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ALFVEN WAVE CYCLOTRON
RESONANCE HEATING

BY

R. B. WHITE, S. YOSHIKAWA,
AND C. OBERMAN

PLASMA PHYSICS LABORATORY

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R. B. White

Shoichi Yoshikawa

C. Oberman

Princeton Plasma Physics Laboratory

Princeton University

Princeton, NJ 08544

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R. B. White, Shoichi Yosikawa, and C. Oberman

Plasma Physics Laboratory, Princeton University

Princeton, New Jersey 08544, U.S.A.

Abstract

The resonance absorption of fast Alfvén waves at the proton cyclotron resonance of a predominately deuterium plasma is investigated. An approximate dispersion relation is derived, valid in the vicinity of the resonance, which permits an exact calculation of transmission and reflection coefficients. For reasonable plasma parameters significant linear resonance absorption is found.

I. Introduction

The propagation of Alfvén waves in plasmas is of considerable interest, since under suitable conditions these waves can lead to significant plasma heating. A method has been proposed for heating a two-component plasma using Alfvén waves at a hybrid resonance frequency, as discussed by Buchsbaum.¹ Experimental results, however, indicate that heating takes place predominately at the cyclotron resonance.² In the experiment of Ref. 2 a slow Alfvén wave was launched at a high magnetic field point and propagated along the field to a region of low magnetic field. This type of heating was called magnetic beach heating.

As the size of the plasma device increases, however, the coupling of the slow wave becomes increasingly difficult, because the phase velocity, v_{ph} , is

too small to satisfy

$$v_{ph} \gg \omega/k_{\perp} \quad (1)$$

where $k_{\perp}$ is the wave number perpendicular to the magnetic field. Thus, it would be more practical to launch a fast wave and have it absorbed by the plasma. The propagation of fast waves in a tokamak geometry has been discussed by many authors,^{3,4} but agreement with experimental results is not yet completely satisfactory.

The dispersion relation for the two component plasma may be expressed as

$$N_{\parallel}^2 = \frac{x+1+v_1}{(1+x)(2+x)} (N_{\perp}^2 + \frac{x-1-v_1}{(1-x)(2-x)}) + \frac{v_1}{1-x} + \frac{2v_2}{4-x^2} = 0 \quad (2)$$

Here N is the normalized wave vector $N^2 = k^2 v_A^2 / \omega^2$, where v_A is the Alfvén wave velocity associated with species 1 and v_1 is the fraction of species 1, and v_2 is that of species 2. For simplicity we will take the mass ratio M_2/M_1 to equal 2, i.e., we have in mind a hydrogen-deuterium plasma. Spatial dependence arises through the varying magnetic field, and we take as spatial variable x , the frequency normalized to Ω_1 , where $\Omega_1/2\pi$ is the cyclotron frequency of the species 1 ion. Obviously $v_1 + v_2 = 1$. Equation (2) has a double root at $x_0 = (1+3v_1)^{1/2}$, $N_{\parallel}^2 = (1/3)$, if $N_{\perp}^2 = 0$. For finite $N_{\perp}^2$ the roots are separated, at $x = x_0$, being $N_{\parallel}^2 = 1/3$ and $N_{\perp}^2 = 1/3 - N_{\parallel}^2$. There is a cutoff ($N_{\perp}^2 = 0$) at $x = \{[(1+v_1)^2 - 2N_{\perp}^2(1+v_1)]/[1 - N_{\perp}^2(2-v_1)]\}^{1/2}$. A sample dispersion relation is shown in Fig. 1. In this figure $N_{\parallel}^2$ can be interpreted either as a function of increasing ω or $1/B$. We concern ourselves with

geometries in which $N_{\perp}$ remains small, and $N_{\parallel}$ varies significantly spatially, probably best realized in mirror configurations, but also applicable to some tokamak heating geometries.

