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**PROCEEDINGS OF THE 1979 WORKSHOP
ON BEAM CURRENT LIMITATIONS IN STORAGE RINGS**

BROOKHAVEN NATIONAL LABORATORY

JULY 16 - 27, 1979

**Edited by
C. PELLEGRINI**

ACCELERATOR DEPARTMENT

**BROOKHAVEN NATIONAL LABORATORY
ASSOCIATED UNIVERSITIES, INC.
UPTON, NEW YORK 11973**

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PREFACE

The Workshop on "Beam Current Limitations in Storage Rings" was held at Brookhaven National Laboratory from July 16 to 27, 1979. The purpose of this Workshop was to discuss the physical mechanisms limiting the beam current or current density in accelerators or storage rings.

Many of these machines are now being built or planned for a variety of applications, such as colliding beam experiments, synchrotron light production, heavy ion beams. This diversity was reflected in the Workshop and in the papers which have been contributed to these Proceedings.

I want to thank M.Q. Barton and J.R. Sanford for their support, which made it possible to hold the Workshop at Brookhaven. I also want to thank Paula Hughes and Edward Dexter for their care and helpfulness in organizing the Workshop. Particular thanks are due to Paula Hughes for her help in the publication of these Proceedings. Special thanks are due Diana Votruba who typed all the manuscripts.

C. Pellegrini

COLLECTIVE EFFECTS ON DCI^{*}

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ORSAY

I. INTRODUCTION

Collective effects already seen on other storage rings such as head-tail, phase instabilities, bunch lengthening have been observed again on DCI. A new collective effect on the betatron Q value seems to have been seen until now only on DCI and PETRA. A few other instabilities seem to have been observed only on DCI, sometimes for obvious reasons (the 4-beam instability for instance).

Most single beam instabilities on DCI were cured although not completely understood. In one case systematic investigations hit on a difficulty: the instability was not always present in apparently similar experimental conditions. However adequate routines were always found to overcome the presence of non-cured instabilities and did not affect the operation of the machine for colliding beams. For this reason little time was spent on systematic studies of these instabilities and search for theoretical models, most of the effort until a few months being devoted to 4-beam studies.

This paper presents some of the results obtained by the ORSAY Storage Ring Group on single beam and beam-beam behavior over the past few years.

II. HEAD-TAIL EFFECT

2.1. Instability Threshold and Cure With Positive Chromaticities

At the energy of 1 GeV, $V_{rf} = 250$ kV and for the operating point $Q_x = 3.8$ $Q_z = 1.8$, the threshold is about 2 mA and goes down with the rf voltage (shorter bunch length). Two sextupole families were used to achieve positive chromaticities of about 0.5 to 1. Stable single bunch up to 450 mA were stored in the machine in these conditions. At this intensity a crude estimate of the bunch lengthening was made: $\sigma_L(450 \text{ mA}) \sim 4 \sigma_L(2 \text{ mA})$. Therefore, we are able on DCI to reach with positive chromaticities a longitudinal density (I/σ_L) 50 times higher than the value obtained with negative chromaticities.

Lack of space on the machine leads us however to put the sextupole coils inside the quadrupoles with the result of having large multipole components which reduce the injection efficiency at $Q_x = 3.8$ $Q_z = 1.8$. At other operating points sextupoles are even more prohibitive for injection.

2.2. Shift Of The Vertical Tune Q_z With The Bunch Intensity

At low current and for small chromaticity only one betatron vertical frequency is observed. When the bunch current is increased,

^{*}Work supported by the "Institut National de Physique Nucléaire et de Physique des Particules."

one observes several well separated frequencies. The shift from the initial value at low current varies with the current and the rf voltage.

Figure 1 shows for the Lower Ring and a flat beam the dependence of Q_z versus $V_{rf}^{\frac{1}{2}}$ at 1 GeV and for $I^+ = 72$ mA. Three frequencies are observed. They were called S (strong), M (moderate), W (weak) according to the response to the same level of excitation. Two of these frequencies S and M seem to present a linear decrease with $V_{rf}^{\frac{1}{2}}$, the weak one W staying constant. Extrapolation at $V_{rf} = 0$ of these frequencies gives the same value to within $\delta Q = \pm 4 \times 10^{-4}$.

Figure 2 shows for the Lower Ring and a flat beam the intensity dependence of Q_z . Only two frequencies were unfortunately measured, the weak one being only deduced at 72 mA from the previous graph.

Fig. 1. Dependence of Q_z versus $V_{rf}^{\frac{1}{2}}$ (Lower Ring, Flat Beams).

Fig. 2. Dependence of Q_z versus Intensity (Lower Ring, Flat Beams).

All frequencies were measured to ± 1 or 2×10^{-4} .

Typically at 1 GeV, $I^+ = 75$ mA and $V_{rf} = 250$ kV the largest frequency shift is 3.5×10^{-3} .

Figure 3 shows for the Upper Ring, a flat beam and for $I^+ = 48$ mA at 1 GeV the variation of the frequencies versus $V_{rf}^{\frac{1}{2}}$. Extrapolation at zero voltage of these frequencies gives the same value to within $\pm 4.5 \times 10^{-4}$. The behavior is somewhat different from the behavior of the Lower Ring from the point of view of the excitation S and M. The frequency shift is also stronger.

Fig. 3. Dependence of Q_z versus $V_{rf}^{\frac{1}{2}}$ (Upper Ring, Flat Beams).

Figure 4 shows for the Lower Ring the dependence of the observed frequencies of excitation of the beam as one crosses the coupling resonance $Q_x - Q_z = 2$. The curve is plotted versus the strength of the Q_4 quadrupole family which mostly changes the Q_z tune and little the Q_x tune. One observes a single frequency for Q_x (which does vary with intensity). Q_z however shows a split in two frequencies, both of them shifted from the low intensity value. There is a difficulty to understand the diagram inherent to the fact that by continuity an horizontal excitation changes in a vertical excitation as one crosses the coupling resonance and vice-versa. Therefore at low current for instance the $Q_x(\Delta)$ becomes $Q_z(\Delta)$.

The intensity and rf voltage dependence of the effect, as well as its order of magnitude, strongly suggest that these phenomena could be due to damped modes of the Head-Tail. The difference between the two rings might be accounted to differences between vacuum chamber components. The effect of the frequency shift must be taken into account at injection closed to the 1/3 resonance, for large intensities.

Fig. 4. Excitation frequencies on the coupling resonance (strong coupling width: $5 \cdot 10^{-3}$) (lower ring).

III. PHASE OSCILLATIONS

Large phase oscillations can be observed, either with a single bunch circulating or one bunch per beam. In the latter both 0 and π modes are present. They are related to the parasitic modes of the accelerating cavity. Such a source leads to a poor reproducibility of the observations.

The 0 mode is easily cured by the existing phase feedback.

