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IN A SHEARED MAGNETIC FIELD

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Theory of Universal Eigenmodes in a Sheared
Magnetic Field

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Collisionless drift-wave eigenmodes in a sheared magnetic field are analyzed using the WKBJ method. It is found that, for $L_s/L_n > (L_s/L_n)_c$, ion-sound dynamics determines the eigenvalues at small k_y^2 and the eigenmodes are damped. However, at large k_y^2 electron dynamics dominates and the eigenmodes become marginally stable. For $L_s/L_n < (L_s/L_n)_c$, the eigenmodes are damped for all values of k_y^2 . The critical value $(L_s/L_n)_c$ scales as $(m_i/m_e)^{1/4}$.

Recently, Ross and Mahajan¹ as well as Tsang et al.² have made an important contribution to the theory of collisionless drift-wave eigenmodes in a slab geometry with finite magnetic shear. In contradiction to previous theoretical predictions,³ they found numerically that, with full (both resonant and non-resonant) electron dynamics taken into account, there is no absolutely unstable drift-wave. They point out that the errors in previous theories are due to retaining only resonant electron dynamics and/or using improper perturbation analyses.

While the numerical results obtained by these authors appear correct, their analytical theories are highly dubious. Specifically, in Ref. 1, the authors employed a variational principle to derive the eigenvalue condition. Since the operators involved are demonstrably non-Hermitian, such an approach lacks a sound basis. Furthermore, the eigenvalue condition thus derived depends critically on the integral $I = \int_{-\infty}^{\infty} dx \phi^2 Z(x_e/|x|)/|x|$, where $x_e = \omega L_s/k_y v_e$, Z is the plasma dispersion function,⁴ and $\phi(x)$ is a trial function chosen to be $\exp(-i\alpha x^2/2)$. Contrary to what the authors have assumed [cf their Eq. (8)], however, we note that α always contain a positive imaginary part and; therefore, the integral I tends to be divergent. In fact, the divergence of I is much more severe at larger k_y^2 where the modes are marginally stable and the corresponding α values are purely positive imaginary! (We have confirmed this by independent numerical work.) Thus, at least for the marginally stable modes the analysis is inconsistent. In Ref. 2, the eigenvalue problem was solved by

the method of matched asymptotic expansions. In the outer region ($|x| > |x_e|$), the authors neglected the electron Z function and, with the outgoing-wave boundary condition, obtained a solution in terms of parabolic cylinder functions. The eigenvalue condition was then derived by matching the slope ϕ'/ϕ of the outer-region solution to that of the inner-region solution using the electron Z function and the "constant- ϕ " approximation. However, in evaluating the inner-region ϕ'/ϕ , only the resonant electron contributions [cf the logarithmic term in their Eq. (2)] are included! This contradicts with their numerical work which indicates that it is necessary to retain the full electron dynamics to obtain marginal stability. Thus, the agreement of this calculation with their numerical work can only be regarded as fortuitous.

In this letter we present an eigenmode analysis employing the WKBJ or Phase-Integral method.⁵ The WKBJ approach is motivated by the large parameter, L_s/L_n . We find there exists a critical shear scale length, $(L_s/L_n)_c$. For $L_s/L_n < (L_s/L_n)_c$, i.e., strong shear, the eigenvalues are determined by the turning points due to the ion-sound waves (i.e., the Pearlstein-Berk turning points)³ and eigenmodes are found to be damped for all k_y^2 . For $L_s/L_n > (L_s/L_n)_c$, i.e., weak shear, there is a critical perpendicular wave number, k_{yc}^2 . At long wavelengths, $k_y^2 < k_{yc}^2$, the Pearlstein-Berk turning points again determine the eigenvalues and there are only damped eigenmodes. For $k_y^2 > k_{yc}^2$, however, we find the eigenvalues are determined by a new pair of turning points induced by the electron dynamics and, in this case,

the eigenmodes are predicted to be marginally stable. For this parameter range, it can be expected that the analytical theories of Ref. 1 and 2 do not properly describe the marginally stable eigenmodes because both the trial function $\phi = \exp(-i\alpha x^2/2)$ and the "constant- ϕ " approximation assumed implicitly that the Pearlstein-Berk turning points determine the eigenmodes.

