Resistive Drift and Alfvén Instabilities

in Sheared Magnetic Fields*

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ABSTRACT

It is shown that, due to finite- β effects, there exist unstable resistive drift and Alfvén eigenmodes in a slab plasma with finite magnetic shear. The corresponding growth rates of both instabilities scale with the electron-ion collision frequency ν_{ei} as $\nu_{ei}^{1/2}$.

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Low-frequency drift waves have been rather extensively studied for many years due to their potential importance to transport processes in magnetically confined plasmas. An outstanding question in these investigations has been the stability of drift wave eigenmodes in a plasma slab with finite magnetic shear. It has been believed in the past that unstable eigenmodes do indeed exist in such a case and they correspond to solutions with outward energy propagating waves as boundary conditions.¹ Recent analytical and numerical work has, however, demonstrated that this result is based on an incomplete description of the electron dynamics. When the electron dynamics is fully and properly accounted for, it can be shown that in the electrostatic limit and slab geometry, neither the collisionless² nor collisional³ drift waves have unstable eigenmodes.⁴ Physically, the inclusion of the complete electron dynamics sufficiently modifies the dielectric properties of the plasma such that the waves suffer an enhanced shear damping, which then overpowers the usual growth. For plasmas with β (= plasma pressure/magnetic pressure) $> m_e/m_i$, however, it is well known that drift waves can couple to shear Alfvén waves so that finite- β electromagnetic corrections are important. It is the purpose of this letter to demonstrate that finite- β effects can qualitatively modify the stability properties of drift wave eigenmodes. Specifically, we find that, with finite- β effects included, resistive drift waves do indeed have unstable eigenmodes. Physically, finite- β effects permit a new localization of the mode over a region in which conventional shear damping plays no role. Furthermore, we also discover a new

type⁵ of lower frequency eigenmode ($\text{Re}\omega \ll \omega_{e*}$), which we term as resistive (shear) Alfvén mode. The growth rates of both types of instabilities are found to scale with the electron-ion collision frequency ν_{ei} as $\nu_{ei}^{1/2}$. Since these instabilities produce magnetic perturbations as well, they are likely to strongly influence plasma transport via the formation of turbulent magnetic islands.⁶ We now proceed with the analysis.

Consider an inhomogeneous (density gradient along $\underline{e}_x$) plasma in a sheared magnetic field $\underline{B} = B [\underline{e}_z + \underline{e}_y (x/L_s)]$. We describe the electrons by a drift-kinetic equation with a Krook-type number conserving collision operator. Ions are fully kinetic and collisionless. Since $\beta \ll 1$, we may ignore compressional Alfvén waves ($b_{||} = 0$) and, hence, use ϕ and $\underline{A} = A_{||} \underline{e}_{||}$ as perturbed potentials. That is, we have $\underline{E} = -\nabla\phi + \partial A_{||} \underline{e}_{||} / c \partial t$ and $\underline{b}_\perp = -\underline{\nabla}_\perp \times A_{||} \underline{e}_{||}$. Linearizing the kinetic equations, we can readily solve for the perturbed distribution functions and, thus density and current perturbations. Substituting the obtained perturbations into the quasi-neutrality condition and Ampere's law, we derive the following set of two coupled equations

$$\left(1 + \frac{\omega_{i*}}{\omega}\right) \rho_s^2 \nabla_\perp^2 \phi = F(x) \left(\phi - \frac{\omega A_{||}}{ck_{||}}\right) \quad (1)$$

and

$$\rho_s^2 \nabla_\perp^2 A_{||} = \frac{\omega c}{k_{||} V_A^2} F(x) \left(\phi - \frac{\omega A_{||}}{ck_{||}}\right) \quad (2)$$

where

$$F(x) = \left(1 - \frac{\omega_{e*}}{\omega}\right) \left(1 - \frac{x^2}{x_T^2}\right) / f(x) \quad , \quad (3)$$

and

$$f(x) = 1 - ix_R^2/x^2 \quad . \quad (4)$$

Here, $\nabla_{\perp}^2 = d^2/dx^2 - k_Y^2$, $\tau = T_e/T_i$, $\omega_{e*} = k_Y c T_e / e B L_n$,

