besides Eqs. (B.11), the substitution of Eq. (B.9) into Eq. (B.8) yields with the help of Eqs. (II.3.10):

$$(\Omega(n) - \epsilon_{m1} + i\hbar\eta) y_{m1}(\omega_c) = \sum_{\bar{n}, \bar{j}} \frac{V_{m1\bar{n}\bar{j}} K_{\bar{n}\bar{j}}^-(\omega_c)}{\Omega(n) - \epsilon_{\bar{n}\bar{j}} + i\hbar\eta} - \sum_{\bar{n}, \bar{j}} \frac{V_{m1\bar{n}\bar{j}} A_{\bar{n}\bar{j}}^-(\omega_c)}{\Omega(n) + \epsilon_{\bar{n}\bar{j}} + i\hbar\eta} \quad (B.12a)$$

and

$$-(\Omega(n) - \epsilon_{m1} + i\hbar\eta) z_{m1}(\omega_c) = \sum_{\bar{n}, \bar{j}} \frac{V_{\bar{n}\bar{j}m1} K_{\bar{n}\bar{j}}^-(\omega_c)}{\Omega(n) - \epsilon_{\bar{n}\bar{j}} + i\hbar\eta} - \sum_{\bar{n}, \bar{j}} \frac{V_{\bar{n}\bar{j}m1} A_{\bar{n}\bar{j}}^-(\omega_c)}{\Omega(n) + \epsilon_{\bar{n}\bar{j}} + i\hbar\eta}. \quad (B.12b)$$

Premultiplying Eqs. (B.12a) with $y_{m1}^*(\omega_c)$ and Eqs. (B.12b) with $z_{m1}^*(\omega_c)$,

adding them and using the normalization condition $\langle c | c \rangle = 1$, we get the

dispersion relation:

$$\Omega(n) - \epsilon_{m1} + i\hbar\eta = \sum_{\bar{n}, \bar{j}} \left\{ \frac{|K_{\bar{n}\bar{j}}^-(\omega_c)|^2}{\Omega(n) - \epsilon_{\bar{n}\bar{j}} + i\hbar\eta} - \frac{|A_{\bar{n}\bar{j}}^-(\omega_c)|^2}{\Omega(n) + \epsilon_{\bar{n}\bar{j}} + i\hbar\eta} \right\}. \quad (B.13)$$

We can approximately solve the dispersion relation (B.13) by assuming that:

$$\left. \begin{array}{l} \text{(a) the shift, } \operatorname{Re}(\Omega(\eta) - \omega_c) = \Delta\omega_c(\eta), \text{ in the collective} \\ \text{frequency, } \omega_c, \text{ is small;} \\ \text{(b) the decay width, } \operatorname{Im}(\Omega(\eta)) = -\Gamma(\eta)/2, \text{ is much smaller} \\ \text{than the collective frequency, } \omega_c, \text{ i.e.,} \\ \Gamma(\eta) \ll 2\omega_c. \end{array} \right\} \quad (\text{B.14})$$

Assumptions (B.14) allow us to make a Taylor expansion of the right-hand-side of Eq. (B.13) around the point $\Omega(\eta) = \omega_c$ and keep only the zeroth order term. Equation (B.13) then yields:

$$\Delta\omega_c(\eta) - i\eta \frac{\Gamma(\eta)}{2} + i\eta \sum_{\vec{n}, \vec{j}} \left\{ \frac{|K_{\vec{n}, \vec{j}}(\omega_c)|^2}{\hbar\omega_c - \epsilon_{\vec{n}, \vec{j}} + i\eta} - \frac{|A_{\vec{n}, \vec{j}}(\omega_c)|^2}{\hbar\omega_c + \epsilon_{\vec{n}, \vec{j}} + i\eta} \right\}. \quad (\text{B.15})$$

When the additional subspace, S_A , is dense, the summations in Eq. (B.15) can be substituted with integrations. Then, with the help of the well-known formula:

$$\lim_{\eta \rightarrow 0^+} \frac{1}{x - e + i\eta} = P \frac{1}{x - e} - i\pi \delta(x - e), \quad (\text{B.16})$$

Eq. (B.15) yields:

$$\Gamma = \lim_{\eta \rightarrow 0^+} \Gamma(\eta) = \frac{2\pi}{\hbar} \iint d\vec{n} d\vec{j} |K(\vec{n}, \vec{j}; \omega_c)|^2 \delta(\hbar\omega_c - \epsilon(\vec{n}, \vec{j})) \quad (\text{B.17})$$

and

$$\Delta\omega_c = \lim_{\eta \rightarrow 0^+} \Delta\omega_c(\eta) = F(\omega_c) - G(\omega_c), \quad (\text{B.18})$$

where the quantities $F(\omega_c)$ and $G(\omega_c)$ are

$$F(\omega_c) = P \iint d\vec{n} d\vec{j} \frac{|K(\vec{n}, \vec{j}; \omega_c)|^2}{\hbar\omega_c - \epsilon(\vec{n}, \vec{j})} \quad (\text{B.19a})$$

$$G(\omega_c) = \iint d\vec{n} d\vec{j} \frac{|A(\vec{n}, \vec{j}; \omega_c)|^2}{\hbar\omega_c + \epsilon(\vec{n}, \vec{j})}. \quad (\text{B.19b})$$

(11) S_A contains additional states of the 2p-2h structure only.

Then, the time-dependent Second RPA Eq. (B.6) yields:

$$\begin{bmatrix} A_{11} & B_{11} & A_{12} & B_{12} \\ -B_{11}^* & -A_{11}^* & -B_{12}^* & -A_{12}^* \\ A_{21} & B_{21} & E_{22} & 0 \\ -B_{21}^* & -A_{21}^* & 0 & -E_{22} \end{bmatrix} \begin{bmatrix} \Pi_1(t) \\ P_1(t) \\ \Pi_2(t) \\ P_2(t) \end{bmatrix} = i\hbar \frac{d}{dt} \begin{bmatrix} \Pi_1(t) \\ P_1(t) \\ \Pi_2(t) \\ P_2(t) \end{bmatrix}, \quad (\text{B.20})$$

when one neglects the couplings among the states of the additional subspace. Except from obvious changes in notation, Eq. (B.20) is solved exactly as Eq. (B.8).

APPENDIX C: GENERAL SOLUTION UNDER SLOW VARIATION ASSUMPTION

In this appendix we obtain a solution of the system of Eqs. (II.3.15) which is more general than the simplified solution (which invokes also assumption (II.3.17b)) discussed in the main body of the paper. Here we utilize only the slow variation assumption (II.3.17a)--i.e., that the coupling matrix elements $|K_{n\bar{n}}(\omega_b)|$, $|A_{n\bar{n}}(\omega_a)|$, and the spacing parameter $d_{n\bar{n}}$, depend slowly on the energies, $\epsilon_{n\bar{n}}$, of the additional subspace, S_A .

Equations (II.3.15c) and (II.3.15d) yield immediately:

$$Y_{n\bar{n}}(\omega_v) = \frac{\sum_a \{f_{va} K_{n\bar{n}}(\omega_a) + f_{v\bar{a}} A_{n\bar{n}}^*(\omega_a)\}}{\hbar\omega_v - \epsilon_{n\bar{n}}}, \quad (C.1)$$

$$Z_{n\bar{n}}(\omega_v) = -\frac{\sum_a \{f_{va} A_{n\bar{n}}(\omega_a) + f_{v\bar{a}} K_{n\bar{n}}^*(\omega_a)\}}{\hbar\omega_v + \epsilon_{n\bar{n}}}. \quad (C.2)$$

Direct substitution of Eqs. (C.1) and (C.2) in Eqs. (II.3.15a) and (II.3.15b) yields the following homogeneous system of equations:

$$\begin{aligned} (\hbar\omega_v - \hbar\omega_b) f_{vb} = \sum_a f_{va} \left\{ \frac{K_{n\bar{n}}^*(\omega_b) K_{n\bar{n}}(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}} - \frac{A_{n\bar{n}}^*(\omega_b) A_{n\bar{n}}(\omega_a)}{\hbar\omega_v + \epsilon_{n\bar{n}}} \right\} \\ + \sum_a f_{v\bar{a}} \left\{ \frac{K_{n\bar{n}}^*(\omega_b) A_{n\bar{n}}^*(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}} - \frac{A_{n\bar{n}}^*(\omega_b) K_{n\bar{n}}^*(\omega_a)}{\hbar\omega_v + \epsilon_{n\bar{n}}} \right\} \end{aligned} \quad (C.3a)$$

and

$$\begin{aligned} -(\hbar\omega_v + \hbar\omega_b) f_{v\bar{b}} = \sum_a f_{va} \left\{ \frac{A_{n\bar{n}}(\omega_b) K_{n\bar{n}}(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}} - \frac{K_{n\bar{n}}(\omega_b) A_{n\bar{n}}(\omega_a)}{\hbar\omega_v + \epsilon_{n\bar{n}}} \right\} \\ + \sum_a f_{v\bar{a}} \left\{ \frac{A_{n\bar{n}}(\omega_b) A_{n\bar{n}}^*(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}} - \frac{K_{n\bar{n}}(\omega_b) K_{n\bar{n}}^*(\omega_a)}{\hbar\omega_v + \epsilon_{n\bar{n}}} \right\}. \end{aligned} \quad (C.3b)$$

*Indices (v,u) run over the dimensions of the extended subspace, S .

Indices (a,b) run over the dimensions of the restricted subspace, S_R .

To every physical eigenvector, $\begin{pmatrix} f_v \\ \bar{f}_v \end{pmatrix}$, with positive energy $\hbar\omega_v$ of the system of Eqs. (C.3) corresponds another unphysical eigenvector, $\begin{pmatrix} f_v \\ \bar{f}_v \end{pmatrix} = \begin{pmatrix} \bar{f}_v^* \\ f_v^* \end{pmatrix}$, with negative energy, $\hbar\omega_v = -\hbar\omega_v$. This is seen by complex conjugating Eqs. (C.3a) and (C.3b). From now on we will concentrate on the physical eigenvectors $\begin{pmatrix} f_v \\ \bar{f}_v \end{pmatrix}$ with the positive energies $\hbar\omega_v$.

In the case of a positive energy, $\hbar\omega_v$ --due to the slow variation implied by condition (II.3.17a)--the term $\sum_{n\bar{n}} \frac{K_{n\bar{n}}^*(\omega_b) K_{n\bar{n}}(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}}$, having the difference, $\hbar\omega_v - \epsilon_{n\bar{n}}$, in the denominator is seen to split approximately in two parts, when one repeats the same steps as in Sec. II.3, from Eq. (II.3.20) to Eq. (II.3.24), i.e.,

$$\begin{aligned} \sum_{n\bar{n}} \frac{K_{n\bar{n}}^*(\omega_b) K_{n\bar{n}}(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}} \approx F(K^*(\omega_b), K(\omega_a); \omega_v) \\ + K_{p\bar{k}}^*(\omega_b) K_{p\bar{k}}(\omega_a) \frac{\pi}{d_v} \cot\left(\frac{\pi \tau_v}{d_v}\right). \end{aligned} \quad (C.4)$$

Here, $(p, \bar{k})$ is the state $(n, \bar{j})$ closest in energy to $\hbar\omega_v$ and $d_v = d_{p\bar{k}}$.

Also $\hbar\omega_v = \epsilon_{p\bar{k}} + \tau_v$. In analogy to Eq. (II.3.24), the first term in the right-hand side of Eq. (C.4) is:

$$F(K^*(\omega_b), K(\omega_a); \omega_v) = \sum_{n\bar{n}} \frac{K_{n\bar{n}}^*(\omega_b) K_{n\bar{n}}(\omega_a)}{\epsilon_{p\bar{k}} + \frac{\tau_v}{2} - \epsilon_{n\bar{n}}}. \quad (C.5)$$

The terms, $\sum_{n\bar{n}} \frac{K_{n\bar{n}}^*(\omega_b) A_{n\bar{n}}^*(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}}$, $\sum_{n\bar{n}} \frac{A_{n\bar{n}}(\omega_b) K_{n\bar{n}}(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}}$, and $\sum_{n\bar{n}} \frac{A_{n\bar{n}}(\omega_b) A_{n\bar{n}}^*(\omega_a)}{\hbar\omega_v - \epsilon_{n\bar{n}}}$, split in two parts in a way analogous to Eq. (C.4) with straightforward modification in the notation.