However a second method is used to combat both modes. It consists in shifting slightly the parasitic modes by adjusting the cavity temperature and the tuner position. This is done with the help of two parallel feedback loops.

As a consequence no limit in current arises from the phase oscillations on DCI.

Typically the residual phase oscillation is 50 ps. In the presence of the 0 mode only, this can be reduced to 5 ps.

IV. INSTABILITY CLOSE TO THE COUPLING RESONANCE $Q_x - Q_z = 2$

Another type of coherent transverse instability is observed for Q values close to the coupling resonance. This instability is strongly dependent on the current. Its location depends on the rf voltage and also on the horizontal vertical coupling (determined by the sextupoles through closed orbit vertical excursions produced by the vertical magnets).

The following graphs show results obtained with three different coupling:

- Fig. 5 - weak coupling: the instability is seen on both sides on the resonance for $\Delta Q = Q_x - Q_z = 3.5 \times 10^{-3}$.

- Fig. 6 - intermediate coupling: the instability is seen on the resonance for $\Delta Q = Q_x - Q_z = 3.5 \times 10^{-3}$.

Fig. 5-6. Instability close to the coupling resonance (Lower Ring).

- Fig. 4 - strong coupling: $\Delta Q > 3.5 \times 10^{-3}$ instability not observed. Notice that for a coupling of 3×10^{-3} , synchrotron resonances of the coupling resonance are observed for weak current (1 mA), $\Delta Q = Q_S$ or $2 Q_S$. They are however non destructive.

The effect of the rf voltage is more mysterious, the instability being not always reproducible. When these instabilities are seen the corresponding ΔQ seems to vary as $V_{rf}^{1/2}$. Are they due to a non-linear resonance of the type $3 Q_x - 3 Q_z = 2 Q_S$ or to parasitic frequencies of the transmitter?

In a special test a bunch of 100 mA was accompanied by another bunch of 3 mA distant by half a turn. When crossing the unstable region, a large part of the intense bunch was knocked off whereas the weak bunch was not affected. This strongly suggests a bunch effect.

V. TRANSVERSE INSTABILITIES RELATED TO PARASITIC FREQUENCIES OF THE rf

It has been shown that parasitic longitudinal fields of the rf system can excite large transverse oscillations. These instabilities occur for discrete values of the horizontal or vertical tune.

Systematic studies of the coherent frequency of the transverse instability as a function of the rf voltage have been made (Fig. 7). It varies linearly with $\sqrt{V_{rf}}$ (slope roughly $\sim 2 Q_S$).

Fig. 7. Instability frequencies versus $V_{rf}^{1/2}$.

The extrapolation to zero gives the parasitic frequency that could be observed on the plate of the tetrode of the transmitter (40 db below the fundamental). By changing transmitter conditions one can make the parasitic frequencies appear or disappear and observes the corresponding coherent frequencies. They verify roughly $F - f_{rev} (n - Q_y - 2 Q_S)$.

Parasitic frequency (kHz_z)	834	922	668
Q_y	0.737	0.709	0.789
$Q_y - 2 Q_s$	0.728	0.700	0.780
$Q_c \left(\begin{smallmatrix} \text{spectrum} \\ \text{analyser} \end{smallmatrix} \right)$	0.732	0.703	0.790

The parasitic frequency is given in term of the distance to the nearest harmonic of the revolution frequency.

VI. CROSS SECTION PULSING IN BEAM-BEAM INTERACTION

When two beams are put in interaction close to the beam-beam limit, they clearly show cross-section pulsing on the TV screen. When observed on a beam profile monitor, this pulsing appears as a series of fast bursts followed by a damping phase, whose time constant is consistent with radiation damping.

The repetition time of the bursts is about 100 ms and it increases when the beams weaken. After an hour or two, the pulsing stops, at lower intensities.

The explosion phase is accompanied by a coherent signal lasting a few milliseconds, with a frequency decreasing in time, as shown on the opposite scheme by the solid lines. If the signal is scanned by a spectrum analyser along the dotted line, the picture is as shown. The distance between the peaks corresponds to the time interval between the bursts, and it depends on the scanning speed.

The values indicated on the graphs correspond to typical beams of $75 \text{ mA}^+ \times 72.7 \text{ mA}^-$, giving a luminosity of 4.7×10^{29} at 0.9 GeV. The ξ is 0.035 per crossing, giving a coherent tune shift of 0.078 per turn. The tune of the bursts varies from 0.120 to 0.109 above the machine tune.

Two hours later, the currents were $64.1 \times 60.5 \text{ mA}^2$, and the luminosity 4.0×10^{29} . The ξ was the same, and the bursts were 0.113 to 0.105 above the machine tune.

On the spectrum analyser, one sees side-bands located 70 kHz on each side of the main figure, which apparently correspond to a modulation of the length of the beams. These side-bands appear in several other occasions, but they have not been studied yet.

VII. BETATRON FEEDBACK

A betatron feedback system has been used on DCI but is still under development. Single beams up to 100 mA have been stored with negative chromaticities (100 mA is not a limit).

The betatron feedback has proved to be successful in 3 cases: against head-tail, against the so-called instability close to the coupling resonance, in the beam-beam interaction. In the latter at 1 GeV, two beams of 65 mA were put in interaction with sextupoles on. The feedback improved the specific luminosity by 10%. Furthermore the stability region was enlarged by more than a factor of 2. It will be investigated whether the luminosity can be increased with feedback.

VIII. BEAM-BEAM BEHAVIOR

7.1. Two-Beam Interaction

The two-beam interaction is characterized by an increase of the transverse cross-section with the beam intensity. In fact, as shown in Fig. 8 for one particular tune $Q_x \approx 3.725$ $Q_z \approx 1.725$, the value of ξ saturates with current. The beam cross-section enlargement leads to a linear dependence of the luminosity versus current. The limit is compatible with the vertical vacuum chamber acceptance.

For the same machine tune and in the energy range 0.8 to 1.2 GeV, the maximum current and luminosity measured over a period of a few months scale respectively like E and E^2 . The values of ξ and of the beam cross-section were found to be constant in this range. However the machine performances are still improving with tune, so it is difficult to conclude about scaling laws.

Investigations in the Q_x, Q_z diagram $3.7 < Q_x < 3.9$, $1.7 < Q_z < 1.9$ along the coupling resonance have shown that the stability domain (non-perturbed tune) is reduced to discrete areas which decrease and shift with current. Close to the limit these areas are located on non-linear resonances $5/7$, $3/4$, $4/5$. (See Fig. 9).

Inside these areas the ξ_{max} varies roughly linearly with the unperturbed tune as shown in Fig. 10 for instance.

It is important to notice that, for $Q_{x,z} = 0.7$, $\xi = 0.03$, $\Delta Q = 0.034$ and that, for $Q_{x,z} = 0.8$, $\xi = 0.041$, $\Delta Q = 0.055$.

