STUDY OF SPIN RESONANCES IN THE
ACCELERATORS WITH SNAKES*

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ABSTRACT

Spin resonances in the circular accelerators with snakes are studied to understand the nature of snake resonances. We analyze the effect of snake configuration, and the snake superperiod on the resonance. Defining the critical resonance strength ϵ_c as the maximum tolerable resonance strength without losing the beam polarization after passing through the resonance, we found that ϵ_c is a sensitive function of the snake configuration, the snake superperiod at the first order snake resonance, the higher order snake resonance conditions and the spin matching condition. Under properly designed snake configuration, the critical resonance strength ϵ_c is found to vary linearly with N_s as $\langle \epsilon_c \rangle = (1/\pi) \sin^{-1}(|\cos \pi v_z|^{1/2}) N_s$, where v_z and N_s are the betatron tune and the number of snakes respectively. We also study the effect of overlapping intrinsic and imperfection resonances. The imperfection resonance should be corrected to a magnitude of insignificance (e.g., $\epsilon \leq 0.1$ for two snakes case) to maintain proper polarization.

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1. INTRODUCTION

Spin Physics has been important in the particle physics research in the past.¹ To maintain the spin polarization up to the highest energy available in the particle accelerator is however a horrendous task.² First one should prepare the intense polarized ion source. The polarized particles have to be accelerated through many depolarizing spin resonances. Many of these resonances can easily depolarize the beam. Presently, the polarized beam has reached 22GeV/c at AGS in BNL. The proton synchrotrons in JAPAN(KEK) has also program to accelerate polarized beam. Most of the proposed hadron facilities such as the European Hadron Facility (EHF), and the TRIUMF II in Vancouver have the provision to accelerate polarized beam.³

In the past, one uses the harmonic correction scheme to correct the imperfection resonances and the tune jump to correct the intrinsic resonances.³⁻⁵ These methods are limited to operation at low energies. The harmonic correction is cumbersome and difficult to handle at high energies due to the current limitation. The tune jump is limited by the available operable tune space and the resonance strength. As the resonance strength ϵ is larger than 0.05, the tune jump method becomes ineffective. This implies that the tune jump method will not be useful for the resonance $\gamma G=60$ -v at AGS, where the resonance strength is expected to be 0.15.

To maintain the polarization up to high energy, a new method has to be employed. Realizing that the spin depolarization is due mainly to the resonant condition that the spin tune (spin precession frequency) equals to the resonance frequency related to particle vertical betatron frequency and closed orbit harmonics in the accelerator, Derbenev and Kondratenko⁶ proposed using a local spin rotator in the accelerator to maintain the spin tune of 1/2. Thereby the spin tune does not sweep through the resonant frequencies. The particle would not encounter resonance condition. Detailed study shows however that a new type of SNAKE RESONANCE appears.^{7,8} The tolerable resonance strength is however drastically increased. When the machine betatron tune is properly chosen, the accelerator installed with snakes can eliminate or restore the spin depolarization after passing through the spin resonance.

There are many excellent review articles and reports on the spin dynamics.²⁻¹⁵ These studies are important to the understanding of maintaining the polarized proton in the circular accelerators, such as AGS, ZGS, KEK, and SATURNE etc. More recently, there are effort to obtain the polarized beam in the high energy machines such as RHIC and SSC. New techniques are needed to correct the spin resonances. Studies of resonances with snakes in the accelerator is an important and timely effort.

When snakes are installed in the accelerator, the snake and spin resonances interact in a special way. There exists snake resonances, which are due to the

beating between the spin tune and the resonant frequency K . A detailed study has been performed^{6,7} to derive the condition for the snake resonances.^{6,7,16,17} There are however many problems remaining to be solved: What happen when two resonances are present near to each other?¹⁸ What is the best snake configuration to maintain the polarization? What is the effect of the snake superperiods? How many snakes is needed for the polarized proton in SSC? There exists partially analysis here and there. In this paper, we shall study these problems systematically. We hope to answer some of these questions.

The paper is organized in the following. Section 2 introduce the basic Thomas-BMT equation⁹ and solve some simple examples. Section 3 evaluated the spin resonance strength and discuss the scaling property of the spin resonance strength. Section 4 reviews the conventional method of correcting the depolarizing resonances and study the effect of a local spin rotator invented by Derbenev and Kondratenko. Section 5 study the effect of the local spin rotator on the spin. Section 6 studies the effect of the perturbed spin tune and the snake resonances for two snakes and the effect of overlapping resonances. Section 7 studies the effect of the snake superperiod for the accelerator with more than two snakes. We study also the critical resonance strength ϵ_c , defined to be the maximum tolerable resonance strength. Conclusion is given in Section 8.

2. BASIC EQUATIONS OF MOTION

The spin precession of a polarized particle in a magnetic fields $\vec{B}$ at the rest frame of the particle is given by the magnetic interaction,

$$\frac{d\vec{S}}{dt} = \vec{\mu} \times \vec{B} \quad (2.1)$$

where $\vec{\mu}$ is the magnetic moment of the proton, i.e. $\vec{\mu} = g \frac{e}{2m} \vec{S}$, with $g=2(1+G)$. Here G is called the Pauli anomalous magnetic moment. For proton $G=(g-2)/2=1.792846$.

Particles moving at relativistic velocities are being accelerated by the electric field of the radiofrequency stations, and are guided in the circular accelerator by the increasing magnetic dipole fields. To describe the spin motion, we shall transform all physical quantities to the rest frame of the moving particle. Let βc be the instantaneous velocity of the particle and $\gamma=(1-\beta^2)^{-1/2}$, the magnetic field seen by the particle is given by

$$\vec{B}_s = \gamma \vec{B}_\perp + \vec{B}_\parallel - \gamma \beta \vec{E} \quad (2.2)$$

Let us represent the particle orbit by the Frenet-Serret curvilinear coordinate in Fig. 1, where s is the distance along the reference orbit, $\rho(s)$ is the local radius of curvature of the reference orbit, $\vec{r}_0(s)$ describes the reference orbit and the unit vectors $\hat{x}$, $\hat{s}$ and $\hat{z}$ form a coordinate system on the reference orbit with

$\hat{s} = d\vec{r}_0/ds$. Due to the curvature of the reference orbit, the vectors $\hat{x}$, $\hat{s}$ and $\hat{z}$ can change with s as

$$\frac{d\hat{x}}{ds} = \frac{\hat{s}}{\rho} ; \quad \frac{d\hat{s}}{ds} = -\frac{\hat{x}}{\rho} ; \quad \frac{d\hat{z}}{ds} = 0 \quad (2.3)$$

Fig. 1 The Frenet-Serret coordinate system, where a particle coordinate can be expressed as $\vec{r}(x,s,z) = \vec{r}_0(s) + x\hat{x} + z\hat{z}$.

The coordinate of a particle can be expressed by

$$\vec{r}(x,s,z) = \vec{r}_0(s) + x\hat{x} + z\hat{z}, \quad (2.4)$$

then the velocity of the particle is given by

$$\begin{aligned} \vec{v}(x,s,z) &= \frac{ds}{dt} \frac{d\vec{r}}{ds} = \frac{ds}{dt} \left(x'\hat{x} + \left(1 + \frac{x}{\rho}\right)\hat{s} + z'\hat{z} \right) \\ &\approx v(x'\hat{x} + \hat{s} + z'\hat{z}), \end{aligned}$$

where $v \approx (1+x/\rho)ds/dt$ and primes represent derivatives with respect to the s . We have also approximate the velocity to the first order in x , x' and z' . Similarly, the change of velocity with respect to s becomes,

$$\vec{v} = v [(x'' - 1/\rho)\hat{x} + x'/\rho \hat{s} + z'' \hat{z}]$$

Using the Lorentz equation, $\frac{d\vec{v}}{dt} = \frac{e}{\gamma mc} \vec{v} \times \vec{B}$, the vertical $\vec{B}$ field can be determined to be

$$\vec{B}_\perp = \frac{-1}{|\vec{v}|^2} (\vec{v} \times (\vec{v} \times \vec{B})) = B\rho(1-x/\rho) \left[-z''\hat{x} + \frac{z'}{\rho} \hat{s} + (x''-1/\rho)\hat{z} \right] \quad (2.5)$$

where $B\rho$ is the magnetic rigidity of the beam. The longitudinal $\vec{B}$ field is determined by the edge effect of the dipole for the requirement of satisfying the Maxwell equation and the component of B field, $\vec{B}_\parallel = -B\rho/\rho \hat{z}$, parallel to the $\vec{v}$, i.e.

$$\vec{B}_\parallel = -B\rho z \left(\frac{1}{\rho} \right)' \hat{s} - B\rho z' \frac{1}{\rho} \hat{s} = -B\rho \left(\frac{z}{\rho} \right)' \hat{s} \quad (2.6)$$

Thus the equation of motion for the spin is given by¹²

$$\frac{d}{d\vartheta} \vec{S} = \vec{S} \times \vec{F} \quad (2.7)$$

where $S = S_1\hat{x} + S_2\hat{s} + S_3\hat{z}$ and $F = F_1\hat{x} + F_2\hat{s} + F_3\hat{z}$, with

$$F_1 = -\rho z''(1+\gamma G)$$

$$F_2 = (1+\gamma G)z' - \rho(1+G) \left(\frac{z}{\rho} \right)'$$

$$F_3 = (\rho x'' - 1)(1+\gamma G) = - (1+\gamma G)$$

and $d\vartheta = ds/\rho$. Using Eq. (2.3), we can express Eq. (2.7) in terms of its components as

$$\begin{aligned} \frac{d}{d\vartheta} S_1 &= -\kappa S_2 - F_2 S_3 \\ \frac{d}{d\vartheta} S_2 &= +\kappa S_1 + F_1 S_3 \\ \frac{d}{d\vartheta} S_3 &= +F_2 S_1 - F_1 S_2 \end{aligned} \quad (2.8)$$

where $\kappa = -(1+F_3) = \gamma G$. Equation (2.8) indicates that the spin precesses around the vertical axis at a precession frequency of γG . The depolarization of the spin is due mainly to the coupling terms F_1 and F_2 . Because $\rho z''$ is much larger than z' term, we expect F_1 to be larger than F_2 . The unitarity of the spin motion is preserved in Eq. (8), i.e. $S_1^2 + S_2^2 + S_3^2 = 1$.

2.2 SPINOR FORMULATION OF THE SPIN EQUATION.

The spin equations, Eq. (2.8), can be casted into the spinor formulation by defining $S_i = \langle \psi | \sigma_i | \psi \rangle$, where σ_i are Pauli spin matrices,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} ; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The spinor equation is not a quantum mechanical description of the spin motion. The spinor wave function satisfies,

$$\frac{d}{d\vartheta} \psi = \frac{i}{2} (F_1 \sigma_1 + F_2 \sigma_2 - \gamma G \sigma_3) \psi = -\frac{i}{2} \begin{bmatrix} \gamma G & -\zeta \\ -\zeta^* & -\gamma G \end{bmatrix} \psi \quad (2.9)$$

where

$$\zeta = -(1+\gamma G)(\rho z'' + iz') + i\rho(1+G) \left(\frac{z}{\rho} \right)' \quad (2.10)$$

characterizes the spin depolarization effect due to the coupling between the up and down components of the spinor wave function. When $\zeta=0$, the spin precesses around the vertical guide field without changing the spin polarization. In general, the depolarization driving terms in Eq. (2.10) are small unless the resonance condition encountered. These spin resonances shall be discussed in the next section.