*It is understood that $(\bar{\epsilon}_v)_b = f_{vb}$, $(\bar{f}_v)_b = f_{v\bar{b}}$.

In analogy with Eq. (II.3.25c) we define

$$G(K(\omega_b), \Lambda(\omega_a); \omega_v) = \sum_{n_j} \frac{K_{n_j}(\omega_b) \Lambda_{n_j}(\omega_a)}{K_{n_j} \omega_v + \epsilon_{n_j}}. \quad (C.6)$$

It is straightforward to extend the notation of Eq. (C.6) to the rest of the terms in Eq. (C.3) with the sum, $K_{n_j} \omega_v + \epsilon_{n_j}$, in the denominator. Thus, for the case of positive eigenenergies, $K_{n_j} \omega_v$, Eqs. (C.3) yield:

$$\begin{pmatrix} A(\omega_v) & B(\omega_v) \\ -C(\omega_v) & -D(\omega_v) \end{pmatrix} \begin{pmatrix} f_v \\ \bar{f}_v \end{pmatrix} + \frac{\pi}{d_v} \cot \left(\frac{\pi \tau_v}{d_v} \right) \Xi(\omega_v) \begin{pmatrix} K_{pk}^* \\ -\Lambda_{pk}^* \end{pmatrix} = K_{pk} \begin{pmatrix} f_v \\ \bar{f}_v \end{pmatrix}, \quad (C.7)$$

where it is understood that $(K_{pk}^*)_b = K_{pk}^*(\omega_b)$ and $(\Lambda_{pk}^*)_b = \Lambda_{pk}^*(\omega_b)$. The submatrices, $A(\omega_v)$, $B(\omega_v)$, $C(\omega_v)$, $D(\omega_v)$, have matrix elements:

$$\begin{aligned} A_{ba}(\omega_v) &= K_{ba} \delta_{ba} + F(K^*(\omega_b), K(\omega_a); \omega_v) - G(\Lambda^*(\omega_b), \Lambda(\omega_a); \omega_v) \\ B_{ba}(\omega_v) &= F(K^*(\omega_b), \Lambda^*(\omega_a); \omega_v) - G(\Lambda^*(\omega_b), K^*(\omega_a); \omega_v) \\ C_{ba}(\omega_v) &= F(\Lambda(\omega_b), K(\omega_a); \omega_v) - G(K(\omega_b), \Lambda(\omega_a); \omega_v) \\ D_{ba}(\omega_v) &= K_{ba} \delta_{ba} + F(\Lambda(\omega_b), \Lambda^*(\omega_a); \omega_v) - G(K(\omega_b), K^*(\omega_a); \omega_v). \end{aligned} \quad (C.8)$$

The constant $\Xi(\omega_v)$ is:

$$\Xi(\omega_v) = \sum_a \{ f_{va} K_{va}(\omega_a) + f_{va} \Lambda_{va}^*(\omega_a) \} = (f_v, -\bar{f}_v) \begin{pmatrix} K_{pk}^* \\ -\Lambda_{pk}^* \end{pmatrix}. \quad (C.9)$$

Observe that

$$A^\dagger(\omega_v) = A(\omega_v); \quad D^\dagger(\omega_v) = D(\omega_v); \quad B^\dagger(\omega_v) = C(\omega_v). \quad (C.10)$$

Solution of Eq. (C.7)

The matrix,

$$\begin{pmatrix} A(\omega_v) - K \Omega & B(\omega_v) \\ -C(\omega_v) & -(D(\omega_v) + K \Omega) \end{pmatrix}, \quad (C.11)$$

where the submatrix, Ω , has matrix elements

$$\Omega_{ba} = \omega_b \delta_{ba}. \quad (C.12)$$

represents the interaction among the base vectors, $\begin{pmatrix} \Delta_b \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \bar{\Delta}_b \end{pmatrix}$, which span the restricted RPA subspace, S_R , arising from their coupling with the states of the additional subspace, S_A . It is understood that $(\Delta_b)_a = \delta_{ba}$ and $(\bar{\Delta}_b)_a = \delta_{ba}$. The vectors, $\begin{pmatrix} \Delta_b \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \bar{\Delta}_b \end{pmatrix}$, with energies $K\omega_b$ and $K\omega_b = -K\omega_b$, respectively, are solutions of the set of equations (II.3.15a) and (II.3.15b) when the S_R -to- S_A coupling is neglected.

We solve Eq. (C.7) in two steps:

(1) In the first step we diagonalize the matrix,

$$\begin{pmatrix} A(\omega_v) & B(\omega_v) \\ -C(\omega_v) & -D(\omega_v) \end{pmatrix}, \quad (C.13)$$

that is, we consider the effect of the interaction matrix (C.11) upon the base vectors $\begin{pmatrix} \Delta_b \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \bar{\Delta}_b \end{pmatrix}$ which span the restricted RPA subspace, S_R . This effect perturbs the vectors $\begin{pmatrix} \Delta_b \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \bar{\Delta}_b \end{pmatrix}$ with energies $K\omega_b$, $K\omega_b$ respectively, and replaces them with new vectors, $\begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix}$, with energies, $K\omega'_s$, which are solutions of the equation,

$$\begin{pmatrix} A(\omega_v) & B(\omega_v) \\ -C(\omega_v) & -D(\omega_v) \end{pmatrix} \begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix} = K\omega'_s \begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix}. \quad (C.14)$$

The eigenenergies, $K\omega'_s$, of Eq. (C.14) are positive as well as negative. The negative eigenenergies and the corresponding eigenvectors will be denoted by $K\omega'_s$ and $\begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix}$, respectively, whenever the need arises to differentiate them from the positive ones. However, $K\omega'_s \neq -K\omega'_s$, and $\begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix} \neq \begin{pmatrix} \bar{u}_s \\ u_s \end{pmatrix}$, unlike the case of the usual RPA equations⁴⁰ of motion in Appendix A.

The replacement of the energies $K\omega_b$ with the energies $K\omega'_s$ corresponds to the replacement of the collective energy $K\omega_c$ with the shifted energy $K\omega_0 = K\omega_c + F - G$ in Sec. II.3.

* Indices (s,t) run over the dimensions of the restricted subspace, S_R .

The interaction matrix (C.11) often constitutes a small perturbation, so that Eq. (C.14) can be solved by perturbation theory. Since the matrix (C.11) depends on the energy $\hbar\omega_v$, the solutions of (C.14) will also be functions of $\hbar\omega_v$.

Before proceeding to the second step in solving Eq. (C.7) we show that the solutions of Eq. (C.14) can be normalized to obey the orthonormality relations,

$$(u_t^*, -\bar{u}_t^*) \begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix} = \delta_{st} \frac{\omega_s'}{|\omega_s'|} \quad (C.15)$$

or

$$\sum_s \{u_{ta}^* u_{sa} - \bar{u}_{ta}^* \bar{u}_{sa}\} = \delta_{st} \frac{\omega_s'}{|\omega_s'|}$$

To prove (C.15) we consider the equation,

$$\begin{pmatrix} u_t^* & \bar{u}_t^* \end{pmatrix} \begin{pmatrix} A(\omega_v) & B(\omega_v) \\ C(\omega_v) & D(\omega_v) \end{pmatrix} \begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix} = \hbar\omega_s' \begin{pmatrix} u_t^* & -\bar{u}_t^* \end{pmatrix} \begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix} \quad (C.15)$$

obtained by premultiplying (C.14) by the row vector $(u_t^*, -\bar{u}_t^*)$.

Interchanging the indices s and t , taking the adjoint and using Eq. (C.10) we find:

$$(\hbar\omega_t' - \hbar\omega_s') (u_t^*, -\bar{u}_t^*) \begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix} = 0. \quad (C.17)$$

Thus,

$$\sum_s \{u_{ta}^* u_{sa} - \bar{u}_{ta}^* \bar{u}_{sa}\} = 0 \text{ for } s \neq t. \quad (C.18)$$

If now in analogy with usual RPA,⁴⁰ we assume that the matrix,

$$\begin{pmatrix} A(\omega_v) & B(\omega_v) \\ C(\omega_v) & D(\omega_v) \end{pmatrix}$$

is positive definite, Eq. (C.16) suggests the normalization (C.15) for $s = t$.

(2) In the second step of the problem we assume that the vectors $\begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix}$ and the energies $\hbar\omega_s'$ have been obtained. Then we seek a solution of Eq. (C.7) in the form:

$$\begin{pmatrix} f_v \\ \bar{f}_v \end{pmatrix} = \sum_s g_{vs} \begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix} + \sum_{\bar{s}} g_{v\bar{s}} \begin{pmatrix} u_{\bar{s}} \\ \bar{u}_{\bar{s}} \end{pmatrix}. \quad (C.19)$$

In (C.19) we have differentiated between vectors $\begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix}$ with positive energies $\hbar\omega_s'$ and vectors $\begin{pmatrix} u_{\bar{s}} \\ \bar{u}_{\bar{s}} \end{pmatrix}$ with negative energies $\hbar\omega_{\bar{s}}'$.

The unknown quantities to be determined now are the coefficients g_{vs} and $g_{v\bar{s}}$. Substituting Eq. (C.19) into (C.7) and using Eq. (C.14), we get:

$$\sum_s \left\{ (\hbar\omega_v - \hbar\omega_s') g_{vs} \begin{pmatrix} u_s \\ \bar{u}_s \end{pmatrix} \right\} + \sum_{\bar{s}} \left\{ (\hbar\omega_v - \hbar\omega_{\bar{s}}') g_{v\bar{s}} \begin{pmatrix} u_{\bar{s}} \\ \bar{u}_{\bar{s}} \end{pmatrix} \right\} = -\frac{\pi}{d_v} \cot \left(\frac{\pi \tau_v}{d_v} \right) \Xi(\omega_v) \begin{pmatrix} K_{pk}^* \\ -\Lambda_{pk}^* \end{pmatrix}. \quad (C.20)$$

Premultiplication of Eq. (C.20) with the row vector $(u_t^*, -\bar{u}_t^*)$ and use of the orthonormality relations (C.15) yields:

$$g_{vt} = \frac{M_{vt} \Xi(\omega_v)}{\hbar\omega_v - \hbar\omega_t'} \frac{\pi}{d_v} \cot \left(\frac{\pi \tau_v}{d_v} \right) \quad (C.21a)$$

where the matrix elements M_{vt} are:

$$M_{vt} = (u_t^*, \bar{u}_t^*) \begin{pmatrix} K_{pk}^* \\ \Lambda_{pk}^* \end{pmatrix} = \sum_s \left\{ K_{pk}^*(\omega_s) u_{ta}^* + \Lambda_{pk}^*(\omega_s) u_{t\bar{a}}^* \right\}. \quad (C.22a)$$

Premultiplication of Eq. (C.20) with the row vector $(u_t^*, -\bar{u}_t^*)$, and use of the orthonormality relations (C.15) yields:

$$g_{v\bar{t}} = -\frac{M_{v\bar{t}} \Xi(\omega_v)}{\hbar\omega_v - \hbar\omega_{\bar{t}}'} \frac{\pi}{d_v} \cot \left(\frac{\pi \tau_v}{d_v} \right), \quad (C.21b)$$

where the matrix elements $M_{v\bar{t}}$ are:

$$M_{v\bar{t}} = (u_t^*, \bar{u}_t^*) \begin{pmatrix} K_{pk}^* \\ \Lambda_{pk}^* \end{pmatrix} = \sum_s \left\{ K_{pk}^*(\omega_s) u_{ta}^* + \Lambda_{pk}^*(\omega_s) u_{t\bar{a}}^* \right\}. \quad (C.22b)$$

Substitution of (C.19) into (C.9) shows that:

$$\Xi(\omega_v) = \sum_s \varepsilon_{vs} M_{vs}^* + \sum_s g_{vs} M_{vs}^* \quad (C.23)$$

