RECEIVED BY TIC JUL 31 1978
JULY 1978

PPPL-1463

DC-20g

NUMERICAL DETERMINATION
OF AXISYMMETRIC TOROIDAL
MAGNETOHYDRODYNAMIC EQUILIBRIA

PLASMA PHYSICS
LABORATORY

MASTER

REPRODUCTION OF THIS DOCUMENT IS UNLIMITED

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

This work was supported by the U. S. Department of Energy
Contract No. ET-76-C-01-3073. Reproduction, translation,
publication, use and disposal, in whole or in part, by or
for the United States Government is permitted.

Numerical Determination of Axisymmetric Toroidal
Magnetohydrodynamic Equilibria

J.L. Johnson,* H.E. Dalhed, J.M. Greene, R.C. Grimm,
Y.Y. Hsieh, S.C. Jardin, J. Manickam, M. Okabayashi,
R.G. Storer,[†] A.M.M. Todd, D.E. Voss, and K.E. Weimer

Plasma Physics Laboratory, Princeton University
Princeton, New Jersey 08540

Numerical schemes for the determination of stationary axisymmetric toroidal equilibria appropriate for modeling real experimental devices are given. Iterative schemes are used to solve the elliptic non-linear partial differential equation for the poloidal flux function Ψ . The principal emphasis is on solving the free boundary (plasma-vacuum interface) equilibrium problem where external current-carrying toroidal coils support the plasma column, but fixed boundary (e.g., conducting shell) cases are also included. The toroidal current distribution is given by specifying the pressure and either the poloidal current or the safety factor profiles as functions of Ψ . Examples of the application of the codes to tokamak design at PPPL are given.

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

124

I. INTRODUCTION

The determination of MHD equilibrium configurations is an integral part of the problem of designing plasma confinement devices and interpreting their operational results. Even in tokamaks with circular plasma cross section, finite pressure effects make the magnetic surfaces nonconcentric so that full solution of the partial differential equation is important [1,2]. Interest [3,4] in tokamaks with noncircular cross sections makes these calculations even more essential. Detailed parametric studies of high- β tokamak properties [5] may determine the criteria for design of the next generation of tokamak devices. Equilibrium calculations play a major role in reactor studies where a knowledge of the plasma configuration (produced, for example, by divertors) provides the basic framework for detailed system design.

Very little can be done on this problem analytically. A few exact equilibria [6] have been obtained in simple closed form, but they are usually associated with nearly constant toroidal current inside the plasma and have little shear. More complicated analytical solutions [7] are only of limited value in detailed tokamak design. Furthermore, analytic solutions give little information for real systems where the currents in coils outside the plasma must be specified. Numerical calculations are ideally suited to these design problems.

Numerous codes have been developed to study this problem. Laing et al., [8] first published results of numerical work on axisymmetric toroidal equilibria; they introduced a large aspect ratio expansion to express the solution in a series of terms associated with a one-dimensional cylindrical problem and numerically evaluated higher order effects [9]. Killeen and Whiteman [10] tackled the full two-dimensional case numerically using ADI to solve the resulting finite difference equations. Marder and Weitzner [11] used similar techniques for solution of the difference equations but introduced a three-level iteration scheme for the nonlinear terms and investigated bifurcation phenomena in the Astron. These methods have been extended by Suzuki et al., [12] for free-boundary tokamak calculations. Callen and Dory [13] and Chu et al., [14] used SOR methods to solve the difference equations and used those codes to investigate noncircularity of the plasma cross section and limitations on pressure.

To our knowledge, the earliest works on the free-boundary cases are those of Sheffield [15], who used a Gauss-Seidel iteration and applied the code to design work for PLT, and Von Hagenow and Lackner (see Feneberg and Lackner [16]), who developed codes similar to those described here. Solution techniques utilizing similar iterative methods have been reported by Cenacchi et al., [17] and Sakurai et al., [18].

This paper is not intended to be a review of previous work. Two particularly good discussions of the equilibrium problem and its solution have recently appeared. MacNamara [19] briefly reviews tokamak equilibrium calculations and emphasizes the computation of 3D equilibria in systems with nonisotropic pressure. Lackner [20] reviews axisymmetric toroidal calculations with particular emphasis on iterative schemes for the nonlinearities and gives an excellent discussion of the treatment of plasma shaping. Our intention is to describe the equilibrium section of the Princeton Equilibrium, Stability and Transport package (PEST) [21] for tokamak calculations, to describe some applications of the code as it has been used for tokamak design at Princeton, and to present recent extensions which make it possible to calculate equilibria with given safety factor profiles or asymmetric cross sections.

The equilibrium problem in a toroidal axisymmetric system consists of solving an elliptic partial differential equation for the poloidal flux function $\Psi(X,Z)$, subject to proper boundary conditions outside the plasma. The typical geometry is shown in Fig. 1. We use a Green's function formulation to determine Dirichlet boundary conditions on the edge of a computational domain and finite difference methods to solve the partial differential equation. Since the problem is highly nonlinear, it is necessary to employ iterative procedures.

The problem is most easily solved when we prescribe the pressure $p(\Psi)$ and a toroidal field function $g(\Psi)$ such that

$B_\phi = RB_0 g(\psi)/X$ with B_0 the externally imposed vacuum toroidal field at $X = R$. The cylindrical coordinates (X, ϕ, Z) are shown in Fig. 1. For completeness, we describe the program using this prescription in the next section. In Sec. III we describe the modifications that are necessary when we prescribe the pressure and the safety factor. This is a useful way to formulate the problem since the safety factor is determined experimentally and used in conceptual designs. Such a formulation has been described by Grad et al. [22].

Sometimes one prefers to prescribe the shape of the plasma boundary rather than the specific coil configuration. We describe in Sec. IV how this can be done. If the equilibrium does not possess mirror symmetry with respect to the midplane of the torus, special care must be taken to achieve convergence. This problem can be solved by using some auxiliary "feedback" coils as described in Sec. V [11].

A partial understanding of the stability properties of the system can be obtained by evaluating the functions D_I and D_R , associated with stability criteria [23] for localized ideal and resistive interchange modes. This involves evaluation of a set of surface integrals on the magnetic surfaces. Thus, although stability properties are outside the scope of this paper, local stability criteria are computationally part of the equilibrium. We treat this in Sec. VI. In Sec. VII we describe some calculations that demonstrate that the code can represent analytic configurations and describe sensitive equilibrium functions. This gives some credence to its predictions.