2.3 SPIN RESONANCES

Due to the repetitive nature of the circular accelerator, the coupling term can be expanded in terms of the Fourier components,

$$\zeta = \sum \varepsilon_i \exp(iK_i \vartheta) \quad (2.11)$$

where K_i is the resonance frequency and ε_i is the resonance strength,

$$\varepsilon_i = \frac{1}{2\pi} \int \zeta \exp(iK_i \vartheta) d\vartheta \quad (2.11a)$$

When the precession frequency γG is near to the resonance frequency of the accelerator, the spin is kicked by the depolarization force coherently every turns around the accelerator (Eq. 2.9). In the following, we study two simple problems with analytic solutions.

2.3a SOLUTION FOR A SINGLE RESONANCE AT CONSTANT γG (SLOW ACCELERATION)

For a single resonance, $\zeta = \varepsilon \exp(iK\vartheta)$, Eq. (2.9) can be solved analytically in the resonance precession frame by defining ψ_K as

$$\psi_K = \exp(iK\vartheta/2 \sigma_3) \psi.$$

The spinor equation for ψ_K becomes

$$\frac{d}{d\vartheta} \psi_K = \frac{i}{2} \begin{bmatrix} \delta & \varepsilon \\ \varepsilon^* & -\delta \end{bmatrix} \psi_K \quad (2.12)$$

where $\delta = K - \gamma G$. The solution of Eq. (2.12) is given by,

$$\psi_K(\vartheta_f) = \exp\left(\frac{i}{2} [\delta \sigma_3 + \varepsilon_R \sigma_1 - \varepsilon_I \sigma_2](\vartheta_f - \vartheta_i)\right) \psi_K(\vartheta_i) \quad (2.13)$$

with ε_R and ε_I as real and imaginary parts of the resonance strength. There are two orthonormal eigenmode of the equation,

$$\psi_K^{(+)} = \frac{1}{((\lambda - \delta)^2 + |\varepsilon|^2)^{1/2}} \begin{bmatrix} \varepsilon \\ -(\lambda - \delta) \end{bmatrix} e^{-i\lambda\vartheta/2} \quad (2.14a)$$

$$\psi_K^{(-)} = \frac{1}{((\lambda - \delta)^2 + |\varepsilon|^2)^{1/2}} \begin{bmatrix} \varepsilon \\ (\lambda + \delta) \end{bmatrix} e^{-i\lambda\vartheta/2} \quad (2.14b)$$

with $\lambda = (\delta^2 + |\varepsilon|^2)^{1/2}$. The general solution for Eq. (2.12) can be expressed as a linear combination of these two eigenmodes of Eqs. (2.14)

$$\psi_K = C_+ \cdot \psi_K^{(+)} + C_- \cdot \psi_K^{(-)}$$

with $|C_+|^2 + |C_-|^2 = 1$. The spin polarization is given by

$$S = \langle \psi | \sigma_3 | \psi \rangle = \frac{\delta}{\lambda} (|C_+|^2 - |C_-|^2) + \frac{|\varepsilon|}{\lambda} |C_+ C_-| \cos(2\lambda\vartheta + \phi)$$

Thus the average polarization is given by

$$P \equiv \langle S \rangle = \frac{\delta}{\lambda} P_0 = \frac{K - \gamma G}{((K - \gamma G)^2 + |\varepsilon|^2)^{1/2}} P_0 \quad (2.15)$$

where $P_0 = |C_+|^2 - |C_-|^2$ is the initial polarization far away from the resonance. Equation (2.15) manifests that $|\epsilon|$ represents the resonance width. At a distance $K - \gamma G = \pm |\epsilon|$ away from the resonance, the polarization is reduced by a factor of $1/\sqrt{2}$. A minimum of $3|\epsilon|$ distance away from the resonance is required to maintain 95% polarization. We can therefore use $3|\epsilon|$ as the width of spin resonance. Outside the width, the spin is in the asymptotic region, where the resonance give little effect to the spin polarization.

From Eq. (2.13), we obtain then the spinor wave function in particle moving frame as,

$$\begin{aligned} \psi(\theta_f) &= e^{-\frac{i}{2} K \theta_f \sigma_3} e^{\frac{i}{2} [\delta \sigma_3 + \epsilon_R \sigma_1 - \epsilon_I \sigma_2] (\theta_f - \theta_i)} e^{\frac{i}{2} K \theta_i \sigma_3} \psi(\theta_i) \\ &\equiv t(\theta_f, \theta_i) \psi(\theta_i) \end{aligned} \quad (2.16)$$

Here the $t(\theta_f, \theta_i)$ is the spin transfer matrix, which shall be used in section 5 for the spin tracking study. The diagonal matrix element of $t(\theta_f, \theta_i)$ manifests the spin precession frequency around the vertical axis and the off diagonal matrix element describes the spin perturbing kick due to the presence of a resonance.

2.3b SOLUTION FOR A SINGLE RESONANCE AT A CONSTANT ACCELERATION RATE

To simplify our discussion, we shall assume a constant acceleration rate $\gamma G = \kappa_0 + \alpha \vartheta$, where κ_0 is a reference precession frequency and α is the acceleration rate. Transforming Eq. (2.9) onto the spin precession frame (interaction picture) as,

$$\psi(\vartheta) = \exp\left(-\frac{i}{2} \int_0^\vartheta \gamma G d\vartheta \sigma_3\right) \psi_I(\vartheta) \quad (2.17)$$

one obtains the equation of motion as,

$$\frac{d}{d\vartheta} \psi_I = \frac{i}{2} \begin{bmatrix} 0 & \xi \\ \xi^* & 0 \end{bmatrix} \psi_I \quad (2.18)$$

where $\xi = \epsilon \exp(i[(\kappa_0 - K)\vartheta + 0.5\alpha\vartheta^2])$. The resonance effect is greatest when the exponent in Eq. (2.18) reaches the stationary phase, i.e. $\gamma G = K$. At the resonance location, the perturbing kicks add up coherently from turn to turn. We can choose the initial condition to be $\kappa_0 = K$ and express the spinor wave function in terms of two components $\psi_I^{(+)}$ and $\psi_I^{(-)}$. Equation (2.18) can be expressed as,

$$\frac{d}{d\vartheta} \psi_I^{(+)} = \frac{i}{2} \epsilon \exp\left(\frac{i}{2} \alpha \vartheta^2\right) \psi_I^{(-)} \quad (2.19)$$

$$\frac{d}{d\vartheta} \psi_1^{(-)} = \frac{i}{2} \varepsilon^* \exp\left(-\frac{i}{2} \alpha \vartheta^2\right) \psi_1^{(+)} \quad (2.20)$$

or

$$\frac{d^2}{d\vartheta^2} \psi_1^{(\pm)} \mp i\alpha \vartheta \frac{d}{d\vartheta} \psi_1^{(\pm)} + \frac{|\varepsilon|^2}{4} \psi_1^{(\pm)} = 0 \quad (2.21)$$

The solution of this equation becomes^{8,15}

$$\psi_1^{(+)}(\vartheta) = A z(\vartheta) - B w(\vartheta) \quad (2.22)$$

where A and B are constants and z and w are given in terms of the confluent hypergeometric functions,

$$z(\vartheta) = M(iq, 1/2; -i\alpha\vartheta^2/2)$$

$$w(\vartheta) = \frac{i}{2} \varepsilon \vartheta \exp\left(\frac{i}{2} \alpha \vartheta^2\right) M(1-iq, 3/2; -i\alpha\vartheta^2/2)$$

with the dimensionless resonance strength q given by $q = |\varepsilon|^2/8\alpha$. Using Eq. (2.19), we obtain also,

$$\psi_1^{(-)}(\vartheta) = A \bar{w}(\vartheta) + B \bar{z}(\vartheta) \quad (2.23)$$

where $\bar{z}$ and $\bar{w}$ are the complex conjugate functions of z and w respectively. We can write then

$$\psi_1 = \begin{bmatrix} z(\vartheta) & -w(\vartheta) \\ \bar{w}(\vartheta) & \bar{z}(\vartheta) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = T(\vartheta) \begin{bmatrix} A \\ B \end{bmatrix} \quad (2.24)$$

Equation (2.24) defines the matrix $T(\vartheta)$, which is unitary and $\det(T(\vartheta))=1$. The constants A and B are determined from the initial conditions as

$$\begin{bmatrix} A \\ B \end{bmatrix} = T^{-1}(\vartheta_i) \psi_1(\vartheta_i)$$

Thus the general solution can be expressed as $\psi_1(\vartheta) = T(\vartheta) T^{-1}(\vartheta_i) \psi_1(\vartheta_i)$. Here $T(\vartheta) T^{-1}(\vartheta_i)$ is called the spin transfer matrix. It transforms the spinor wave function from one side to the other side of the resonance. The essential idea is similar to the wave propagation matrix method in the semiclassical theory¹⁹ or the transfer matrix method in the lattice calculation.²⁰

When $\vartheta \gg 0$ and $\vartheta_i \ll 0$, away from the resonance location we can use asymptotic expansion to the confluent hypergeometric functions in Eq. (2.24) to obtain²¹

$$\psi_I(\vartheta) = e^{-iq \ln\left(\frac{\alpha\vartheta^2}{2}\right)} \sigma_3 U(q) e^{iq \ln\left(\frac{\alpha\vartheta_i^2}{2}\right)} \sigma_3 \psi_I(\vartheta_i) \quad (2.25)$$

with

$$U(q) = \begin{bmatrix} e^{-2\pi q} & C(q) \\ -C^*(q) & e^{-2\pi q} \end{bmatrix} \quad (2.25a)$$

$$C(q) = -\frac{i}{4} \varepsilon \sqrt{\frac{2}{\alpha}} \frac{\Gamma\left(\frac{1}{2} + iq\right)}{\Gamma(1 - iq)} e^{i\pi/4} e^{-\pi q} (e^{\pi q} + e^{-\pi q})$$

It is easy to check $\det(U(q)) = 1$. The final polarization is given by

$$P = \langle \psi_I | \sigma_3 | \psi_I \rangle = 2 e^{-4\pi q} - 1 \quad (2.26)$$

This is the celebrated Froissart-Stora formula¹⁰ for the final polarization after passing through an isolated resonance with fixed acceleration rate. The present derivation of using the spin transfer matrix is useful for dealing the problem with many resonances and analytic evaluation of tune jump method.¹⁵

3. RESONANCE STRENGTH

We have discussed the effect of crossing a single resonance on the polarization of the beam. It is important to know the resonance strength ε . Following the basic beam dynamics in the circular accelerator,²⁰ the resonance strength defined in Eq. (2.11a) can be approximated by,

$$\begin{aligned} \varepsilon &= \frac{-1}{2\pi} \int_0^{2\pi} \left[(1 + \gamma G)(\rho z'' + i z') - i \rho (1 + G) \left(\frac{z}{\rho} \right)' \right] \exp(iK\vartheta) d\vartheta \\ &\approx \frac{\gamma G}{2\pi} \int_0^{2\pi R} z'' \exp(iK\vartheta) ds = \frac{\gamma G}{2\pi} \int_0^{2\pi R} \frac{\partial B_z / \partial x}{B\rho} \cdot z \cdot \exp(iK\vartheta) ds. \end{aligned} \quad (3.1)$$

The amplitude z of the beam can be decomposed into two parts: (1) the closed orbit amplitude due to errors in the dipole field or quadrupole misalignment and (2) the betatron amplitude of the particle, i.e. $z = z_{co} + z_\beta$. Basic beam dynamics in the circular accelerator gives us,²⁰

$$z_{co}(s) = \beta_z^{1/2}(s) \sum \frac{v_z^2 f_k \exp(ik\phi(s))}{v_z^2 - k^2} \quad (3.2a)$$

$$z_\beta(s) = \left[\frac{\beta_z \varepsilon_N}{\pi \gamma} \right]^{1/2} \cos(v_z \phi + \chi) \quad (3.2b)$$

where v_z , ϵ_N are the vertical betatron tune of the machine and the normalized emittance of the beam respectively. The betatron phase $\phi(\vartheta)$ and the Fourier amplitude f_k are given by

$$\phi(\vartheta) = \int ds/v_z \beta_z$$

$$f_k = \frac{1}{2\pi v_z} \int_0^{2\pi R} \beta_z^{1/2} \frac{\Delta B}{B\rho} \exp(-ik\phi) ds$$

There are two types of resonances: (1) the imperfection resonances due to z_{co} of closed orbit error and (2) the intrinsic resonances due to the betatron motion z_β of the beam. For a perfect machine, the strengths of the imperfection resonances are zero. However the intrinsic resonance is relatively independent of the alignment of the machine. The strength of the intrinsic resonance, depending on the betatron amplitude, z_β , is proportional to the $(\gamma\epsilon_N)^{1/2}$ of the normalized emittance ϵ_N . In the following we shall discuss these resonance strengths.