The dispersion relation for the unknown energies $\hbar\omega_v$ results from a combination of (C.23) and (C.21):

$$\frac{\pi}{d_v} \cot\left(\frac{\pi\tau_v}{d_v}\right) \left\{ \sum_s \frac{|M_{vs}|^2}{\hbar\omega_v - \hbar\omega_s^*} - \sum_s \frac{|M_{vs}|^2}{\hbar\omega_v - \hbar\omega_s} \right\} = 1. \quad (C.24)$$

Calculation of the constant $\Xi(\omega_v)$

The constant $\Xi(\omega_v)$ is determined by using the normalization condition $\langle v|v \rangle = 1$, or

$$\sum_b \{ |\varepsilon_{vb}|^2 - |\varepsilon_{vb}|^2 \} + \sum_{m1} \{ |Y_{vm1}(\omega_v)|^2 - |Z_{vm1}(\omega_v)|^2 \} = 1. \quad (C.25)$$

Using the expansion (C.19) and the orthonormality relations (C.15), one finds:

$$\begin{aligned} \sum_b \{ |\varepsilon_{vb}|^2 - |\varepsilon_{vb}|^2 \} &= (\varepsilon_v^*, -\varepsilon_v^*) \begin{pmatrix} \varepsilon_v \\ \varepsilon_v \end{pmatrix} \\ &= \sum_{st} g_{vs}^* g_{vt} (u_s^*, -u_s^*) \begin{pmatrix} u_t \\ u_t \end{pmatrix} + \sum_{s\bar{t}} g_{vs}^* g_{v\bar{t}} (u_s^*, -u_s^*) \begin{pmatrix} u_{\bar{t}} \\ u_{\bar{t}} \end{pmatrix} \\ &= \sum_{st} g_{vs}^* g_{vt} \delta_{st} - \sum_{s\bar{t}} g_{vs}^* g_{v\bar{t}} \delta_{s\bar{t}} \\ &= \sum_s |g_{vs}|^2 - \sum_{\bar{s}} |g_{v\bar{s}}|^2. \end{aligned} \quad (C.26)$$

Thus, the normalization condition (C.25) becomes:

$$\sum_s |g_{vs}|^2 - \sum_{\bar{s}} |g_{v\bar{s}}|^2 + \sum_{m1} \{ |Y_{vm1}(\omega_v)|^2 - |Z_{vm1}(\omega_v)|^2 \} = 1. \quad (C.27)$$

The coefficients $Y_{vm1}(\omega_v)$ given in Eq. (C.1) are rewritten as:

$$Y_{vm1}(\omega_v) = \frac{\Xi_{vm1}(\omega_v)}{\hbar\omega_v - \varepsilon_{m1}}, \quad (C.28)$$

where,*

$$\Xi_{vm1}(\omega_v) = (\varepsilon_v, \bar{\varepsilon}_v) \begin{pmatrix} K_{m1} \\ \Lambda_{m1}^* \end{pmatrix} = \sum_a \{ \varepsilon_{va} K_{m1}(\omega_a) + \varepsilon_{v\bar{a}} \Lambda_{m1}^*(\omega_a) \}. \quad (C.29)$$

The coefficients $Z_{vm1}(\omega_v)$ are given by Eq. (C.2).

Following the same steps as in Sec. II.3--from Eq. (II.3.26) to Eq. (II.3.27)--we find for the second term in the right-hand side of Eq. (C.27):

$$\sum_{m1} \{ |Y_{vm1}(\omega_v)|^2 - |Z_{vm1}(\omega_v)|^2 \} \left(\frac{\pi |\Xi(\omega_v)|}{d_v} \right)^2 \operatorname{cosec}^2\left(\frac{\pi\tau_v}{d_v}\right). \quad (C.30)$$

The coefficients g_{vs} , $g_{v\bar{s}}$ are given by Eqs. (C.21).

Combining the normalization condition (C.27) with Eqs. (C.21) and (C.30), one finds for the constant $\Xi(\omega_v)$:

$$\begin{aligned} \Xi(\omega_v) &= \left\{ \frac{\pi^2}{d_v^2} \cot^2\left(\frac{\pi\tau_v}{d_v}\right) \left[\sum_s \frac{|M_{vs}|^2}{(\hbar\omega_v - \hbar\omega_s^*)^2} - \sum_{\bar{s}} \frac{|M_{v\bar{s}}|^2}{(\hbar\omega_v - \hbar\omega_{\bar{s}})^2} \right] \right. \\ &\quad \left. + \frac{\pi^2}{d_v^2} + \frac{\pi^2}{d_v^2} \cot^2\left(\frac{\pi\tau_v}{d_v}\right) \right\}^{-1/2}, \end{aligned} \quad (C.31)$$

where the quantity $\frac{\pi}{d_v} \cot\left(\frac{\pi\tau_v}{d_v}\right)$ is determined from the dispersion relation (C.24).

Solution of the system of Eqs. (II.4.9)

Apart from obvious changes in notation, the system of Eqs. (II.4.9) is solved exactly as the system of Eqs. (II.3.15).

*Accordingly, for the quantity $\Xi(\omega_v)$ defined in Eq. (C.9) one has

$$\Xi(\omega_v) = \Xi_{pk}(\omega_v).$$

APPENDIX D: SUFFICIENT MATHEMATICAL CONDITIONS FOR WEAK COUPLING SOLUTION

In this appendix we derive the conditions under which the generalized solution of Appendix C reduces to the one used in the main body of the paper. This reduction is carried out in two steps:

(1) In the first step we assume that:

$$\Gamma_{vs} = \frac{\text{def.}}{\hbar} \frac{2\pi}{d_v} \frac{|M_{vs}|^2}{d_v} \ll \frac{|\omega'_s - \omega'_t|}{2} \text{ and } \Gamma_{vs} = \frac{\text{def.}}{\hbar} \frac{2\pi}{d_v} \frac{|M_{vs}|^2}{d_v} \ll \frac{|\omega'_s - \omega'_t|}{2} \quad (D.1)$$

for all indices v and for all indices $(s, \bar{s}, t)$ denoting positive and negative eigenvalues of Eq. (C.14) with $(s \neq t)$.

With condition (D.1) and the slow variation condition (II.3.17a), the dispersion relation (C.24) simplifies approximately to:

$$\left. \begin{aligned} \hbar\omega_v - \hbar\omega'_s &\approx \frac{\pi|M_{vs}|^2}{d_v} \cot\left(\frac{\pi\tau_v}{d_v}\right), \text{ when } |\omega_v - \omega'_s| \leq 10\Gamma_s, \\ \cot\left(\frac{\pi\tau_v}{d_v}\right) &\approx 0 \text{ or } \tau_v = 0, \text{ otherwise.} \end{aligned} \right\} \quad (D.2)$$

where $\Gamma_s = \Gamma_{vs}(\hbar\omega_v \approx \hbar\omega'_s)$.

With the simplified dispersion relation (D.2), the constant $\Xi(\omega_v)$ of Eq. (C.31) becomes:

$$\Xi(\omega_v) = \left\{ \begin{aligned} &|M_{vs}| \left\{ 1 + \frac{\pi^2|M_{vs}|^2}{d_v^2} + \frac{(\hbar\omega_v - \hbar\omega'_s)^2}{|M_{vs}|^2} \right\}^{-1/2} \\ &\quad \text{when } |\omega_v - \omega'_s| \leq 10\Gamma_s \\ &0, \text{ otherwise} \end{aligned} \right\} \quad (D.3)$$

With the dispersion relation (D.2) and the constant $\Xi(\omega_v)$ given by (D.3), in conjunction with conditions (D.1), Eq. (C.21) yields for the coefficients g_{vt} and $g_{v\bar{t}}$:

$$|g_{vt}|^2 \approx \left\{ \begin{aligned} &1 - \frac{\pi^2|M_{vt}|^2}{d_v^2} + \frac{(\hbar\omega_v - \hbar\omega'_t)^2}{|M_{vt}|^2} - 1 \\ &\quad \text{when } |\omega_v - \omega'_t| \leq 10\Gamma_t \\ &0, \text{ otherwise} \end{aligned} \right\} \quad (D.4a)$$

$$g_{v\bar{t}} \approx 0 \quad (D.4b)$$

(2) In the second step we assume that the interaction matrix (C.11) represents such a small perturbation that the zeroth order correction in the perturbation expansion is sufficient. Thus, for the physical eigenvectors of Eq. (C.14), we approximately set

$$(c_{sa} \approx \delta_{sa} \delta_{ba}; u_{s\bar{b}} \approx 0), \quad (D.5)$$

where the index a runs over the physical eigenvalues of Eq. (II.3.10).

With solution (D.5) and the simplified coefficients g_{vs} , $g_{v\bar{s}}$ of Eq. (D.4), expansion (C.19) yields:

$$(f_{va} = g_{va} \delta_{ba}; f_{v\bar{b}} = 0). \quad (D.6)$$

When $a=c$ —where the index c denotes the collective RPA state in the restricted subspace, $\mathcal{E}_R$ —the solution specified by (D.6) is identical with solution (II.3.18). Substitution of Eq. (D.6) into Eq. (C.14) yields the zeroth order correction for the energies $\hbar\omega'_a$:

$$\hbar\omega'_a = \hbar\omega_a + F(K^*(\omega_a), K(\omega_a); \omega_v) - G(A^*(\omega_a), A(\omega_a); \omega_v). \quad (D.7)$$

When $a=c$, Eq. (D.7) is identical with (II.3.25b).^{*} With Eq. (D.5) and (D.4) the matrix elements M_{vs} (Eq. (C.22a)) become:

$$M_{vs} = K_{vs}^*(\omega_a) \delta_{sa}. \quad (D.8)$$

^{*} ω'_a and ω_0 denote the same quantity.

Then Eq. (D.4a) yields, when $a=c$:

$$|g_{vc}|^2 = \left\{ 1 + \frac{\pi^2 |K_{pk}(\omega_c)|^2}{d_v^2} + \frac{(\hbar\omega_v - \hbar\omega'_c)}{|K_{pk}(\omega_c)|^2} \right\}^{-1}$$

when $|\omega_v - \omega'_c| \leq 10\Gamma_c$. Equation (D.9) is identical with Eq. (II.3.28).

APPENDIX E: MATRIX ELEMENTS USED IN SECT. IV.2 IN THE CALCULATION OF THE UNDAMPED MEAN FIELD

In this appendix we calculate certain matrix elements used in Sect. IV.2 in the calculation of the undamped mean field, specifically those in Eqs. (IV.2.8), (IV.2.11) and (IV.2.13).

Equation (IV.2.8)

One has

$$\begin{aligned} \langle c^{(n)} | a_a^\dagger a_a | c^{(n)} \rangle &= \frac{1}{n!} S_R \langle 0 | Q_c^n a_a^\dagger a_a (Q_c^\dagger)^n | 0 \rangle_{S_R} \\ &= \frac{1}{n!} S_R \langle 0 | [Q_c^n, a_a^\dagger a_a] (Q_c^\dagger)^n | 0 \rangle_{S_R}. \end{aligned} \quad (E.1)$$

The last step in (E.1) was taken because $Q_c | 0 \rangle_{S_R} = 0$. One has

$$\begin{aligned} S_R \langle 0 | [Q_c^n, a_a^\dagger a_a] (Q_c^\dagger)^n | 0 \rangle_{S_R} \\ = S_R \langle 0 | [Q_c^n, a_a^\dagger a_a] (Q_c^\dagger)^n | 0 \rangle_{S_R} + S_R \langle 0 | a_a^\dagger a_a [Q_c^n, (Q_c^\dagger)^n] | 0 \rangle_{S_R}. \end{aligned} \quad (E.2)$$

For the commutator $[Q_c^n, a_a^\dagger a_a]$ one has

$$\begin{aligned} [Q_c^n, a_a^\dagger a_a] &= \sum_{k=0}^{n-1} Q_c^k [Q_c, a_a^\dagger a_a] Q_c^{n-k-1} \\ &= n [Q_c, a_a^\dagger a_a] Q_c^{n-1}. \end{aligned} \quad (E.3)$$

The last step in (E.3) was taken because the commutator $[Q_c, a_a^\dagger a_a]$ is a c-number in the RPA.