Let us consider, for simplicity, cold ions and no temperature gradients. The corresponding eigenmode equation is given by

$$[d^2/dx^2 + Q(x)]\phi(x) = 0, \quad (1)$$

where

$$Q(x) = \lambda + x^2/4 + (\mu\epsilon/|x|)Z(\epsilon/|x|), \quad (2)$$

$$\lambda = -(\omega L_s/2\omega_* L_n)(k_y^2 \rho_s^2 + 1 - \omega_*/\omega), \quad (3)$$

$$\mu = (L_s/2L_n)(1 - \omega/\omega_*), \quad (4)$$

$$\epsilon = (\omega m_e L_s / \omega_* m_i L_n)^{1/2}, \quad (5)$$

and the rest of the notations are standard. The analytical continuation of this expression into the complex plane, $t = x + iy$, is provided by $|x| \rightarrow (t^2)^{1/2}$. The Riemann structure consists of two cuts originating at $t = 0$, which we take along the positive and negative imaginary axes, respectively. There are two sheets, which we refer to as the physical sheet,

$(x^2)^{1/2} > 0$, and the nonphysical sheet, $(x^2)^{1/2} < 0$.

There are three pairs of turning points in the complex t plane, each pair being symmetrically located about the origin. The first pair corresponds to the usual ion-sound (Pearlstein-Berk) turning points;

$$\pm P \approx \mp (2i\lambda^{1/2} - i\sqrt{\pi}\mu\epsilon/2\lambda) . \quad (6)$$

The second pair, which also plays a role in the WKBJ analysis, is induced by the electron dynamics via the Z function and its location is approximately given by

$$\pm E \approx \mp i\sqrt{\pi}\mu\epsilon/\lambda . \quad (7)$$

The third pair normally is found further out in the complex plane and plays no essential role in the WKBJ analysis. Its appropriate location is given by

$$\pm B \approx \mp (2i\lambda^{1/2} + i\sqrt{\pi}\mu\epsilon/2\lambda) . \quad (8)$$

In deriving Eqs. (6) to (8), we have assumed $|\mu\sqrt{\pi}| > |\lambda|$ and $|2\lambda^{3/2}| > |\sqrt{\pi}\mu\epsilon|$, which are usually satisfied. We note also that $\pi/2 > \arg(\lambda) \geq 0$ and, since $|\omega_r| \gg |\omega_i|$, μ and ϵ are predominantly real.

For small k_y^2 we have $\arg(\lambda) > 0$. $\pm P$ are located in the fourth and second quadrants of the physical sheet, respectively.

$\pm B$ and $\pm E$ are on the nonphysical sheet. The corresponding Stokes diagram is shown in Fig. 1. For this Stokes pattern the outgoing-wave boundary condition at large $|x|$ corresponds to taking dominant solutions (with respect to $\pm P$) in regions (i) and (i)' and, hence, subdominant solutions in (ii) and (ii)'. The eigenvalue condition is then determined by the physical Pearlstein-Berk turning point; i.e.,

$$\int_0^P [Q(t)]^{1/2} dt = (2n+1)\pi/4 ; n = 0, 1, \dots \quad (9)$$

The Stokes diagrams and eigenvalues presented here have been determined by a code which uses the eigenvalue condition [e.g., Eq. (9)] to iteratively search the complex frequency plane. Once the frequency has been determined, the resulting Stokes diagram indicates the possible choices of boundary conditions for the solution.⁵ We note that, for the present case of small k_y^2 , the eigenmodes are found to be damped. As k_y^2 is increased, $\arg(\lambda)$ decreases and all three pairs of turning points migrate toward the imaginary (y) axis. (This tendency can be readily seen using a perturbation theory.) The properties of the eigenvalue problem, however, depend critically on whether the magnetic shear is weak or strong.

In the weak-shear case, $L_s/L_n > (L_s/L_n)_c$, there exists a critical value of k_y^2 , k_{yc}^2 , at which the Pearlstein-Berk pair $\pm P$ and the electron-induced pair $\pm E$ coalesce at $\pm C$. The value of k_{yc}^2 can be determined by the coalescence conditions;

$Q(C) = Q'(C) = 0$, and the eigenvalue condition $\int_0^C [Q(t)]^{1/2} dt = (2n+1)\pi/4$. In the present analysis, k_{yc}^2 depends on m_e/m_i and L_n/L_s . For example, with $m_i/m_e = 1837$, we find $k_{yc}^2 \rho_s^2 = 0.13$ for $L_s/L_n = 50$ and $k_{yc}^2 \rho_s^2 = 0.045$ for $L_s/L_n = 100$. We note that C lies on the imaginary (y) axis. Since $Z(iv) = i\sqrt{\pi} \exp(v^2) \operatorname{erfc}(y)^4$, it then easily follows that $\operatorname{Im} \omega = 0$ and the eigenmode is marginally stable.

