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A free boundary problem arising in
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Abstract

This article is concerned with free boundary problems for the differential equation

$$u'' + (1/x)u' + f(u) = 0, \quad x > 0.$$

In particular, it addresses the questions of existence and uniqueness of a finite point P and a solution u satisfying the conditions

$$u'(0) = 0; \quad u(x) > 0, \quad 0 < x < P; \quad u(P) = u'(P) = 0.$$

Among the admissible nonlinearities are functions of the type $f(u) = u^p - u^q$ for certain combinations (p, q) , where $0 \leq q < p \leq 1$, including the special case $f(u) = \sqrt{u} - 1$, which has been proposed in plasma physics as a simple model for Tokamak equilibria with magnetic islands.

1 Introduction

In this article we study free boundary problems for the differential equation

$$u'' + (1/x)u' + f(u) = 0, \quad x > 0. \quad (1)$$

We prove that, for a broad class of functions f , there exists a finite point P and a solution u of (1) satisfying the conditions

$$u'(0) = 0; \quad u(x) > 0, \quad 0 < x < P; \quad u(P) = u'(P) = 0. \quad (2)$$

The class of admissible nonlinearities includes functions of the type $f(u) = u^p - u^q$ for certain combinations (p, q) , where $0 \leq q < p \leq 1$. The special

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case $f(u) = \sqrt{u} - 1$, proposed recently by Miller, Faber, and White [1] in plasma physics as a very simple model of Tokamak equilibria with magnetic islands, provided the motivation for the present investigation.

Let F be the integral

$$F(u) = - \int_0^u f(s) ds, \quad u > 0. \quad (3)$$

Throughout this article we assume that f satisfies the following conditions.

[F1] f is continuous on $[0, \infty)$ and locally Lipschitz on $(0, \infty)$.

[F2] $F(u) > 0$ for $0 < u < \beta$ and $f(u) > 0$ for $u \geq \beta$, for some $\beta > 0$.

[F3] $\lim_{\epsilon \downarrow 0} \int_{\epsilon}^{\beta} F^{-1/2}(u) du$ exists and is finite.

[F4] $\liminf_{u \rightarrow \infty} f(u) > 0$.

[F5] $u \mapsto \frac{f(u)}{u - \beta}$ is nonincreasing for $u > \beta$.

Condition [F1] imposes reasonable continuity requirements on f ; the Lipschitz continuity guarantees the existence and uniqueness of solutions of initial value problems whenever the initial data are positive.

Conditions [F2] and [F3] are necessary for the existence of a solution of the free boundary problem. Condition [F2] requires, first of all, that f be more negative than positive near the origin and, furthermore, that f eventually become positive. These properties certainly hold if f has exactly one zero, provided $f(u)$ is negative near $u = 0$ and positive for large u . Condition [F3] is satisfied, for instance, if $f(u)$ behaves like $-u^q$ ($0 < q < 1$) as $u \rightarrow 0$, or if $f(0) \neq 0$. If $f(u) = u^2 - u$, the free boundary problem does not have a solution, which indicates that [F3] is close to being necessary and sufficient. Condition [F3] was first introduced by Peletier and Serrin [2].

Condition [F4] guarantees that the nonlinear term $f(u)$ retains sufficient strength to bend the solution of (1) down toward the x -axis as the initial height at $x = 0$ increases.

Condition [F5] expresses the fact that f grows sublinearly beyond β . The condition is sufficient for the existence of a solution; however, it is not necessary (see [7]).

In Section 2 we prove the existence of a solution of the free boundary problem. The proof is based on the use of the shooting method. In Section

3 we briefly discuss the uniqueness of the solution of the free boundary problem. The uniqueness is essentially an immediate consequence of a theorem of Peletier and Serrin [2], as extended by the authors in [3] or, more specifically, in [4]

The results of this article have been extended to free boundary problems for the more general equation $u'' + g(x)u' + f(u) = 0$; see [7].