3.1 THE IMPERFECTION RESONANCE STRENGTH

Substituting z_{co} of Eq. (3.2a) into Eq. (3.1), we obtain

$$\epsilon_K = -\frac{\gamma G}{2\pi} \sum \frac{v_z^2 f_k}{v_z^2 - k^2} \int_0^{2\pi R} \frac{\partial B_z / \partial x}{B\rho} \cdot \beta_z^{1/2}(s) \cdot \exp(i(k\phi + K\vartheta)) ds. \quad (3.3)$$

To study the main characteristic of the resonance, we shall evaluate the integral in a thin lense approximation. We further assume that the accelerator is composed of P superperiods and M FODO cells in each superperiod with an insertion section to connect the superperiod. Each FODO cell has equal length dipole with phase advance $2\pi\mu$ per cell. The insertion section contains only quadrupoles. The integral in Eq. (3.3) can be evaluated easily in the above approximation as,

$$\epsilon_K = \frac{\gamma G}{2\pi} \sum \frac{v_z^2 f_k}{v_z^2 - k^2} e^{i \frac{P-1}{P} (k+K)\pi} \zeta_P \left(\frac{k+K}{P} \right) \times \left\{ \zeta_M \left(\frac{K+k-\frac{v_B}{v}}{MP} \right) \right.$$

$$\left. \times \left(g_D \beta_z^{1/2}(D) - g_F \beta_z^{1/2}(F) e^{-i \left(k \frac{v_B}{v} + K \right) \pi / MP} \right) e^{i \left(k \frac{v_B}{v} + K \right) \pi / P} + X_I \right\} \quad (3.3b)$$

where the function $\zeta_N(x)$ is given in the Appendix A and $v_B = MP\mu$ is the phase advance accumulated through all dipole cells, which affect the spin precession and X_I represents the contribution to the integral from the insertion region. g_F and g_D are quadrupole strengths. $\beta_z(F)$ and $\beta_z(D)$ are the betatron amplitude

function evaluated at the horizontal focusing and defocusing quadrupole locations respectively.

It is known that the most important harmonic for the closed orbit is at $k=\pm[v_z]$, the nearest integer to $\pm v_z$. The enhancement factor $\zeta_p\left(\frac{K+k}{P}\right) \equiv P$ is located at $K=mP\pm[v_z]$, where m is an integer. This enhancement is due to the fact that each superperiod contribute coherently in phase to the spin kicks.

The enhancement factor $\zeta_M((K+kv_B/v)/MP)$ equals to M at $K=mPM\pm[v_B]$, where m is an integer and $[v_B]$ is the nearest integer of $[v]v_B/v$. The contribution of each FODO cell adds up to give the factor M . When m is an odd integer, the effect is specially enhanced due to the addition between the focusing and defocusing quadrupoles, i.e.,

$$g_D\beta_z^{1/2}(D)-g_F\beta_z^{1/2}(F)e^{-i\left(k\frac{v_B}{v}+K\right)\pi/MP} = \begin{cases} g_D\beta_z^{1/2}(D)+g_F\beta_z^{1/2}(F) & m=\text{odd} \\ g_D\beta_z^{1/2}(D)-g_F\beta_z^{1/2}(F) & m=\text{even} \end{cases}$$

at $K=mPM\pm[v_B]$. At the odd integer m , the focusing and defocusing quadrupoles contribute additively to the spin kick. Figure 2 shows the reduced resonance strength $\epsilon_K/\sqrt{K}$ for intrinsic resonance and ϵ_K/K for imperfection resonance for AGS, where $P=12$, $M=5$, $v_B/v=8.75$, $\beta_z(D)=22m$ and $\beta_z(F)=11m$. Note here that the reduced imperfection resonance strength at $K=60-[v]$ should be approximately 6 times larger than that at $K=[v]$. The large resonance

Fig. 2 The reduced resonance strength for AGS, i.e. $\epsilon_K/\sqrt{K}$ for intrinsic resonance and ϵ_K/K for imperfection resonance. The feature indicate that the primary intrinsic and imperfection resonance occurs at $60\pm v$ and $0\pm v$ discussed in the text.

strength at $K=mP\pm[v]$ or $K=mPM\pm[v_B]$ are due essential to the large orbit error at $k=[v]$ harmonic, they can also be corrected by the orbit correction of $k=[v]$ harmonic. For example, the large resonance strength at $\gamma G=27=36-[v]$ in AGS can be corrected by the harmonic 9 orbit correction.

In the high energy machine, closed orbit correction is essential in the dynamical aperture. There are various orbit correction schemes, e.g. harmonic or global corrections. These schemes eliminate the integer error harmonics nearest to $[v]$ in Eq. (3.2a). The final closed orbit error would behave like

$$Z_{co}(s) = \beta_z^{1/2}(s) \sum Z_k e^{ik\phi(s)} \quad (3.4)$$

where the resonance structure of Eq. (3.2a) has disappeared from Eq. (3.4). The amplitude $|Z_k|$ is expected to be nearly constant at $|k-v| \sim 0$ and $|Z_k| \equiv 1/k^2$ at $|k-v| \gg 0$. Thus the dominant closed orbit harmonic would be $|k-v| \leq v/2$ in Eq. (3.4). The resulting statistical orbit deviation would be

$$\langle Z_{co}^2 \rangle^{1/2} = \beta_z^{1/2}(s) \sqrt{v} \langle Z_k^2 \rangle^{1/2} = \frac{1}{\sqrt{2}} \beta_z^{1/2}(s) \sqrt{v} Z_{k,max} \quad (3.5)$$

The resonance strength generated from the closed orbit deviation of Eq. (3.4) is given by

$$\begin{aligned} \epsilon_k = \frac{\gamma G}{2\pi} \sum_k Z_k e^{i \frac{P-1}{P}(k+K)} \zeta_P\left(\frac{k+K}{P}\right) & \left\{ \zeta_M\left(\frac{K+k \frac{v_B}{v}}{MP}\right) \right. \\ & \times \left(g_D \beta_z^{1/2}(D) - g_F \beta_z^{1/2}(F) e^{-i\left(K+k \frac{v_B}{v}\right)\pi/MP} \right) e^{i\left(K+k \frac{v_B}{v}\right)\pi/P} + X_I \left. \right\} \end{aligned} \quad (3.6)$$

Using the estimate of Eq. (3.5), we obtain

$$\epsilon_{k,max} \sim \frac{\gamma G}{2\pi} M \frac{\sqrt{2} \langle Z_{co}^2 \rangle^{1/2} g_D}{\sqrt{v}} \left(1 + \left(\frac{\beta_z(F)}{\beta_z(D)} \right)^{1/2} \right) \quad (3.7)$$

Putting the SSC value, we expect $\epsilon_{k,max} \cong 0.2$ at $\langle Z_{co}^2 \rangle^{1/2} \cong 0.1$ mm. This closed orbit error may be difficult to achieve. Using similar estimate for AGS, we obtain $\epsilon_{k,max} \cong 6 \times 10^{-3}$ at $\gamma G=60-[v]$ region. This estimation is a factor of 3 larger than the resonance strength obtained from a realistic calculation with DEPOL program¹² and MICADO correction method. If a similar situation

happens in SCC, we expect $\varepsilon_{k, \max} \cong 0.2$ at $\langle Z_{co}^2 \rangle^{1/2} \cong 0.3$ mm in a realistic calculation.

3.2 INTRINSIC RESONANCE

Substituting the betatron amplitude z_β of Eq. (3.2b) into Eq. (3.1), one obtains easily the intrinsic resonance strength in the following with the assumptions: (1) thin lense quadrupoles, (2) P superperiods, with M FODO cells per superperiod, (3) no dipole in the insertion regions and (4) $v_B = MP\mu$ where $2\pi\mu$ is the phase advance in each FODO cell.

$$\varepsilon_K = -\frac{G}{4\pi} \left(\frac{\gamma \varepsilon_N}{\pi} \right)^{1/2} \left\{ E_P^+ \left[E_M^+ (g_F \beta_z^{1/2}(F) - g_D \beta_z^{1/2}(D)) e^{i(K+v_B)\pi/MP} \right] + X_{ins}^+ \right\} \\ + E_P^- \left[E_M^- (g_F \beta_z^{1/2}(F) - g_D \beta_z^{1/2}(D)) e^{i(K-v_B)\pi/MP} \right] + X_{ins}^- \right\}$$

where g_F and g_D are the quadrupole strengths. $\beta_z(F)$ and $\beta_z(D)$ are respectively the vertical betatron amplitude function at horizontal focusing and defocusing quadrupoles. $X_{ins}^\pm$ are contribution from the insertions. $E_P^\pm$ and $E_M^\pm$ are the enhancement factors due to the superperiod P and M FODO cells respectively, i.e.

$$E_P^\pm = \frac{1 - \exp(i2\pi(K \pm v))}{1 - \exp(i2\pi(K \pm v)/P)} ; \quad E_M^\pm = \frac{1 - \exp(i2\pi(K \pm v_B)/P)}{1 - \exp(i2\pi(K \pm v_B)/PM)}$$

where $E_P^\pm$ and $E_M^\pm$ are the enhancement factors due to the superperiods and the M FODO cells respectively. Thus the resonance strength is enhanced by a factor of P at $K=mP \pm v$ and by a factor of M at $K=mPM \pm v_B$. For most of the large accelerator, $M \gg P$, we therefore expect important resonances located at $K=mPM \pm v_B$. Similar to that of the imperfection resonances, the important resonance corresponds to $m=\text{odd integer}$, where the focusing and defocusing quadrupoles add up coherently to the spin kick.

Figure 3 shows the resonance strengths of various accelerators calculated from the DEPOL program¹² for the primary family, $K=mPM \pm v_B$ with $m=\text{odd}$ for the normalized emittance of $10\pi\text{mm-mrad}$. Although these accelerators are designed differently, the spin resonance strength following a simple scaling law of $\sqrt{\gamma}$ nicely.

Fig. 3 Compilation of spin resonance strength of various accelerators in the world. The intrinsic resonance strength follows the scaling of $\sqrt{\gamma}$ while the imperfection resonance follows the scaling of γ .