The objective of this paper is to explore the possibility of transmission from the fast wave (A in Fig. 1), which has a cutoff at the hybrid frequency, to the resonant branch (B in Fig. 1) which is absorbed at $\omega/\omega_1 = 2$ (i.e., at the cyclotron resonance of the lighter ions). Equation (2) does not readily admit an analysis of transmission and reflection properties, and we thus simplify it in order to obtain a model equation valid in the vicinity of the resonance.

11. Approximate Dispersion Relation

Equation (2) is not convenient as it stands because the second term, proportional to $N_{\perp}^2$, also vanishes at x_0 . Rewrite Eq. (2) in the form

$$N_{\parallel}^2 - x_1 - \alpha_1 N_{\parallel}^2 + \frac{\beta}{2-x} - \frac{N_{\perp}^4}{4} \\ + N_{\perp}^2 - \alpha \frac{2^{\frac{1}{2}} 3 (1+x_1)}{(1-x)(2-x)} + \alpha (x^3) = 0 \quad (3)$$

where

$$x_1 = -\frac{N_{\perp}^2}{2} \quad (4)$$

$$x_2 = -\frac{N_{\perp}^2 (1+x)}{2} \quad (5)$$

$$\gamma = - \frac{(1 - v_1)}{(1 + v_1)} \frac{N_1^2}{2} (1 - x) \quad (6)$$

$$\alpha = \frac{x + 1 + v_1}{(1 + x)(2 + x)} \quad (7)$$

$$\beta = \frac{1 + \epsilon_2 [x - (1 + v_1)(1 + \epsilon_3)]}{(1 - x)} \quad (8)$$

We are interested in Eq. (3) in the vicinity of the resonance and cutoff, both near $x = 1$ (see Fig. 1). In this region both α and β are slowly varying and can be evaluated at some particular point. Furthermore, the $N_1^2 - \alpha$ term is negligible as is readily shown. In the neighborhood of x_0 and moving away from x_0 on the asymptotically propagating branches of the dispersion relation (see Fig. 1), $N_1^2 - \alpha$ is of order N_1^2 , making the whole term of order N_1^4 . Thus, on these branches of the dispersion relation, for which the N_1^4 term in Eq. (3) is relevant, which will be seen to determine the reflection and transmission properties of the waves. On the other branches of the dispersion relation, since α, β are of order unity, the $N_1^2 - \alpha$ term produces corrections of order N_1^2 to N_1^2 , which are not relevant.

Since the α, β and δ are slowly varying in the vicinity of resonance and cutoff we evaluate them at $x = x_0$, giving for $N_1^2 \ll 1$ the dispersion relation

$$N_1^2 - g^2 N_1^2 + u \left(1 + \frac{A}{x - 2} \right) - \frac{N_1^4}{4} = 0 \quad (9)$$

with $g^2 = \epsilon_1 + \frac{1}{3}$, $u = \frac{1 + \epsilon_2}{x_0 - 1}$, $A = 1 - v_1 - \epsilon_3 - v_1 \epsilon_3$, all evaluated at $x = x_0$.

In a real experiment ω is kept constant and the magnetic field is a function of position. We have defined x to be the frequency normalized to the cyclotron frequency of species 1, or

$$x = \frac{B_0 \omega}{B_1} = \frac{B_0 \omega}{B_1 l_B} = \frac{B_0 \omega}{B_1} \frac{V_A}{l_B \omega} \quad \epsilon = 2 \frac{r}{r_0} \quad (10)$$

Here B_0 is the magnetic field for which the mass 1 (i.e., D^+) ion resonates with the applied field, l_B is the scale length for the field gradient, V_A the Alfvén velocity, and B_1 the field for which the cyclotron frequency agrees with the wave frequency, i.e., $B_0 = B_1$. Introducing the variable $\tau = \frac{l_B}{u} (t - t_0)$ finally reduces Eq. (9) to the form

$$x^2 + 1 + \frac{a}{\tau} = -k^2 + f^2 + \lambda^2 = 0, \quad (11)$$

with

$$a = \frac{1}{x_0} - \frac{N_1^2}{2} \frac{x_0^2}{1 + x_0^2}, \quad (12)$$

$$k^2 = \frac{1 + \frac{1}{2} l_B^2 l_B}{x_0 - 1} \frac{1}{V_A}, \quad 1 = 1 - \frac{1}{2} \lambda = \frac{1}{2} \lambda, \quad (13)$$