One has

$$[Q_c, (Q_c^\dagger)^n] | 0 \rangle_{S_R} = n! \delta_{SR}. \quad (E.4)$$

Thus,

$$\begin{aligned} \langle c^{(n)} | a_a^\dagger a_a | c^{(n)} \rangle &= \sqrt{n} [Q_c, a_a^\dagger a_a] \langle c^{(n-1)} | c^{(n-1)} \rangle + S_R \langle 0 | a_a^\dagger a_a | 0 \rangle_{S_R} \\ &= S_R \langle 0 | a_a^\dagger a_a | 0 \rangle_{S_R}, \end{aligned} \quad (E.5)$$

where we used the orthogonality property of the n-phonon states.

Since we work in the RPA, the contraction $S_R \langle 0 | a_{\alpha}^{\dagger} a_{\alpha} | 0 \rangle_{S_R}$ is calculated over the Hartree-Fock ground state:

$$S_R \langle 0 | a_{\alpha}^{\dagger} a_{\alpha} | 0 \rangle_{S_R} \approx \langle HF | a_{\alpha}^{\dagger} a_{\alpha} | HF \rangle = \begin{cases} \delta_{\alpha\alpha'}, & \text{when } \alpha=h, \alpha'=h' \\ 0, & \text{otherwise} \end{cases} \quad (E.6)$$

The index h here denotes a hole.

Equation (IV.2.11)

Equation (IV.2.11) gives the pp and hh elements of the one-body density matrix when the wave packet (IV.2.9) is used as a wave function. One has

$$\begin{aligned} \rho_{\alpha\alpha'} &= \langle \psi(t) | a_{\alpha}^{\dagger} a_{\alpha'} | \psi(t) \rangle \\ &= e^{-\epsilon^2} \sum_{s=0}^{\infty} \frac{\epsilon^{s+r}}{\sqrt{s!r!}} e^{i(s-r)(\omega_c t + \phi)} \langle c^{(s)} | a_{\alpha}^{\dagger} a_{\alpha'} | c^{(r)} \rangle. \end{aligned} \quad (E.7)$$

Repeating the same steps as from (E.2) to (E.4), one finds:

$$\langle c^{(s)} | a_{\alpha}^{\dagger} a_{\alpha'} | c^{(r)} \rangle = \sqrt{s} [Q_c, a_{\alpha}^{\dagger} a_{\alpha'}] \langle c^{(s-1)} | c^{(r)} \rangle + S_R \langle 0 | a_{\alpha}^{\dagger} a_{\alpha'} | 0 \rangle_{S_R} \delta_{sr}. \quad (E.8)$$

Due to the RPA boson commutation relations (A.29a) one has:

$$[Q_c, a_{\alpha}^{\dagger} a_{\alpha'}] = 0 \begin{cases} \text{when } \alpha'=n, \alpha=m \text{ (both indices particles)} \\ \text{when } \alpha'=j, \alpha=i \text{ (both indices holes)} \end{cases} \quad (E.9)$$

Thus,

$$\begin{aligned} \langle c^{(s)} | a_{n m}^{\dagger} a_{n m} | c^{(r)} \rangle &= S_R \langle 0 | a_{n m}^{\dagger} a_{n m} | 0 \rangle_{S_R} \delta_{sr} \\ &\approx \langle HF | a_{n m}^{\dagger} a_{n m} | HF \rangle \delta_{sr} = 0 \end{aligned} \quad (E.10)$$

and

$$\begin{aligned} \langle c^{(s)} | a_{j i}^{\dagger} a_{j i} | c^{(r)} \rangle &= S_R \langle 0 | a_{j i}^{\dagger} a_{j i} | 0 \rangle_{S_R} \delta_{sr} \\ &\approx \langle HF | a_{j i}^{\dagger} a_{j i} | HF \rangle \delta_{sr} = \delta_{ij} \delta_{sr}. \end{aligned} \quad (E.11)$$

Substituting Eqs. (E.10) and (E.11) in Eq. (E.7) one gets

$$\rho_{nn} = 0 \quad (E.12)$$

and

$$\rho_{ij} = e^{-\epsilon^2} \left(\sum_{r=0}^{\infty} \frac{\epsilon^{2r}}{r!} \right) \delta_{ij} = \delta_{ij}. \quad (E.13)$$

Equation (IV.2.13)

One has using the packet (IV.2.9):

$$\langle \psi(t) | Q_a^{\dagger} | \psi(t) \rangle = e^{-\epsilon^2} \sum_{s=0, r=0}^{\infty} \frac{\epsilon^{s+r}}{\sqrt{s!r!}} e^{i(s-r)(\omega_c t + \phi)} \langle c^{(s)} | Q_a^{\dagger} | c^{(r)} \rangle. \quad (E.14)$$

But,

$$\begin{aligned} \langle c^{(s)} | Q_a^{\dagger} | c^{(r)} \rangle &= \frac{1}{\sqrt{s!r!}} S_R \langle 0 | [Q_c^{\dagger}, Q_a^{\dagger} (Q_c^{\dagger})^r] | 0 \rangle_{S_R} \\ &= \sqrt{s} \delta_{ac} \delta_{s,r+1}. \end{aligned} \quad (E.15)$$

Thus,

$$\begin{aligned} \langle \psi(t) | Q_a^{\dagger} | \psi(t) \rangle &= e^{-\epsilon^2} \left(\sum_{r=0}^{\infty} \frac{\epsilon^{2r+1}}{r!} \right) e^{i(\omega_c t + \phi)} \delta_{ac} \\ &= \epsilon e^{i(\omega_c t + \phi)} \delta_{ac}. \end{aligned} \quad (E.16)$$

APPENDIX F: MATRIX ELEMENTS USED IN THE CALCULATION OF

COLLECTIVE VARIABLES AND COLLECTIVE ENERGY IN SECTS. IV.3 AND IV.4

In this appendix we calculate the matrix elements utilized in determining the collective variables and collective energy in Sects. IV.3 and IV.4, specifically those in Eqs. (IV.3.22), (IV.4.11), (IV.4.25) and (IV.4.27).

Equation (IV.3.22)

Using the packet (IV.3.8), one has

$$\begin{aligned}
 \langle \Psi(t) | \sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger} | \Psi(t) \rangle &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+n}}{m!n!} S \langle 0 | \left(\sum_{\kappa} f_{\kappa c} e^{i(\omega_{\kappa} t + \phi)} \hat{Q}_{\kappa} \right)^m \left(\sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger} \right) \left(\sum_{\lambda} f_{\lambda c}^* e^{-i(\omega_{\lambda} t + \phi)} \hat{Q}_{\lambda}^{\dagger} \right)^n | 0 \rangle_S \\
 &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+n}}{m!n!} \sum_{\kappa_1, \dots, \kappa_m, \nu, \lambda_1, \dots, \lambda_n} f_{\kappa_1, c} \dots f_{\kappa_m, c} f_{\nu c}^* f_{\lambda_n, c}^* \dots f_{\lambda_1, c}^* \\
 &\quad \times e^{i(\omega_{\kappa_1} t + \dots + \omega_{\kappa_m} t - \omega_{\lambda_n} t - \dots - \omega_{\lambda_1} t)} e^{-i(m-n)\phi} S \langle 0 | \hat{Q}_{\kappa_1} \dots \hat{Q}_{\kappa_m} \hat{Q}_{\nu}^{\dagger} \hat{Q}_{\lambda_n}^{\dagger} \dots \hat{Q}_{\lambda_1}^{\dagger} | 0 \rangle_S. \quad (F.1)
 \end{aligned}$$

The expectation value of the product of boson operators in (F.1) is different from zero only when the number of lowering and raising operators is the same, that is, only when $m = n+1$. Then using Eq. (IV.3.10), one gets:

$$\begin{aligned}
 \langle \Psi(t) | \sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger} | \Psi(t) \rangle &= e^{-\epsilon^2} \sum_{n=0}^{\infty} \frac{\epsilon^{2n+1} e^{i\phi}}{n!} \\
 &\quad \times \sum_{\kappa_1, \dots, \kappa_n, \nu} |f_{\kappa_1, c}|^2 \dots |f_{\kappa_n, c}|^2 (|f_{\nu c}|^2 e^{i\omega_{\nu} t}). \quad (F.2)
 \end{aligned}$$

Remembering that

$$\sum_{\kappa} |f_{\kappa c}|^2 = \langle c | c \rangle = 1, \quad (F.3)$$

and that [see Eq. (IV.3.12)]

$$\sum_{\nu} |f_{\nu c}|^2 e^{-i\omega_{\nu} t} = F(t), \quad (F.4)$$

Eq. (F.2) yields

$$\begin{aligned}
 \langle \Psi(t) | \sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger} | \Psi(t) \rangle &= e^{-\epsilon^2} \epsilon e^{i\phi} F^*(t) \sum_{n=0}^{\infty} \frac{\epsilon^{2n}}{n!} \\
 &= \epsilon e^{i\phi} F^*(t). \quad (F.5)
 \end{aligned}$$

Equation (IV.4.11)

Using the packet (IV.3.8), one has

$$\begin{aligned}
 \langle \Psi(t) | (1 \sum_{\nu} \omega_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger}) | \Psi(t) \rangle &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{i\epsilon^{m+n}}{m!n!} S \langle 0 | \left(\sum_{\kappa} f_{\kappa c} e^{i(\omega_{\kappa} t + \phi)} \hat{Q}_{\kappa} \right)^m \left(\sum_{\nu} \omega_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger} \right) \left(\sum_{\lambda} f_{\lambda c}^* e^{-i(\omega_{\lambda} t + \phi)} \hat{Q}_{\lambda}^{\dagger} \right)^n | 0 \rangle_S \\
 &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{i\epsilon^{m+n}}{m!n!} \sum_{\kappa_1, \dots, \kappa_m, \nu, \lambda_1, \dots, \lambda_n} \omega_{\nu} f_{\kappa_1, c} \dots f_{\kappa_m, c} f_{\nu c}^* f_{\lambda_n, c}^* \dots f_{\lambda_1, c}^* \\
 &\quad \times e^{i(\omega_{\kappa_1} t + \dots + \omega_{\kappa_m} t - \omega_{\lambda_n} t - \dots - \omega_{\lambda_1} t)} e^{i(m-n)\phi} S \langle 0 | \hat{Q}_{\kappa_1} \dots \hat{Q}_{\kappa_m} \hat{Q}_{\nu}^{\dagger} \hat{Q}_{\lambda_n}^{\dagger} \dots \hat{Q}_{\lambda_1}^{\dagger} | 0 \rangle_S. \quad (F.6)
 \end{aligned}$$

The expectation value of the product of boson operators in (F.6) is different from zero only when the number of lowering operators equals the number of raising operators, that is, only when $m = n+1$. Then using Eq. (IV.3.10) one gets:

$$\begin{aligned}
 \langle \Psi(t) | (1 \sum_{\nu} \omega_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger}) | \Psi(t) \rangle &= e^{-\epsilon^2} \sum_{n=0}^{\infty} \frac{\epsilon^{2n+1} e^{i\phi}}{n!} \sum_{\kappa_1, \dots, \kappa_n, \nu} |f_{\kappa_1, c}|^2 \dots |f_{\kappa_n, c}|^2 (1 \omega_{\nu} |f_{\nu c}|^2 e^{i\omega_{\nu} t}). \quad (F.7)
 \end{aligned}$$

Taking into account Eqs. (F.3) and (F.4), Eq. (F.7) yields

$$\begin{aligned}
 \langle \Psi(t) | (1 \sum_{\nu} \omega_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger}) | \Psi(t) \rangle &= e^{-\epsilon^2} \epsilon e^{i\phi} \frac{dF^*(t)}{dt} \sum_{n=0}^{\infty} \frac{\epsilon^{2n}}{n!} \\
 &= \epsilon e^{i\phi} \frac{dF^*(t)}{dt}. \quad (F.8)
 \end{aligned}$$

Equation (IV.4.25)