$L_n = |d \ln n(x) / dx|^{-1}$, $\omega_{i*} = \omega_{e*} / \tau$, $\rho_s = c_s / \omega_{ci}$, $c_s^2 = T_e / m_i$,

$k_{\parallel} = k'_{\parallel} x$, $k'_{\parallel} = k_Y / L_s$, $x_R^2 = \omega v_{ei} / k'_{\parallel}{}^2 v_e^2$ and x_T^2

$= \omega (\omega - \omega_{e*}) / [k'_{\parallel}{}^2 c_s^2 (\omega + \omega_{i*})]$. In deriving Eqs. (1) and (2), we have taken the small ion-Larmor-radius expansion; (i.e., $|\rho_s^2 / \tau| \nabla_{\perp}^2| \ll 1$) and ignored ion-Landau damping. Furthermore, we have assumed the resistive limit; $v_{ei} > |\omega|$.

We now proceed to solve Eqs. (1) and (2). First, we assume the solutions to be fast varying in x such that $|d^2/dx^2| \gg |k_Y^2|$. Equations (1) and (2) can then be reduced to the following second order differential equation in terms of the variable $E \equiv d\phi/dx$;

$$\rho_s^2 \left(1 + \frac{\omega_{i*}}{\omega}\right) \frac{d}{dx} \left[\frac{dE/dx}{F(x)} \right] + \left(\frac{x_A^2}{x^2} - 1 \right) E = \frac{c_1}{x^2} \quad (5)$$

where $x_A^2 = \omega (\omega + \omega_{i*}) / k'_{\parallel}{}^2 v_A^2$ and c_1 is an integration constant. Second, we seek solutions of $E(x)$ which are spatially localized in x such that $|x^2 E(x)| \rightarrow 0$ as $|x| \rightarrow \infty$; i.e., we set $c_1 = 0$. Here, it is instructive to note that $c_1 \neq 0$ corresponds to usual resistive tearing modes;⁷ which require external MHD support. Also,

we assume the eigenmodes are localized well within x_T and approximate $F(x) \approx (1 - \omega_{e*}/\omega)/f(x)$. This is one of the main points of departure from conventional drift wave treatments. Equation (5) can then be written as

$$\frac{d}{dx} \left[f(x) \frac{dE}{dx} \right] + \left(\frac{\Lambda}{x^2} - \frac{1}{x_0^2} \right) E(x) = 0, \quad (6)$$

where $x_0^2 = \rho_s^2 (\omega + \omega_{i*}) / (\omega - \omega_{e*})$ and $\Lambda = x_A^2 / x_0^2 = \omega(\omega - \omega_{e*}) / k_{||}^2 v_A^2 \rho_s^2$.

Assuming $|x_A^2| \gg |x_R^2|$, we solve the eigenvalue problem as posed by Eq. (6) by matching solutions in the outer and inner regions. In the outer region, $|x|^2 \gg |x_R|^2$, $f(x) \approx 1$ and Eq. (6) has the solution

$$E_0(x) = A_0 (x/x_0)^{1/2} K_\mu(x/x_0), \quad (7)$$

where K_μ is the modified Bessel function and $\mu = (1/2)(1 - 4\Lambda)^{1/2}$. We note that in deriving Eq. (7), we have taken $\text{Re} x_0 > 0$ and, therefore, $E_0(x)$ properly decays away as $x \rightarrow \infty$. Note also since the solutions have either even or odd parities, only $x \geq 0$ half space need to be considered. In the inner region, $|x| \sim |x_R| < |x_A|$ and we may neglect the last (x_0^{-2}) term in Eq. (6). Letting $t = (x/x_R) \exp(-i\pi/4)$ and

$$\psi(t) = (t^2 - 1) \frac{1}{t} \frac{dE}{dt}, \quad (8)$$

Eq. (6) then becomes

$$\left(\frac{d^2}{dt^2} + \frac{\lambda}{t^2-1} \right) \psi_I(t) = 0, \quad (9)$$