Finally, to show the capability of the code, application is made to several tokamak configurations in Sec. VIII.

II. SOLUTION PROCEDURE WITH $p(\Psi)$ AND $g(\Psi)$ SPECIFIED

A simple description of a tokamak configuration is given by the MHD equations [24] for force balance,

$$\nabla p = \mathbf{J} \times \mathbf{B} \quad , \quad (1)$$

$$\mathbf{J} = \nabla \times \mathbf{B} \quad , \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad . \quad (3)$$

In this model, $\mathbf{B}$ and $\mathbf{J}$ lie on the nested toroidal surfaces of constant pressure. Since the configuration is axisymmetric, it is convenient to write

$$\mathbf{B} = \frac{1}{2\pi} \nabla \phi \times \nabla \Psi + R \mathbf{B}_0 g \nabla \phi \quad (4)$$

where $\Psi = (1/2\pi) \int \mathbf{B} \cdot \nabla \phi d\tau$ measures the poloidal magnetic flux between the axis of symmetry and the magnetic surface of interest. It is related to the ϕ component of the magnetic vector potential $\mathbf{A}$ through

$$\Psi = 2\pi \times A_\phi \quad . \quad (5)$$

Here ϕ is the ignorable coordinate, and ψ is an angle coordinate which increases by 2π in going once the short way around the torus. Since Eq. (1) implies that $\vec{B} \cdot \nabla p = 0$, we have $p = p(\psi)$ only.

The equilibrium equation, obtained from Eqs. (1), (2), and (4) is

$$X \frac{\partial}{\partial X} \frac{1}{X} \frac{\partial \psi}{\partial X} + \frac{\partial^2 \psi}{\partial Z^2} = 2\pi X J_\phi, \quad (6)$$

with

$$J_\phi = -2\pi \left(X \frac{dp}{d\psi} + \frac{R^2 B_o^2}{2X} \frac{dg^2}{d\psi} \right). \quad (7)$$

When the plasma is surrounded by a vacuum region, J_ϕ is nonzero only in the interior of the plasma where $\psi \leq \psi_\ell$, and $p(\psi) = 0$ for $\psi \geq \psi_\ell$. The self-consistent determination of this plasma-vacuum interface, $\psi = \psi_\ell$, is an important part of the solution. It introduces a fundamental nonlinearity into the problem, even if $p(\psi)$ and $g(\psi)$ are chosen to depend linearly on ψ . The physical boundary conditions for Eq. (6) are that ψ be constant on some contour if the system is surrounded by a conducting wall, or that ψ vanishes at $X = 0$ and ∞ .

The iterative procedure employed to solve the problem is illustrated schematically in Fig. 2. It consists basically of two nested loops; an outer loop (labeled m) in which approximate Dirichlet boundary conditions for ψ are determined for the computational domain R , and an inner loop (labeled n) which

iterates over the nonlinear "source" term, XJ_ϕ , and, hence, converges to a solution of Eq. (6) with fixed boundary values on ∂R .

The iteration procedure is conveniently divided into the following steps:

Step A: Initialization

Choosing a good initial approximation to the solution is an important aspect of the problem, both with regard to minimization of the computer time required and because multiple (bifurcated) equilibria sometimes exist with this formulation of the problem. Such alternate solutions, although they satisfy the same boundary conditions, will usually differ with respect to some equilibrium property such as the location of the magnetic axis or the value of ψ_ℓ . By careful tailoring of the initial conditions, the iteration procedure can be directed towards finding the equilibrium of interest.

In the usual finite difference notation (subscripts i, j represent grid points, superscripts m, n represent iteration number labels) we write the initial current approximation as

$$J_{i,j}^{(1,0)} = \begin{cases} \frac{c}{X} \left(1 - \frac{r^2}{D^2} \right) & \text{if } r^2 < D^2 \\ 0 & \text{if } r^2 \geq D^2 \end{cases} ,$$

where $r^2 \equiv (x_{i,j} - x_g)^2 + (z_{i,j} - z_g)^2$ and x_g, z_g , and D define input parameters specifying the center and radius of the initial current distribution. The factor c is adjusted to fix the given total toroidal current. Alternatively, $\psi_{i,j}^{(1,0)}$ can be specified as the initial conditions.

Step B: Computation of Boundary Flux Values $\psi_b^{(m)}$

We divide ψ on the boundary ∂R of the computational domain R into its two contributing parts

$$\psi_b^{(m)} = \psi_b^{\text{coils}} + \psi_b^{(m,0)}, \quad \{b\} \in \partial R \quad (8)$$

where ψ_b^{coils} represents the contribution from the currents flowing in external toroidal coils, and $\psi_b^{(m,0)}$ represents the contributions from the plasma currents using the latest approximation to J_ϕ as the calculation proceeds.

Both terms are computed using the appropriate Green's function for a toroidal current source [25];

$$G(X, Z | X', Z') = (2\pi\kappa)^{-1} (XX')^{1/2} [(2 - \kappa^2)K(\kappa^2) - 2E(\kappa^2)] \quad (9)$$

where

$$\kappa^2 \equiv 4XX' / [(X + X')^2 + (Z - Z')^2],$$

and $K(\kappa^2)$ and $E(\kappa^2)$ are the usual elliptic integrals. Thus

$$\begin{aligned}\psi_b^{\text{coils}} &= 2\pi \sum_{\ell} I_{\ell} G(x_b, z_b | x_{\ell}, z_{\ell}) , \\ \psi_b^{(m,0)} &= 2\pi \Delta X \Delta Z \sum_{i=1,I} \sum_{j=1,J} J_{\phi i,j}^{(m-1)} G(x_b, z_b | x_{i,j}, z_{i,j}) .\end{aligned}\tag{10}$$

The contribution from the external coils in Eq. (8) is computed only at the beginning of the calculation. Likewise, the matrices of the Green's functions of Eq. (9) need only be evaluated initially and stored for use later in the computation. Updating of $\psi_b^{(m)}$ is then reduced to a simple matrix multiplication.