(1) We show that

$$\langle \Psi(t) | \hat{\sigma}_c^2 | \Psi(t) \rangle = \langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle^2 + S \langle \hat{\sigma}_c^2 | 0 \rangle_S, \quad (F.9)$$

that is, the expectation value of the square of $\hat{\sigma}_c$ with the packet $|\Psi(t)\rangle$ is expressed through the square of the expectation value plus the zero point fluctuations. Using the packet (IV.3.8) and the definition (IV.4.1)

for the position coordinate $\hat{\sigma}_c$, one has

$$\begin{aligned} \langle \Psi(t) | \hat{\sigma}_c^2 | \Psi(t) \rangle &= \langle \Psi(t) | (\hat{Q}_c^+ + \hat{Q}_c)^2 | \Psi(t) \rangle \\ &= \langle \Psi(t) | \left\{ \left(\sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^+ \right)^2 + \left(\sum_{\nu} f_{\nu c} \hat{Q}_{\nu} \right)^2 + \sum_{\mu\nu} f_{\mu c}^* f_{\nu c} (\hat{Q}_{\mu}^+ \hat{Q}_{\nu} + \hat{Q}_{\nu} \hat{Q}_{\mu}^+) \right\} | \Psi(t) \rangle. \end{aligned} \quad (F.10)$$

The first term in the right-hand-side of (F.10) is:

$$\begin{aligned} \langle \Psi(t) | \left(\sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^+ \right)^2 | \Psi(t) \rangle &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+n}}{m!n!} S \langle 0 | \left(\sum_{\kappa} f_{\kappa c} e^{i(\omega_{\kappa}t+\phi)} \hat{Q}_{\kappa} \right)^m \left(\sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^+ \right)^2 \left(\sum_{\lambda} f_{\lambda c}^* e^{-i(\omega_{\lambda}t+\phi)} \hat{Q}_{\lambda}^+ \right)^n | 0 \rangle_S \\ &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+n}}{m!n!} \sum_{\substack{\kappa_1, \dots, \kappa_m, \\ \nu, \mu, \\ \lambda_1, \dots, \lambda_n}} f_{\kappa_1 c} \dots f_{\kappa_m c} f_{\nu c}^* f_{\mu c}^* f_{\lambda_1 c} \dots f_{\lambda_n c} \\ &\quad \times e^{i(\omega_{\kappa_1} + \dots + \omega_{\kappa_m} - \omega_{\lambda_1} - \dots - \omega_{\lambda_n})t} e^{i(m-n)\phi} S \langle 0 | \hat{Q}_{\kappa_1} \dots \hat{Q}_{\kappa_m} \hat{Q}_{\nu}^+ \hat{Q}_{\mu}^+ \hat{Q}_{\lambda_1}^+ \dots \hat{Q}_{\lambda_n}^+ | 0 \rangle_S. \end{aligned} \quad (F.11)$$

The expectation value of the product of boson operators in (F.11) is different from zero only when $m = n+2$. Then using Eq. (IV.3.10), one gets:

$$\begin{aligned} \langle \Psi(t) | \left(\sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^+ \right)^2 | \Psi(t) \rangle &= e^{-\epsilon^2} \sum_{n=0}^{\infty} \frac{\epsilon^{2n+2} e^{2i\phi}}{n!} \sum_{\kappa_1, \dots, \kappa_{n+2}, \nu, \mu} |f_{\kappa_1 c}|^2 \dots |f_{\kappa_{n+2} c}|^2 |f_{\nu c}|^2 |f_{\mu c}|^2 e^{i(\omega_{\nu} + \omega_{\mu})t}. \end{aligned} \quad (F.12)$$

Taking into account Eq. (F.3) and (F.4), Eq. (F.12) yields:

$$\begin{aligned} \langle \Psi(t) | \left(\sum_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^+ \right)^2 | \Psi(t) \rangle &= e^{-\epsilon^2} \sum_{n=0}^{\infty} \frac{\epsilon^{2n+2} e^{2i\phi}}{n!} \left(\sum_{\nu} |f_{\nu c}|^2 e^{i\omega_{\nu}t} \right)^2 \\ &= e^{-\epsilon^2} \epsilon^2 e^{2i\phi} F^{*2}(t) \sum_{n=0}^{\infty} \frac{\epsilon^{2n}}{n!} \\ &= \epsilon^2 e^{2i\phi} F^{*2}(t). \end{aligned} \quad (F.13)$$

We now calculate the term $\langle \Psi(t) | \left(\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} (\hat{Q}_{\mu}^+ \hat{Q}_{\nu} + \hat{Q}_{\nu} \hat{Q}_{\mu}^+) \right) | \Psi(t) \rangle$ in (F.10). One has:

$$\begin{aligned} \langle \Psi(t) | \left(\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} (\hat{Q}_{\mu}^+ \hat{Q}_{\nu} + \hat{Q}_{\nu} \hat{Q}_{\mu}^+) \right) | \Psi(t) \rangle &= \langle \Psi(t) | \left(\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} (2\hat{Q}_{\mu}^+ \hat{Q}_{\nu} + \delta_{\mu\nu}) \right) | \Psi(t) \rangle \\ &= 2 \langle \Psi(t) | \left(\sum_{\mu \neq \nu} f_{\mu c}^* f_{\nu c} \hat{Q}_{\mu}^+ \hat{Q}_{\nu} \right) | \Psi(t) \rangle + \sum_{\nu} |f_{\nu c}|^2. \end{aligned} \quad (F.14)$$

In Eq. (F.14) we used the boson commutation relation, $[\hat{Q}_{\nu}, \hat{Q}_{\mu}^+] = \delta_{\mu\nu}$.

Using the fact that $\hat{Q}_{\nu}|0\rangle_S = 0$, one can see that the zero point fluctuations of $\hat{\sigma}_c$ are given by:

$$\begin{aligned} S \langle 0 | \hat{\sigma}_c^2 | 0 \rangle_S &= S \langle 0 | \left(\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} \delta_{\mu\nu} \right) | 0 \rangle_S \\ &= \sum_{\nu} |f_{\nu c}|^2 = 1. \end{aligned} \quad (F.15)$$

One has:

$$\begin{aligned} \langle \Psi(t) | \left(\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} \hat{Q}_{\mu}^+ \hat{Q}_{\nu} \right) | \Psi(t) \rangle &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+n}}{m!n!} S \langle 0 | \left(\sum_{\kappa} f_{\kappa c} e^{i(\omega_{\kappa}t+\phi)} \hat{Q}_{\kappa} \right)^m \left(\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} \hat{Q}_{\mu}^+ \hat{Q}_{\nu} \right) \left(\sum_{\lambda} f_{\lambda c}^* e^{-i(\omega_{\lambda}t+\phi)} \hat{Q}_{\lambda}^+ \right)^n | 0 \rangle_S \\ &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+n}}{m!n!} \sum_{\substack{\kappa_1, \dots, \kappa_m, \\ \mu, \nu, \\ \lambda_1, \dots, \lambda_n}} f_{\kappa_1 c} \dots f_{\kappa_m c} f_{\mu c}^* f_{\nu c} f_{\lambda_1 c}^* \dots f_{\lambda_n c}^* \\ &\quad \times e^{i(\omega_{\kappa_1} + \dots + \omega_{\kappa_m} - \omega_{\lambda_1} - \dots - \omega_{\lambda_n})t} e^{i(m-n)\phi} S \langle 0 | \hat{Q}_{\kappa_1} \dots \hat{Q}_{\kappa_m} \hat{Q}_{\mu}^+ \hat{Q}_{\nu} \hat{Q}_{\lambda_1}^+ \dots \hat{Q}_{\lambda_n}^+ | 0 \rangle_S. \end{aligned} \quad (F.16)$$

The expectation value of the product of boson operators in (F.16) is different from zero only when the number of lowering operators equals the number of raising operators, that is only when $m=n$. Then using Eq.

(III.3.2) one gets

$$\begin{aligned} & \langle \Psi(t) | (\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} \hat{Q}_{\mu}^{\dagger} \hat{Q}_{\nu}) | \Psi(t) \rangle \\ &= e^{-\epsilon^2} \sum_{n=1}^{\infty} \frac{\epsilon^{2n}}{(n-1)!} \sum_{\kappa_1, \dots, \kappa(n-1), \mu, \nu} |f_{\kappa_1, c}|^2 \dots |f_{\kappa(n-1), c}|^2 |f_{\mu c}|^2 |f_{\nu c}|^2 e^{i(\omega_{\mu} - \omega_{\nu})t}. \end{aligned} \quad (F.17)$$

Taking into account (F.3) and (F.4), Eq. (F.17) yields

$$\begin{aligned} & \langle \Psi(t) | (\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} \hat{Q}_{\mu}^{\dagger} \hat{Q}_{\nu}) | \Psi(t) \rangle \\ &= e^{-\epsilon^2} \epsilon^2 (\sum_{\mu} |f_{\mu c}|^2 e^{i\omega_{\mu}t}) (\sum_{\nu} |f_{\nu c}|^2 e^{-i\omega_{\nu}t}) \sum_{n=1}^{\infty} \frac{\epsilon^{2(n-1)}}{(n-1)!} = \epsilon^2 |\Psi(t)|^2. \end{aligned} \quad (F.18)$$

Combining Eqs. (F.10), (F.13), (F.14), (F.15) and (F.18), one gets:

$$\langle \Psi(t) | \hat{\sigma}_c^2 | \Psi(t) \rangle = \epsilon^2 [e^{i\phi} F^*(t) + e^{-i\phi} F(t)]^2 + S \langle 0 | \hat{\sigma}_c^2 | 0 \rangle_S. \quad (F.19)$$

Taking into consideration Eq. (IV.4.9) for the expectation value of $\hat{\sigma}_c$,

Eq. (F.19) yields:

$$\langle \Psi(t) | \hat{\sigma}_c^2 | \Psi(t) \rangle = \langle \Psi(t) | \hat{\sigma}_c | \Psi(t) \rangle^2 + S \langle 0 | \hat{\sigma}_c^2 | 0 \rangle_S. \quad (F.20)$$

(ii) We show that

$$\langle \Psi(t) | \hat{p}_c^2 | \Psi(t) \rangle = \langle \Psi(t) | \hat{p}_c | \Psi(t) \rangle^2 + S \langle 0 | \hat{p}_c^2 | 0 \rangle_S, \quad (F.21)$$

that is the expectation value of the square of the momentum operator with the packet $|\Psi(t)\rangle$ is equal to the square of the expectation value plus the zero point fluctuations. Using the packet (IV.3.8) and the definition for the collective momentum operator $\hat{p}_c$ (Eqs. (IV.4.3) and (IV.4.5), one has:

$$\begin{aligned} \langle \Psi(t) | \hat{p}_c^2 | \Psi(t) \rangle &= M_c^2 \langle \Psi(t) | \hat{\sigma}_c^2 | \Psi(t) \rangle \\ &= -M_c^2 \langle \Psi(t) | \{ (\sum_{\mu\nu} \omega_{\mu} f_{\nu c}^* \hat{Q}_{\mu}^{\dagger})^2 + (\sum_{\mu\nu} \omega_{\nu} f_{\mu c} \hat{Q}_{\nu})^2 \\ &\quad - \sum_{\mu\nu} \omega_{\mu} \omega_{\nu} f_{\mu c}^* f_{\nu c} (2\hat{Q}_{\mu}^{\dagger} \hat{Q}_{\nu} + \delta_{\mu\nu}) \} | 0 \rangle_S \}. \end{aligned} \quad (F.22)$$

The first term in the right-hand-side of (F.22) is:

$$\begin{aligned} & \langle \Psi(t) | (\sum_{\mu\nu} \omega_{\mu} f_{\nu c}^* \hat{Q}_{\mu}^{\dagger})^2 | \Psi(t) \rangle \\ &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+2n}}{m!n!} S \langle 0 | (\sum_{\kappa} f_{\kappa c} e^{i(\omega_{\kappa}t+\phi)} \hat{Q}_{\kappa})^m (\sum_{\nu} \omega_{\nu} f_{\nu c}^* \hat{Q}_{\nu}^{\dagger})^2 (\sum_{\lambda} f_{\lambda c}^* e^{-i(\omega_{\lambda}t+\phi)} \hat{Q}_{\lambda}^{\dagger})^n | 0 \rangle_S \\ &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+2n}}{m!n!} \sum_{\kappa_1, \dots, \kappa_m, \mu, \nu, \lambda_1, \dots, \lambda_n} \omega_{\mu} \omega_{\nu} f_{\kappa_1, c} \dots f_{\kappa_m, c} f_{\mu c}^* f_{\nu c}^* f_{\lambda_n, c} \dots f_{\lambda_1, c} \\ &\quad \times e^{i(\omega_{\kappa_1} + \dots + \omega_{\kappa_m} - \omega_{\lambda_n} - \dots - \omega_{\lambda_1})t} e^{i(m-n)\phi} S \langle 0 | \hat{Q}_{\kappa_1} \dots \hat{Q}_{\kappa_m} \hat{Q}_{\mu}^{\dagger} \hat{Q}_{\nu}^{\dagger} \hat{Q}_{\lambda_n}^{\dagger} \dots \hat{Q}_{\lambda_1}^{\dagger} | 0 \rangle_S. \end{aligned} \quad (F.23)$$