2 Existence

We use the shooting method, replacing the boundary problem by an initial value problem,

$$u'' + (1/x)u' + f(u) = 0, \quad x > 0, \quad (4)$$

$$u(0) = A, \quad u'(0) = 0. \quad (5)$$

Here, A is a positive constant. We denote a solution of this initial value problem by u_A . According to the classical theory of initial value problems, u_A exists in a neighborhood of the origin $x = 0$. We consider u_A only as long as it is strictly positive or until it reaches its first zero.

If, for some $A > 0$, the solution u_A of (4), (5) is such that $u_A(x) > 0$ for $0 \leq x < P_A$ and $u_A(P_A) = u'(P_A) = 0$ for some finite point P_A , then (P_A, u_A) is a solution of the free boundary problem (1), (2).

With any solution u_A of (4), (5) we associate a function E_A by the expression

$$E_A(x) = \frac{1}{2}[u'_A(x)]^2 - F(u_A(x)), \quad x \geq 0. \quad (6)$$

Since u_A satisfies (4), $E'_A(x) = -(1/x)[u'_A(x)]^2 \leq 0$, so E_A is nonincreasing along a solution curve.

Lemma 1 *Every solution u_A of (4), (5) is bounded.*

Proof. Since $F(u_A(x)) \geq -E_A(x) \geq -E_A(0) = F(A)$ for all $x \geq 0$, F is bounded along any solution curve. According to Condition [F4], the $\liminf$ of f at infinity is (strictly) positive, so $F(u_A)$ remains bounded if and only if u_A is bounded. ■

It follows from Lemma 1 that either any solution u_A of (4), (5) extends (as a positive-valued function) to the entire half-line $[0, \infty)$ or there is a finite point P_A where u_A reaches the x -axis. (In the latter case, we consider only the interval $[0, P_A]$.)

Let $\mathcal{O}$ denote the set of values $A > 0$ for which u_A extends to $[0, \infty)$. Clearly, only starting values A that do not belong to $\mathcal{O}$ can lead to solutions of the free boundary problem.

Lemma 2 *If $0 < A < \beta$, then $A \in \mathcal{O}$.*

Proof. If $0 < A < \beta$, $E_A(0) = -F(A) < 0$. Because E_A is nonincreasing, $E_A(x) < 0$ for all $x \geq 0$. On the other hand, it follows from the definition (6) that $E_A(x) \geq 0$ whenever $u_A(x) = 0$. Hence, if $0 < A < \beta$, u_A cannot reach the x -axis, so it must be the case that u_A extends to $[0, \infty)$. ■

Because of Lemma 2, only starting values $A \geq \beta$ can lead to solutions of the free boundary problem.

Lemma 3 *If $A > \beta$, then u_A is monotone strictly decreasing, at least until it reaches the value β .*

Proof. Letting x tend to 0 and applying l'Hospital's rule, we deduce from (4) that $2u_A''(0) + f(u_A(0)) = 0$. Because $f(u_A(0)) = f(A) > 0$ if $A > \beta$, we have $u_A''(0) < 0$, so u_A' is strictly decreasing near the origin. But $u_A'(0) = 0$, so it follows that $u_A'(x) < 0$ for all sufficiently small x .

If X is the first point after the origin where u_A' vanishes, then u_A must have a local minimum or an inflection point at X . In either case, $u_A''(X) \geq 0$. Then we deduce from (4) that $f(u_A(X)) \leq 0$, which is compatible with Condition [F2] if and only if $u_A(X) < \beta$. It must therefore be the case that X is beyond the point where u_A first reaches the value β . ■

Lemma 4 *If $E_A(X) < 0$ for some $X \geq 0$, then $A \in \mathcal{O}$.*

Proof. If E_A is negative at X , then it is negative for all $x \geq X$, because E_A is nonincreasing. On the other hand, $E_A(x) \geq 0$ whenever $u_A(x) = 0$, so it must be the case that u_A extends to $[0, \infty)$. ■

Lemma 2 is a special case of Lemma 4 ($X = 0$). The following lemma is a converse of Lemma 4.

Lemma 5 *If $A \in \mathcal{O}$, then $\lim_{x \rightarrow \infty} E_A(x) < 0$.*

Proof. The lemma is certainly true if $0 < A < \beta$, because $E_A(0) < 0$.