4. COMPENSATION SCHEMES FOR THE SPIN DEPOLARIZATION RESONANCES

When the particle is accelerated through the resonance, the final polarization is given by $P/P_0 = 2\exp(-\pi|\epsilon|^2/2\alpha) - 1$, the Froissart-Stora formula in Eq. (2.26). To maintain the polarization larger than 95%, the resonance strength should be smaller than $0.127\sqrt{\alpha}$, or $|\epsilon| < 0.00088$ for AGS, where $\alpha = 4.86 \cdot 10^{-5}$. Figure 4 shows that the global orbit correction scheme is not effective for $\gamma > 20$, where the corrected resonance strength is too large to maintain spin polarization. Traditional methods of correcting these spin resonances are based on either decreasing the resonance strength ϵ or increasing/decreasing the spin resonance transition rate α .^{4,5,21}

Fig. 4 The spin resonance strength of AGS. The imperfection resonance strength increases linearly with energy. When the orbit is globally corrected to 0.1mm, the spin would be substantially depolarized at γ larger than 20. This indicates that global orbit correction can not be used effectively in AGS, for the correction of imperfection resonances.

4.1 HARMONIC ORBIT CORRECTION SCHEME

By using a set of dipoles, a specific harmonic k in the closed orbit error can be corrected. Figure 5 shows an example of tuning the cosine and sine components of an imperfection spin resonance in AGS. The scheme requires large number of correcting dipoles where the exciting current would increase with energy. The harmonic orbit correction scheme is an adiabatic process. It shall not cause beam blow up. The method works only when the resonances are not overlapping. Extensive review of the method is given in ref. 5. For a strong imperfection resonance, the spin polarization can be reversed. SATURNE in Saclar France used spin flip to correct the imperfection resonance.¹⁴

Fig. 5 An example of the imperfection resonance correction in AGS for $G\gamma=9$ where the amplitudes of the 9th harmonic are adjusted to give good polarization.

Fig. 6 The final spin vs the fast quadrupole firing time for $G\gamma=v$ resonance in AGS is plotted. The $\theta_1=0$ corresponds to $G\gamma=v$ resonance condition. The plateau in region I corresponds to the time jump space, $\Delta v=0.25$.

4.2 TUNE JUMP (FAST TRANSITION THROUGH THE RESONANCE)

Since the dipole correction scheme can not change the betatron motion of the particle, this method can not be used to correct the intrinsic resonances. However, the resonance position depends on the tune of the machine, the spin particle can pass through the resonance very fast by changing the tune of the machine within a short time (1-2 turns of revolution). Figure 6 shows an example of polarization correction for the intrinsic resonance correction in AGS, where 10 quadrupoles are pulsed within one revolution ($\approx 2.7\mu\text{sec}$) to change the tune by 0.25-0.3. The upper point of Fig. 6 shows the final measured polarization as a function of the time (Gauss clock), or revolution angle that these 10 quadrupoles are pulsed. The numerical calculation, using the method of Eq. (2.24), is also shown in the lower part of Fig. 6.

To avoid the half integer resonances, it is difficult to change the vertical tune of the machine by more than 0.3. Section 2.3a showed that the effect of the spin resonance on the polarization is given by $P/P_0 = (K - \gamma G) / ((K - \gamma G)^2 + |\epsilon|^2)^{1/2}$. We expect a 5% depolarization for resonance strength 0.049 at tune jump $\Delta\nu_z = 0.3$. Figure 7 shows the tracking results of the final polarization as a

Fig. 7 The final polarization for $\Delta\nu=0.25$ (upper) and $\Delta\nu=0.3$ (lower) are shown as a function of the resonance strength ϵ . The curve $N=1$ corresponds to the tune jumping within one turn of particle revolution while $N=4$ indicates the tune jumping within 4 revolutions.

function of the resonance strength for $\Delta\nu_z=0.25$ and 0.3 respectively. It is clear that the tune jump scheme becomes impractical at a resonance strength of 0.05 or larger.

From the orbital beam dynamic point of view, tune jump is a nonadiabatic process. When the quadrupoles are excited to change the betatron tune within 1-2 turns of particle revolution, the betatron amplitude function is increased depending on the configuration and the excitation currents of the tune jump quadrupoles. The increase of the beta-function is translated into the increase in the beam emittance. For each tune jump of 0.25 in AGS, we expect that the emittance will increase by about 10% due to the nonadiabatic increase of the beta-functions.²² Besides, the misalignment in the tune jump quadrupoles is translated into closed orbit error, which gives rise to emittance growth. Figure 9, taken from ref. 18, shows the measured normalized emittance as a function of energy for AGS. The result indicates that proper alignment of the tune jump quadrupoles is important to control the emittance growth in the tune jump process. Using the first order orbit theory, the emittance increment is given by

Fig. 8 The measured normalized vertical emittance as a function of the momentum of the particle taken from ref 19. The increment in the vertical emittance due to the tune jump is corrected by properly aligning the fast quadrupoles (shown in the curve 1988). The remaining emittance increase is due to the unavoidable betatron mismatch in the nonadiabatic tune jump process, where the expected emittance growth is about 10% per jump (see ref. 22).

$\Delta\epsilon_N = 2(\epsilon_N \beta \gamma / \beta_z)^{1/2} \Delta z_{co}$, where E_N, β_z are respectively the normalized emittance and vertical betatron amplitude functions. This feature is shown evidently in the 1986 run (see Fig. 8).

4.3 ALTERNATIVE METHODS OF RESONANCE CORRECTION

Besides the conventional methods used to correct the spin resonance discussed in previous sections. Derbenev and Kondratenko et al.⁶ suggested that a local spin rotator, called Siberian snake by Courant, gives rise to an energy independent spin tune and thus avoids the depolarization due to the resonance crossing.

Assuming that there is no depolarization driving force on the spin, the spinor wave function can be expressed as (see Eq. (2.9)),

$$\psi(\vartheta_f) = \exp\left(-\frac{i}{2}\gamma G(\vartheta_f - \vartheta_i) \sigma_3\right) \psi(\vartheta_i). \quad (4.1a)$$

Let us assume that there is a local spin rotator, which rotates the spin by an angle ϕ around an axis $\hat{n}_s = (\cos\phi_s, \sin\phi_s, 0)$ on the horizontal plane. The local spin rotator would transform the spinor wave function according to

$$\psi(\vartheta_+) = \exp\left(\frac{i}{2} \phi \hat{n}_s \cdot \vec{\sigma}\right) \psi(\vartheta_-). \quad (4.1b)$$

The spinor wave function for ONE TURN around the accelerator becomes,

$$\begin{aligned} \psi(\vartheta) &= \exp(-i\gamma G\vartheta/2 \sigma_3) \cdot \exp(i\phi/2 \hat{n}_s \cdot \vec{\sigma}) \cdot \exp(-i\gamma G(2\pi - \vartheta)/2 \sigma_3) \psi(\vartheta - 2\pi) \\ &= \exp(i\pi\nu_s \hat{n}_s(\vartheta) \cdot \vec{\sigma}) \psi(\vartheta - 2\pi) \end{aligned} \quad (4.2)$$

The spin tune ν_s is given by the trace of the spin transfer matrix, i.e. the ONE TURN MAP(OTM).

$$\cos \pi\nu_s = \cos(\gamma G\pi) \cos(\phi/2) \quad (4.3)$$

Figure 9 shows the spin tune as a function of γG between two consecutive integers. When the spin rotation angle is 180° , the spin tune becomes $1/2$ and independent of energy, i.e. there is no spin tune spread for the bunch of particles as well. The spin closed orbit vector $\hat{n}(\vartheta)$ can be expressed as the directional cosines of the coordinate system as, $\hat{n}(\vartheta) = (\cos\eta_x, \cos\eta_s, \cos\eta_z)$ with

Fig. 9 The spin tune as a function of γG for various snake excitation angles. A 40° degree snake means that the spin is rotated only 40° about a snake axis on the horizontal plane after passing through the snake.

$$\cos \eta_z(\vartheta) = -\sin(\gamma G \pi) \cdot \cos(\phi/2) / \sin(\pi \nu_s)$$

$$\cos \eta_x(\vartheta) = \sin(\phi/2) \cdot \cos(\phi_s - \gamma G(\pi - \vartheta)) / \sin(\pi \nu_s) \quad (4.4)$$

$$\cos \eta_y(\vartheta) = \sin(\phi/2) \cdot \sin(\phi_s - \gamma G(\pi - \vartheta)) / \sin(\pi \nu_s)$$

Figure 10 shows $\hat{n}(\vartheta=\pi)$ for a 40° snake with $\phi_s=0$. The variation of the closed orbit $\hat{n}$ vector arises from the variation of ν_s on γG . The spin orientation depends on the location where the snake is turned on and off.

At $\phi=180^\circ$, the $\hat{n}$ vector lies entirely on the horizontal plane, $\cos \eta_z=0$ and $\eta_x=\phi_s - \gamma G(\pi - \vartheta)$. The spin axis depends on the location in the accelerator and depends on the energy. A special point of interest is the location $\vartheta=\pi$ from the snake location, the symmetric point, where we have $\eta_x=\phi_s$ or $\hat{n}(\vartheta=\pi)=\hat{n}_s$. The energy dependent of the η_x and η_z arises from the dependent of the spin tune on the energy (Eq. (4.3)).

If the spin tune ν_s is always a half integer during the acceleration period, the resonance condition would never encountered. The spin depolarization may be eliminated.

Fig. 10 Measured at the symmetry point, $\vartheta = \pi$ from the snake location, the spin closed orbit function $\hat{n}(\vartheta)$ is plotted as a function of γG for a 40° snake with snake axis $\varphi_s = 0$, or $\hat{n} = \hat{x}$. The snake is adiabatically turned on at $\gamma G = 2.5$ and off at $\gamma G = 5.5$. For a fully excited snake, 180° snake, we obtain the spin closed orbit $\hat{n} = \hat{n}_s$ and independent of energy.

5. A SPIN ROTATOR OR THE SNAKE IN THE ACCELERATOR

Last section, we have analyze the effect of a snake on the spin tune and the spin closed orbit function $\hat{n}(\vartheta)$ in the case of zero spin depolarizing resonance. In this section, we shall study the effect of a single snake on the spin in the presence of a spin resonance.

First the spin transfer matrix of a single resonance is given by Eq. (2.16),

$$\psi(\vartheta_f) = t(\vartheta_f, \vartheta_i) \psi(\vartheta_i), \quad (5.1)$$

where the components of the matrix $t(\vartheta_f, \vartheta_i)$ are

$$\begin{aligned}
t_{11}(\vartheta_f, \vartheta_i) &= a \cdot \exp(i[c - K(\vartheta_f - \vartheta_i)/2]) \\
t_{12}(\vartheta_f, \vartheta_i) &= ib \cdot \exp(-i[d + K(\vartheta_f + \vartheta_i)/2]) \\
t_{21}(\vartheta_f, \vartheta_i) &= -t_{12}^*(\vartheta_f, \vartheta_i) \quad ; \quad t_{22}(\vartheta_f, \vartheta_i) = t_{11}^*(\vartheta_f, \vartheta_i)
\end{aligned} \tag{5.2}$$

with

$$\begin{aligned}
b &= \frac{|\varepsilon|}{\lambda} \sin[\lambda(\vartheta_f - \vartheta_i)/2] = (1 - a^2)^{1/2} \\
\lambda &= (\delta^2 + |\varepsilon|^2)^{1/2} \\
\delta &= K - \gamma G \\
c &= \arctan\left[\frac{\delta}{\lambda} \tan(\lambda(\vartheta_f - \vartheta_i)/2)\right] \\
d &= \arg \varepsilon^*
\end{aligned} \tag{5.3}$$

Note here that the factor b depends on the number of snakes for the accelerator by $b = (|\varepsilon|/\lambda) \cdot \sin(\lambda\pi/N_s)$, where N_s is the number of equally spaced snakes. Thus t_{12} , the depolarizing driving term of the spin transfer matrix, will become smaller for a higher number of snakes. In the following, we shall discuss the situation of a single snake.