The expectation value of the product of boson operators in (F.23) is different from zero only when $m=n+2$. Then using Eq. (IV.3.10) one gets:

$$\begin{aligned} & \langle \Psi(t) | (\sum_{\mu\nu} \omega_{\mu} f_{\nu c}^* \hat{Q}_{\mu}^{\dagger})^2 | \Psi(t) \rangle \\ &= e^{-\epsilon^2} \sum_{n=0}^{\infty} \frac{\epsilon^{2n+2} e^{2i\phi}}{n!} \sum_{\kappa_1, \dots, \kappa_n, \mu, \nu} \omega_{\mu} \omega_{\nu} |f_{\kappa_1, c}|^2 \dots |f_{\kappa_n, c}|^2 |f_{\mu c}|^2 |f_{\nu c}|^2 e^{i(\omega_{\mu} + \omega_{\nu})t}. \end{aligned} \quad (F.24)$$

Taking into account Eqs. (F.3) and (F.4), Eq. (F.24) yields:

$$\begin{aligned} \langle \Psi(t) | (\sum_{\mu\nu} \omega_{\mu} f_{\nu c}^* \hat{Q}_{\mu}^{\dagger})^2 | \Psi(t) \rangle &= -e^{-\epsilon^2} \sum_{n=0}^{\infty} \frac{\epsilon^{2n+2} e^{2i\phi}}{n!} (i \sum_{\nu} \omega_{\nu} |f_{\nu c}|^2 e^{i\omega_{\nu}t})^2 \\ &= -e^{-\epsilon^2} \epsilon^2 e^{2i\phi} \left(\frac{dF^*(t)}{dt} \right)^2 \sum_{n=0}^{\infty} \frac{\epsilon^{2n}}{n!} \\ &= -\epsilon^2 e^{2i\phi} \left(\frac{dF^*(t)}{dt} \right)^2. \end{aligned} \quad (F.25)$$

We now calculate $\langle \Psi(t) | (\sum_{\mu\nu} \omega_\mu \omega_\nu f_{\mu c}^* f_{\nu c} \hat{Q}_\mu^\dagger \hat{Q}_\nu) | \Psi(t) \rangle$. One has

$$\begin{aligned}
 & \langle \Psi(t) | (\sum_{\mu\nu} \omega_\mu \omega_\nu f_{\mu c}^* f_{\nu c} \hat{Q}_\mu^\dagger \hat{Q}_\nu) | \Psi(t) \rangle \\
 &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+n}}{m!n!} S \langle 0 | (\sum_{\kappa} f_{\kappa c} e^{i(\omega_\kappa t + \phi)} \hat{Q}_\kappa)^m (\sum_{\mu\nu} \omega_\mu \omega_\nu f_{\mu c}^* f_{\nu c} \hat{Q}_\mu^\dagger \hat{Q}_\nu) (\sum_{\lambda} f_{\lambda c}^* e^{-i(\omega_\lambda t + \phi)} \hat{Q}_\lambda^\dagger)^n | 0 \rangle_S \\
 &= e^{-\epsilon^2} \sum_{m=0}^{\infty} \frac{\epsilon^{m+n}}{m!n!} \sum_{\substack{\kappa_1, \dots, \kappa_m, \\ \mu, \nu \\ \lambda_1, \dots, \lambda_n}} \omega_\mu \omega_\nu f_{\kappa_1 c} \dots f_{\kappa_m c} f_{\mu c}^* f_{\nu c} f_{\lambda_n c}^* \dots f_{\lambda_1 c}^* \\
 & \quad \times e^{i(\omega_{\kappa_1} + \dots + \omega_{\kappa_m} - \omega_{\lambda_n} - \dots - \omega_{\lambda_1})t} e^{i(m-n)\phi} S \langle 0 | \hat{Q}_{\kappa_1} \dots \hat{Q}_{\kappa_m} \hat{Q}_{\mu}^\dagger \hat{Q}_{\nu}^\dagger \dots \hat{Q}_{\lambda_1}^\dagger | 0 \rangle_S.
 \end{aligned}
 \tag{F.26}$$

The expectation value of the product of boson operators in (F.26) is different from zero only when $m=n$. Then using Eq. (III.3.2) one gets

$$\begin{aligned}
 & \langle \Psi(t) | (\sum_{\mu\nu} \omega_\mu \omega_\nu f_{\mu c}^* f_{\nu c} \hat{Q}_\mu^\dagger \hat{Q}_\nu) | \Psi(t) \rangle \\
 &= e^{-\epsilon^2} \sum_{n=1}^{\infty} \frac{\epsilon^{2n}}{(n-1)!} \sum_{\substack{\kappa_1, \dots, \kappa_{(n-1)}, \\ \mu, \nu}} \omega_\mu \omega_\nu |f_{\kappa_1 c}|^2 \dots |f_{\kappa_{(n-1) c}}|^2 |f_{\mu c}|^2 |f_{\nu c}|^2 e^{i(\omega_\mu - \omega_\nu)t}.
 \end{aligned}
 \tag{F.27}$$

Taking into account Eqs. (F.3) and (F.4), Eq. (F.27) yields:

$$\begin{aligned}
 & \langle \Psi(t) | (\sum_{\mu\nu} \omega_\mu \omega_\nu f_{\mu c}^* f_{\nu c} \hat{Q}_\mu^\dagger \hat{Q}_\nu) | \Psi(t) \rangle \\
 &= e^{-\epsilon^2} \epsilon^2 (i \sum_{\mu} \omega_\mu |f_{\mu c}|^2 e^{i\omega_\mu t}) (-i \sum_{\nu} \omega_\nu |f_{\nu c}|^2 e^{-i\omega_\nu t}) \sum_{n=1}^{\infty} \frac{\epsilon^{2(n-1)}}{(n-1)!} \\
 &= \epsilon^2 \frac{dF^*(t)}{dt} \frac{dF(t)}{dt}.
 \end{aligned}
 \tag{F.28}$$

Using the fact that $\hat{Q}_\nu |0\rangle_S = 0$, one can see that the zero point fluctuations of $\hat{p}_c$ are:

$$\begin{aligned}
 {}_S \langle 0 | \hat{p}_c^2 | 0 \rangle_S &= M_c^2 S \langle 0 | (\sum_{\mu\nu} \omega_\mu \omega_\nu f_{\mu c}^* f_{\nu c} \delta_{\mu\nu}) | 0 \rangle_S \\
 &= M_c^2 S \langle 0 | (\sum_{\nu} \omega_\nu^2 |f_{\nu c}|^2) | 0 \rangle_S.
 \end{aligned}
 \tag{F.29}$$

With Eqs. (F.25), (F.28) and (F.29), Eq. (F.22) yields:

$$\langle \Psi(t) | \hat{p}_c^2 | \Psi(t) \rangle = M_c^2 \epsilon^2 [e^{1\phi} \frac{dF^*(t)}{dt} + e^{-1\phi} \frac{dF(t)}{dt}]^2 + S \langle 0 | \hat{p}_c^2 | 0 \rangle_S.
 \tag{F.30}$$

Taking into consideration Eqs. (IV.4.12) and (IV.4.9) for the expectation value of $\hat{p}_c$, one finds:

$$\langle \Psi(t) | \hat{p}_c^2 | \Psi(t) \rangle = \langle \Psi(t) | \hat{p}_c | \Psi(t) \rangle^2 + S \langle 0 | \hat{p}_c^2 | 0 \rangle_S.
 \tag{F.31}$$

Equation (IV.4.27)

Using the expansion (IV.3.4b), one has:

$$\langle \Psi(t) | \hat{Q}_c^\dagger \hat{Q}_c | \Psi(t) \rangle = \langle \Psi(t) | (\sum_{\mu\nu} f_{\mu c}^* f_{\nu c} \hat{Q}_\mu^\dagger \hat{Q}_\nu) | \Psi(t) \rangle.
 \tag{F.32}$$

Equation (F.32) is identical to Eq. (F.18).

APPENDIX G: CALCULATION OF THE PROPORTIONALITY COEFFICIENT, γ_{limit}
USING EQ. (VI.2.14)

In this appendix we calculate the proportionality coefficient γ given by Eq. (VI.2.14) in the large radius/infinite square well potential limit. One has from Eq. (VI.2.14)

$$\gamma = \frac{\pi}{\omega_0} \iint d\vec{m} d\vec{i} D^2(\vec{m}, \vec{i}) \delta(\mathcal{H}_{\omega_c} - \epsilon(\vec{m}, \vec{i})). \quad (G.1)$$

With assumption (VI.2.3c) that the shell model potential is a spherical square well, the bound single particle wave functions are:

$$\psi(r, \theta, \phi) = C_L j_L(kr) Y_{LM}(\theta, \phi), \quad r \leq R, \quad (G.2)$$

where $j_L(kr)$ are the spherical Bessel functions, C_L is the normalization constant, and R is the radius of the spherical potential well.

We adopt the convention that unprimed indices denote a particle and primed ones a hole, i.e.,

$$\vec{m} \leftrightarrow (k, L, M) \quad \text{and} \quad \vec{i} \leftrightarrow (k', L', M'). \quad (G.3)$$

With Eqs. (VI.2.3b), (VI.2.3c) and (G.2), the matrix elements $D(\vec{m}, \vec{i})$ become:

$$D(\vec{m}, \vec{i}) = -u_0 R^3 C_L C_{L'} j_L(kR) j_{L'}(k'R) \int Y_{LM}^* Y_{L'M'} Y_{\lambda 0} d\Omega. \quad (G.4)$$

In deriving (G.4) we used the fact that the derivative of the step function, $\delta(r-R)$, is a Dirac delta function, $\delta(r-R)$. The integral in Eq. (G.4) over the three spherical harmonics expresses the conservation of parity and angular momentum.

Monopole Vibrations

We first examine the case of a monopole surface vibration, $\lambda=0$.