Also, we have implemented the Lackner and Von Hagenow formulation, which uses Green's theorem, to replace the computation of $\psi_b^{(m,0)}$ of Eq. (10) by an integral over the boundary of the computational mesh [20];

$$\psi_b^{(m,0)}(x', z') = \oint \frac{d\ell}{X} G(x, z | x', z') \frac{\partial u(x, z)}{\partial n} .\tag{11}$$

Here u is the solution of Eq. (6) with $u = 0$ on ∂R , which is obtained in Step D. This can reduce machine time significantly, especially when large meshes are used.

It is sometimes desirable to describe the external vacuum field in a more abstract form. An approach which has been used successfully (see Sec. VIII, C) consists of employing the first few terms of a multipole moment expansion for the vacuum external flux; we take

$$\begin{aligned} \psi_b^{\text{coils}} = & 2\pi \left[\psi_0 + \frac{1}{2} B_D (X^2 - R^2) + \frac{1}{4R^2} B_Q [4X^2 Z^2 - (X^2 - R^2)^2] \right. \\ & \left. + \frac{B_H}{6R^4} [8X^2 Z^4 - 12(X^2 - R^2)X^2 Z^2 + (X^2 - R^2)^3] \right] \end{aligned} \quad (12)$$

with

$$\psi_0 = \left(B_D + \frac{1}{2} B_Q + \frac{1}{3} B_H \right) R^2 / 2 .$$

The external vacuum field is thus characterized by three parameters; B_D , B_Q , and B_H -- the dipole, quadrupole, and hexapole moments, respectively. In practice it seems reasonable to neglect higher order moments since to adjust these experimentally would require the installation of poloidal field coils close to the plasma surface. In the calculation, we then use Eq. (12) in place of Eq. (10) to determine the contribution to the boundary flux from the external vacuum field.

Step C: Updating the Current Distribution J_ϕ

As the iteration proceeds, the latest values of $\psi_{i,j}^{(m,n)}$ are used to compute the values of the inhomogeneous right-hand side of Eq. (6) using Eq. (7). Thus, at each grid point lying inside the closed contour $\psi_l^{(m,n)}$, we take

$$J_{i,j}^{(m,n)} = -2\pi \left[x_i \left(\frac{dp}{d\psi} \right)_{i,j}^{(m,n)} + \frac{R^2 B_Q^2}{2x_i} \left(\frac{dg^2}{d\psi} \right)_{i,j}^{(m,n)} \right] , \quad (13)$$

where the notation used for the quantities in parentheses indicates evaluation of $dp/d\psi$ and $dg^2/d\psi$ using the latest known values $\psi_{i,j}^{(m,n)}$. The simplest forms we take for p and g are

$$p = p_0 \psi_s^\alpha, \quad (14)$$

and

$$g = 1 - \gamma \psi_s^\beta, \quad (15)$$

where

$$\psi_s \equiv (\psi_\ell - \psi) / (\psi_\ell - \psi_a).$$

Here p_0 , the plasma pressure at the magnetic axis, α , and β are the input parameters, and ψ_a is the value of ψ at the magnetic axis, which changes during the iteration, as does ψ_ℓ . The value of γ is adjusted at each step of the calculation to keep the total toroidal current I constant [26]. Evaluating the integral over the plasma cross section in $I = \int_p J_\phi dS$ by quadratures using Eqs. (13) through (15) leads to a simple quadratic equation for γ . We always take the root with smallest absolute value.

This step is bypassed on the first cycle through the inner loop. The determination of ψ_ℓ in Eqs. (14) and (15) is given in Step E below.

Step D: Solution of the Finite Difference Equations

We use centered difference equations for representing the partial derivatives in Eq. (6),

$$\begin{aligned} \frac{(\psi_{i+1,j}^{(m,n)} - 2\psi_{i,j}^{(m,n)} + \psi_{i-1,j}^{(m,n)})}{\Delta X^2} - \frac{1}{X_i} \frac{(\psi_{i+1,j}^{(m,n)} - \psi_{i-1,j}^{(m,n)})}{2\Delta X} \\ + \frac{(\psi_{i,j+1}^{(m,n)} - 2\psi_{i,j}^{(m,n)} + \psi_{i,j-1}^{(m,n)})}{\Delta Z^2} = 2\pi X_i J_{i,j}^{(m-1,n)} \end{aligned} \quad (16)$$

with $\psi_{i,j}^{(m,n)} = \psi_b^{(m)}$ on the boundaries $\{i,j\} \in \partial R$. These equations are solved using double cyclic reduction [27] in a form only slightly different from the usual application to Poisson's equation in cylindrical coordinates [28].

Step E: Computing Ψ on the Plasma-Vacuum Interface

To fully determine $p(\Psi)$ and $g(\Psi)$ in Eqs. (14) and (15) or a similar prescription, the current value of ψ_ℓ must be determined at each iteration. This is accomplished by introducing material or magnetic limiters.

Material limiters are specified geometrically as straight line sections, and the plasma-vacuum interface is defined to be the maximum $\Psi = \text{constant}$ contour which makes contact with any limiter. From a practical point of view, it is preferable to limit the solution with horizontal (rather than vertical) limiters unless some special procedure is used to fix the position of the magnetic axis during the iterations. Vertical limiters couple the iterative changes in major and minor plasma radius, and this can lead to nonconvergence [17]. With horizontal limiters, a large change in the radial position of the plasma column (as may occur with a bad initial guess) can be accommodated without changing the minor radius.

Locating a magnetic limiter or separatrix, if one exists, is more involved. It is accomplished by searching for the Ψ values associated with stagnation (or saddle) points in $\Psi(X, Z)$ [29]. A cubic Lagrange interpolating polynomial is used to define Ψ between mesh points. The minimum Ψ over all such

stagnation points can be taken as the limiting flux surface. Frequently, the value for the plasma-vacuum interface, ψ_ℓ , is set slightly smaller than this value so that the separatrix surface lies in the vacuum region.

Step F: Convergence of the Inner Loop

To test for convergence of the inner iteration loop, we compute

$$A^{(m,n)} = \|\psi^{(m,n)} - \psi^{(m,n-1)}\|_1 / (\psi_\ell - \psi_a) \quad (17)$$

and compare it against the relative error in the $(m-1)^{st}$ boundary outer iteration $B^{(m-1)}$, discussed in Step G. If

$$A^{(m,n)} \leq 0.1 B^{(m-1)},$$

then the inner loop convergence test has been passed. The rationale for comparing against the boundary error instead of a fixed number is to avoid wasting computer time by iterating in one part of the calculation to a very fine tolerance while a substantial error exists in an outer iteration loop.