At the symmetry point of a single snake, i.e. $\vartheta = \pi$, the ONE TURN MAP (OTM) is obtain from the products of spin transfer matrices as,

$$t_{11}(\vartheta_0 + 2\pi, \vartheta_0) = 2ab \cdot \exp(i[c - K\pi]) \cos(d + K\vartheta_0 + K\pi - \phi_s) \tag{5.4a}$$

$$t_{12}(\vartheta_0 + 2\pi, \vartheta_0) = ie^{-i\phi_s} (1 - 2b^2 e^{-i(d + K\vartheta_0 + K\pi - \phi_s)}) \cos(d + K\vartheta_0 + K\pi - \phi_s) \tag{5.4b}$$

$$t_{21}(\vartheta_f, \vartheta_i) = -t_{12}^*(\vartheta_f, \vartheta_i) \quad ; \quad t_{22}(\vartheta_f, \vartheta_i) = t_{11}^*(\vartheta_f, \vartheta_i)$$

where ϕ_s is the angle of the snake axis with respect to the $\hat{x}$, i.e. $\hat{n}_s = (\cos\phi_s, \sin\phi_s, 0)$. The perturbed spin tune Q_s is given by the trace of the OTM.

$$\cos\pi Q_s = -2ab \cdot \cos(c - K\pi) \cdot \cos(d + K\vartheta_0 + K\pi - \phi_s) \tag{5.5}$$

Note here that the deviation of the spin tune from $1/2$ is in first order proportional to $|\varepsilon|/\lambda$. As γG approaches the resonance position K , i.e. $\delta \rightarrow 0$, the spin tune spread becomes,

$$\Delta Q_s \rightarrow -\frac{1}{\pi} \sin^{-1}(\sin(2\pi\epsilon)\cos(K\pi)\cos(d+K\vartheta_0+K\pi-\phi_s)) \quad (5.6)$$

Thus the spin tune could have a large spread when the resonance width $|\epsilon|$ is larger than 0.25. However, a small spin tune spread, e.g. $|\epsilon| \leq 0.01$, does not necessarily warrant polarization to be preserved in passing through the resonance. In the following, we shall examine the conditions of spin depolarization. If we can avoid these conditions in the design of the accelerator and snake configuration, the polarization of the beam can be maintained.

We first assume that the spin is vertically polarized. The OTM reverses the polarization every turn due to the local spin rotator. To measure the proper polarization, we shall study the TWO TURN MAP(TTM). From Eq. (5.4), one obtains the components of TTM, $t(\vartheta_0+4\pi, \vartheta_0)$, up to the linear order in $|\epsilon|$ as,

$$t_{11}(\vartheta_0+4\pi, \vartheta_0) = -1 + O(b^2) + O(b^4) \quad (5.7a)$$

$$t_{12}(\vartheta_0+4\pi, \vartheta_0) = -4iab \cdot \exp(-i\phi_s) \left[\cos K\pi \cdot \cos(c-K\pi) \cdot \cos(d+K\vartheta_0-\phi_s) \right. \\ \left. - i \sin K\pi \cdot \sin(c-K\pi) \cdot \sin(d+K\vartheta_0+K\pi-\phi_s) \right] + O(ab^3) \quad (5.7b)$$

The n-th TTM can be solved iteratively to the SPIN TRACKING EQUATION,

$$T(\vartheta_{n+1}) = t(\vartheta_{n+1}, \vartheta_n) \cdot T(\vartheta_n) \quad (5.8)$$

To the leading order, the off-diagonal element of this equation is given by,

$$T_{12}(\vartheta_{n+1}) = (-1)^{n+1} 4iab \exp(-i\phi_s) \left[\cos K\pi \cdot \cos(c-K\pi) \cdot \cos(d+K\vartheta_0+K\pi-\phi_s) \right. \\ \left. - i \sin K\pi \cdot \sin(c-K\pi) \cdot \sin(d+K\vartheta_0+K\pi-\phi_s) \right] \cdot \zeta_{n+1}(2K) \quad (5.9)$$

where the enhancement function ζ is given in Appendix A. The polarization measured from the nth-TTM is given by

$$P = \langle S \rangle = 1 - 2|T_{12}|^2 \quad (5.10)$$

When T_{12} becomes large, the spin will be depolarized. At a small ϵ , the off diagonal matrix element can be large due to the resonance condition, $2K = \text{integer}$, where the factor $\zeta_{n+1}(2K) = (n+1)$ at the resonance. This is due to

the additive coherent effect of each TTM. Thus the vertical polarization can not be maintained in the accelerator with a single snake at $2K=\text{integer}$. We can easily choose the resonance position away from the half-integer with proper betatron tune, but we can not eliminate the imperfection resonance at $K=\text{integer}$. Even a small resonance strength ϵ , the coherent effect would result in depolarization. Tracking results confirm the analysis.

This is not surprising because the vertical spin does not conform the spin closed orbit $\hat{n}$ vector of the accelerator with a single snake. At the symmetry point, $\vartheta=\pi$, the $\hat{n}$ vector should coincide with snake axis $\hat{n}_s$. Since $\hat{n}_s$ lies on the horizontal plane, the stable spin direction should also lie on the horizontal plane.

To facilitate the spin along the $\hat{n}$ vector, we can easily make a unitary transformation to the OTM of Eq. (5.4),

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\phi_s} & e^{-i\vartheta_s} \\ 1 & -1 \end{bmatrix} \quad (5.11)$$

where $AA^* = A^*A = 1$. The resulting OTM becomes $\tau(\vartheta_0+2\pi, \vartheta_0) = A^* \tau(\vartheta_0+2\pi, \vartheta_0) A$ with

$$\begin{aligned} \tau_{11}(\vartheta_0+2\pi, \vartheta_0) &= i - 2ab \cos(c - K\pi) \cos(d + K\vartheta_0 + K\pi - \phi_s) - \\ &\quad - 2ib^2 \cos^2(d + K\vartheta_0 + K\pi - \phi_s) \end{aligned} \quad (5.12a)$$

$$\begin{aligned} \tau_{12}(\vartheta_0+2\pi, \vartheta_0) &= -2iab \sin(c - K\pi) \cos(d + K\vartheta_0 + K\pi - \phi_s) + \\ &\quad + b^2 \sin(2(d + K\vartheta_0 + K\pi - \phi_s)) \end{aligned} \quad (5.12b)$$

$$\tau_{21}(\vartheta_f, \vartheta_i) = -\tau_{12}^*(\vartheta_f, \vartheta_i) \quad ; \quad \tau_{22}(\vartheta_f, \vartheta_i) = \tau_{11}^*(\vartheta_f, \vartheta_i)$$

Naturally, the spin tune is still given by Eq. (5.5). The n th-OTM can be calculated again through the spin tracking equation of Eq. (5.8), where we obtain to the first order,

$$\begin{aligned} T_{12}(\vartheta_{n+1}) &= (-i)^{n+1} 4iab \sin(c - K\pi) \cos\left(d + K\vartheta_0 + K\pi - \phi_s + n \frac{2K-1}{2} \pi\right) \times \zeta_{n+1} \left(K - \frac{1}{2}\right) \\ &\quad + (-i)^n b^2 \sin(2d + 2K\vartheta_0 + 2K\pi - 2\phi_s + n(2K - 1/2)\pi) \times \zeta_{n+1} \left(2K - \frac{1}{2}\right) \end{aligned} \quad (5.13)$$

where $\zeta_{n+1}(x)$ is given in the Appendix A. The polarization, obtained from Eq. (5.10), results important depolarization at the conditions $K-1/2=\text{integers}$ or $2K-1/2=\text{integers}$. These depolarization can easily be avoided by choosing vertical machine betatron tune away from these resonance conditions. The resonance condition of $K=\text{integer}$ has disappeared in the present spin orientation.

An interesting observation on Eq. (5.13) is that the depolarization driving term τ_{12} vanishes as the spin precession frequency γG is approaching an imperfection resonance $K=\text{integer}$, i.e.

$$\sin(c-K\pi) \rightarrow 0 \text{ at } K=\text{integer and } c \rightarrow 0 \text{ as } K-\gamma G \rightarrow 0.$$

When the spin precession frequency γG is approaching the integer resonance position K , the strength coefficient ab increases, but the effective resonance strength becomes zero at the resonance position due to the $\sin(c-K\pi)$ factor. The argument that the snake is better for the imperfection resonance is based essential on this reduction of the effective strength. Figure 11 shows the polarization calculated from a spin tracking program as the particle being accelerated through an imperfection resonance. The importance of the $\sin^2(c-K\pi)$ factor is seen clearly from the numerical simulation. Figure 12 shows the polarization as a function the resonance position K for resonance strength $\xi=0.2$. We observe that depolarization occurs easily at $K=1/2$ and $3/4$ as predicted earlier. There are also many higher order resonances, which shall be discussed in next section.

Fig. 11 The polarization calculated from a spin tracking for one snake is compared with the envelope functions, $1-8a^2b^2$ and $1-8a^2b^2 \sin^2(c-K\pi)$. Numerical simulation shows clearly the effect of the form factor.

where $b=(|\epsilon|/\lambda)\cdot\sin(\pi\lambda/2)$, $a=(1-b^2)^{1/2}$ and the quantities c and d are defined in Eq. (5.3). Again, the perturbed spin tune Q_s is obtained from the trace of the OTM,

$$\cos \pi Q_s = -\cos(\phi_2-\phi_1)+2b^2\cos(K\vartheta_0+d+K\pi-\phi_2)\cos(K\vartheta_0+d+K\pi-\phi_1) \quad (6.3)$$

Choosing $\phi_2-\phi_1=\pi/2=\pi\nu_s$, we find that the tune spread due the presence of the resonance is proportional to $\sin^2(\pi|\epsilon|/2)$. This is a great improvement over the result obtained for the accelerator with a single snake (Eq. 5.5). Figure 13 shows the perturbed tune spread as a function of the resonance ϵ . Note here that the tune spread becomes 1 at $\epsilon=1,3,5,\dots$. We thus expect the depolarization would occur at any K value. This means that the snakes can not be used to avoid the resonance at a resonance strength $\epsilon=1$. The important issue is however the snake resonance condition due to the beating of the snake tune ν_s and the spin resonance precession frequency K .

Using the spin iterative equation, Eq. (5.8), we obtain the linear order of the n th-iterative OTM as,

Fig. 13 The upper graph shows the envelope of the perturbed spin tune as a function of ϵ (Eq. 6.3). The lower graph shows the polarization after passing through a resonance at $K = 0.81$ as a function of ϵ . The depolarization at $\epsilon = 1,3,5,\dots$ is due to the large spin tune spread.

$$T_{12}(\vartheta_{n+1}) = (-1)^{n+1} 2iabe^{-i(c-K\pi+\phi_2+2n\pi v_s)}$$

$$\frac{1}{2} \left(e^{i(K\vartheta_0+d+K\pi-\phi_1+n\pi K)} \zeta_{n+1}(K-v_s) + e^{-i(K\vartheta_0+d+K\pi-\phi_1+n\pi K)} \zeta_{n+1}(K+v_s) \right) \quad (6.4)$$

where $\zeta_{n+1}(x)$ is given in Eq. (A.1) in the appendix. Note here that when $K \pm v_s = \text{integers}$, the factor $\zeta_{n+1}(K \pm v_s)$ becomes $n+1$. The depolarization driving term grows linearly with the number of circulation in the accelerator. This is called the first order snake resonance.