One has

$$Y_{0,0}(\theta) = \frac{1}{\sqrt{4\pi}} \quad (G.5)$$

and

$$\int Y_{LM}^* Y_{L'M'} Y_{0,0} d\Omega = \frac{1}{\sqrt{4\pi}} \delta_{LL'} \delta_{MM'}. \quad (G.6)$$

Because of Eq. (G.6), Eq. (G.4) yields:

$$D(\vec{m}, \vec{i}) = -\frac{u_0 R^3}{\sqrt{4\pi}} C_L C_{L'} j_L(kR) j_{L'}(k'R) \delta_{LL'} \delta_{MM'}. \quad (G.7)$$

Expression (G.7) simplifies further in the limit of an infinitely deep square well potential, when $u_0 \rightarrow \infty$. To see this we first need to calculate the normalization constant C_L in the limit $u_0 \rightarrow \infty$. From Ref. 73 one gets the following relation for an arbitrary Bessel function,

$$Z_p(ax): \int x [Z_p(ax)]^2 dx = \frac{x^2}{2} \{ [Z_p(ax)]^2 - Z_{p-1}(ax) Z_{p+1}(ax) \}. \quad (G.8)$$

The spherical Bessel functions $j_L(kr)$ are related to the Bessel functions $J_{L+1/2}(kr)$, i.e.:

$$j_L(kr) = \sqrt{\pi/2kr} J_{L+1/2}(kr). \quad (G.9)$$

Substituting (G.9) into (G.8), one gets:

$$\int r^2 [j_L(kr)]^2 dr = \frac{r^3}{2} \{ [j_L(kr)]^2 - j_{L-1}(kr) j_{L+1}(kr) \}. \quad (G.10)$$

In the limit $u_0 \rightarrow \infty$, the particles are completely restricted inside the potential well. Hence;

$$j_L(k_L R) = 0. \quad (G.11)$$

Using Eqs. (G.2), (G.10) and (G.11), one finds

$$C_L^2 = \left[\int_0^R r^2 [j_L(k_L r)]^2 dr \right]^{-1} \frac{1}{u_0^2} = \frac{2}{R^3} \frac{1}{j_{L-1}(k_L R) j_{L+1}(k_L R)}. \quad (G.12)$$

We now use the following recurrence relations:⁶⁴

$$j_{L-1}(\rho) = \left[\frac{d}{d\rho} + \frac{L+1}{\rho} \right] j_L(\rho) \quad (G.13)$$

and

$$j_{L+1}(\rho) = \left[-\frac{d}{d\rho} + \frac{L}{\rho} \right] j_L(\rho), \quad (G.14)$$

where $\rho = kr$. With the help of Eqs. (3.13), (G.14) and the boundary condition (G.11), one gets

$$j_{L-1}(k_L R) = \frac{1}{k_L} \frac{dj_L(k_L r)}{dr} \quad (G.15)$$

and

$$j_{L+1}(k_L R) = -\frac{1}{k_L} \frac{dj_L(k_L r)}{dr} \quad (G.16)$$

With Eqs. (G.15) and (G.16), Eq. (G.12) yields

$$c_L \frac{dj_L(k_L R)}{dr} \bigg|_{u_0 \rightarrow \infty} = \frac{k_L \sqrt{2}}{R^{3/2}} \quad (G.17)$$

Because of the continuity conditions, all single-particle wave functions satisfy the following relation at the boundary in the limit of an infinitely deep square well potential:

$$\sqrt{2\mu u_0/\hbar^2} \psi(r, \theta, \phi) \bigg|_{u_0 \rightarrow \infty} = -\frac{\partial}{\partial r} \psi(r, \theta, \phi) \bigg|_{r=R} \quad (G.18)$$

where μ is the nucleon mass. Using Eq. (C.2) for the wave function,

Eq. (G.18) yields:

$$\sqrt{2\mu u_0/\hbar^2} j_L(k_L R) \bigg|_{u_0 \rightarrow \infty} = -\frac{dj_L(k_L R)}{dr} \quad (G.19)$$

Combining Eqs. (G.7), (G.19) and (G.17), one finds:

$$D(\tilde{m}, \tilde{l}) \bigg|_{u_0 \rightarrow \infty} = -\frac{1}{\sqrt{4\pi}} \frac{\hbar^2}{\mu} k_L k_L' \delta_{LL'} \delta_{MM'} \quad (G.20)$$

Substituting (G.20) into (G.1), one gets

$$\gamma_{\text{limit}} = \frac{\hbar^4}{4\mu^2 \omega_0} \iiint dn dn' dL(2L) (k k')^2 \delta(\hbar\omega_c - \epsilon(\tilde{m}, \tilde{l})), \quad (G.21)$$

where we dropped the index L from the k_L and k_L' . n and n' denote the number of single-particle states inside the spherical box with a given set of angular momentum numbers (L, M) (the number of zeroes for a given spherical Bessel function). We have

$$\epsilon(\tilde{m}, \tilde{l}) = \frac{\hbar^2}{2\mu} (k^2 - k'^2) = \frac{\hbar^2}{2\mu R^2} (\rho^2 - \rho'^2), \quad (G.22a)$$

where

$$0 \leq \rho' \leq k_F R \quad (\text{a hole}) \quad (G.22b)$$

and

$$k_F R \leq \rho \quad (\text{a particle}) \quad (G.22c)$$

k_F is the Fermi wave vector.

For the density in $d\rho$ of the zeroes of a given spherical Bessel function, we use the following expression:^{*}

$$\frac{dn}{d\rho} = \begin{cases} \frac{1}{\pi\rho} \sqrt{\rho^2 - L^2} & , \rho \geq L \\ 0 & , \rho < L \end{cases} \quad (G.23)$$

With Eq. (G.23) the integral (G.21) is rewritten:

$$\gamma_{\text{limit}} = \frac{\hbar^4}{4\pi^2 \mu^2 R^4 \omega_0} \iiint c_F d\rho' dL(2L) \rho \rho' \sqrt{\rho^2 - L^2} \sqrt{\rho'^2 - L^2} \delta(\hbar\omega_c - \epsilon(\tilde{m}, \tilde{l})). \quad (G.24)$$

To evaluate (G.24), we first carry the partial integration over ρ' .

We use well-known properties of the Dirac delta function⁶⁴ to get

$$\begin{aligned} \delta(\hbar\omega_c - \epsilon(\tilde{m}, \tilde{l})) &= \delta(\hbar\omega_c - \frac{\hbar^2}{2\mu R^2} (\rho^2 - \rho'^2)) \\ &= \frac{2\mu R^2}{\hbar^2} \delta((k_F R)^2 \eta - (\rho^2 - \rho'^2)) \\ &= \frac{\mu R^2}{\hbar^2} \frac{\delta(\rho' - \sqrt{\rho^2 - (k_F R)^2 \eta})}{\sqrt{\rho^2 - (k_F R)^2 \eta}}, \end{aligned} \quad (G.25)$$

where

$$\eta = \frac{\hbar\omega_c}{\epsilon_F} \quad (G.26)$$

ϵ_F is the Fermi energy

We calculate γ_{limit} for both cases, $\eta \leq 1$ and $\eta > 1$. When $\eta \leq 1$, the partial integration in (G.24) over ρ' yields:

$$\begin{aligned} \gamma_{\text{limit}} &= \frac{\hbar^4}{4\pi^2 (k_F R)^2} \frac{1}{\eta} \left[\int_0^{(k_F R)\sqrt{1-\eta}} \frac{dL(2L) d\rho(2\rho) \sqrt{\rho^2 - L^2} \sqrt{\rho^2 - L^2 - (k_F R)^2 \eta}}{k_F R} \right. \\ &\quad \left. + \int_{(k_F R)\sqrt{1-\eta}}^{(k_F R)} \frac{dL(2L) d\rho(2\rho) \sqrt{\rho^2 - L^2} \sqrt{\rho^2 - L^2 - (k_F R)^2 \eta}}{\sqrt{(k_F R)^2 \eta + L^2}} \right]. \end{aligned} \quad (G.27)$$

* Expression (G.23) yields the correct answer for the density of states per unit volume in a large spherical box⁷⁴ (see Appendix H for details).

The integration limits in (G.27) have been determined from the delta function, $\delta(h\omega_c - c(\tilde{m}, \tilde{l}))$, and the density (G.23) for the zeroes of a spherical Bessel function. In writing (G.27), we used the approximate equality $\omega_0 \approx \omega_c$.

We now use the following integral:⁷³

$$\int \sqrt{R} dz = \frac{(2cz+b)\sqrt{R}}{4c} + \frac{\Delta}{8c} \int \frac{dz}{\sqrt{R}}, \quad (G.28)$$

where

$$R = a + bz + cz^2, \quad \Delta = 4ac - b^2. \quad (G.29)$$

The quantity $\sqrt{R}$ in Eq. (G.27) is:

$$\sqrt{R} = \sqrt{\rho^2 - L^2} \sqrt{\rho^2 - L^2 - (k_F R)^2 \eta} = \sqrt{\rho^4 - (2L^2 + (k_F R)^2 \eta) \rho^2 + L^2(L^2 + (k_F R)^2 \eta)}. \quad (G.30)$$

From (G.30) we find

$$z = \rho^2, \quad c = 1, \quad b = -(2L^2 + (k_F R)^2 \eta), \quad a = L^2(L^2 + (k_F R)^2 \eta), \quad (G.31)$$

and

$$\Delta = -(k_F R)^4 \eta^2. \quad (G.32)$$

We also use the integral,⁷³

$$\int \frac{dz}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln(2\sqrt{cR} + 2cz + b), \quad [c > 0]. \quad (G.33)$$

After carrying with the help of (G.28) and (G.33) the ρ integration in (G.27), one obtains

$$\begin{aligned} \gamma_{\text{limit}} = \frac{\hbar(k_F R)^4}{4\pi^2 \eta} & \left\{ \frac{1}{4} \int_0^{-1} dx (2x - \eta) \sqrt{x(x - \eta)} \right. \\ & - \frac{1}{4} \int_{-\eta}^{-1} dx (2x + \eta) \sqrt{x(x + \eta)} + \frac{\eta^2}{8} \int_{-1}^{-\eta} dx \ln \left[\frac{2\sqrt{x(x + \eta)} - 2x - \eta}{2\sqrt{x(x - \eta)} - 2x + \eta} \right] \\ & \left. + \frac{\eta^2}{8} \int_{-\eta}^0 dx \ln \left[\frac{\eta}{2\sqrt{x(x - \eta)} - 2x + \eta} \right] \right\}, \quad (G.34) \end{aligned}$$

where

$$x = \frac{L^2}{(k_F R)^2} - 1. \quad (G.35)$$

We now use the following integral:⁷³

$$\int x \sqrt{R} dx = \frac{\sqrt{R}^3}{3c} - \frac{b'}{2c} \int \sqrt{R} dx, \quad (G.36)$$

where

$$R = a' + b'x + c'x^2. \quad (G.37)$$

In Eq. (G.34) one has $R = x^2 \pm \eta x$, and thus

$$a' = 0; \quad b' = \pm \eta; \quad c' = 1. \quad (G.38)$$

Using Eqs. (G.36), (G.37) and (G.38), one finds:

$$\frac{1}{4} \int_0^{-1} dx (2x - \eta) \sqrt{x^2 - \eta x} = \frac{(x^2 - \eta x)^{3/2}}{6} \Big|_0^{-1} = \frac{(1 - \eta)^{3/2}}{6}. \quad (G.39)$$

Likewise,

$$\frac{1}{4} \int_{-\eta}^{-1} dx (2x + \eta) \sqrt{x^2 + \eta x} = \frac{(x^2 + \eta x)^{3/2}}{6} \Big|_{-\eta}^{-1} = \frac{(1 - \eta)^{3/2}}{6}. \quad (G.40)$$

Thus, Eq. (G.34) yields:

$$\begin{aligned} \gamma_{\text{limit}} = \frac{\hbar(k_F R)^4}{8\pi^2} \frac{1}{\eta} & \left\{ \frac{(1 - \eta)^{3/2} - (1 - \eta)^{3/2}}{3} + \frac{\eta^2}{4} \int_{-1}^{-\eta} dx \ln \left[\frac{2\sqrt{x^2 + \eta x} - 2x - \eta}{2\sqrt{x^2 - \eta x} - 2x + \eta} \right] \right. \\ & \left. + \frac{\eta^2}{4} \int_{-\eta}^0 dx \ln \left[\frac{\eta}{2\sqrt{x^2 - \eta x} - 2x + \eta} \right] \right\}. \quad (G.41) \end{aligned}$$

From Eq. (G.41) one sees that the proportionality coefficient γ_{limit} depends upon the frequency of the collective motion, since $\eta = \hbar\omega_c/\epsilon_F$. For the low frequency limit, however, one finds:

$$\gamma_{\text{limit}}^{\eta \rightarrow 0} = \frac{\hbar(k_F R)^4}{8\pi^2}. \quad (G.42)$$

When $n > 1$, the integration of (G.24) yields:

$$\begin{aligned} \gamma_{\text{limit}} = \frac{\hbar}{4\pi^2 (k_F R)^2 \eta} & \int_0^{k_F R} \int \frac{k_F R \sqrt{1 + \eta}}{\sqrt{(k_F R)^2 \eta + L^2}} dL (2L) d\rho (2\rho \sqrt{\rho^2 - L^2} \sqrt{\rho^2 - L^2 - (k_F R)^2 \eta}) \\ & - \frac{\hbar(k_F R)^4}{8\pi^2 \eta} \left\{ \frac{(1 + \eta)^{3/2}}{3} + \frac{\eta^2}{4} \int_{-1}^0 dx \ln \left[\frac{\eta}{2\sqrt{x^2 - \eta x} - 2x + \eta} \right] \right\}. \quad (G.43) \end{aligned}$$

Observe that the limiting value of (G.43) for large η is zero.