Consider $A \geq \beta$. According to Lemma 3, u_A is monotone strictly decreasing, at least until it reaches the value β . Suppose $u'_A(x) < 0$ for all $x > 0$. Then u_A is decreasing everywhere. Since u_A is also bounded from below, $u_A(x)$ must have a finite limit as $x \rightarrow \infty$. Let $\alpha = \lim_{x \rightarrow \infty} u_A(x)$. Then $\alpha \geq 0$.

Because $u_A(x)$ has a finite limit, the same is true for $F(u_A(x))$. Also, because E_A is nonincreasing and bounded below, $E_A(x)$ has a finite limit. Then it follows from (6) that $u'_A(x)$ has a finite limit. This limit is necessarily zero; otherwise, $u_A(x)$ would eventually become negative.

By letting x tend to infinity in (4), we see that $\lim_{x \rightarrow \infty} u''_A(x) = -f(\alpha)$. Obviously, $u''_A(x)$ cannot have a non-zero limit, so α must be a zero of f . Because of Condition [F2], we must have $0 \leq \alpha < \beta$.

Suppose $\alpha = 0$. Then $\lim_{x \rightarrow \infty} E_A(x) = 0$, because $\lim_{x \rightarrow \infty} u'_A(x) = 0$. But E_A is nonincreasing, so we conclude that $E_A(x) \geq 0$ and, hence, $[u'_A(x)]^2 \geq 2F(u_A(x))$ for all $x \geq 0$. But this inequality leads to a contradiction with Condition [F3], since it implies that $-F^{-1/2}(u_A(x))u'_A(x) \geq \sqrt{2}$ for all sufficiently large x , where $u_A(x) < \beta$. The expression in the left member remains integrable as $x \rightarrow \infty$ (i.e., $u_A(x) \rightarrow 0$), whereas the (constant) lower bound is clearly not integrable at infinity. Hence, it must be the case that $\alpha > 0$.

Because $0 < \alpha < \beta$, it follows that $\lim_{x \rightarrow \infty} E_A(x) = -F(\alpha) < 0$, as asserted.

Finally, if u'_A vanishes at some point X , it must be the case that $u_A(X) < \beta$, so $E_A(X) < 0$. Because E_A is nonincreasing, the assertion of the lemma follows. ■

The following lemma is an easy consequence of Lemmas 4 and 5.

Lemma 6 *The set $\mathcal{O}$ is open.*

Proof. Let $A \in \mathcal{O}$. Then u_A is defined on $[0, \infty)$ and $u_A(x) > 0$ for all $x \geq 0$. Lemma 5 implies that $E_A(X) < 0$ for some X large enough. Let X be thus determined. Because the solution depends continuously on the initial height, there is a neighborhood of A such that, for all B in this neighborhood, $u_B(x) > 0$ for all $0 \leq x \leq X$ and $E_B(X) < 0$. By Lemma 4, $B \in \mathcal{O}$. ■

In the proofs of the following lemmas we make use of Hadamard's inequality,

$$\|y'\|^2 \leq 2\|y\| \|y''\|. \quad (7)$$

Here, $\|\cdot\|$ denotes the usual sup-norm in the Banach space of continuous functions. The inequality (7) was first observed by Hadamard for C^2 -functions defined on $(-\infty, \infty)$ [5]. In [6] it was shown that the same inequality holds for C^2 -functions y that are defined on the half-infinite interval $[a, \infty)$ and satisfy the condition $y(a) = 0$, and for C^2 -functions y that are defined on the bounded interval $[a, b]$ and satisfy the conditions $y(a) = y'(b) = 0$.

Lemma 7 *The set $\mathcal{O}$ is a proper subset of $(0, \infty)$.*

Proof. The proof is by contradiction, where we suppose that $\mathcal{O}$ coincides with $(0, \infty)$. Then u_A extends (as a positive-valued function) to the entire interval $[0, \infty)$ for each $A > 0$. The contradiction is reached in several steps.

Step 1. Let $\mathcal{P}(x)$ denote the set $\{u_A(x) : A > 0\}$. This set is well defined for every $x \geq 0$. We claim that $\mathcal{P}(x)$ is unbounded for every $x \geq 0$.