On the other hand, the factor $\zeta_{n+1}(K \pm v_s)$ vanishes when $K = \text{integer}$ and $n+1 = \text{even number}$. This implies that the linear depolarizing driving term cancels each other for the imperfection resonance every two turns around the accelerator at all precession frequencies. This feature is the fundamental design concept of snakes in the accelerator. Similarly, the factor $\zeta_{n+1}(K \pm v_s)$ vanishes when K is a quarter integral and $n+1$ is equal to integer multiple of 4. The depolarizing driving terms cancel each other every four turns around the accelerator.

The resulting polarization $P = \langle S \rangle = 1 - 2|T_{12}|^2$ would normally be bounded by the envelope function, $\langle S \rangle$,

$$\langle S \rangle = 1 - 8a^2b^2 \quad (6.5)$$

with $b = (|\epsilon|/\lambda) \cdot \sin(\pi\lambda/2)$, $a = (1 - b^2)^{1/2}$ and $\lambda = (|\epsilon|^2 + \delta^2)^{1/2}$. Appendix B discusses the envelope function. Figure 14 shows the tracking result of the polarization as a function of γG around the resonance region. The envelope function is also shown for comparison. The agreement between the numerical simulation and the linear response analysis of Eq. (6.5) is good. An interesting feature of the tracking result is that the polarization is restored to 1 at the NODE location where $ab=0$. Around the nodal location the depolarization driving term is small, or the off-diagonal matrix of OTM is zero.

Figure 15 shows the final polarization as a function of the spin resonance position, K , for $\epsilon=1.93$. The first order snake resonance has a width of 0.1 due to the unperturbed spin tune shown in Fig. 13. The perturbed spin tune spread is about 0.1. At any vertical tune within the perturbed tune spread, the spin is easily depolarized. Besides the first order snake resonances, $K = \text{integer} \pm v_s$, there are many higher order resonances at the fractional number of K . These higher order resonances have a much smaller width.

Fig. 14 The polarization at each turn around the accelerator as the particle is accelerated through spin resonance. The envelope function is shown for comparison.

Fig. 15 The polarization after passing through a single spin resonance at strength $\epsilon=1.93$ is shown as a function of the fractional part of the spin resonance. Several higher order resonances are seen easily.

6.1 HEURISTIC APPROACHES TO THE HIGH ORDER SNAKE RESONANCE

We have found that the two snake system enables the spin depolarizing driving terms to cancel each other every two turns around the accelerator for the imperfection resonances and to cancel each other every four turns around the accelerator for the quarter integer resonances. The spin is however easily depolarized at some fractional numbers. To understand the physics of these higher order resonances, we have to solve the iterative equation to higher order in the strength parameter b .

The complete solution of the iterative equation Eq. (5.8) is complicated. We shall evaluate the iterative equation according the order of importance. This is encouraged by the agreement between the linear order analysis and the spin tracking simulation of Fig. 14. To understand the higher order resonance, we shall concatenate 2-OTM, i.e. $t^{(2)}(\vartheta_0+4\pi, \vartheta_0) = t(\vartheta_0+4\pi, \vartheta_0+2\pi) t(\vartheta_0+2\pi, \vartheta_0)$, where the off diagonal matrix element is given by,

$$t_{12}^{(2)}(\vartheta_0+4\pi, \vartheta_0) = 2iab e^{-i(c-K\pi+\phi_2)} \left[e^{i\pi\nu_1} \cos\Phi + e^{-i\pi\nu_1} \cos(\Phi+2K\pi) \right] - 4iab^3 e^{-i(c-2K\pi+\phi_2)} \cos\Phi \cos(\Phi+2K\pi) \cos(\Phi+K\pi+\pi\nu_1) \quad (6.6)$$

with $\Phi = K\vartheta_0 + d + K\pi - \phi_1$. The term linear in ab presents nothing new (Eq. 6.4). The next higher order is proportional to ab^3 , which results from the product of term linear in ab and the term proportional to b^2 of the diagonal matrix element.

Using the nonlinear term, ab^3 , as the driver of the spin iterative equation Eq. (5.8), and neglecting the nonlinear contribution from the diagonal matrix element, we obtain the final off-diagonal matrix element of $T_{12}(\vartheta_{2(n+1)})$ proportional to

$$ab^3 \zeta_{n+1} (2[v_s \pm 3K]) \quad \text{and} \quad ab^3 \zeta_{n+1} (2[v_s \pm K])$$

Thus the snake resonances would occur at conditions $2[v_s \pm 3K] = \text{integer}$ and $2[v_s \pm K] = \text{integer}$, i.e. $K = 1/6, 1/3, 1/2, 2/3$ and $5/6$. Note here that the linear driving terms cancel each other every six turns around the accelerator at these fractional K values. These resonances are therefore less important. However, when the spin tracking matrix of the concatenated three OTM is given by,

$$t^{(3)}(\vartheta_0+6\pi, \vartheta_0) = t(\vartheta_0+6\pi, \vartheta_0+4\pi) t^{(2)}(\vartheta_0+4\pi, \vartheta_0) \\ = 0(ab) + 4iab^3 e^{-i(c-K\pi+\phi_2)} \left[f_3^+ e^{i3\Phi} + f_1^+ e^{i\Phi} + f_1^- e^{-i\Phi} + f_3^- e^{-i3\Phi} \right] + 0(ab^5) \quad (6.7)$$

where $\Phi = K\vartheta_0 + d + K\pi - \phi_1$. Figure 16 shows $|f_1^\pm|$ and $|f_3^\pm|$ as a function of the resonance position K of Eq. (6.7). We found that $f_3^- = 0$ at $K = 2/6$ and $4/6$. Thus the resonance position at $K = 2/6$ and $4/6$ would be less important. Using $t^{(3)}$ as a driver for the iterative equation, we obtain the resonance condition as,

$$3(v_s \pm K) = \text{integer} \quad \text{and} \quad 3(v_s \pm 3K) = \text{integer} \quad (6.7a)$$

The resonances of Eq. 6.7a are then located at $1/6$ and $5/6$ due to $f_1^\pm$ and $1/18, 3/18, \dots, 17/18$ due to $f_3^\pm$. Figure 16 indicates that the resonance driver amplitude would be larger at $11/18$ than $13/18$. Similar feature is seen in the tracking result in Fig. 15.

Similarly, using the term proportional to ab^5 in $t^{(3)}$ as a driver for the iterative equation, we obtain the resonance condition,

$$3(v_s \pm \ell K) = \text{integer} ; \quad \ell = 1, 3, 5 .$$

These new resonances are located at fractional K equal to $1/30, 1/10, 1/6, 7/30, \dots, 29/30$.

In general, using the concatenated spin transfer matrix, we can derive the snake resonance condition for the higher order resonances. The general condition is given by,

$$m(v_s \pm \ell K) = \text{integer} ; \quad m = 1, 3, 5 ; \quad \ell = 1, 3, \dots, (2m-1) \quad (6.8)$$

Fig. 16 The amplitudes of the second order term of the concatenated three turn map is plotted as a function of the spin resonance K . Note that amplitude at $K = 11/18$ is larger than that at $13/18$ is reflected in Fig. 15.

where the missing even m order is due mainly to the cancellation of the linear driving term after m turns around the accelerator.

6.2 HIERARCHY EQUATIONS FOR THE SPIN TRACKING

We have studied heuristically the snake resonance conditions (Eq. 6.8). The approach is however neither systematic nor complete. We can predict the importance of the first few order of resonances. But we can not foresee the relative importance of snake resonances of higher orders, e.g. $K=17/18$ vs. $K=29/30$ etc. The problem can of course be obtained from the numerical simulation studies. To obtain a systematic solution to the problem, the set of hierarchy equations can be derived and solved. We shall set up the hierarchy equation without solving them in this section.

The n-th iterative spin transfer matrix $T(\vartheta_n)$ of Eq. (5.8) can be decomposed according the order of the strength of the parameter b,

$$T_{11}(\vartheta_n) = T_{11}^{(0)}(\vartheta_n) + T_{11}^{(1)}(\vartheta_n) + T_{11}^{(2)}(\vartheta_n) + \dots \quad (6.8a)$$

$$T_{12}(\vartheta_n) = T_{12}^{(1)}(\vartheta_n) + T_{12}^{(2)}(\vartheta_n) + \dots \quad (6.8b)$$

where $T_{11}^{(m)} = O(b^{2m})$ and $T_{12}^{(m)} = O(ab^{2m-1})$. The iterative equation of Eq. (5.8) can be solved by the following set of hierarchy equations,

$$T_{11}^{(0)}(\vartheta_{n+1}) = -e^{i(\phi_1 - \phi_2)} T_{11}^{(0)}(\vartheta_n) \quad (6.9a)$$

$$T_{12}^{(1)}(\vartheta_{n+1}) = -e^{i(\phi_1 - \phi_2)} T_{12}^{(1)}(\vartheta_n) - 2iab e^{-i(c - K\pi + \phi_2)} \cos(K\vartheta_n + d + K\pi - \phi_1) T_{11}^{(0)*}(\vartheta_n) \quad (6.9b)$$

$$T_{11}^{(1)}(\vartheta_{n+1}) = -e^{i(\phi_1 - \phi_2)} T_{11}^{(1)}(\vartheta_n) + 2b^2 e^{i(K\vartheta_n + d + K\pi - \phi_2)} \cos(K\vartheta_n + d + K\pi - \phi_1) T_{11}^{(0)}(\vartheta_n) \\ + 2iab e^{-i(c - K\pi + \phi_2)} \cos(K\vartheta_n + d + k\pi - \phi_1) T_{12}^{(1)*}(\vartheta_n) \quad (6.9c)$$

$$T_{12}^{(2)}(\vartheta_{n+1}) = -e^{i(\phi_1 - \phi_2)} T_{12}^{(2)}(\vartheta_n) + 2b^2 e^{i(K\vartheta_n + d + K\pi - \phi_2)} \cos(K\vartheta_n + d + K\pi - \phi_1) T_{12}^{(1)}(\vartheta_n) \\ + 2iab e^{-i(c - K\pi + \phi_2)} \cos(K\vartheta_n + d + k\pi - \phi_1) T_{11}^{(1)*}(\vartheta_n) \quad (6.9d)$$

This set of hierarchy equations can be solved step by step from the low order to high order in the resonance strength b. The sum of each order shall reveal the resonance structure discussed in the previous section.

6.3 OVERLAPPING RESONANCES

Analytic solution of a single resonance in Eq. (2.16) indicates that the spin will be influenced by the resonance at $\delta = K - \gamma G \equiv 3|\epsilon|$, away from the resonance. With the introduction of the local spin rotator, the effect remains unchanged with strength parameter, b . For the high energy accelerators, the resonance strength is of the order 1 or larger, the resonances may overlap each other. When two resonances overlap each other, the interference of these two resonances can give rise to some interesting phenomena. Unfortunately, there is no analytic solution yet to tackle the problem of overlapping resonance. Our analysis is based on single isolated resonance and numerical simulation. However, the analysis can explain some interesting phenomena of the overlapping resonances.

6.3.1 OVERLAPPING INTRINSIC RESONANCES

We first note that the locations of nodes with $ab=0$ plays an important role on the spin restoration after passing through the isolated spin resonance (see section 6.1). The polarization around the resonance region follows in general the envelope function $\langle S \rangle$ of Eq. (6.5). The location that the polarization is restored to 1, called the NODE, located at either $b=0$ or 1, i.e. $\sin[\pi\lambda/N_s]=0$ or 1, where $\lambda=(\delta^2+|\epsilon|^2)^{1/2}$; $\delta=K-\gamma G$ and N_s is the number of snakes in the accelerator. Away from the resonance location, $\delta \gg |\epsilon|$, we expect that the nodes are separated at a distance of $\Delta\delta=N_s$. For two snakes, they are separated by 2 unit in γG (see Fig.14 and appendix B).