Vibrations of Arbitrary Multipolarity $(\lambda, 0)$

We now examine the case when the multipolarity of the surface vibration is arbitrary, i.e., $(\lambda, 0)$. Instead of Eq. (G.21), one now has:

$$Y_{\text{limit}} = \frac{\pi \hbar^4}{2 \mu^2 \omega_0} \iiint d\Omega d\Omega' dL dL' (k_L k_{L'})^2 \sum_{MM'} |Y_{LM}^* Y_{L'M'} Y_{\lambda 0}|^2 \delta(\hbar \omega_c - \epsilon(\tilde{m}, \tilde{l})). \quad (\text{G.44})$$

The sum over indices M and M' is calculated with the help of the following relation of the Wigner 3-j symbols:⁷⁵

$$\int Y_{L,-M} Y_{L',M} Y_{\lambda 0} d\Omega = \left[\frac{(2\lambda+1)(2L+1)(2L'+1)}{4\pi} \right]^{1/2} \begin{pmatrix} L & L' & \lambda \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & L' & \lambda \\ -M & M & 0 \end{pmatrix} \quad (\text{G.45})$$

and

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (2j_3+1)^{-1} \delta_{j_3 j_3} \delta_{m_3 m_3} \delta(j_1 j_2 j_3), \quad (\text{G.46})$$

where $\delta(j_1 j_2 j_3) = 1$ if j_1, j_2, j_3 satisfy the triangular condition, and is zero otherwise.

Using Eqs. (G.45) and (G.46), one finds

$$\begin{aligned} \sum_M |Y_{L,-M} Y_{L',M} Y_{\lambda 0}|^2 &= \frac{(2\lambda+1)(2L+1)(2L'+1)}{4\pi} \begin{pmatrix} L & L' & \lambda \\ 0 & 0 & 0 \end{pmatrix}^2 \sum_M \begin{pmatrix} L & L' & \lambda \\ -M & M & 0 \end{pmatrix}^2 \\ &= \frac{(2L+1)(2L'+1)}{4\pi} \begin{pmatrix} L & L' & \lambda \\ 0 & 0 & 0 \end{pmatrix}^2 \delta(L L' \lambda). \end{aligned} \quad (\text{G.47})$$

For $\lambda \ll k_F R$, the large values of L and L' , which mainly contribute to the integral (G.44), are very close so that

$$\frac{dn'}{d\rho'} = \frac{1}{\pi \rho'} \sqrt{\rho'^2 - L'^2} \approx \frac{1}{\pi \rho'} \sqrt{\rho'^2 - L^2}, \quad (\text{G.48})$$

that is the densities of zeroes of the spherical Bessel functions, $j_L(\rho)$

and $j_{L'}(\rho')$ are approximately equal. With Eq. (G.48) and (G.47), Eq. (G.44)

yields:

$$\begin{aligned} Y_{\text{limit}} &= \frac{\pi \hbar^4}{4\pi^2 \mu^2 \omega_0 R^4} \iiint d\rho d\rho' dL (\rho \rho') \sqrt{\rho'^2 - L^2} \sqrt{\rho'^2 - L^2} (2L+1) \\ &\quad \times \sum_{L'} (2L'+1) \begin{pmatrix} L & L' & \lambda \\ 0 & 0 & 0 \end{pmatrix}^2. \end{aligned} \quad (\text{G.49})$$

To carry the summation over L' , we utilize the following orthogonality relation for the Wigner 3-j symbols:⁷⁵

$$\sum_{j_3 m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}. \quad (\text{G.50})$$

Considering the large values $2L \gg 1$, we find

$$Y_{\text{limit}} = \frac{\pi \hbar^4}{4\pi^2 \mu^2 \omega_0 R^4} \iiint d\rho d\rho' dL (2L) (\rho \rho') \sqrt{\rho'^2 - L^2} \sqrt{\rho'^2 - L^2} \delta(\hbar \omega_c - \epsilon(\tilde{m}, \tilde{l})). \quad (\text{G.51})$$

Equation (G.52) is identical with Eq. (G.24). From Eq. (G.52) one then proceeds as in the case of monopole surface vibrations.

APPENDIX H: ZEROES OF THE SPHERICAL BESSEL FUNCTIONS AND THE NUMBER OF STATES IN A SPHERICAL FERMI GAS

In this appendix, we justify the use of expression (G.23) for the density of zeroes of a spherical Bessel function by showing that (G.23) provides the correct answer to the following problem:⁷⁴

Problem: To count the number of states with $k \leq k_F$ in a spherical box of large radius, R .

(1) For a cubical box, the answer is straightforward:

$$\frac{N}{(\text{volume})} = \iiint dn_x dn_y dn_z = \iiint \frac{dk_x dk_y dk_z}{(2\pi)^3} = \int_0^{k_F} \frac{k^2 dk}{2\pi^2} = \frac{k_F^3}{6\pi^2}, \quad (\text{H.1})$$

where

$$\text{volume} = L_x L_y L_z, \quad (\text{H.2})$$

and L_x , L_y , and L_z are the three dimensions of the cubical box.

(2) For a sphere, we first count eigenfunctions $j_L(kr)Y_{LM}(\theta, \phi)$ defined by $j_L(kR) = 0$. Let S_L be the number of radial eigenfunctions $j_L(kr)$ with $0 \leq k \leq k_F$ for a given L . S_L is also the number of zeroes of $j_L(kr)$ for $0 < r < R$. Then the desired result is:

$$N = \sum_{L=0}^{L_{\max}} (2L+1)S_L. \quad (\text{H.3})$$

(3) The McMahon formula for the location of the zeroes of $J_\nu(kr)$ is given in Abramowitz.⁷⁶ According to this formula, the n^{th} zero of $j_L(kr)$, where

$$j_L(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{L+1/2}(kr), \quad (\text{H.4})$$

is located at $z_{L,n}$ given by:

$$z_{L,n} = \left[\beta - \frac{L(L+1)}{2\beta} + \dots\right], \quad \text{where } \beta = \pi\left(n + \frac{L}{2}\right). \quad (\text{H.5})$$

Then, setting $z_{L,n} = k_F R$ gives an estimate of $S_L = n$ via:

$$k_F R = \pi S_L + \frac{L\pi}{2}. \quad (\text{H.6})$$

Whence,

$$N = \sum_{L=0}^{L_{\max}} (2L+1) \left[\frac{k_F R}{\pi} - \frac{L}{2} \right] \approx \left[\frac{k_F R}{\pi} \int_0^{L_{\max}=2k_F R/\pi} 2L dL \right] - \int_0^{2k_F R/\pi} L^2 dL \\ = \left(\frac{k_F R}{\pi}\right)^3 \left(4 - \frac{8}{3}\right) = (\text{volume}) \frac{k_F^3}{\pi^4}. \quad (\text{H.7})$$

The result (H.7) does not agree with the result (H.1) for a cubical box.

The result (H.7) underestimates the total number of states.

(4) The fault lies in the McMahon expression for the zeroes; it is not accurate enough to give a good integral. We find a better expression in the Royal Society Mathematical Tables⁷⁷ as a differential form for the number of zeroes per unit $kr = \rho$:

$$\frac{dn}{d\rho} = \begin{cases} \frac{1}{\pi\rho} \sqrt{\rho^2 - L^2}, & \rho \geq L \\ 0, & \rho < L \end{cases} \quad (\text{H.8})$$

Then,

$$S_L = \int_0^{S_L} dn = \int_L^{k_F R} \frac{dn}{d\rho} d\rho = \frac{1}{\pi} \int_L^{k_F R} \frac{d\rho}{\rho} \sqrt{\rho^2 - L^2}. \quad (\text{H.9})$$

We use the following integral:⁷³

$$\int \frac{dx}{x} u = u - \sqrt{-a} \arccos \frac{1}{x} \sqrt{-a/c}, \quad [a < 0, c > 0], \quad (\text{H.10})$$

where

$$u = \sqrt{a + cx^2}. \quad (\text{H.11})$$

From (H.9), one has

$$x = \rho, \quad c = 1 > 0, \quad a = -L^2 < 0. \quad (\text{H.12})$$

Thus,

$$S_L = \frac{1}{\pi} \left[\sqrt{(k_F R)^2 - L^2} + L \arcsin \left(\frac{L}{k_F R} \right) - \frac{L\pi}{2} \right]. \quad (\text{H.13})$$

Using Eq. (H.13), one gets

$$N = \sum_{L=0}^{k_F R} (2L+1) S_L \approx \int_0^{k_F R} \frac{2L dL}{\pi} \left[\sqrt{(k_F R)^2 - L^2} + L \arcsin \left(\frac{L}{k_F R} \right) - \frac{L\pi}{2} \right]. \quad (\text{H.14})$$

* Equation (H.8) is the limit of the expression in Ref. 77 in the limit of large L .

The integral I_1 , where

$$I_1 = \int_0^{k_F R} dL L \sqrt{(k_F R)^2 - L^2}, \quad (H.15)$$

is calculated by setting $L = k_F R \sin \theta$. One has:

$$I_1 = -(k_F R)^3 \int_0^{\pi/2} \cos^2 \theta d(\cos \theta) = \frac{(k_F R)^3}{3}. \quad (H.16)$$

The integral I_2 , where

$$I_2 = \int_0^{k_F R} dL L^2 \arcsin\left(\frac{L}{k_F R}\right), \quad (H.17)$$

is calculated using the following relations:⁷³

$$\int x^2 \arcsin \frac{x}{a} dx = \frac{x^3}{3} \arcsin \frac{x}{a} - \frac{1}{3} \int \frac{x^3 dx}{\sqrt{a^2 - x^2}}, \quad (H.18)$$

and

$$\int \frac{x^3 dx}{\sqrt{a^2 - x^2}} = \frac{(a^2 - x^2)^{3/2}}{3} - a^2 \sqrt{a^2 - x^2}. \quad (H.19)$$

Here we have

$$a = k_F R, \quad x = L. \quad (H.20)$$

One finds

$$\begin{aligned} I_2 &= \frac{L^3}{3} \arcsin\left(\frac{L}{k_F R}\right) \Big|_0^{k_F R} - \frac{1}{9} (k_F R)^2 - L^2)^{3/2} \Big|_0^{k_F R} \\ &\quad + \frac{(k_F R)^2}{3} \sqrt{(k_F R)^2 - L^2} \Big|_0^{k_F R} \\ &= (k_F R)^3 \left(\frac{\pi}{6} - \frac{2}{9}\right). \end{aligned} \quad (H.21)$$

The integral I_3 is calculated:

$$I_3 = -\frac{\pi}{2} \int_0^{k_F R} dL L^2 = -\frac{\pi}{6} (k_F R)^3. \quad (H.22)$$

Finally, the number of states is found to be

$$N = \frac{2}{\pi} (I_1 + I_2 - I_3) = \frac{2}{9\pi} (k_F R)^3 = (\text{volume}) \frac{k_F^3}{6\pi}. \quad (H.23)$$

The result (H.23) agrees exactly with the cubical box result (H.1).

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TABLE OF ABBREVIATIONS AND SYMBOLS

SHO	Simple Harmonic Oscillator
RPA	Random Phase Approximation
TDA	Tamm-Dancoff Approximation
TDHF	Time-Dependent Hartree-Fock
$c>$	One-phonon Collective State
$v>$	Eigenstates in Extended Subspace
$Q_c^\dagger$	One-phonon Collective State Creation Operator
Q_c	One-phonon Collective State Annihilation Operator
$Q_v^\dagger$	Creation Operators for Eigenstates in Extended Subspace S
Q_v	Annihilation Operators for Eigenstates in Extended Subspace S
S_R	Restricted Subspace
S_A	Additional Subspace
$S = S_R + S_A$	Extended Subspace
$ \phi(t)>$	Packet for Decay of One-phonon Collective State
$ \phi^{(n)}(t)>$	Packet for Decay of the N-phonon Collective States
$ \varphi(t)>$	Packet for Decay of Mean Field Oscillations
$ \psi(t)>$	Packet for Undamped Mean Field Oscillations
$\mathcal{H}_F$	Frictional Hamiltonian
$\mathcal{H}_c$	Generalized Collective Hamiltonian
$\mathcal{H}_c$	Decoupled Collective Hamiltonian
$\mathcal{H}$	Total Hamiltonian

Fig. 1