7.2. Four-Beam Interaction

The same investigation in the Q_x, Q_z diagram has shown that the non-linear resonances still strongly affect the 4-beam interaction. For weaker currents per bunch as compared to the e^+e^- case the stable areas are much smaller and they are located roughly at the same distance of two non-linear resonances.

Fig. 8. ξ versus intensity of each beam.

Fig. 9. Stability diagram with 2 beams and 4 beams ($E = 800$ MeV).

The maximum ξ value obtained is 0.018, as in the e^+e^- case for the same tune. However, up to the limit, no-beam cross-section increase is observed and the current limit is much weaker. As a consequence the maximum luminosity with four beams is less than with two beams.

Fig. 10. ξ versus unperturbed tune ($E = 1$ GeV).

Theoretical studies of collective instabilities of compensated colliding beams made by Y. Derbenev predicted that the behavior should be strongly dependent on the operating point in the Q diagram. Following this model a more accurate calculation was worked out for DCI by Nguyen Ngoc Chau and D. Potaux assuming single non-linear resonances up to the 10th order and neglecting machine resonances. Their results can be summarized as follows:

Case	Q between two crossing points	$\xi_{\max}$
1	0.87	0.075
2	0.73	0.15
3	0.39	0.22

The experiments made with round and flat beam in case 1 and 2 corresponding respectively to the present operating conditions with two and one crossing point have shown no real differences.

Case 3 has been compared with case 1 only with flat beam and in these conditions the same current limitation has been observed.

However, with four beams colliding, coherent signal with frequencies above and below the operating point have been observed which point out in favor of collective coherent instabilities.

Betatron feedback will be tested soon with four beams; tune shift of the four beams and frequency spread with octupole may perhaps give some damping but this remains to be checked.

7.3. Three-Beam Interaction

A weak beam (2 to 5 mA) used as a probe meets two strong companion beams. This experiment was made in order to check the compensation space charge of the incoherent effect.

With the experimental conditions $E = 800$ MeV, $\bar{Q} = 0.725$, one observes two cases for companion beam intensities > 20 mA.

Upper ring	$\leftarrow \overset{I_1}{\rule{1cm}{0.4pt}}$	$I_2 > I_1$ stable
Lower ring	$\xrightarrow{i} \leftarrow \underset{I_2}{\rule{1cm}{0.4pt}}$	$I_2 < I_1$ unstable
Upper ring	$\xrightarrow{i} \leftarrow \overset{I_1}{\rule{1cm}{0.4pt}}$	$I_1 > I_2$ stable
Lower ring	$\leftarrow \underset{I_2}{\rule{1cm}{0.4pt}}$	$I_1 < I_2$ unstable

The stable case is obtained with a small unbalance of the two companion beam intensities of sign opposite to the polarity of the weak beam (e^+e^- overall interaction).

In the stable case the 3-beam cross-section stays approximately constant and equal to the natural cross-section of a single beam up to 55 mA, whereas the e^+e^- cross-section rises by a factor 2.5 at 25 mA.

For the e^+e^- case the ξ_M is 0.018. For the three-beam case, ξ can be similarly defined from the intensity and the cross-section of the strong companion beams. Its maximum values is then 0.1 and we can therefore say that a compensation factor of 5.5 has been reached.

The three-beam unstable case is characterized by important halos. No model is presently available to explain the two different behaviors. They could be two aspects of the same coherent phenomena with different conditions of Landau damping.

BEAM STABILITY IN THE ISR

A. Hofmann, CERN

I. ISR PARAMETERS

The ISR consists of two rings with a radius $R \approx 150$ m intersecting in 8 points. The maximum energy is 31.4 GeV obtained by phase displacement acceleration from the maximum injection energy of 26 GeV. The beam is accumulated by stacking in longitudinal phase space. A pulse with 70 - 150 mA in 20 bunches is injected from the CPS, trapped with the ISR rf-system and accelerated across the aperture towards the already circulating beam where it is debunched. The maximum current obtained is 50 A and the maximum luminosity about $0.6 \cdot 10^{32} \text{ cm}^{-2} \text{ s}^{-1}$ in the low beta intersection.

There are 3 effects which limit the current in the ISR:

The gas desorption by the beam produced ions leads to a pressure rise at a certain beam current. To increase this current limit the vacuum system has been improved continuously which resulted in a maximum beam current of 50 A.

The microwave instability leads to a dilution of the longitudinal phase space density during acceleration of the bunches across the chamber and during debunching. This limits the longitudinal density of the final stack and therefore the total current which can be accumulated in the given aperture.

The transverse instability of the coasting beam represents another potential limitation of the beam current. This effect is controlled by Landau damping provided by the betatron frequency spread and by feedback systems.

The ion induced gas desorption represents the lowest current limit at the top energies. However the other two limitations are not far away and they depend on the proper adjustment of many machine parameters.

II. IMPEDANCE ESTIMATE

During the construction of the ISR the impedance of each element to be installed in the vacuum chamber has been measured and any sharp resonances have been damped with resistors. As a result of this precaution all impedances with long memory (resonators) are small and coupled bunch instabilities have a small growth rate in the ISR. However single bunch effects can still be important; they depend on the broad band impedance which is not reduced by damping resistors. This impedance is caused by bellows, aperture changes etc. and can be approximated by a smooth impedance $Z(f)$ which has a flat maximum around the cut-off frequency of the vacuum chamber (1 - 2 GHz). At very low frequencies it is dominated by the skin effect of the smooth chamber, thereafter it is mainly inductive with a nearly constant $|Z/n| \approx 20 \text{ Ohms}$ ($n = f/f_{\text{rev}}$) over the frequency range of the bunch spectrum. For higher frequencies the impedance becomes more resistive and $|Z/n|$ decreases later. The transverse impedance Z_T is related to the longitudinal one by

$$Z_T \approx \frac{2}{b} \frac{R}{2} Z/n$$

where $b \approx 32$ mm is the effective radius of the vacuum chamber.

III. LONGITUDINAL SINGLE BUNCH EFFECTS

III.1 Microwave Instability (Turbulent Bunch Lengthening).

This instability leads to a dilution of the longitudinal phase space density while the beam is bunched or being debunched. We describe this with the coasting beam model which applies the longitudinal coasting beam stability criterium to the instantaneous values of the momentum spread Δp and the current I in the bunch

$$|Z/n| \leq F \frac{E|\eta|}{e} \frac{(\Delta p/p)^2}{I}, \quad (1)$$

with

$$|\eta| = \frac{|\Delta f_{\text{rev}}/f_{\text{rev}}|}{|\Delta p/p|},$$

E = energy of the beam,

F = form factor.