When two resonances are near to each other, the spin is influenced by these two resonances. Figure 17 shows the tracking result of the polarization around the region of two resonances and the corresponding envelope functions. When the nodes of these two resonances coincide, the polarization is expected to be restored after crossing the resonances. Figure 18 shows the final polarization as a function of the spacing between these two resonances, ΔK . We observe that $\Delta K=1,3$ would result in depolarization due to the combined effect of these two resonances. In reality, the important spin resonances are separated by a distance of the superperiod of the machine P or $2\nu_B$, where ν_B is the total betatron tune across the dipole cells in the accelerator (see Section 3). The effect of two overlapping intrinsic resonances may not be important.

Fig. 17 The polarization obtained from a tracking result for two overlapping resonances at $\Delta K=2$ and 3 are compared with the envelope functions. Note that the nodal points coincide at $\Delta K=\text{even}$ for two snakes. The polarization can be restored easily. The resonance strength are both 0.745.

Fig. 18 The final polarization plotted as a function of the spacing between two resonances ΔK . When $\Delta K = \text{odd}$ for two snakes, the polarization is easily lost due to the combined effect of these overlapping resonances. At $\Delta K \geq 5$ for $\epsilon = 0.745$, these two resonances can be considered as isolated resonances.

6.3.2 OVERLAPPING INTRINSIC AND IMPERFECTION RESONANCES

In Section 3, we have found that the important imperfection resonance located near to the leading intrinsic resonances. It is therefore important to evaluate the effect of overlapping intrinsic and imperfection resonances.

The snake is very effective in the correction of an isolated imperfection resonances due to the cancellation of depolarization driving terms every two turns around the accelerator. This feature disappears when an intrinsic resonance is introduced nearby. Thereby the even order effect may be important. Thus the polarization may depend sensitively on the strength of the imperfection resonance strength. Figure 19 shows the final polarization as a function of the strength of the imperfection resonance. Note here that the imperfection resonance should be less than 0.1 when the intrinsic resonance strength is 0.75 for the accelerator with two snakes.

The implication is that the orbit correction is also very important for the large accelerator. The imperfection resonance strength should be corrected to a magnitude of insignificance. Here global closed orbit correction scheme can be used.

Fig. 19 The final polarization of a tracking calculation for an intrinsic resonance a $K_2=4.826$ $\epsilon_2=0.745$, and an imperfection resonance $K_1=3$ is plotted as a function of the imperfection resonance strength ϵ_1 . Note that ϵ_1 should be less than 0.1 to maintain the spin polarization.

7. ACCELERATORS WITH FOUR OR MORE SNAKES

There is a limit of resonance strength that two snakes can effectively be used to maintain the polarization to high energy. The first condition is of course the perturbed spin tune should not cross the resonance line. Since the perturbed spin tune spread is given by (Eq. 6.3)

$$\Delta Q_s = Q_s - \frac{1}{2} \approx \frac{1}{\pi} \sin^{-1} \left[\frac{|\varepsilon|^2}{\delta^2 + |\varepsilon|^2} \sin^2 \left(\frac{\pi(\delta^2 + |\varepsilon|^2)^{1/2}}{N_s} \right) \right]$$

At $|\varepsilon| = N_s/2$, the perturbed spin tune spread would cover the entire tune space (see Fig. 13). Therefore the spin would be depolarized at any spin resonance K value. Recent calculations by Yokoya¹⁶ demonstrate this effect in two snake situation clearly. Besides the condition of spin tune spread, the limiting resonance strength would depend on the resonance frequency K . Away from the snake resonance, the tolerable resonance strength would be larger. Figure 3 shows however that the resonance strength is expected to be about 5-8 at the SSC energy. These high energy accelerators may need many more snakes. In this section, we shall study the minimum requirement of the snake configuration and analyze the effect of many snakes on the polarization, including the snake superperiodicity.

7.1 BASIC REQUIREMENTS FOR THE SNAKE CONFIGURATION

Let us assume that there are four snakes installed in the accelerator. Let the snake axes be ϕ_1, ϕ_2, ϕ_3 , and ϕ_4 respectively w.r.t. to the $\hat{x}$ axis, i.e. $\hat{n}_1 = (\cos\phi_1, \sin\phi_1, 0)$ etc. The spin OTM for the accelerator without the spin resonance is given by

$$e^{i[2\gamma G(\pi - \vartheta_{12} - \vartheta_{34}) + \phi_4 - \phi_3 + \phi_2 - \phi_1] \sigma_3}$$

where ϑ_{12} and ϑ_{34} are the orbit rotation angle between the snakes 1,2 and snakes 3,4 respectively. Thus the conditions of obtaining (1) an energy independent spin tune is $\vartheta_{12} + \vartheta_{34} = \pi$, (2) a spin tune of $j + 1/2$ is $\phi_4 - \phi_3 + \phi_2 - \phi_1 = (2j+1)\pi/2$; $j = \text{integer}$.

In general, the accelerator with N_s snakes requires the following basic conditions:

$$\sum_{k=\text{odd}}^{N_s} \vartheta_{k,k+1} = \sum_{k=\text{even}}^{N_s} \vartheta_{k,k+1} = \pi \quad (7.1a)$$

$$\pi v_s = \sum_{k=1}^{N_s} (-1)^k \phi_k = (2j+1)\pi/2 \quad j = \text{integer} \quad (7.1b)$$

where $\vartheta_{k,k+1}$ is the orbit bending angle between the k-th and (k+1)-th snakes. The condition in Eq. (7.1a) ensure that the spin tune ν_s to be independent of the spin precession frequency γG . The condition in Eq. (7.1b) guarantees that the spin tune ν_s is an half integer.

7.2 EXAMPLE OF FOUR SNAKES

Although the conditions in Eq. (7.1) is rather simple, it is not easy to analyze the effect of the snake on the polarization. The strength parameter, b, in the spin transfer matrix of Eq. (5.1) depends on the rotational angle between the adjacent snakes. The OTM of the spin transfer matrix is rather complicated. To simplify the analytic analysis, we shall assume that the snakes in the accelerator are equally spaced along the 2π rotational angle. Thus the strength parameter b of Eq. (5.1) is the same for all sections between the snakes, i.e. $\vartheta_{12}=\vartheta_{23}=\vartheta_{34}=\vartheta_{41}=\pi/2$ for four snake example.

We first consider these four snakes satisfies $\phi_4-\phi_3+\phi_2-\phi_1=\pi\nu_s$. The OTM $t(\vartheta_0+2\pi, \vartheta_0)$ becomes,

$$\begin{aligned} t_{11} = & e^{i\pi\nu_s} \left(1 - 2b^2 e^{i(\chi+K\pi-\phi_3)} \cos(\chi+K\pi-\phi_3) \right) \left(1 - 2b^2 e^{i(\chi+K\pi-\phi_1)} \cos(\chi+K\pi-\phi_1) \right) \\ & - 4a^2 b^2 e^{i(\phi_2-\phi_4)} \cos(\chi+K\pi-\phi_3) \cos(\chi-\phi_1) \end{aligned} \quad (7.2a)$$

$$\begin{aligned} t_{12} = & 2iab e^{i(c-K\pi+\pi\nu_s)} \left[e^{i\phi_1} \cos(\chi+K\pi-\phi_3) \left(1 - 2b^2 e^{i(\chi-\phi_1)} \cos(\chi+K\pi-\phi_1) \right) \right. \\ & \left. + e^{i\phi_1} \cos(\chi-\phi_1) \left(1 - 2b^2 e^{i(\chi+K\pi-\phi_3)} \cos(\chi+K\pi-\phi_3) \right) \right] \end{aligned} \quad (7.2b)$$

where $b=(|E|/\lambda)\sin(\pi\lambda/4)$, $\chi=K\vartheta_0+d+K\pi/2$. A few interesting observation can be drawn from the OTM in Eq. (7.2).

(1) By choosing ϕ_1 and ϕ_3 properly, we can eliminate some imperfection resonances, e.g. $K=\text{even}$ or $K=\text{odd}$, in the first order.

(2) The perturbed spin tune Q_s is given by

$$\begin{aligned} \cos \pi Q_s = & \cos \pi \nu_s - 2b^2 \cos(\chi+K\pi-\phi_4-\phi_2+\phi_1) \cos(\chi+K\pi-\phi_3) - \\ & - 2b^2 \cos(\chi+K\pi-\phi_4+\phi_3-\phi_2) \cos(\chi-\phi_1) - 4b^4 \cos(2\chi+K\pi-\phi_4-\phi_2 \\ & \cos(\chi+K\pi-\phi_3) \cos(\chi-\phi_1) \\ & - 4a^2 b^2 \cos(\phi_2-\phi_4) \cos(\chi+K\pi-\phi_3) \cos(\chi-\phi_1) \end{aligned} \quad (7.3)$$

We observe that the spin tune is of the order of b^2 . Because the strength parameter b is smaller in the four snakes situation, the deviation of Q_s from ν_s is also smaller.

(3) At the imperfection resonances, $K=\text{integer}$, the linear depolarization driving terms cancel each other every two turns around the accelerator. The conditions for the first order snake resonances from Eq. (7.2b) are given by

$$\nu_s \pm K = \text{integer}.$$

Besides, the OTM contains the term proportional to ab^3 with the snake resonance conditions,

$$\nu_s \pm K = \text{integer} \quad \text{and} \quad \nu_s \pm 3K = \text{integer}$$

For $\nu_s=1/2$, the snake resonance will be more important at $K=1/6$ or $5/6$ in the four snake situation than the two snake situation.

Figure 20 shows the final polarization after passing through a resonance located at K as a function of the fractional part of K for $|\epsilon|=1.0$. The width of the resonance at $K=0.5$ is also about 0.1, the same as that of two snake situation. Thus the four snake configuration does not offer much advantage over the situation with two snakes.

Fig. 20 Final polarization as a function of K for 4 snakes organized in $P_s=1$ and $\nu_s=1/2$. The resonance strength is $|\epsilon|=1.0$.

7.3 SNAKE SUPERPERIODICITY

Four or more snakes in an accelerator can be organized in many different configurations. For example, we can organize four snakes with $(\phi_1, \phi_2, \phi_3, \phi_4)$ equal to $(0, \pi/4, \pi/2, 3\pi/4)$ or $(0, \pi/4, 0, \pi/4)$. The first organization gives rise to the snake superperiod 1 and the second configuration give us the snake superperiod 2. For higher number of snakes, the possibility of snake configuration is even greater. In this section, we shall examine the effect of the snake superperiod.

Let us consider the example of $N_s/2$ pairs (ϕ_1, ϕ_2) snakes in the accelerator. The snake superperiod is then $P_s = N_s/2$. The spin tune is $\pi\nu_s = N_s(\phi_2 - \phi_1)/2$. Thus $\phi_2 - \phi_1 = \pi\nu_s/P_s$. The concatenated spin transfer matrix through the snake superperiod is then,

$$t_{11}\left(\vartheta_0 + \frac{2\pi}{N_s}, \vartheta_0\right) = -e^{-i\pi\nu_s/P_s} \left[1 - 2b^2 e^{i(K\vartheta_0 + \eta)} \cos(K\vartheta_0 + \eta) \right] \quad (7.4)$$

$$t_{12}\left(\vartheta_0 + \frac{2\pi}{N_s}, \vartheta_0\right) = -2iab e^{i(c - K\pi + \phi_2)} \cos(K\vartheta_0 + \eta) \quad (7.5)$$

where $b = (|E|/N_s) \sin(\pi\lambda/N_s)$ and $\eta = d + K\pi/N_s - \phi_1$ and a, c, d, λ are defined in Eq. (5.3).