Step G: Convergence of the Outer Loop

The boundary values are considered to have converged when

$$\left| \frac{\psi_{i,j}^{(m,0)} - \psi_{i,j}^{(m-1,0)}}{\psi_\ell - \psi_a} \right| \leq \epsilon_b, \{i, j\} \in \partial R, \quad (18)$$

where ϵ_b is an input tolerance (usually 10^{-4}). Also computed at this stage is the quantity $B^{(m)}$ used in Step F, which is the simple average over all the boundary grid points of the left-hand

side of Eq. (18).

The effectiveness of these numerical procedures is, of course, problem dependent. Over the five to six years during which the code has been applied extensively to many different tokamak configurations, it has been modified frequently to handle special cases, and the present version, which incorporates the features of this section, is capable of handling a wide variety of applications. Typical calculations, using a 65×65 mesh, involve the order of 5 to 30 inner and outer iterations for 4 digit convergence. A complete calculation, including local stability calculations (see Sec. VI), usually can be carried out in 30 seconds on a CDC 7600 computer, provided a sensible initial approximation is used. Frequently, production calculations can be achieved in half this time.

III. SOLUTION PROCEDURE WITH $p(\Psi)$ AND $q(\Psi)$ SPECIFIED

In typical tokamaks $g(\Psi)$ does not vary much inside the plasma and is not amenable to measurement. We describe a modification in which the safety factor $q(\Psi)$, a function that is readily inferred from experimental measurements and used in most reactor design considerations, is prescribed. Further, it is the appropriate quantity to specify when computing a series of equilibria from flux conserving principles [5].

We find it convenient to work in a ψ, θ, ϕ coordinate system [21] in which the magnetic field lines appear straight. Thus,

we introduce a new flux label $\psi \equiv (\int d\tau/x^2)/(\int_p d\tau/x^2)$ and a coordinate, such that

$$\Psi = 2\pi B_0 \int_0^\psi f(\psi) d\psi + \Psi_a, \quad (19)$$

and

$$J \equiv (\nabla\psi \times \nabla\zeta \cdot \nabla\phi)^{-1} = v x^2 / 2\pi R \quad (20)$$

where

$$v \equiv (R/2\pi) \int_p d\tau/x^2.$$

Then,

$$f(\psi) = v g(\psi) / 2\pi f(\psi). \quad (21)$$

Integrating Eq. (7) over a magnetic surface gives a differential constraint [22] which the equilibrium solution must satisfy on each magnetic surface,

$$2\pi R^2 g(\psi) g'(\psi) = - \frac{R}{v B_0^2} V'(\psi) p'(\psi) - f(\psi) [\kappa(\psi) f(\psi)]', \quad (22)$$

where the prime denotes differentiation with respect to ψ .

Here $V(\psi)$ is the volume inside the surface and

$$\kappa(\psi) \equiv \oint |\nabla\psi|^2 d\zeta.$$

Using Eq. (21) to eliminate $g(\psi)$ from Eq. (22) gives a first-order differential equation for $f(\psi)$ in terms of $p(\psi)$ and $q(\psi)$,

$$f'(\psi) = - \frac{(2\pi R_0 V' / B_0) dp/d\psi + v^2 f \kappa' + 16\pi^4 R^2 f^2 q B_0 dq/d\psi}{8\pi^3 R^2 q^2 + v^2 \kappa} \quad (23)$$

with the boundary condition

$$f(1) = v/2\pi q(1). \quad (24)$$

Similarly, using Eq. (22) to eliminate g from Eq. (7) gives

$$J_\phi = \frac{2\pi}{X} \frac{dp}{d\psi} \left(\frac{RV'}{2\pi v} - X^2 \right) + \frac{B_0}{2\pi X} (\kappa f)' \quad (25)$$

This form is particularly instructive since it clearly separates the Pfirsch-Schlüter current [30] (first term) from the Ohmic heating current (second term). Since the net current inside a surface is implicitly determined by the specification of the safety factor q , the total discharge current $I = B_0 (v/2\pi R) \kappa(1) f(1)$ must be allowed to change during the course of the iteration, and the position of the magnetic axis will move. We prevent such motion by scaling the current in the vertical field coils. We also adjust the current in the divertor coils to position the separatrices.

The incorporation of specified safety factor profiles within the framework of Fig. 2 is straightforward. In Step C we use Eq. (25) rather than Eq. (7). The latest approximation for f must be renormalized at each iteration in Step E to maintain $\psi_b - \psi_a$ fixed. After exiting from the inner loop in Step F, we call a mapping routine [21] to determine $v, V(\psi)$, and $\kappa(\psi)$. We

then solve the coupled Eqs. (19) and (23) iteratively, using a Runge-Kutta technique, to determine $\Psi(\psi)$ and $f(\psi)$. In practice $f(\psi)$ is under-relaxed to damp oscillations in the iteration procedure. If the position of the magnetic axis or the location of a separatrix has changed significantly, we modify the current in the vertical field coils or the divertor coils after this step. In the convergence tests of Step G, we test that the $q(\Psi)$ returned by the mapping procedure agrees with the initial specification.

All of the surface functions $\Psi(\psi)$, $f(\psi)$, $q(\psi)$, $\kappa(\psi)$, invariably converge much faster than the solution itself. A typical free-boundary calculation requires 50 to 100 outer iterations taking about 7 minutes on a CDC 7600. The large increase in time over cases with $q(\Psi)$ specified is due to the addition of the mapping procedure and the need for more outer loops to find the appropriate external field. Fixing the position and the shape of the plasma boundary can reduce the computing time by a factor of 3 to 5 by providing rapid convergence of the outer loop.

IV. SOLUTIONS WITH PLASMA BOUNDARY SPECIFIED

For many purposes it is useful to prescribe the shape of the plasma by making $\Psi(X, Z)$ a constant, Ψ_s , on a specified surface. The implementation of such a boundary condition has been handled directly using finite difference equations [13]. Since in practice the plasma is shaped by external fields produced by currents flowing in external conductors, it is of considerable interest to determine the positions and the currents in the coils as part of the solution. We describe a simple

extension of the procedure of Sec. II which allows us to do this rapidly by imposing boundary conditions on the edge of a rectangular domain and employing double cyclic reduction to solve Eq. (16).