This criterion (1) indicates that for a fixed phase space area a short bunch is more stable than a long one. A large rf-voltage would therefore be desirable. However this is in conflict with the requirement that the bucket has to fit tightly around the bunch while approaching the already stacked beam to avoid dilution of the latter. The situation can be improved by injecting bunches with the same given phase space density but with a larger area and therefore a larger current. These bunches are more stable according to the criterion (1). Furthermore the critical phase of debunching is done inside the tail of the already accumulated stack. This helps since the wall impedance sees mainly the difference between the bunched current and the surrounding continuous current.

III.2 Inductive Wall Effect (Potential Well Bunch Lengthening).

The longitudinal impedance is predominantly inductive over the frequency range contained in the bunch spectrum and the voltage induced in the wall is proportional to the derivative of the bunch current

$$V_{\text{ind}} \propto \frac{dI}{dt} \propto \frac{dI}{ds} \approx \text{const. } s,$$

where s is the longitudinal coordinate. The bunches in the ISR have in good approximation a parabolic current distribution and the induced voltage is proportional to s . This voltage has to be added to the external rf-voltage leading to a reduction of the slope of the total voltage seen by the particles inside the bunch. This leads to a reduction of the incoherent phase oscillation frequency but leaves the coherent dipole oscillation frequency unchanged since the induced potential

moves with the coherent motion. This coherent frequency can now easily be outside the incoherent frequency distribution which means a loss of Landau damping. Longitudinal coupled bunch mode instabilities can now occur. They have a relatively small growth rate but can become disturbing with the large injected currents used to fight the microwave instability. To cure the coupled bunch mode instability a Landau cavity has been installed. It operates at the third harmonic of the rf-frequency in the bunch lengthening mode (the slope of the total rf-voltage is zero in the bunch center).

From the phase oscillation frequency shift (without the Landau cavity) due to the inductive wall we can deduce the inductive impedance at low frequencies

$$Z_i/n \approx 20 \text{ Ohms.}$$

The measured microwave instability gives through (1) an estimate of the absolute impedance at medium frequencies

$$|Z/n| \approx 14 \text{ Ohms.}$$

IV. TRANSVERSE EFFECTS

IV.1 Transverse Instability Of The Coasting Beam.

This instability is partly damped by the Landau damping provided by the betatron frequency shift and partly by feed-back systems.

The betatron oscillations of the particles in the beam show up as side bands of the revolution frequency harmonics in the spectrum of the beam as seen by a stationary observer (transverse pick-up station or wall impedance)

$$f_f = f_{\text{rev}}(n_f + \nu) \quad (\text{fast wave})$$

$$f_s = f_{\text{rev}}(n_s - \nu) \quad (\text{slow wave}).$$

Both the revolution frequency and the betatron wave number ν depend on the momentum p of the particle. The first one through the parameter b and the second one through the chromaticity of the machine lattice. The spread of the betatron frequencies can be increased by increasing the chromaticity in order to get more Landau damping. This procedure is limited by the space available in the ν -diagram between low order, non-linear resonances. In order to keep the background for physics experiments low any such resonance of order 6 and lower has to be avoided in the ISR.

A transverse feed-back system operating between ~ 0 and 1 MHz is used in each plane to further increase the transverse stability of the beam. An additional vertical feed-back system with 50 MHz bandwidth has been built but is not yet used routinely.

IV.2. Observation Of The Transverse Stability.

The large beam currents, achieved presently, are rather close to

the stability limit. The working line $\nu = \nu(\Delta p)$, which determines the betatron frequency spread, has to be controlled very carefully. In particular the effects of space charge have to be corrected.

To monitor the stability the response in phase and amplitude of the beam to a transverse excitation (beam transfer function) is measured. This is done by exciting the beam with white noise and comparing its response with this excitation using a fast Fourier transform processor. This response becomes large if the stability limit is approached. The frequencies for which this transfer function is large indicate directly which part of the stack is in danger to become unstable. Based on this information the working line can be changed locally to increase the frequency spread in the critical region. The polar representation of the inverse transfer function can also be displayed which is nothing else than the stability diagram of the beam including the effects of wall impedance and feed-back systems.

A beam, being by itself close to the stability limit, can become unstable due to the fields created by the other beam. This phenomenon, which is different from the classical beam-beam limit, is specially pronounced during luminosity optimization when the two beams are vertically displaced with respect to each other. The transfer function shows under this condition a strong response at frequencies which correspond to non-linear resonances.

A SURVEY OF BEAM-BEAM LIMITATIONS

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I. INTRODUCTION

The effect of beam-beam interaction is not normally felt in the ISR and will probably not be a serious problem for ISABELLE, at least at high energy. However, it is known to limit the luminosity of electron-positron storage rings and will, no doubt, limit the proton-antiproton collision scheme for the SPS. While theorists are struggling to explain this phenomenon it is more instructive to list their failures than their rather limited successes, in the hope that experiments may emerge which will direct their endeavors.

The search for a description of a nonlinear system as it approaches the limit in which ordered motion breaks down, is the nub of the problem. It has engaged many fine mathematical intellects for decades and will no doubt continue to do so long after ISABELLE, the $p\bar{p}$ and LEP are past achievements.

Empirical scaling laws are emerging which relate electron machines to each other but their extrapolation to proton machines remain a very speculative exercise. Experimental data on proton limits is confined to one machine, the ISR, which does not normally suffer the beam-beam effect and where it must be artificially induced or simulated. This machine is also very different in important ways from the $p\bar{p}$ collider.

The gloomy picture which has emerged recently is that the fixed limits which were conventionally assumed for proton and electron machines can only be said to be valid for the machines which engendered them - the best guess that could be made at the time. They are very difficult to extrapolate to other sets of parameters.

II. THE CONVENTIONAL LIMITS

First it should be said that the beam-beam limit as seen in electron machines appears as a phenomenon which increases the beam size of one or other of the beams as soon as they collide. Luminosity and lifetime suffer and above a critical current the situation becomes rapidly worse. In proton machines the beam-beam forces cause a steady dilation of emittance and beam loss over many hours.

A year ago it was commonly held that the maximum values for the linear Δv (used as an index for the nonlinear forces which cause the phenomenon) were different for electron and proton machines but independent of design energy and other machine parameters. The conventional limit for e^+e^- machine was taken to be between $\Delta v = 0.03$ and 0.06 . Designers, hoping for improvements in understanding as their machines were built, took the higher figure for future projects.

The corresponding limit for proton machines was assumed to be an order of magnitude smaller ($\Delta v = 0.005$). Estimates had been made of how the limit depended upon the growth time to which one was sensitive. This suggested that if at $\Delta v = 0.05$ the growth time is a few milliseconds and sufficient to overcome the radiation damping of an electron machine it would extend to many hours at a lower limit of $\Delta v = 0.005$ and then be tolerable for an undamped proton machine.

