**A NOTE ON THE RELATIONSHIP BETWEEN
THE EMITTANCE, THE BETA FUNCTION
AND THE ENERGY IN A LINEAR COLLIDER**

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Consider the scaling laws for linear colliders for the case of laterally round Gaussian beams and for the case that mutual pinching of the beams can be ignored.^[1] We choose these simple cases to make a point. Later we shall alter the laws to take account of the pinch.

$$L = \frac{fN^2}{A} , \quad (1)$$

$$P_b = fNE , \quad (2)$$

$$D = \frac{4\pi r_e m_e c^2 \sigma_z N}{AE} , \quad (3)$$

$$\delta = 2.71 \frac{r_e^2 N^2 E}{m_e c^2 A \sigma_z} , \quad (4)$$

where L is the luminosity, P_b is the power in each beam, E is the energy of each beam, D is the disruption parameter and δ is the mean fractional energy spread of particles that have passed through the opposing bunch due to their emission

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of classical radiation in the collective field of that bunch. (We shall demonstrate later that the classical approximation is valid in the cases with which we shall be dealing.) Further, f is the frequency of bunch collisions, N is the number of particles in a bunch, A is the effective interaction area, r_e is the classical radius of the electron and σ_z is the bunch length (standard deviation). The meanings of the other symbols are obvious.

These equations may be solved for A , N , f and σ_z (the accelerator parameters) in terms of E , L , D , δ and P_b (the performance and feasibility parameters), giving a unique specification for the former set of variables once the latter set is chosen. In this note we shall concentrate on one of the equations so obtained,

$$A(E, L, D, \delta, P_b) ,$$

and require the luminosity to vary as E^3 . We shall find as a result that the collider's energy is rather strongly restricted by the beam-forming technology we can hope to achieve.

The effective interaction area is

$$A = 4\pi\sigma_r^2 , \quad (5)$$

where σ_r is the radial standard deviation of the transverse distribution of the interacting beams. In terms of the beta function at the interaction region, β^* , and the normalized emittance, ϵ_n , of the beams from the accelerators assuming no pinch it is

$$A = 4\pi m_e c^2 \epsilon_n \beta^* / E , \quad (6)$$

where E is the beam energy.

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Equations (1) through (4) can be solved for A , f , N and σ_E , and in particular the result for A is⁽¹⁾

$$A = \frac{1}{2.71(4\pi)} \left(\frac{P_b^3 D \delta}{r_e^4 E^3 L^3} \right) . \quad (7)$$

The luminosity must scale as E^2 , so the interaction area varies as E^{-2} , a powerful dependence! Taking this fact together with Eq. (6) and Eq. (7), we may write the equation

$$\epsilon_n \beta^* = \frac{1}{2.71(4\pi)^2 r_e^4 m c^2} \left(\frac{P_b^3 D \delta}{L_{TeV}^3 E^2} \right) , \quad (8)$$

where L_{TeV} is the luminosity for a beam energy of 1 TeV (E_{cm} of 2 TeV). The value required for the product $(\epsilon_n \beta^*)$ varies as E^{-2} once we have fixed the factors in the numerator at values as large as the law or the traffic permits.

Conversely, the energy we can reach with the specified values of L_{TeV} , E , P_b , D and δ varies only as the eighth root of the smallest value of $(\epsilon_n \beta^*)$ we can achieve in practice. This is a very weak dependence. It implies that even if we are able to reduce the attainable value of $(\epsilon_n \beta^*)$ to one thousandth of the value we can now reach, we shall increase the collider energy by only a factor of 2.4. Whether such a weak dependence creates a serious threat to future colliders depends on the numbers we can reasonably attach to the variables, but before we investigate that question, we should correct Eq. (8) to take account of the pinch.

Consider the simplified picture of beams pinching each other shown in Fig. 1. The symbol A is meant to stand for the actual effective interaction area — that which would result from measuring the luminosity and solving Eq. (1) for A . A_0 is the effective interaction area which would prevail if the pinch were "turned off".⁽¹⁾ We might call it the incoming area. It is the area determined by the emittance and beta function produced by the accelerators of the collider. In other words, Eq. (6) should be replaced by

$$A_0 = 4\pi m_e c^2 \epsilon_n \beta^* / E . \quad (9)$$

Also the disruption parameter is customarily defined in terms of this area.

$$D_o = \frac{4\pi r_e m_e c^2 \sigma_s N}{A_o E} . \quad (10)$$

This equation replaces Eq. (3).

The enhancement factor, H , is¹²⁾

$$H(D_o) = \frac{A_o}{A} , \quad (11)$$

where the argument of H is specified as D_o because the predictions of H are presented in terms of it. We can now replace Eq. (8) with the improved form

$$\epsilon_n \beta^* = \frac{1}{2.71(4\pi)^2 r_e^4 m c^2} \left(\frac{P_b^3 D_o \delta H^2(D_o)}{L_{TeV}^3 E^3} \right) . \quad (12)$$

Solving this equation for E and inserting numerical values for the physical constants, we get

$$E = 3.39 \times 10^{11} \left(\frac{P_b^3 D_o \delta H^2(D_o)}{L_{TeV}^3 \epsilon_n \beta^*} \right)^{1/3} , \quad (13)$$

where E is in TeV, P_b is in megawatts, ϵ_n is in cm-radians, β^* is in cm and L_{TeV} is in c-g-s units (the usual).

To get a feeling for the practical magnitude of E , we can use the values of emittance and beta function from the SLC.

$$\epsilon_n \beta^* = 1.5 \times 10^{-3} \text{ cm}^2$$

Next we choose large values for the variables in the numerator: $P_b = 1$ MW, $\delta = 0.1$ and $D_o = 10$ which gives $H(D_o) = 5$.¹³⁾ Finally we choose a value for L_{TeV} which would make it possible to pursue a versatile experimental program. We choose $L_{TeV} = 10^{34} \text{ cm}^{-2} \text{s}^{-1}$. The result is

$$E = 0.2 \text{ TeV} .$$

The luminosity of this collider would be $4 \times 10^{32} \text{ cm}^{-2} \text{s}^{-1}$.

If we were able to reduce the product $(\epsilon_n \beta')$ by a divisor of one hundred, we could reach only

$$E = 0.35 \text{ TeV} .$$

Note that the factors used in the numerator are all at least several times the corresponding quantities in the SLC. The beam power is fourteen times that for which the SLC is designed for example. To seek to build a collider for a center-of-mass energy of even 0.7 TeV would compel us to make substantial improvements in the feasible values of many accelerator parameters in one generation.

At the outset we assumed that our calculations would place us in a region of variables where the classical beamstrahlung formula holds. This can be verified for the examples given.

Finally, an additional degree of freedom could be introduced by considering non-round beams. Equation (13) could be elaborated in terms of a "flatness parameter". Perhaps this should be done, but it does not seem likely that the upper limit on energy would be changed much.

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REFERENCES

1. J. Rees, SLAC-PUB-4073, (Sep 1986).
2. R. Hollebeek and A. Minton, Collider Note-302, SLAC Internal Memo, June 10, 1985.

FIGURE CAPTION

Fig. 1. The incoming area, A_0 , and the effective interaction area, A , in the case of a pinch.

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Fig. 1