We define the plasma surface by a set of M points X_j, Z_j which are usually distributed with equal increments of arc length. At each step of the iteration, we compute $\psi_p^{(m)}(X_j, Z_j)$, the values of ψ at each of these points. In general $\psi_p^{(m)}(X_j, Z_j)$ will not equal the desired solution ψ_s . We modify the external field to produce an additional contribution on the plasma boundary

$$\psi_{\text{ext}}^{(m)}(X_j, Z_j) = \psi_s - \psi_p^{(m)}(X_j, Z_j) . \quad (26)$$

To determine the necessary external currents, we solve the overdetermined problem obtained by finding the least square error in the poloidal flux at the M points, X_j, Z_j produced by currents in N coils at set locations X_i, Z_i with $M \geq N$. Since the flux at X_j, Z_j due to the current I_i in the coil at X_i, Z_i is $\psi_i(X_j, Z_j) = G(X_i, Z_i | X_j, Z_j) I_i$, we minimize

$$\begin{aligned} \bar{\epsilon} = & \sum_{j=1}^M \left| \sum_{i=1}^N \psi_i(X_j, Z_j) - \psi_{\text{ext}}(X_j, Z_j) \right|^2 \\ & + \gamma L \sum_{i=1}^N I_i^2 + \lambda L \sum_{i=1}^N I_i , \end{aligned} \quad (27)$$

with respect to I_i, ψ_s , and λ . The regularization parameter γ [19,31] is introduced to stabilize the solution procedure against

large oscillations in I_i . The parameter L provides a normalization between the terms. The last term can be used to constrain the total current in all the coils to zero with λ an undetermined Lagrange multiplier. If we do not require this constraint, we can set $\lambda = 0$. Similarly, we can specify Ψ_s , reducing the number of unknowns and variational equations by one. We invert the matrix equation which minimizes Eq. (27) to obtain I_i , Ψ_s , and λ . In practice it is necessary to iterate this step to reduce the second term in Eq. (27). This is achieved by using a sequence of values for γ ,

$$\gamma_{k+1} = \gamma_k (\epsilon_b / \epsilon_k)^{1/2}, \quad (28)$$

where

$$\epsilon_k \equiv || \psi_{\text{ext}}^k(x_j, z_j) - [\psi_s - \psi_p^k(x_j, z_j)] || / |\psi_l - \psi_a|.$$

The iteration is stopped when $\epsilon_k \leq \epsilon_b$.

To implement this, we use the standard free-boundary code, incorporating Step B into the inner loop and calculating the I_i 's in Step C each time the flux, and thus the plasma current, is updated. The procedure converges rapidly so that we can get solutions with deviations of Ψ from Ψ_s on the surface of order 10^{-4} in less than ten iterations.

V. NON-Z SYMMETRIC EQUILIBRIA

Several modifications of the code are necessary for the computation of free surface equilibria which are not symmetric about the plane $Z = 0$.

Determination of the location of the magnetic axis and the positions of the separatrices in Step E of Fig. 2 requires a more general procedure than that discussed in Sec. II. Each grid point of interest must be tested against its nearest neighbors to locate the regions where $|\nabla\psi|^2$ is a local minimum. At these points, we compute the centered difference approximation to

$$S(X_i, Z_i) = \left(\frac{\partial^2 \psi}{\partial X^2} \right) \left(\frac{\partial^2 \psi}{\partial Z^2} \right) - \left[\frac{\partial^2 \psi}{\partial X \partial Z} \right]^2. \quad (29)$$

If $S > 0$, a magnetic axis exists in the vicinity; if $S < 0$, there is a nearby separatrix. The location of the axis or separatrix is determined by quadratic interpolation.

A more significant modification to the standard procedure of Fig. 2 is needed to prevent the magnetic axis from drifting off in the Z direction as the iterations proceed. This problem is avoided in symmetric calculations where the magnetic axis is constrained to lie on $Z = 0$. It can be solved by incorporating some additional external coils which provide a field in the plasma region that is essentially horizontal. If, in the course of iterating, the magnetic

axis tends to drift from a preselected vertical position, the currents in these coils are adjusted and new boundary values ψ_b calculated (Step B of Fig. 2). In practice, the feedback algorithm

$$\Delta I_{\text{coil}} = - \text{Sgn}(Z_{\text{coil}}) [C_1(z^{(n)} - Z_G) + C_2(z^{(n)} - z^{(n-1)})] I, \quad (30)$$

where Z_G is the desired vertical position of the magnetic axis, $z^{(n)}$ and $z^{(n-1)}$ are the current and old positions, and I is the total plasma current, has proven effective. The values of the constants C_1 and C_2 are chosen by trial and error. This procedure has the attribute that it not only determines interesting equilibria, but it provides information concerning the locations and the magnitudes of feedback currents that would be required to maintain the axis at any desired location.

This use of feedback coils bears some resemblance to the procedure described in Sec. IV where additional external coils are employed to shape the plasma surface. Indeed, we have used the techniques of that section to compute arbitrarily shaped non-Z-symmetric equilibria. In this case the position of the magnetic axis can not be prescribed.

VI. CALCULATION OF LOCAL STABILITY CRITERIA

Having calculated an equilibrium, one is interested in its properties. One such property, the spectrum of small amplitude

oscillations, including instabilities, can be calculated by a finite element procedure as discussed elsewhere [32]. However, some local criteria can be evaluated from integrals over the flux surfaces as part of the equilibrium calculation.

Those localized stability criteria, for stability against interchange modes that exist in an ideal model of the plasma [33,34] and for modes whose existence depends on resistivity [35], have been derived analytically on the assumption that the modes are strongly localized near some particular magnetic surface, the singular surface. This localization makes analysis feasible, but makes finite element calculations much more difficult. Thus, the numerically evaluated analytic criteria complement finite element treatments of stability.