$$\lambda = \frac{x_0 - 1}{1 + \frac{1}{2} l_B^2 l_B} \frac{N_1^2}{2}, \quad (14)$$

and $k^2 = N_1^2/u$. This dispersion relation is shown in Fig. 2. It is seen to consist of a coupled fast wave ($k^2 = f^2$), and a damped Budden-like wave⁵ ($k^2 = -1 - a/\tau$). Since $N_1^2 \ll 1$ and $x_0 \approx 1/3$, $\lambda \ll 1$. The parameter a can vary

greatly depending on magnetic field gradient and wave frequency.

III. Linear Resonance Absorption

Using Laplace's method we write the wave amplitude as

$$\psi(\cdot) = \frac{1}{2\pi i} \int_{\gamma} dk \, \psi(k) e^{ik\eta}, \quad (15)$$

where the function $\psi(\eta)$ satisfies the differential equation

$$\left(\frac{d}{d\eta} \right)^2 + 1 + \frac{a}{c} \left(\frac{d}{d\eta} \right)^2 + f^2 + \lambda^2 \psi(\eta) = 0. \quad (16)$$

We are interested in the boundary conditions shown in Fig. 2, namely an incoming fast wave (dark arrow) and transmitted (labeled T) and reflected (labelled R) fast waves, with the generation of damped Budden waves.

The transmission and reflection properties for very small and very large λ are readily apparent from the dispersion relation. The case $\lambda = 0$ degenerates into a fast wave ($k^2 = f^2$) and a damped Budden branch ($k^2 = -1 - (a/c)$). For $\lambda > 0$ the branches are coupled as shown in Fig. 2 b,c. The turning point ($k^2 = 0$) occurs at $\eta_t = -a/(1 + c^2 f^2)$. For very large λ (Fig. 2c) we find two Budden branches, one propagating and one damped, but $\chi_c \neq 0$, and thus a wave will propagate through the resonance with no absorption or reflection. Thus, we expect complete transmission for $\lambda \neq 0$ and λ very large.

Substituting Eq. (15) into Eq. (16) we readily find the general solution

$$\psi(\cdot) = \frac{1}{2\pi i} \int_{\gamma} dk \, e^{ik\eta} + \int_{\gamma} k dk \, G(k') \quad (17)$$

with

$$G(k) = \frac{2k(f^2 - 1 - 2k^2) - ia(f^2 - k^2)}{(k^2 + 1)(k^2 - f^2) - \lambda^2}, \quad (18)$$

and integration contour c chosen to satisfy desired boundary conditions.

There are four integration contours, giving the four independent solutions for $n > 0$, and similarly four contours for $n < 0$. These are shown in Fig. 3. The contours circle the singularities of $G(k)$, given by $k = k_p$,

$$k_p^2 = \frac{f^2 - 1 \pm [f^2 + 1][1 + 4\lambda^2/(f^2 + 1)^2]^{1/2}}{2} \quad (19)$$

where the labels $p = 1 - 4$ are used for $n < 0$ and $p = 5 - 8$ for $n > 0$. (They are the same roots, the different labels are to distinguish the contours and the solutions valid in the two regimes.) Since $\lambda^2 > 0$, k_p^2 is real, with two real roots and conjugate imaginary roots. The small displacements from the axes shown are infinitesimal ones prescribed by causality.