and the solutions are given by

$$\psi_I(t) = (t^2 - 1)^{1/2} \left[A_I P_\nu^1(t) + B_I Q_\nu^1(t) \right], \quad (10)$$

where $\nu = \mu - 1/2$. P_ν^1 and Q_ν^1 are the associated Legendre functions. Matching the large-argument expansion ($|t| \gg 1$) of $\psi_I(t)$ ⁸ with the small-argument limit ($|x/x_0| \ll 1$) of $\psi_0(t) = t dE_0/dt$, we obtain the following eigenvalue condition

$$\frac{\nu \Gamma(\nu + 3/2) \Gamma(\nu + 1/2) \Gamma(-1-\nu)}{(\nu + 1) \Gamma(1/2 - \nu) \Gamma(-\nu-1/2) \Gamma(\nu)} = \left(\frac{x_R}{4x_0} \right)^{1+2\nu} \left(1 + \pi \frac{B_I}{A_I} \cot \pi \nu \right) \cdot \exp[i\pi(1 + 2\nu)/4], \quad (11)$$

where the ratio B_I/A_I is determined by the parity conditions at $x = 0$; i.e., $\psi(0) = 0$ for odd- ψ modes and $\psi'(0) = 0$ for even- ψ modes. We note that while for the odd- ψ (E) modes the corresponding potentials (or radial displacements) $\phi(x) = \int_{-\infty}^x E(x) dx$ are localized in x , this is not generally true for the even- ψ (E) modes. In fact, one can show that even- ψ (E) modes give rise to perturbations of the type of internal $m = 1$ kink modes.⁹ We, thus, only consider the odd- ψ (E) modes for which $(B_I/A_I)_0$ is given by

$$\left(\frac{B_I}{A_I}\right)_0 = -\frac{2}{\pi} \sin\left(\frac{\nu\pi}{2}\right) \exp\left(-i\frac{\nu\pi}{2}\right)$$

In obtaining Eq. (12), we have taken $\text{Im}[x_R \exp(i\pi, \text{Equation (11) along with Eq. (12) is thus the con}$
value condition.

While in general the eigenvalue condition is solved numerically, analytic progress, however, noting that $|x_R/x_0| < 1$. Equation (11) thus has a solution given by

$$\nu = n + 1/2 - \delta_n,$$

where $n = 0, 1, \dots$ and $|\delta_n| \ll 1$. For the lowest ($n=0$) mode,⁵ we have

$$\delta_0 = \pm i^{3/2} (3x_R/8x_0).$$

Noting $2\mu = (1-4\Lambda)^{1/2} = 2\nu + 1$, the corresponding dispersion relation is given by

$$\omega(\omega - \omega_{e*}) + (3/4)(k_{||}^2 v_{As}^2)^2 [1 \pm i^{3/2} (x_R/x_0)] = 0$$

For drift waves of interest here, we assume $|\omega_{e*}| > \beta_e > 2(L_n/L_s)^2$; Eq. (15) then has two unstable solutions. The resistive drift mode, has ω close to ω_{e*} ;

$$\frac{\omega - \omega_{e*}}{\omega_{e*}} \approx 1 - \left(\frac{3}{2\beta_e}\right) \left(\frac{L_n}{L_s}\right)^2 \left[1 - \frac{3^{1/2}}{2} \left(1 + i\right) \left(\frac{\tau}{\tau + 1}\right)^{1/2} \beta_e^{-1/2}\right]$$

The other mode, which we call resistive Alfvén mode, has ω lower than ω_{e*} :

$$\frac{\omega_2}{\omega_{e*}} \approx \frac{3}{2\beta_e} \left(\frac{L_n}{L_s} \right)^2 \left[1 + 2^{-1/2} (i + 1) \left| \frac{L_s}{L_n} \right| \left| \frac{v_{ei}}{\omega_{e*}} \frac{m_e}{m_i} \right|^{1/2} \right] \quad (17)$$

One can readily verify that these two modes satisfy the consistency requirements; $\text{Re}(x_0), \text{Im}[x_R \exp(i\pi/4)] > 0$.