In the notation of Glasser et al. [23], localized ideal instabilities occur if

$$D_I > 0 \quad (31)$$

where

$$D_I \equiv E + F + H - 1/4 ,$$

and localized resistive instabilities, if

$$D_R > 0 \quad (32)$$

where

$$D_R \equiv E + F + H^2 ;$$

$$E \equiv - \frac{p'v'}{q'} \left(\oint \frac{d\zeta}{B \cdot \nabla \zeta} \frac{B^2}{|\nabla \Psi|^2} \right) \left(\frac{v''}{v'} - q' \frac{2\pi Rg}{\oint d\zeta (B^2/B \cdot \nabla \zeta)} \right),$$

$$F \equiv \frac{4\pi^2 p'^2}{q'^2} \left[\oint \frac{d\zeta}{B \cdot \nabla \zeta} \frac{B^2}{|\nabla \Psi|^2} \oint \frac{d\zeta}{B \cdot \nabla \zeta} \frac{x^2}{|\nabla \Psi|^2} - R^2 g^2 \left(\oint \frac{d\zeta}{B \cdot \nabla \zeta} \frac{1}{|\nabla \Psi|^2} \right)^2 \right],$$

$$H \equiv \frac{2\pi Rg p'}{q'} \left(\oint \frac{d\zeta}{B \cdot \nabla \zeta} \frac{1}{|\nabla \Psi|^2} - v' \oint \frac{d\zeta}{B \cdot \nabla \zeta} \frac{B^2}{|\nabla \Psi|^2} / \oint \frac{d\zeta}{B \cdot \nabla \zeta} B^2 \right),$$

$$v' \equiv \oint \frac{d\zeta}{B \cdot \nabla \zeta}, \quad q \equiv \frac{Rg}{2\pi} \oint \frac{d\zeta}{B \cdot \nabla \zeta} \frac{1}{x^2},$$

and primes denote derivatives with respect to the poloidal flux Ψ .

The flux surface contour integrals in these expressions are evaluated as part of the mapping calculation [21] mentioned earlier in Sec. III. There we find it convenient for the purpose of contouring to introduce the special coordinate system in which surfaces of constant r are concentric circular toruses, and θ is a polar angle. Then

$$B \cdot \nabla \zeta = \frac{1}{2\pi rX} \left(\frac{\partial \Psi}{\partial X} \cos \theta - \frac{\partial \Psi}{\partial Z} \sin \theta \right), \quad (33)$$

and we evaluate the integrals as ordinary differential equations in θ . The quantities v' and q must be differentiated; cubic spline interpolation is adequate in most cases.

VII. CODE VALIDATION

An essential part of any code construction is the testing to determine its internal consistency and the accuracy with which it can represent a given set of physical conditions. This type of study is an ongoing process. We comment on two specific studies which have been done that give us some confidence in the efficacy of the code.

A. Calculation of an Analytic Equilibrium

There are very few useful toroidal equilibria for which exact analytic solutions can be obtained. Those that do exist have sharp discontinuities in the current distribution. Thus, they provide a severe test for a numerical program.

A useful model was found by Solov'ev [6]:

$$\psi = -\frac{2\pi^2 p'}{\alpha^2 + 1} \left[X^2 Z^2 + \frac{\alpha^2}{4} (X^2 - R^2)^2 \right] \quad (34)$$

with $p' \equiv dp(\psi)/d\psi$ a constant inside the plasma region and $g(\psi)$ of Eq. (4) constant. This has been studied using the code formulation described in Sec. IV to investigate the adequacy of the fixed-boundary formalism. For a typical case with $p' = -1/\pi^2$, $\alpha = 1$, $R = 2$, and the plasma boundary specified by setting $\psi = 1$, calculated on a 65×65 mesh using a convergence test such that ϵ_b of Eq. (18) is 10^{-4} , the maximum difference in the plasma region between the calculated and analytic values of ψ , normalized to the maximum current, is 10^{-4} . We can conclude that the error

associated with the use of a difference form for the differential operator is at least this small.

B. Shafranov Shift

A major concern in the calculation of free-boundary equilibria is that the magnetic axis find its proper location. The measurement of the shift of the centers of the magnetic surface with respect to this axis in a tokamak provides a sensitive test of this. For a low-beta large aspect ratio, nearly circular tokamak, the shift is given by [2]

$$\Delta(\psi) = \frac{U}{4\pi R} \int_0^\psi \frac{d\psi}{\psi^2 f^2} \int_0^\psi \psi f \left(f - \frac{2UR^2}{B_0} \frac{dp}{d\psi} \right) d\psi, \quad (35)$$

where we have used the coordinates and notation employed in Sec. III. Comparison of the calculated values of Δ with those obtained from Eq. (35) is given in Fig. 3 for a typical PLT (Princeton Large Torus) equilibrium. The agreement is remarkably good. On the other hand, similar calculations for very high-beta configurations or systems with strongly noncircular cross sections show larger discrepancies. These are largely due to the violation of the assumptions used in the derivation of Eq. (35).

These studies provide some feeling that the code can produce accurate equilibria, even for extreme conditions. Nevertheless, care must be taken in the interpretation of any application, particularly in the studies of high-beta tokamaks where the toroidal currents in the plasma are large and are far from constant on a

magnetic surface.

VIII. APPLICATIONS

We give here a few typical results obtained from the code. These calculations were made for specific configurations that have been built or are being considered.

A. The Princeton Large Torus (PLT)

The calculation of equilibrium configurations for the PLT is a good illustration of the use of the basic code when the position of the operative limiter and $p(\psi)$ and $g(\psi)$ are specified. We use the exact poloidal field coil configuration of the device, and the profiles of Eqs. (14) and (15) for p and g with the parameters inferred from Thompson scattering measurements of the density and temperatures together with the total current and the value of q at the axis. Flux contours corresponding to a typical Ohmically heated discharge are shown in Fig. 4 for a case where $\beta_p = 0.23$ with q varying from 1.05 at the magnetic axis to 5.3 at the limiter. Evaluation of the localized stability functions D_I and D_R as well as studies of global modes with the PEST code [21] indicates that this equilibrium should be stable with respect to ideal MHD modes. Detailed comparisons of the numerical results with analytic calculations [1,2] for this case have been carried out as part of an effort to improve the positioning of the plasma column during experimental operation [36].