Analysis of ISR behavior in situations where a nonlinear lens, intrabeam scattering and partial neutralization were severally thought to simulate features of the beam-beam interaction seemed to confirm the order of magnitude of $\Delta v = 0.005$. Further confirmation came when the beam-beam limit was induced in the ISR by running at low energy and with a high beta configuration on one of the beams to enhance Δv .

Later we shall show how experimental evidence from electron machines leads one to believe the electron limit is not fixed but energy and machine dependent, while the ensuing deeper examination of the beam-beam mechanism gives good reasons for believing the ISR proton limit may be too optimistic for the bunched beams of the $p\bar{p}$ collider.¹

III. CLASSIFICATION OF MACHINES AND EFFECTS

Before embarking on this classification a few very qualitative words about the dynamical model of the effect are in order for those who are not conversant with the theory. It is believed that the diffusion or beam dilation occurs when nonlinear forces are strong enough to produce a chaotic motion which we shall for the moment refer to as stochastic. Since the beam-beam force is linear for small betatron displacements the chaotic motion characteristic of the stochastic limit will occur only for particles whose betatron amplitudes are above some lower limit. We can think of the beam-beam interaction as defining a "stochastic boundary" in the betatron phase plane such that motion "inside" the boundary is stable, and that particles which once cross this boundary enter the land of stochastic motion where they begin a random walk towards death. As the strength of the beam-beam interaction increases we expect the "radius" of the stochastic boundary to decrease and the so-called beam-beam limit will occur when the boundary begins to encroach significantly upon the phase area occupied by the stored beam. Nonlinear forces generate islands of bounded motion in phase space. One may think of the stochastic boundary as the point where these islands merge covering a majority of the available phase space.

The strength of the nonlinear forces necessary to cause the islands to merge depends on the number of degrees of freedom of the system. If there are more degrees of freedom the number of islands and possible resonant conditions is increased. They need not therefore be as strong before they merge. It is instructive to classify the various machines in terms of the number of degrees of freedom.

At first sight ISR and ISABELLE appear to have only one degree of freedom. Their beams are unbunched and are cross at a finite angle where they are wide and flat. The interaction should be almost entirely in the vertical plane. Any small horizontal forces tend to

cancel as the beams cross. However, if one calculates the $\Delta\nu$ limit for one degree of freedom it turns out to be 0.2 and much larger than the empirical 0.005 for these machines. To reconcile theory with experiment one must believe there is after all some coupling into a second degree of freedom and, in addition, that other processes like intrabeam scattering hasten the journey of particles to the stochastic region. This situation is at best unsatisfactory.

Electron-positron machines and the $p\bar{p}$ collider are clearly machines in which the beam-beam forces act in two coupled degrees of freedom (vertical and radial) as their beams clash head on. The $\Delta\nu$ which produces the stochastic limit is lower for the more numerous resonant conditions of a coupled system and has been calculated to be $\Delta\nu = 0.05$. Moreover in contrast to one degree systems, Arnold diffusion is possible. This is a nonlinear diffusive process which promotes the migration of particles even well within the stochastic boundary and effectively lowers the $\Delta\nu$ limit.

The bunched beam machines, e^+e^- and $p\bar{p}$ also have a third degree of freedom; the longitudinal plane, which must aggravate the situation still further. Synchro-betatron resonances are possible adding further to the number of islands so that one would expect the $p\bar{p}$ or hypothetical undamped e^+e^- machine to have a lower $\Delta\nu$ limit than the ISR's $\Delta\nu = 0.005$. Another way of convincing ourselves that this is the case is to remember that switching on the rf speeds beam loss if nonlinear resonances are near enough to be crossed in the course of synchrotron motion.

Electron machines of course have synchrotron radiation damping which takes care of beam-beam diffusion if it is lower than the few milliseconds damping time constant. However they have two other (perhaps related effects) which give us hope that they are worse than the $p\bar{p}$ collector. The noise induced by quantum emission probably makes the stochastic diffusion worse and in an electron machine, particles are continually sampling the edges of phase space. In a proton machine betatron and synchrotron amplitudes remain invariable for a given particle and there are no quantum jumps.

There have been extensive attempts to simulate electron machines with computers (e.g. Close at Berkeley) and these confirm that synchrotron motion is an essential ingredient in simulating their beam-beam effects.

While it seems reasonably safe to extrapolate ISR experience to a similar machine like ISABELLE and while as we shall see electron machines are beginning to fit into a pattern, the $p\bar{p}$ is so dissimilar from existing machines and the theory so incomplete that it is difficult to say more than its $\Delta\nu$ limit will be lower than the ISR and higher than an electron machine with a damping constant of many hours. However since it bears more similarity to the electron machines let us explore what one may learn from them.

IV. EXPERIMENTAL EVIDENCE FROM ELECTRON POSITRON MACHINES

Very recent and yet unpublished measurements from the SPEAR Group, clearly confirm the energy dependence of the small amplitude beam-beam tune shift $\Delta\nu$ (previously observed at ACO, ADONE and SPEAR itself) down to energies as low as 0.6 GeV.

Data from existing electron machines now extend to damping times as long as ~ 1 s, a small but significant step in the direction of undamped proton machines.

Although the dynamics of the beam-beam interaction is far from being understood, existing data on the maximum tolerable Δv exhibit common distinctive features.

Δv is a function of energy below a certain threshold energy E_0 , depending on the particular machine and exhibits the expected energy independent behavior above threshold.

The energy dependence is of the form E^α with α in between 1 and 2.²

It has been suggested, on the basis of simple models postulating the effect of beam-beam interaction to be equivalent to a random excitation in the betatron phase space,³⁻⁵ that the energy dependence comes in through the scaling parameter

$$\sqrt{n} \tau_D \propto \sqrt{\frac{b\rho}{E^3}}$$

b = bunch number
 ρ = bending radius

the number of interactions in a damping time.

The square root dependence on the number of bunches is well established for those machines (ACO and ADONE) capable of operating with different bunch numbers (b). Experiments with many bunches have also been performed at the CEA,⁶ where this type of behavior was not observed. However the Δv was extremely small and the beam configuration so different from that of the other space charge limited rings that we do not think they fall in the same class as the others. The prediction of the "threshold" energy is still not entirely satisfactory but some degree of success has been obtained by a number of authors^{3,4,7} in trying to unify the data from different machines. If the picture should prove to be essentially correct, it is somewhat alarming to find the Δv limit for SPEAR when the damping time is as short as 1 second, is already as low as that assumed for $p\bar{p}$ which must preserve emittance for many hours.

If one can extrapolate the existing data from electron machines to the number of crossings corresponding to a 24 h life-time in the SPS, one would predict the tolerable Δv for $p\bar{p}$ to be much lower than 10^{-3} .

Such an extrapolation over two orders of magnitude in $\sqrt{nT}$ is, of course, very questionable and almost certainly pessimistic since it ignores the points mentioned above in favor of proton machines.