The assertion is trivially true for $x = 0$, as $\mathcal{P}(0) = \{A : A > 0\}$.

Let a be fixed, $a > \beta$, and let u_a be the solution of (4) that satisfies the initial conditions $u_a(0) = a$, $u'_a(0) = 0$. Let x_a be such that $u_a(x) \geq \beta$ for $0 \leq x \leq x_a$; clearly, $x_a > 0$. It follows from Condition [F5] that $f(u_A)/(u_A - \beta) \leq f(u_a)/(u_a - \beta)$ as long as $u_A(x) \geq u_a(x) > \beta$. A standard Sturm comparison argument gives

$$\frac{u_A(x) - \beta}{A - \beta} \geq \frac{u_a(x) - \beta}{a - \beta}, \quad 0 \leq x \leq x_a. \quad (8)$$

Hence, $u_A(x)$ grows at least as fast as A , as A increases while x is being kept fixed. This shows that $\mathcal{P}(x)$ is unbounded for $0 \leq x \leq x_a$.

Suppose that $\mathcal{P}(x)$ is bounded for some $x_b > x_a$: $u_A(x_b) \leq C$ for all $A > 0$, say. We show that this supposition leads to a contradiction.

Take any $A > 0$. Without loss of generality we may assume that $u'_A(x_b) < 0$. Then either $u'_A(x) < 0$ for all $x \geq x_b$ or there is a finite point x_c , where u_A ceases to be monotone. In the former case, we put $x_c = \infty$. We apply Hadamard's inequality to the function y , defined by the expression $y(x) = u_A(x) - u_A(x_b)$ on the interval (x_b, x_c) . As $\|y\| \leq C$, $\|y'\| = \|u'_A\|$, and $\|y''\| \leq (1/x_b)\|u'_A\| + K$, where $K = \max\{|f(u)| : 0 \leq u \leq C\}$, we have

$$\|u'_A\|^2 \leq 2C \left(\frac{1}{x_b} \|u'_A\| + K \right). \quad (9)$$

This quadratic inequality shows that $\|u'_A\|$ is bounded by a constant independent of A . In particular, $|u'_A(x_b)| \leq D$ for some constant D .

Given that both u_A and u'_A are bounded at x_b by constants that are independent of A , it is not difficult to show that u_A is bounded at x_a . If $L = \max\{-f(u) : u \geq 0\}$ (notice that L is finite), then

$$u_A(x) \leq v(x), \quad x_a \leq x \leq x_b, \quad (10)$$

where v is the solution of the (backwards) initial value problem

$$v'' + (1/x_a)v' - L = 0, \quad x < x_b; \quad v(x_b) = C, \quad v'(x_b) = -D. \quad (11)$$

Since v is independent of A , it follows in particular that $u_A(x_a)$ is bounded by a constant that does not depend on A . But this contradicts our earlier finding that $\mathcal{P}(x)$ is unbounded at x_a . It must therefore be the case that $\mathcal{P}(x)$ is unbounded for all $x > x_a$.

Since we already know that $\mathcal{P}(x)$ is unbounded for $0 \leq x \leq x_a$, we conclude that $\mathcal{P}(x)$ is unbounded for all x , as claimed.

Step 2. Let $\mu = \frac{1}{2} \liminf_{u \rightarrow \infty} f(u)$; because of Condition [F4], μ is positive. Let γ be fixed, such that $f(u) \geq \mu$ for all $u \geq \gamma$; without loss of generality we may assume that $\gamma \geq \beta$.

Let ϵ be an arbitrarily small positive constant, and let $x_0 = 1/\epsilon$.

Since the set $\mathcal{P}(x) = \{u_A(x) : A > 0\}$ is unbounded for all x , there is an $A > 0$ for which $u_A(x_0) = \mu/\epsilon^2$. Let A be so determined, and let x_1 be the point where u_A reaches the value γ . We claim that we can make $|u'_A(x_1)|$ arbitrarily large by choosing ϵ sufficiently small.