Similar studies, using the q-solver formalism of Sec. III, to investigate the effect of rapid plasma heating on equilibrium and stability have been made for the Princeton Toroidal Fusion Test Reactor (TFTR) [37].

B. The Princeton Divertor Experiment (PDX)

The main application of the code to configurations where the plasma surface is determined by a magnetic separatrix has been in the design of the PDX device. A major problem in the design has been to insure that the plasma can be formed in a small region and heated to an interesting plasma regime such that the field lines inside the separatrix are kept away from any material wall for the duration of the experiment. Studies such as the one shown in Fig. 5 have made significant impact on the design of the poloidal field coil configuration in the device. Figure 5a shows the shapes of the magnetic surfaces early in the discharge when the plasma is cold; Fig. 5b describes the configuration while the plasma is being heated; and Fig. 5c represents the system during the high-current, quasi-steady-state period. These studies have shown how this can be achieved with the ratios of the currents in the different coils fixed so that elaborate timing to control the currents in each separate coil is not needed.

It is possible to use the PDX device to investigate the effect of the shape of the plasma column on the behavior of the plasma. Indeed, an early set of experiments will consist of using different magnitudes of the currents in the poloidal-field

coils such that the divertor is tightly tied to the inner set of coils, the outer ones, or to both. The equilibrium results shown in Fig. 6 demonstrate that this can be done. Evaluation of D_I and D_R shows that the system is stable with respect to localized criteria for the first two cases. It is interesting that "outside D's" like that of Fig. 6b can be found that have favorable D_I and D_R . Studies of the stability properties of these equilibria with respect to axisymmetric modes have been made [38].

C. High- β D-Shaped Tokamaks

The equilibrium codes described here have been used extensively for the conceptual design of toroidal devices with non-circular cross section plasmas. Here we give an example of a study of D-shaped plasma configurations, which could be operated in PDX size devices, where the toroidal field strength B_T is 25 kG at $R = 1.5$ m.

The numerical results with flux conserving constraints [5] are shown in Figs. 7 and 8. This sequence was obtained by increasing the parameter p_0 in Eq. (14), thus keeping the pressure profile the same. Here the pressure parameter α in Eq. (14) is 1.2, the external vacuum field moment parameters of Eq. (12) are $B_D = 1.0$, $B_Q = -0.12$, $B_H = -0.04$, and the safety factor q is 1.2 at the magnetic axis and 3.5 at the plasma-vacuum interface. The plasma major radius is kept at $R = 1.5$ m by multiplying the vacuum poloidal equilibrium field by a constant factor. This permits a slight change in the plasma cross section during the

sequence of equilibria described by Fig. 8. We see that, as pointed out recently [5,39], equilibria can be found with $\beta \sim 20\%$ with moderate increase in plasma poloidal beta β_p . The required field is increased nearly proportionally to the plasma pressure at higher β_p , as is the case in simple circular cross section plasmas [1,2].

An interesting result from these calculations is the large amount of inductive poloidal flux (volt sec) which must be produced by the Ohmic heating coil system. We note that a significant poloidal flux is produced by the increase of the vacuum equilibrium field needed to center the plasma. Thus, if flux conservation adequately describes plasma properties during the heating process, the actual flux due to the Ohmic heating coils must be reduced in order to maintain the total poloidal flux at the plasma boundary. Otherwise, considerable surface eddy currents may be excited during the heating process. These results suggest that a careful study of the poloidal field coil system should be carried out, including the impedance of the entire poloidal field coil system.

D. Asymmetric Cross Sections in PDX

The PDX device could be used to investigate the behavior of asymmetric configurations if the poloidal field coils are so arranged. A study of the equilibrium and stability properties, with special emphasis on the control of axisymmetric instabilities, is most essential. Such an equilibrium is illustrated in Fig. 9.

IX. DISCUSSION

Accurate and efficient numerical codes for the determination of magnetohydrodynamic equilibria, where the prescription of the boundary conditions and plasma parameters accurately represent the actual device, are necessary for the design of the large, expensive experimental devices needed to make progress in fusion research. Furthermore, the results of a calculation must be in a form that is amenable to physical interpretation as well as suitable for further study utilizing stability and transport codes. In this paper we have described the formulation of such a code that is readily accessible. It has been implemented on the United States Department of Energy, Office of Fusion Energy, Computer network as part of a complete stability package [21]. Here we have presented the algorithm used to solve the standard equilibrium problem where $p(\psi)$, $g(\psi)$, and the currents in the external coils are prescribed. We then showed how the code is changed to treat the case with $p(\psi)$ and $q(\psi)$ specified. Modifications to handle problems with a fixed plasma surface or equilibria that are not symmetric in Z were then given. We have applied the code to two known problems which show that the code can treat both fixed and free-boundary problems with some reliability. Even so, more work is necessary to investigate the quantitative dependence of the results on the error limits built into the code and on the mesh size. Several applications are given here which show that the code has already impacted on the design of experimental devices in the magnetic fusion energy program. It will be

used in the data reduction and analysis when the PDX and TFTR devices become operational.

ACKNOWLEDGMENTS

The program described here grew out of an equilibrium code developed by G. V. Sheffield. We are indebted to many members of the Princeton University Plasma Physics Laboratory staff, in particular, D. Meade, P. H. Rutherford, H. P. Furth, K. Bol, and R. Hawryluk for useful discussions of the applications. One of us (R.G.S.) thanks M. B. Gottlieb for the hospitality accorded him during his stay in Princeton.

This work was supported by the U.S. Department of Energy Contract EY-76-C-02-3073.

REFERENCES

*On loan from Westinghouse Research and Development Center,
Pittsburgh, PA.

†The Flinders Univ. of S. Australia, Bedford Park, South Australia.