V. CONCLUSIONS ON $p\bar{p}$ COLLIDER

It would be unwise to close ones eyes to the reasons for now suspecting that the Δv limit is lower than that derived from ISR experience with an unbunched beam. However there is encouraging evidence from the ISR that at $\Delta v = 0.0005$ per crossing the dynamics resolve into ordered motion again.⁸ Beam-beam induced resonances up to eighth order can be clearly seen at discrete momenta in the ISR stack with a transverse Schottky scan. If these resonances are the

major destructive mechanism of the beam-beam effect one can hope to steer the much smaller $\Delta p/p$ of the SPS between them. Should this prove to be the case it will still be possible to reach the lower end ($L = 10^{29}$) of the range of luminosities expected from the $p-\bar{p}$ scheme. Provided one is prepared to inflate transverse beam dimensions rather than reduce bunch intensity, the Δv per encounter could then be as low as 0.0004 for 6×10^{11} particles in each beam.

How far one may expect to progress in the direction of higher luminosity must depend on the application of ingenuity to the problems of beam separation and exploration of the working diagram. To improve our understanding of these phenomena before we come to meet them face to face we suggest a number of experiments which are relevant to $p\bar{p}$ and future e^+e^- projects.

VI. PROJECTED EXPERIMENTS

Clearly, of the machines studied, the $p\bar{p}$ is the one most sensitive to the uncertainties in the theory because no comparable bunched proton collider exists. Experiments to find Δv for bunched beams might be carried out in the ISR to narrow the considerable window of uncertainty between the conventional limit to that extrapolated directly from electron machines. Experiments simulating nonlinear behavior with the SPS proton beam and the ISR nonlinear lens could also throw some light on the harmful effect of synchrotron motion.

Electron machines encouraged by the emergence of a semi-empirical fit will no doubt extend the measurements of energy dependence. This is especially urgent at the long damping time end for $p\bar{p}$ and at the high energy end for LEP. PETRA and PEP can provide valuable information on this and on the bunch number dependence.

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COMPENSATION FOR LONGITUDINAL SPACE CHARGE FORCES
AS INDUCTIVE WALL IMPEDANCES

G. H. Rees

I. INTRODUCTION

In particle accelerators and storage rings the longitudinal coupling impedance between the beam and its environment arises from the longitudinal space charge forces and the vacuum chamber component impedance.

The longitudinal coupling impedance is written $Z(n\omega)/n$, where ω is the beam angular revolution frequency, and $n\omega$ is the angular mode frequency. The longitudinal space charge contribution to $Z(n\omega)/n$ is $-jgZ_0/2\beta\gamma^2$, where g is a function of the beam transverse distribution and the wall geometry, Z_0 is the free-space impedance, $Z_0 = 377\Omega$ and $\gamma = (1 - \beta^2)^{-1/2}$ is the ratio of the particle mean energy to its rest energy.

The space charge term is small for large γ -values, but is the dominant term at low β -values. Examples are:

1. ISR (at 26 GeV); the vacuum chamber wall discontinuities contribute an inductance of approximately $10\ \mu\text{H}$ at frequencies up to 1 GHz whereas the equivalent inductance (negative) due to the longitudinal space charge forces is approximately $0.5\ \mu\text{H}$.

2. BNL, 10 GeV U^+ storage ring, proposed for heavy ion fusion, where $\beta = 0.29$; the equivalent negative inductance due to the space charge is estimated at 2.75 mH.

In this note a scheme is considered for compensating the total longitudinal coupling impedance within a restricted frequency range. For the low- β case, when the net longitudinal impedance is a negative inductance, compensation is achieved by means of adding either passive inductive wall elements or actively excited inductive elements. In the latter case, the excitation is by current pulses which exactly simulate amplified beam current pulses, in shape and sign. For the high- γ case, when the net longitudinal impedance is a positive inductance, an active inductance wall element is again introduced but the exciting signal is reversed in sign from the previous case.

Compensation in the low β -case is useful to improve the trapping efficiency of an initially unbunched beam. As the trapping proceeds the longitudinal space charge forces increase and, in the absence of compensation, this usually results in enhanced trapping losses and/or dilution of the longitudinal phase space density. For both low- β and high- γ machines the compensation scheme is worth evaluating to see if the longitudinal stability of the beam is enhanced.

In general, compensation is for large inductance values. This infers compensating elements with low resonant frequencies. The resonances must be damped and the useful frequency range is restricted to frequencies typically below 5 to 20 MHz.

As examples, reference is made to the SNS (Rutherford, U.K.) and to the BNL 10 GeV U^+ accumulated ring. Both are examples of the low- β case.

The variation of the parameter g as a function of the transverse coordinates of beam particles is noted and the possible effect on the compensation.

II. COMPENSATION SCHEME

Schematically, the system is:

o is the machine orbit.

J_b is the beam current.

P is a wideband beam pick-up signal.

T is an adjustable time delay.

A is an adjustable gain stage.

PA is a class A power amplifier.

L is a ferrite loaded cavity, using high Q ferrite.

The signal transit time from P to L equals the particle transit-time from P to L. The lowest resonant frequency of the cavity must lie above the harmonic components of interest in J_b .

III. RELEVANT FORMULAE

$$Z_{sc}/n = -jgZ_0/2\beta\gamma^2 \quad (1)$$

$$-L_{sc} = gR/2\xi_0\gamma^2\beta c^2 \quad (2)$$

$$g_u = 1 + 2 \ln(b/a) \quad (3)$$

$$g_p = 1.5 + 2 \ln(b/a) \quad (4)$$

$$\hat{\lambda} = \hat{\lambda} [1 - (t/t_0)^2] \quad (5)$$

$$\hat{I} = 3 Ne/4t_0 \quad (6)$$

$$\Delta E_{sc} = \pm 3 Ne (L_{sc})/2t_0^2 \quad (7)$$

Z_{sc}/n is the longitudinal space charge contribution to $Z(n\omega)/n$,

$-L_{sc}$ is the equivalent negative inductance to Z_{sc}/n ,

g_u is the g parameter at the center of a cross-section of a beam of uniform density, ρ , in the plane transverse to the beam direction,

g_p is the g parameter at the center of a cross-section of a beam of parabolic density, ρ_p , in the plane transverse to the beam direction,

λ is the line charge density of the beam, assumed parabolic in form,

$\hat{\lambda}$ is the peak value of λ ,

$\hat{I}$ is the peak instantaneous beam current corresponding to $\hat{\lambda}$,

ΔE_{sc} is the maximum energy gain/loss per machine revolution for particles at the extremities of the assumed parabolic bunch,

ξ_0 equals $10^{-9}/36\pi$ Farad m^{-1} ,

R is the mean radius of the storage ring/accelerator,

b is the radius of the machine vacuum chamber, assumed circular,

a is the radius of a beam transverse cross-section, assumed circular,
 t_0 is half the time duration of the beam bunch,
 Ne is the total charge in the beam bunch,
 ρ equals $\lambda/\pi a^2$, and
 ρ_p equals $(\lambda/\pi a^2) (2) [1 - (r/a)^2]$ at a beam radius, r .
 ΔE_{sc} includes only the contribution due to the longitudinal space charge forces.