As k_y^2 is increased beyond k_{yc}^2 , P and E again separate. All three pairs of turning points stay, however, on the imaginary axis. With increasing k_y^2 , E slowly migrates toward the origin and P and B move in the opposite direction. The Stokes diagram corresponding to this regime of large k_y^2 is shown in Fig. 2. Again, the outgoing-wave boundary condition along x dictates that the solutions be dominant in regions (i) and (i)' and, thus, subdominant in (ii) and (ii)'. Therefore, the eigenvalue condition is determined by E instead of P ; i.e., we have

$$\int_0^E [Q(t)]^{1/2} dt = (2n+1)\pi/4 ; n = 0, 1, \dots \quad (10)$$

Since E lies on the y axis, the eigenmodes continue to be marginally stable. Here, we should remark that while the full Z function is well behaved along the y (imaginary) axis, the resonant-electron contribution alone $i\sqrt{\pi} \exp(-\omega^2/k_{||}^2 v_e^2 t^2)$, is not. Thus, the coalescence of turning points as well as the eigenvalue condition, Eq. (10), is pertinent to the case with the full electron dynamics included.

In the strong-shear case, $L_s/L_n < (L_s/L_n)_c$, P and E , however, do not coalesce for any value of k_y^2 . The corresponding Stokes diagram is similar to that shown in Fig. 1. The eigenvalue condition is then given by Eq. (9) and the eigenmodes are damped for all k_y^2 . We note that, in the present analysis, $(L_s/L_n)_c$ is a function of m_i/m_e only. We have computed $(L_s/L_n)_c$ for the $n = 0$ eigenmode and a wide range of m_i/m_e . The results are shown in Fig. 3 using a log-log plot. Interestingly, the results indicate that $(L_s/L_n)_c$ scales as $(m_i/m_e)^{1/4}$. In fact, the following is a closely-fit scaling law

$$(L_s/L_n)_c = 3. (m_i/m_e)^{1/4}. \quad (11)$$

Here, it is interesting to contrast the scaling law of $(L_s/L_n)_c$ given by Eq. (11) with the $(m_i/m_e)^{1/3}$ scaling of L_s/L_n found in previous perturbative³ and convective amplification treatments.⁶

The WKBJ results presented here have also been verified by direct numerical integration of Eq. (1). Taking $m_i/m_e = 1837$, $L_s/L_n = 50$, and using the WKBJ eigenvalue conditions [Eqs. (9) and (10)], we have computed ω of the $n = 0$ eigenmode for a range of k_y^2 . Figure 4 shows the analytical (WKBJ) results as well as those obtained numerically using a shooting scheme identical to that in Ref. 1. The agreement is sufficiently good. We have also examined the case with $L_s/L_n = 100$ and obtained better agreement, as is expected because the WKBJ approximation becomes better as the large parameter L_s/L_n increases. Finally,

we have also confirmed numerically that for $L_s/L_n < (L_s/L_n)_c$ there is no transition to a marginally stable mode; the eigenmodes are always damped. Using a value of $L_s/L_n = 10$, which is less than the critical value of $(L_s/L_n)_c \approx 19$, ($m_i/m_e = 1837$), we have run the numerical shooting code for k_y^2 up to a large value, $k_y^2 \rho_s^2 = 20$. Throughout this entire range, the $n = 0$ eigenmode remains damped and the damping rate asymptotically approaches a constant value.

In conclusion, we have shown that only for sufficiently weak shear $L_s/L_n > (L_s/L_n)_c \sim (m_i/m_e)^{1/4}$, do the collisionless drift-wave eigenmodes become marginally stable for large k_y^2 . (In the opposite limit, the modes are always damped.) An interesting consequence might be that for $L_s/L_n > (L_s/L_n)_c$, convective amplification of noise perturbations in the plasma could be enormous and limited only by nonlinear effects.

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R E F E R E N C E S

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Figure Captions

Fig. 1 Stokes diagram for the case with damped eigenmodes. The solid (dashed) line is Stokes on the physical (nonphysical) sheet of the branch cut.

Fig. 2 Stokes diagram for the case with marginally stable eigenmodes. The rest is

Fig. 3 A log-log plot of $(L_s/L_n)_c$ vs.

Fig. 4 Plot of ω_r and ω_i vs. $k_y^2 \rho_s^2$.

Fig. 1. 782100

Fig. 2. 782111

Fig. 3. 782061

Fig. 4. 782060