1. V. S. Mukhavatov and V. D. Shafranov, Nucl. Fusion 11
(1971) 605.
2. J. M. Greene, J. L. Johnson, and K. E. Weimer, Phys. Fluids
14 (1971) 671.
3. L. A. Artsimovich and V. D. Shafranov, Pis'ma Zh. Eksp. Teor.
Fiz. 15 (1972) 72 [Sov. Phys.--JETP Lett. 15 (1972) 51].
4. H. P. Furth, Nucl. Fusion 15 (1975) 487.
5. Y.-K. M. Peng, R. A. Dory, D. B. Nelson, and R. D. Sayer,
Phys. Fluids 21 (1978) 467.
6. L. S. Solov'ev, Zh. Eksp. Teor. Fiz. 53, (1967) 626 [Sov.
Phys. JETP 26 (1968) 400].
7. E. Mazzucato, Phys. Fluids 18 (1975) 536.
8. E. W. Laing, S. J. Roberts, and R. T. P. Whipple, J. Nucl.
Energy C (Plasma Physics) 1 (1959) 49.
9. J. M. Greene and J. L. Johnson, Phys. Fluids 4 (1961) 875.
10. J. Killeen and K. J. Whiteman, Phys. Fluids 9 (1966) 1846.
11. B. Marder and H. Weitzner, Plasma Phys. 12 (1970) 435.
12. Y. Suzuki, A. Kameari, H. Ninomiya, M. Masuzaki, and H.
Tayana, in "Plasma Physics and Controlled Nuclear Fusion
Research," Vol. 1, p. 411, IAEA, Vienna, 1975.
13. J. S. Callen and R. A. Dory, Phys. Fluids 15 (1972) 1523.
14. M. S. Chu, D. Dobrott, T. H. Jensen, and T. Tamaro, Phys.
Fluids 17 (1974) 1183.
15. G. V. Sheffield (private communication).
16. W. Feneberg and K. Lackner, Nucl. Fusion 13 (1973) 549.

17. G. Cenacchi, R. Galvao, and A. Taroni, Nucl. Fusion 16 (1976) 457.
18. K. Sakurai, Y. Tanaka, and T. Okuda, Plasma Phys. 17 (1975) 261.
19. B. McNamara, in "Methods in Computational Physics," Vol. 16, p. 211, Academic Press, New York, 1976.
20. K. Lackner, Computer Phys. Commun. 12 (1976) 33.
21. R. C. Grimm, J. M. Greene, and J. L. Johnson, in "Methods in Computational Physics," Vol. 16, p. 253, Academic Press, New York, 1976.
22. H. Grad, P. N. Hu, and D. C. Stevens, Proc. Math. Acad. Sci. 72 (1975) 3789.
23. A. H. Glasser, J. M. Greene, and J. L. Johnson, Phys. Fluids 18 (1976) 875.
24. M. D. Kruskal and R. M. Kulsrud, Phys. Fluids 1 (1958) 265.
25. J. D. Jackson, "Classical Electrodynamics," p. 141, J. Wiley, New York, 1962.
26. G. Cenacchi, A. Taroni, and A. Sestero, Il. Nuovo Cimento 25B (1975) 279.
27. O. Buneman, Stanford Univ. Institute for Plasma Research Rept. SUIPR-294 (1968); unpublished.
28. R. W. Hockney, in "Methods in Computational Physics," Vol. 9, p. 135, Academic Press, New York, 1970.
29. S. C. Jardin, Ph.D. dissertation Princeton Univ. (1976) p. 252
30. L. Spitzer, Jr., Phys. Fluids 1 (1958) 253.
31. V. D. Shafranov and L. E. Zakharov, Nucl. Fusion 12 (1972) 599.

32. M. S. Chance, J. M. Greene, R. C. Grimm, J. L. Johnson, J. Manickam, W. Kerner, D. Berger, L. C. Bernard, R. Gruber, and F. Troyon, J. Comput. Phys. (in press).
33. C. Mercier, Nucl. Fusion Suppl. 2 (1962) 801.
34. J. M. Greene and J. L. Johnson, Phys. Fluids 5 (1962) 510.
35. J. L. Johnson and J. M. Greene, Plasma Phys. 9 (1967) 611.
36. D. E. Voss, K. Bol, and J. L. Johnson, Bull. Am. Phys. Soc. II 22 (1977) 1118.
37. A. M. M. Todd, M. S. Chance, J. M. Greene, R. C. Grimm, J. L. Johnson, and J. Manickam, Phys. Rev. Lett. 38 (1977) 826.
38. J. L. Johnson, M. S. Chance, J. M. Greene, R. C. Grimm, S. C. Jardin, W. Kerner, J. Manickam, and K. E. Weimer, in "Plasma Physics and Controlled Nuclear Fusion Research," Vol. 2, p. 395, IAEA, Vienna, 1977.
39. Y.-K. M. Peng and R. A. Dory, Nucl. Fusion 17 (1977) 21.

LEGENDS FOR FIGURES

Fig. 1 Computational domain R .

Fig. 2 Code flow chart. The dashed section is inserted when $p(\Psi)$ and $q(\Psi)$, rather than $p(\Psi)$ and $g(\Psi)$ are specified.

Fig. 3 Comparison of analytic and numerical calculations of the Shafranov shift for the PLT equilibrium of Fig. 4.

Fig. 4 A typical PLT equilibrium with $\beta_p = 0.23$ and $1.05 < q < 5.3$. The solid curve marks the position of the vacuum vessel. The pluses and minuses denote the poloidal field coils.

Fig. 5 Evolution of the flux surfaces as the current evolves in a PDX discharge; $\alpha = \beta = 1$ in Eqs. (14) and (15). (a) Low current, $I = 100$ kA, configuration early in the ignition. (b) Medium current, $I = 250$ kA, during the build up. (c) High current, $I = 500$ kA, steady-state equilibrium.

Fig. 6 Typical PDX equilibria, showing that the plasma can be attached to (a) the inside divertor coils or (b) the outside divertor coils. Here $\alpha = \beta = 2$.

Fig. 7 A typical D-shaped tokamak cross section used for studies of flux-conserving equilibria. This corresponds to $p_0 = 0.6 \times 10^6$ in Fig. 8.

Fig. 8 Equilibrium parameters for the flux-conserving tokamak sequence containing the equilibrium of Fig. 7. Here $\beta \equiv 2\langle p \rangle / \mu_0 B^2$, $\beta^* = 2\langle p \rangle^{1/2} / \mu_0 B^2$, and B_D is the dipole.

Fig. 9 A typical equilibrium, asymmetric in Z , that could be contained in the PDX device.

Fig. 1. 782150

Fig. 2. 782157

Fig. 3. 782151

Fig. 4. 782153

Fig. 5. 782155

Fig. 6. 782158

Fig. 7. 782154

Fig. 8. 782156

Fig. 9. 782152