Because u_A is monotone, at least until it reaches the value β , it is certainly monotone on $[x_0, x_1]$; we denote the inverse function by $u \mapsto x(u)$ and define R by the expression

$$R(u) = [u'_A(x(u))]^2, \quad u_A(x_0) \geq u \geq \gamma. \quad (12)$$

Notice that R is strictly positive on its interval of definition, because u_A is decreasing on $[x_0, x_1]$. This function satisfies the initial value problem

$$\frac{dR}{du} = 2 \left(\frac{1}{x} \sqrt{R(u)} - f(u) \right), \quad u < u_A(x_0); \quad R(u_A(x_0)) = (u'_A(x_0))^2. \quad (13)$$

We compare R with the solution r of the initial value problem,

$$\frac{dr}{du} = 2 \left(\epsilon \sqrt{r(u)} - \mu \right), \quad u < u_A(x_0); \quad r(u_A(x_0)) = 0, \quad (14)$$

From the theory of differential inequalities we conclude that $R(u) \geq r(u)$ for all $u_A(x_0) \geq u \geq \gamma$. In particular, $R(\gamma) \geq r(\gamma)$.

The solution of (14) is most easily found by going back to the associated second-order problem,

$$u'' + \epsilon u' + \mu = 0, \quad x > 0, \quad u(0) = \mu/\epsilon^2, \quad u'(0) = 0. \quad (15)$$

The solution u is monotone; and, if $x \mapsto x(u)$ denotes the inverse function, then r , defined by $r(u) = (u'(x(u)))^2$, satisfies (14). (Recall that $u_A(x_0) = \mu/\epsilon^2$.) Thus we find that

$$u(x) = (\mu/\epsilon^2)(1 - \epsilon x - (1 - e^{-\epsilon x})), \quad (16)$$

$$u'(x) = -(\mu/\epsilon)(1 - e^{-\epsilon x}), \quad (17)$$

and therefore,

$$r(u) = (\mu/\epsilon)^2(1 - e^{-\epsilon x(u)})^2. \quad (18)$$

In particular,

$$r(\gamma) = (\mu/\epsilon)^2(1 - e^{-\epsilon x(\gamma)})^2, \quad (19)$$

where $x(\gamma)$ is determined by the equation

$$(\mu/\epsilon^2)(1 - \epsilon x(\gamma) - (1 - e^{-\epsilon x(\gamma)})) = \gamma. \quad (20)$$

Since γ and μ are both fixed, it follows that $1 - \epsilon x(\gamma) - (1 - e^{-\epsilon x(\gamma)}) = O(\epsilon^2)$ and, hence, $x(\gamma) = O(\epsilon^{-1})$ as $\epsilon \rightarrow 0$. Therefore, $r(\gamma) = O(\epsilon^{-2})$ as $\epsilon \rightarrow 0$.

Since $R(\gamma) \geq r(\gamma)$, we conclude that $R(\gamma) = O(\epsilon^{-2})$ as $\epsilon \rightarrow 0$. This proves the claim that we can make $|u'_A(x_1)|$ arbitrarily large by taking ϵ sufficiently small.

Step 3. We now show that the conclusion of Step 2 leads to a contradiction by applying Hadamard's inequality beyond x_1 .

Either $u'_A(x) < 0$ for all $x \geq x_1$ or there is a finite point x_2 , where u_A ceases to be monotone. In the former case, we put $x_2 = \infty$. The function y , defined on (x_1, x_2) by the expression $y(x) = u_A(x) - u_A(x_1)$, satisfies the norm estimates $\|y\| \leq \gamma$, $\|y'\| = \|u'_A\|$, and $\|y''\| \leq \epsilon\|u'_A\| + L$, where $L = \max\{-f(u) : u \geq 0\}$, so

$$\|u'_A\|^2 \leq 2\gamma(\epsilon\|u'_A\| + L). \quad (21)$$

It follows that $\|u'_A\|$ is bounded if ϵ is sufficiently small; in particular, $|u'_A(x_1)|$ is bounded.