The asymptotic behavior of the eight solutions $\psi_p(n)$, $p = 1, 2, \dots, 8$ for large $|n|$ is readily calculated, the major contribution to the integrals coming from the vicinity of the root k_p . In the vicinity of a root $k = k_p$ we find

$$G(k) \approx \frac{-1 + i\epsilon_p}{k - k_p} \quad (20)$$

with

$$\epsilon_p = \frac{a(f^2 - k_p^2)}{2k_p(f^2 - 1 - 2k_p^2)} \quad (21)$$

and thus

$$\psi(\eta) = \frac{1}{2\pi i} \int_{C_p} dk \frac{e^{ik\eta}}{(k - k_p)^{1-i\delta_p}} \quad (22)$$

For $\eta < 0$ write $k - k_p$ as $re^{(-\pi i/2)}$ for the upward part of the contour, and as $re^{(3\pi i/2)}$ for the downward part (Fig. 3). We then find for $p = 1 - 4$

$$\psi_p(\eta) = -\frac{(-\eta)^{i\delta_p} \Gamma(i\delta_p)}{2\pi i} e^{(-3\pi i/2)\delta_p} - e^{(\pi i/2)\delta_p} e^{ik_p \eta} \quad (23)$$

In passing to $\eta > 0$ the contours are rotated counterclockwise in the k -plane, and we thus write $k - k_p$ as $re^{(\pi i/2)}$ for the downward part of the contour and as $re^{(5\pi i/2)}$ for the upward part. For large $\eta > 0$ ($p = 5 - 8$) we then have

$$\psi_p(\eta) = \frac{(-\eta)^{i\delta_p} \Gamma(i\delta_p)}{2\pi i} e^{-5\pi i/2\delta_p} - e^{-\pi i/2\delta_p} e^{ik_p \eta} \quad (24)$$

To calculate the transmission and reflection coefficients we continue analytically from $\eta < 0$ to $\eta > 0$ as is done for the Rudden problem.^{3,6} Requiring that for $\eta < 0$ the solution be purely outgoing, we write

$$f(\eta) = A\psi_4(\eta) + B\psi_7(\eta) \quad (25)$$

Deforming the integration contours as η is rotated in the complex plane to the positive real axis we pass through the sequence shown in Fig. 4 to find for $\eta > 0$

$$\begin{aligned} \psi(\eta) = & A \psi_8(\eta) + [B + A(-1 + e^{2\pi\delta_1})] \psi_7(\eta) \\ & + \{B(-1 + e^{2\pi\delta_2}) + e^{2\pi\delta_2} A(-1 + e^{2\pi\delta_1})\} \psi_6(\eta) \end{aligned}$$

$$+ \{ B(-1 + e^{2\pi\delta_2}) + e^{2\pi\delta_2} A(-1 + e^{2\pi\delta_1}) \} \psi_5(\eta) . \quad (26)$$

Setting the coefficient of $\psi_7(\eta)$ equal to zero (exponentially increasing wave) determines the ratio A/B . Normalizing with respect to the incoming fast wave and discarding the damped Budden waves we find for large $|\eta|$, $\eta < 0$

$$\psi(\eta) = (-\eta)^{i\delta_1} T e^{-ik_1\eta} \quad (27)$$

and for $\eta > 0$

$$\psi(\eta) = n^{i\delta_1} e^{-ik_1\eta} + R n^{-i\delta_1} e^{ik_1\eta} \quad (28)$$

where

$$T = e^{-\pi\delta_1} \quad (29)$$

$$R = \frac{\Gamma(i\delta_1)}{\Gamma(-i\delta_1)} e^{-\pi\delta_1} - e^{-2\pi\delta_1} . \quad (30)$$

Just as in the Budden problem^{5,6} there is absorption at the resonance.

Shown in Fig. 5 are the transmitted and reflected intensities $|T|^2$ and $|R|^2$, and the absorbed intensity $|A|^2 = 1 - |T|^2 - |R|^2$, as a function of λ . Their analytic expressions are

$$|T|^2 = e^{-2\pi\delta_1} \quad (31)$$

$$|R|^2 = e^{-2\pi\epsilon} + 2e^{-4\pi\epsilon} + e^{-6\pi\epsilon} \quad (32)$$

$$|A|^2 = 1 + 2e^{-2\pi\epsilon} + 2e^{-4\pi\epsilon} + e^{-6\pi\epsilon} \quad (33)$$

where

$$\epsilon = \frac{1}{2} \left(\frac{1 + \sqrt{1 + 4k_1^2}}{2k_1^2} - 1 \right) \quad (34)$$

and

$$k_1^2 = \frac{1}{2} \left(1 + (f^2 + 1) \left[1 + 4f^2/(f^2 + 1)^2 \right]^{1/2} \right)^{1/2} \quad (35)$$