IV. EXAMPLE 1. SNS

The SNS is a pulsed spallation neutron source based on a high current, rapid-cycling (50 Hz) proton synchrotron. The energy range is 70.44 MeV to 800 MeV; the beam radius is 26 m.

Step discontinuities in the vacuum chamber wall are estimated to contribute a longitudinal inductive impedance of $2.5 \mu H$.

At 70.44 MeV, $\beta = 0.367$, $g = 1.5$, $-L_{sc} = 157.5 \mu H$

At 800 MeV, $\beta = 0.842$, $g = 2.45$, $-L_{sc} = 16.5 \mu H$

Between 70.44 MeV and 800 MeV, the net inductance varies from $-155 \mu H$ to $-14 \mu H$.

It is planned to install two inductive cavities of impedance $7 \mu H$, each. Thus there is compensation at 800 MeV without any additional cavity excitation. Compensation at 70.44 MeV is achieved by exciting the cavities by a current pulse ten times the beam current pulse in addition to the excitation provided by the beam pulse itself.

The synchrotron has rf harmonic number 2.

At 70.44 MeV, for $t_0 = 162.5$ ns, $2N = 2.7 \cdot 10^{13}$, $\Delta E_{sc} = \pm 19.35$ keV

At 800 MeV, for $t_0 = 50$ ns, $\Delta E_{sc} = \pm 21.41$ keV

V. EXAMPLE 2. BNL 10 GeV U^+ ACCUMULATOR RING

The 10 GeV U^+ accumulator ring, as proposed, has:

$R = 200$ m

$\beta = 0.2896$

$N = 6 \cdot 10^{14}$

$(\Delta p/p) \ell = 0.04$ m (= momentum spread $\times$ bunch length)

$f = 69.14$ kHz (= revolution frequency)

Assume $g = 2$ (restricting b/a)

Then $-L_{sc} = 2.745$ mH

The value of L_{sc} indicates that large longitudinal space charge forces are to be expected as the beam is compressed in phase extent in the ring. If the ring is filled initially with U^+ ions the required compression is from ± 7232 ns to ± 450 ns, a factor of 16. At a bunch extent of ± 450 ns the transverse space charge detuning reduces the incoherent betatron tunes of the U^+ ions from 12.25 to approximately 6. Additional compression to ± 50 ns is envisaged in the 1 km transfer line to the pellet chamber, external to the accumulator ring.

The estimates for ΔE_{sc} in the U^+ accumulator ring are:

t_0 (ns)	ΔE_{sc}
5000	± 15.81 keV
1000	± 395.25 keV
450	± 1.952 MeV
50*	± 158.1 MeV (* not envisaged)

A pulsed rf system is proposed for the ring with compression in 20 to 50 turns. The rf system must compensate for the maximum ΔE_{sc} values of ± 1.952 MeV (major requirement) and provide an appropriate waveform for the compression.

The inductive compensation scheme described in this note is worth considering if the conventional rf system is found to give too large a longitudinal emittance in the bunch compression process. The scheme would require the installation in the U^+ ring of a very large number of passive inductive cavities of total inductive value 2.745 mH (eg. 200, 14 μ H units, each containing ~ 7 large ferrite toroids). It is not practical to consider using a smaller number of active units because a parabolic U^+ bunch of ± 450 ns duration has an $\hat{I}$ value of 160 A. Together with the compensating units is required a modest rf system capable of giving the bunch compression in the absence of space charge. Compensation cannot be exact, as indicated in the next section and the effect of inexact compensation would need to be computed.

VI. PARAMETERS g_u AND g_p

The exact expressions for g_u and g_p are:

$$g_u = 1 - (r/a)^2 + 2 \ln(b/a) \quad (8)$$

$$g_p = (3/2) - 2(r/a)^2 + (r/a)^4/2 + 2 \ln(b/a) \quad (9)$$

r is particle radius in a transverse circular section of beam.

	g_u	g_p
$r = 0$	$1 + 2 \ln(b/a)$	$(3/2) + 2 \ln(b/a)$
$r = a/2$	$(3/4) + 2 \ln(b/a)$	$(33/32) + 2 \ln(b/a)$
$r = a$	$2 \ln(b/a)$	$2 \ln(b/a)$

The effect on the longitudinal motion of the variation of g with particle radius requires a detailed computational study. Compensation by the use of inductive units is only exact for $t = 0$. At other values of r the space charge force is reduced and there is over compensation of the longitudinal focusing.

IMPLICATIONS OF TRANSVERSE INSTABILITY CRITERIA IN THE DESIGN OF HIGH CURRENT, MULTIBUNCH ELECTRON STORAGE RINGS

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I. INTRODUCTION

Sacherer¹ has given a very clear analysis and explanation of instability phenomena arising with bunched beams. This paper uses his results for transverse instabilities and attempts to outline their special application to electron storage rings with low emittance, multibunch beams as are used or proposed for synchrotron light sources.

The general expression given by Sacherer for the growth rate of an instability of order m is:

$$\Delta\omega_m = \frac{1}{1+m} \frac{j}{2Q\omega_o} \frac{e\beta}{\gamma m_o} \frac{I_b}{L} \left[\frac{\sum Z_T(\omega) h_m(\omega - \omega_\xi)}{\sum h_m(\omega - \omega_\xi)} \right], \quad (1)$$

for a system with identical bunches where I_b is the bunch current and L is the bunch 'length', here considered as a definite length rather than a Gaussian distribution. $Z_T(\omega)$ is the transverse impedance of the ring as a function of frequency, ω , and $\omega_\xi = (\xi/\eta) Q\omega_o$ is the frequency of oscillations within the bunch arising for non-zero chromaticity.

The behavior of the system is very much dependent on the expression in square brackets which represents the interaction of the power spectrum of the bunch with the transverse impedance function. Reference 1 gives the behavior of $h_m(\omega)$ for the case of a fixed bunch length which is cosinusoidal. However for electron bunches we must use a Gaussian distribution for the bunch, with a standard deviation in length of σ .

The power spectrum of a Gaussian bunch² is

$$h_o(\omega) = \frac{\sigma e}{\sqrt{\pi}c} e^{-(\omega^2 \sigma^2 / c^2)}, \quad \int_{-\infty}^{\infty} h_o(\omega) d\omega = 1, \quad (2)$$

for the zero mode, $m = 0$. The effect of finite chromaticity is to shift the whole spectrum by ω_ξ . For the present, attention is restricted to zero chromaticity, which is the usual condition of operation in electron storage rings.