Solving the spin iterative equation Eq. (5.8), we obtain the first order snake condition as

$$\frac{1}{P_s} (\nu_s \pm K) = \text{integer} \quad (7.6)$$

Note here that there are fewer snake resonance in this case, but the snake resonance width would also increase by P_s times. This feature is similar to the betatron resonance where the spacing between the half integer stop bands increase linearly with the superperiodicity of the machine. The problem with the spin polarization is that the particle precession frequency is sweeping through many resonances. If the spin encounter a resonance with large width, the polarization may be lost easily.

Similar to previous discussions, at the imperfection resonance, $K = \text{integer}$, the linear depolarization driving term cancels each other every two turns around the accelerator. At the quarter integer, they cancel each other every four turns around the accelerator.

7.4 NUMERICAL EXAMPLE OF SIX SNAKES

The effect of the snake superperiodicity can be tested in the numerical study of the accelerator with six snakes. We consider the following three possible configurations:

(A) $P_s=3$ with $\phi_1=0$ and $\phi_2=\pi/2$; $v_s=3/2$

(B) $P_s=3$ with $\phi_1=0$ and $\phi_2=\pi/6$; $v_s=1/2$

(C) $P_s=1$ with $\phi_1=0$, $\phi_2=\pi/6$, $\phi_3=\pi/3$, $\phi_4=\pi/2$, $\phi_5=2\pi/3$, $\phi_6=5\pi/6$; $v_s=1/2$

All other variant can be reduced into one of the above configurations or the essential physics will not differ from the above configurations.

The first order snake resonance is clearly given by the condition in Eq. (7.6). The configuration (C) gives rise to snake resonance at every half integer K . The configurations (A) and (B) encounter snake resonance only at $K=3\text{-integer} \pm v_s$. For example $K=456.5$, $459.5 \dots$ are not a first order snake resonance for the configuration A, likewise $K=457.5$, $460.5 \dots$ are not a first order snake resonance for the configuration B.

Let us now define the critical resonance strength ϵ_c as the maximum spin resonance strength that the polarization is not lost in passing through the resonance.

Figure 21 shows the critical resonance strength ϵ_c vs $K=460+\Delta$ for the configurations A (upper part) and B (lower part). Note that $K=460.5$ is a first order snake resonance for the configuration A. The critical resonance strength at $K=460.5$ is $\epsilon_c=2 \cdot 10^{-3}$. The snake resonance is very wide with width about 0.3 due to the snake superperiod. On the other hand, $K=460.5$ is not a first order snake resonance of the configuration B (lower part of Fig. 21). The critical resonance strength is about 1.2. The higher order resonances become prominent in this case.

A special interesting feature in Fig. 21 for the large critical resonance strength at $K=460.75$ is due to the previously mentioned cancellation of the linear depolarization term every four turns around the accelerator for the quarter integer resonance.

Outside the width of the first order snake resonance, the critical resonance strength is determined by the higher order resonances. There the critical resonance strength does not depend on the snake configuration. Figure 22 shows the width of the next order resonance $K=5/6$ for configurations A, B and C respectively. Note here that the width of the resonance is about 0.01. The tune control is very important even in the high order snake resonances. The critical resonance strength is insensitive to the snake configurations. Machine tune at 0.81 present a reasonable choice of spin and orbital operation point.

Fig. 21 The critical resonance strength ϵ_c is plotted as a function of the resonance frequency $K=460+\Delta$. At $\Delta=0.5$, the snake configuration in the upper graph corresponds to a first order snake resonance with three snake superperiod. On the other hand, $\Delta=0.5$ is not a first snake superperiod for the snake configuration in the lower graph.

Fig. 22 The critical resonance strength near a higher order snake resonance is plotted vs. the distance from the snake resonance at $K=5/6$. The width is about 0.01 and independent of the snake configuration.

7.5 SCALING PROPERTY OF THE ϵ_c VS. N_s

We have learned that $K=0.81$ is a reasonable location for the machine tune. In the following, we shall study the scaling property with the spin resonance $K=0.81 + \text{integers}$.

Figure 23 shows the critical resonance strength at $K=0.81 + \text{integers}$ as a function of the number of snakes, N_s . We organized the snake in a superperiod $P_s=1$, so that the first order snake resonance is not important at this time. The choice of $K = 0.81$ is also sufficiently far away from the higher order snake resonances. Note here that ϵ_c falls on a linear line of the maximum tune spread as

$$\langle \epsilon_c \rangle = \frac{\sin^{-1} (|\cos \pi K|^{1/2})}{\pi} N_s \quad (7.7)$$

At the expected critical resonance strength $\langle \epsilon_c \rangle$, the perturbed tune spread will encounter the spin resonance position K . The polarization would be lost easily.

Equation (7.7) indicates that the critical resonance strength would be larger at a betatron tune, or K , nearer to integer. The stability of orbital motion prefers however K not too close to an integer. With a proper choice of betatron tune to avoid snake resonances, the critical resonance strength can be simply expressed by Eq. (7.7).

Fig. 23 The critical resonance strength is plotted as a function of N_s for two snake configurations at $K=0.81$.

One interesting feature of Fig. 23 is that the critical resonance strength at 16 snakes falls below the average curve of Eq. (7.7). This is related to the spin anti-matching condition¹⁷ with $(K \pm \nu_s)/N_s = \text{integer or half integer}$ (Eq. 7.6). Figure 24 shows the spin tracking result for different resonance position with snake superperiod $P_s=1$. Figure 25 summarizes the critical resonance strength $\varepsilon_c < \varepsilon_c$ as a function $(K - \nu_s)/N_s$. This feature is independent of the snake configuration.

8. CONCLUSION

We have studied the effect of resonances on the polarization of proton in the circular accelerators.

The important spin depolarization resonance position and strength are derived in a simple analytical formula. We found that the important intrinsic resonances are separated by $\pm 2\nu_B$, where ν_B is the betatron phase advance accumulated through all dipole cells. The intrinsic resonance strength of various accelerators can be expressed by the following equation.

$$\varepsilon \approx \left(\frac{E}{\text{TeV}} \right)^{1/2} \cdot \left(\frac{\varepsilon_N}{10\pi\mu\text{m}} \right)^{1/2}$$

The important imperfection resonances are also found to appear near to the intrinsic resonances. The strength depends on the closed orbit error, z_{co} . With proper orbit correction, the imperfection resonance should be corrected to a magnitude of insignificance (e.g. $\varepsilon \leq 0.2$ for SSC).

We study the requirement of the snake configuration and the effect of the snake resonances, which is due to the beating between the spin tune ν_s and the spin resonance frequency K . There are a few important considerations on the snake configurations, the number of snakes, the snake superperiod, and the spin resonance positions, K , due to the lattice design of the accelerators.

(1) When large snake superperiod P_s is used, the spin resonance positions K should be chosen properly to avoid the first order snake resonance condition in Eq. (7.6). Once one avoids the first order snake resonance, the snake configuration is not important. The critical resonance strength ε_c , defined as the maximum tolerable resonance strength, is independent of the snake configuration.

(2) The fractional part of vertical betatron tune ν_z should be chosen to avoid higher order snake resonances (Eq. 6.8). Away from the higher order snake resonances, the critical resonance strength ε_c scales linearly with N_s given by Eq. (7.7), where the perturbed tune, Q_s , would encounter the spin resonance K , i.e., $Q_s \pm K = \text{integer}$ (see Eq. 6.3 and Fig. 13). The slope of ε_c vs N_s depends on the fractional part of betatron tune K (see Eq. 7.7). It is preferable to choose K nearer to an integer.

Fig. 24 Spin tracking result of various resonance position κ with $N_s = 16$, $v_s = 1/2$. Note here that when $(\kappa - v_s)/N_s = \text{integer or half integer}$, the polarization is strongly influenced by the spin resonance. This corresponds to the first order snake resonance condition.

(3) The number of snakes, N_s , required can be obtained from Eq. (7.7). For example, when $\epsilon=5$ is encountered in SSC, we shall choose a minimum of 14 snakes for $K=0.81 + \text{integer}$. Besides the criterion of the perturbed spin tune in Eq. (7.7), the number of snakes should be chosen to avoid spin antimatching at $(K-v_s) = \text{integer or half integer}$ (see Fig. 25). The condition can be avoided by choosing N_s or K properly.

Properly designed snake configuration, the tolerable resonance strength ϵ_c agrees well with the expected $\langle \epsilon_c \rangle$ from the perturbed tune spread consideration. For SSC, a resonance strength of the order 5 ~ 8 is expected from Fig. 3, one may need 12 ~ 24 snakes. For RHIC, the maximum resonance strength is about 0.5, two snakes seem suffice.

We have studied in this paper mainly the effect of a single spin resonance on the polarization. The effect of overlapping intrinsic resonances (see Section 6.3) is discussed in terms of the envelope function for a single resonance. When the intrinsic and imperfection resonances overlaps with each other, the polarization can be lost easily. This is probably due to the even order effect discussed in Section (6.3). The imperfection resonance strength should be corrected to a magnitude of insignificance. For two snakes, the imperfection resonance strength should be corrected to less than 0.1 when the nearby intrinsic resonance strength is $\epsilon=0.75$. The problem related to overlapping resonances remains a difficult problem to be tackled.

Fig. 25 The critical resonance strength $\epsilon_c / \langle \epsilon_c \rangle$ is shown as a function of K/N_s . Note here that when $(K-v_s)/N_s \equiv \text{integer or half-integer}$, $\epsilon_c / \langle \epsilon_c \rangle$ falls below 1.

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APPENDIX A. The enhancement factor $\zeta_n(x)$

The enhancement factor $\zeta_n(x)$ discussed in various sections in this paper is defined to be

$$\zeta_n(x) = \frac{\sin nx\pi}{\sin x\pi} \quad (\text{A.1})$$

This arrives from the coherent contribution of n identical sections. At a special phase advance of each section, the sum of n identical can be added or cancelled. This corresponds to the resonance condition in optical diffraction. These resonance condition corresponds to

$$\zeta_n(x) \rightarrow n \quad \text{at } x = \text{integers}$$

APPENDIX B. The envelope function

Based on the linear response solution to the interactive equation, the polarization around a resonance will oscillate within an envelope function

$$\langle\langle S \rangle\rangle = 1 - 8a^2b^2 \quad (\text{B.1})$$

where

$$b = \frac{|\epsilon|}{\sqrt{\delta^2 + |\epsilon|^2}} \sin \frac{\pi(\delta^2 + |\epsilon|^2)^{1/2}}{N_s} = (1 - a^2)^{1/2}$$

$$\delta = K - \gamma G$$

and N_s is the number of snakes in the accelerator. In the upper part of Fig. B1 shows the envelope function for two snakes as a function of γG for various resonance strengths. Note here that the nodal locations, $ab=0$, are separated by two unit of γG in the two snakes case. In the lower part of Fig. A1, similar plot is drawn for $N_s=12$ and 6 respectively. Note that the nodal locations are separated by N_s unit of γG .

The spin tracking calculation conforms well with the envelope function when the snake resonance condition is avoided. At the snake resonance condition, the spin perturbing kick is enhanced over the envelope function.

Fig. B1 The envelope functions for two snakes (upper) and 6,12 snakes (lower) are shown for various resonance strength.