These expressions are valid solutions to the problem posed by Eq. (16) for all values of the parameters. It is easily shown that there is a value of λ giving maximum absorption, $\lambda = \lambda_m$, with

$$\lambda_m^2 = \lambda_m^2 - 1 + k_1^2 \quad (36)$$

and

$$k_1^2 = \frac{5f^2 + 1 + \sqrt{17f^4 + 16f^2 + 1}}{4} \quad (37)$$

and $\lambda_m^2 > 3/4$.

In the derivation of Eq. (16), however, we used $\lambda \ll 1$, and thus for the physical problem posed in Sec. I the maximum absorption point is not attainable. Shown in Fig. 6 are the transmitted, reflected, and absorbed

intensities for $\lambda = 0.1$ and $f^2 = 1/3$ as a function of a . It is apparent that absorption of more than 10% of the incident intensity can be attained for reasonable values of the parameters. For $\lambda \ll f$ we can obtain simplified expressions for the intensities. We have from Eqs. (34), (35)

$$k_1 = f + \frac{\lambda^2}{2f(1+f^2)} \quad (38)$$

$$\kappa_1 = \frac{\pi \lambda^2}{2f(1+f^2)^2} \quad (39)$$

and provided also that $\delta_1 \ll 1$ we have

$$|R|^2 = 1 - \frac{\pi \lambda^2 a}{f(f^2 + 1)^2} \quad (40)$$

$$|R|^2 = \frac{\pi \lambda^4 a^2}{f^2(f^2 + 1)^4} \quad (41)$$

$$|A|^2 = \frac{\pi \lambda^2 a}{f^2(f^2 + 1)^2} \quad (42)$$

IV. Conclusion

The two species cold plasma dispersion relation can be replaced, in the vicinity of the cyclotron resonance for the light ions, with an approximate dispersion relation. The obtained dispersion relation, consisting of an acoustic-like branch and a Budden-like branch, can be analyzed exactly, giving closed form expressions for wave reflection and transmission. This analysis may have a wider applicability for mode conversion problems in general.

From an analysis of the linear resonance absorption at the cyclotron resonance point for a minority ion species, we find that reasonable plasma heating can be expected through this mechanism. In an actual device, repeated transits through the resonance point can be expected because of reflection of the fast wave, giving rise to even more heating.

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81T0005

Fig. 1. Dispersion relation for minority light ion concentration of 5%, and $N_1^2 = k_{\perp}^2 v_A^2 / \omega^2 = 0.1$ and 0.

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Fig. 2. The approximate dispersion relation, shown schematically for $\lambda = 0$, λ small, and λ large. For $\lambda = 0$ it consists of a fast wave branch, and a damped Budden branch. For $\lambda \neq 0$ the two branches are coupled.

81T0002

(a) k -Plane
 $\eta < 0$

(b) k -Plane
 $\eta > 0$

Fig. 3. The contours in the k -plane for the four independent solutions for $\eta < 0$, and those for $\eta > 0$.

81T0003

(a) k -Plane
 $\eta < 0$

(b) k -Plane
 $\eta \sim i$

(c) k -Plane
 $\eta > 0$

Fig. 4. The deformation of the integration contours as η is rotated from the negative real axis to the positive real axis.

FIGURE 1. The squared magnitudes of the transmission, reflection, and absorption coefficients as a function of the parameter λ . The curves are calculated for a lossless medium with a constant refractive index $n = 1.5$ and a constant thickness $d = 0.1$ cm.

81T0001

Fig. 6. Transmission, reflection, and absorption coefficients as a function of the parameter α for $\epsilon = 1$ and $\mu = 1$.