Equations (16) and (17) are our basic results. They show that unstable eigenmodes with $\text{Re}\omega < \omega_{e*}$ do exist; in contrast to the results in the electrostatic limit.³ It is seen that β_e has to exceed a critical value $\sim 2(L_n/L_s)^2$ for the existence of these unstable eigenmodes. (More detailed conditions are given below). Both modes have eigenfunctions well localized within the ion-sound turning point x_T . The basic point here is that when the Alfvén layer thickness x_A is greater than the resistive layer thickness x_R but less than x_T [which translates to the condition $|L_n/L_s| > \beta > (m_e/m_i) (v_{ei}/\omega_{e*})$], a new localization of drift waves is possible and, in this case, the usual shear damping plays little role. This localization arises entirely because of the coupling between drift waves and shear Alfvén waves. The corresponding eigenfunctions are very different from those obtained in the electrostatic limit³ or the conventional finite- β treatments.¹⁰ Finally, the growth rates of both modes scale with the electron-ion collision frequency as $v_{ei}^{1/2}$ and can be substantial.

Having established the existence of unstable modes, we must check the validity limit of our results as imposed by the following assumptions: (i) The separation of two regions requires $|x_A| = |\kappa^{1/2} x_0| \approx (3/4)^{1/2} |x_0| > |x_R|$. (ii) The small ion-Lamor-radius expansion is valid if $|x_R|^2 > \rho_s^2/\tau$. (iii) The assumption that the eigenmodes are localized within x_T requires the mode amplitudes decay sufficiently before x_T is reached; i.e., $\text{Re}(|x_T|/x_0) > 1$ or $|x_T x_{0R}| > |x_0|^2$. Finally, (iv) the assumption $|d^2/dx^2| > k_y^2$ requires $k_y^2 \ll |x_0|^{-2}$. For the resistive drift mode, these conditions can be summarized to $1 > |L_n/L_s| > \beta > |v_{ei} m_e / \omega_{e*} m_i| > |L_n/L_s|^2$ and $|k_y \rho_s|^2 < \beta^{-1} |L_n/L_s|^2 < 1$; where we have taken $\tau \sim 0(1)$. Meanwhile, for the resistive Alfvén mode, we require $\tau = T_e/T_i > \beta |L_s/L_n|^2 > 1$, $1 > \beta > |L_n/L_s|^2 > |v_{ei} m_e / \omega_{e*} m_i| > \beta/\tau, |L_n/L_s|^4$, and $|k_y \rho_s|^2 < \beta |L_s/L_n|^2$. We note that some of the above limitations are due to the analytical treatment only. Thus, a numerical solution of Eqs. (1) and (2) should demonstrate the existence of these modes in a broader region of parameter space. Such an investigation is in progress.

In conclusion, (i) we have demonstrated that in contrast to the electrostatic results there do exist unstable resistive drift eigenmodes in a slab plasma with sheared magnetic fields when finite- β effects are included. (ii) We have discovered the existence of an unstable resistive Alfvén eigenmode with frequencies lower than ω_{e*} . It is fairly certain that collisionless analogues of both of these modes exist; this is under further investigation. Finally, the parameter regime of these

instabilities, appears to be such that they should play an important role in current and future tokamak experiments.

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⁴ The absence of unstable eigenmodes is related to the Presence of shear damping. As shown in Ref. 3, if shear damping is nullified by either toroidal couplings or sharp ω_{e*} variations, then unstable eigenmodes do exist.

⁵ The two modes discussed here are the eigenmodes corresponding to the two modes obtained by Mikhailovsky in the local theory of drift wave and Alfvén wave coupling in a resistive plasma. The local theory gives growth rates for both modes proportional to v_{ei} , in contrast to the $v_{ei}^{1/2}$ scaling predicted here for the $n = 0$ eigenmode. For higher eigenmodes the growth rates indeed scale as v_{ei} but those modes are weaker and less important. In the text, we, therefore, only discuss the $n = 0$ eigenmode in detail.

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