The conclusions of Step 2 and Step 3 are clearly incompatible, so the supposition underlying the arguments of the proof, namely that $\mathcal{O}$ is all of $(0, \infty)$, is untenable. We conclude that $\mathcal{O}$ is indeed a proper subset of $(0, \infty)$, as asserted. ■

Now we consider the complement of the set $\mathcal{O}$. Let $\mathcal{C}$ be the set of initial heights A for which u_A crosses the x -axis at some finite point P_A with a (strictly) negative slope, $u'_A(P_A) < 0$.

Lemma 8 *The sets $\mathcal{C}$ and $\overline{\mathcal{O}}$, the closure of $\mathcal{O}$, are disjoint.*

Proof. The sets $\mathcal{C}$ and $\mathcal{O}$ are disjoint by definition, so it suffices to show that a point of $\mathcal{C}$ cannot be a boundary point of $\mathcal{O}$.

Let A be a point in $\mathcal{C}$. Given any small positive number ϵ , we can find a point x_0 to the left of the point P_A , but close enough that $u_A(x_0) = \epsilon$ and $|u'_A(x_0)| > \frac{1}{2}|u'_A(P_A)|$. Suppose that A is a boundary point of $\mathcal{O}$. Since the solution depends continuously on the initial height, there is a point $B \in \mathcal{O}$ sufficiently close to A that $u_B(x_0) < 2\epsilon$ and $|u'_B(x_0)| > \frac{1}{2}|u'_A(P_A)|$. Now, u_B does not reach the x -axis, so either u_B is monotone on $[x_0, \infty)$, or there is a first point x_1 beyond x_0 where u_B ceases to be monotone. In the former case, we set $x_1 = \infty$. We apply Hadamard's inequality to the function $y(x) = u_B(x_0) - u_B(x)$ on (x_0, x_1) . Since $\|y\| < 2\epsilon$, $\|y'\| = \|u'_B\|$, and $\|y''\| \leq (1/x_0)\|u'_B\| + N$, where $N = \max\{|f(u)| : 0 < u < 2\epsilon\}$, we obtain

$$\|u'_B\|^2 \leq 4\epsilon((1/x_0)\|u'_B\| + N). \quad (22)$$

This quadratic inequality implies that $\|u'_B\|$ is bounded. But then we can make the expression in the right member of (22) and, hence, $\|u'_B\|$ arbitrarily small by decreasing ϵ . But now we have a contradiction, since $|u'_B(x_0)|$ is at least equal to $\frac{1}{2}|u'_A(P_A)|$. ■

We are now ready to prove the existence result.

Theorem 1 *If f satisfies the conditions [F1] – [F5], then there exists a solution of the free boundary problem (1), (2).*

Proof. The set $\mathcal{O}$ is open and does not coincide with the entire ray $(0, \infty)$. Hence, it must have a finite boundary point A , which is neither in $\mathcal{O}$ nor in $\mathcal{C}$ (Lemma 8). The solution u_A of (4), (5) that starts with the initial value $u_A(0) = A$ reaches the x -axis at P_A with a horizontal slope, $u'_A(P_A) = 0$. Thus, u_A defines a solution of the free boundary problem, with $P = P_A$. ■

3 Uniqueness

Uniqueness results for boundary value problems for the Emden-Fowler equation are hard to establish. The most comprehensive result for equations with a sublinear (for large u) term $f(u)$ is due to Peletier and Serrin [2]. These authors were concerned with ground states, i.e., nonnegative, nontrivial solutions of the equation

$$u'' + \frac{n-1}{x}u' + f(u) = 0, \quad x > 0, \quad (23)$$

for which $\lim_{x \rightarrow \infty} u(x) = 0$. They established the uniqueness of a ground state when $n \leq 2$ under Conditions [F1], [F2], and [F5] and when $n > 2$ under the additional condition that f is negative near the origin. These results of Peletier and Serrin were subsequently improved by the authors [3], who showed that Conditions [F1], [F2], and [F5] suffice for *all* values of n ($n \geq \frac{3}{2}$). It turns out that the same proof works for solution of free boundary problems of the type studied in the present article. We state the result and omit the proof.

Theorem 2 *If f satisfies the conditions [F1] – [F5], then the free boundary problem (1), (2) has one and only one solution.*

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