The power spectrum for the $h = 1$ mode varies as $\omega^2 e^{(-\omega^2 \sigma^2 / c^2)}$. The ω^2 term has the effect of peaking the distribution where $\omega \sim c/\sigma$. It has been shown⁴ that it is sufficiently accurate to assume a Gaussian distribution even when a third harmonic rf system is used to give bunch lengthening.

II. SOME SIMPLE CONSIDERATIONS

The general expression (Eq. 1) for the growth rate can be reduced to the following expression for $m = 0$, $\xi = 0$.

$$\Delta\omega_o = \frac{j\beta c}{4\pi Q} \frac{I}{E} \left[\frac{\sum Z_T h_o(\omega)}{B \sum h_o(\omega)} \right] \quad (3)$$

when I is now the average beam current

E is the electron energy

B is the bunching factor, $ML/2\pi R$,

M being the number of bunches.

Leaving, for the moment, the implications of the expression in square brackets, it becomes obvious that the growth rate is higher, the higher the average beam current, the lower the electron energy and the lower the Q -value of the ring.

The implication of the energy dependence is that one is making conditions more difficult by injecting at a lower energy than the operating energy. The situation is made even worse since the chief instability counteracting mechanism, radiation damping, varies as E^3 . Both the Daresbury SRS and the BNL x-ray ring have this problem to face and the implications of this will be considered later.

III. DISCUSSION OF IMPEDANCE

Investigations at SPEAR² and CERN³ have revealed the type of impedance distribution encountered in a storage ring. At SPEAR the longitudinal impedance, Z_{11} was determined to be approximately linearly rising from zero frequency to a peak at 1.3 GHz and thereafter falling as $\omega^{-0.68}$. Investigation on the P.S. at CERN supported this general picture. The impedance is usually normalised as Z_{11}/n , where n is the frequency divided by the orbital frequency. For SPEAR Z_{11}/n maximised at about 9 ohms, but by the stringent use of the smooth-wall philosophy it is hoped to reduce this on modern machines to about 1 ohm. This broad-band impedance is due to aperture changes, bellows, etc. and is due to the sum effect of resonances in cavity-like insertions. As will be seen later it is not a major contribution to multi-bunch transverse instabilities.

The general shape of the impedance curve has been likened⁴ to that of a very low Q resonator with a resonance about 1.3 GHz, a frequency roughly determined by the lowest made cut-off of the vacuum vessel. Below this frequency the impedance is predominantly inductive. A reactive impedance cannot in itself cause an instability since it gives rise to a real frequency shift. However by moving the coherent betatron oscillation frequency outside the band of incoherent frequency spread it can negate Landau damping within the beam and leave the beam a prey to other causes of instability. It will be shown later that, even with very smooth walls, this effect can be dominant in typical storage rings.

The dominant causes of transverse instability are the resistive wall effect and narrow band impedances, usually arising at higher mode frequencies of the rf cavities, but which could arise similarly from other cavity-like insertions in the ring. The smooth spectrum shown in Fig. 1 will, in practice, be dominated in the range above the normal rf by such sharp resonant spikes. The resistive wall impedance (Z_{11}/n) falls with frequency as $\omega^{-\frac{1}{2}}$ and is dominated by the skin effect.

The transverse impedance is usually expressed in terms of the longitudinal impedance by the expression

$$Z_T = (2R/b^2)(Z_{11}/n)$$

which is strictly only accurate for a round pipe of radius b .

Fig. 1. Bunch power spectrum envelopes for SRS at injection (600 MeV).

- (a) with third harmonic cavity, $m = 0$;
- (b) with third harmonic cavity, $m = 1$;
- (c) main rf only, natural bunch length, $m = 0$;
- (d) assumed broad band impedance;
- (e) range of significant cavity resonances.

For an elliptical pipe an equivalent radius b' may be found⁴:

$$b' = (h/2) [2(\xi_1 - \epsilon_1)]^{-1/3}$$

where ξ_1 and ϵ_1 are functions given by Laslett.

Even for a flat ellipse b' is not very much greater than the half-height $h/2$. Thus by choosing the smallest possible beam aperture to minimize the cost of bending magnets one pays the penalty of enhanced transverse impedance. If clearing electrodes are inserted in a vacuum vessel they will form the effective boundary and increase the transverse impedance. The practical implications of this are discussed later.

For cavity modes it is usual to assess by measurement the transverse impedance. For troublesome modes in the DORIS cavities⁵, the shunt impedances were of the order of $1 \text{ M}\Omega$ for the TM_{110} and TM_{111} modes. In relating shunt impedance to the transverse impedance the radius, b , has been taken as that of the outer wall of the cavity. A better empirical agreement with experimental results is obtained for assuming b is half this radius.

IV. THE BUNCH POWER SPECTRUM

For narrow band impedances, which only interact with a single line of the spectrum at ω_n the expression in square brackets in Eq. 3 may be reduced to:

$$F'_o = \frac{1}{B} \frac{h_o(\omega_n)}{\sum h_o(\omega)} = \text{const.} \frac{h_o(\omega_n)}{h_o(0)} \quad (4)$$

Figure 2 gives the envelope of the curve given by the expression for $h_o(\omega)$ (Eq. 2) as applied to the SRS at injection both for the natural bunch length and for the length obtainable with a third harmonic cavity. Within this envelope there occur lines at frequencies given by:

$$\omega_p = (n + kM + Q)\omega_U \quad (5)$$

where n is the multibunch mode number and k is any integer.

Fig. 2. Typical bunch power spectra for SRS at injection (600 MeV).
 (a) for natural bunch length;
 (b) with third harmonic cavity.

It is clear that, for a given mode number, and with all bunches filled, there will only be a few lines spaced at the rf. Positive frequencies represent stable lines, negative ones unstable. This is only a convenient form of presentation. A spectrum analysis would see, if a given mode were excited, a set of double lines separated by $2 Q\omega_0$. Due to the fact that n ranges in integer steps up to the full number of bunches, M , there can be dangerous frequencies at spacings of ω_0 .

Table I gives the frequencies for which $\omega = c/\sigma$ for various storage rings. At this frequency the $m = 0$ envelope has fallen to 0.37 of maximum but the $m = 1$ envelope is at its maximum value.

TABLE I.

	2σ	$f = c/(2\pi\sigma)$
SRS ⁶	0.023 m 0.076*	4.15 GHz 1.29
BNL X-ray ⁷	0.030 0.156*	3.18 0.61
BNL VUV ⁸	0.078 0.258*	1.22 0.37
BESSY ⁹	0.036** 0.072	2.65 1.32
ESRF ¹⁰	0.020 0.0474*	4.77 2.01

* Represents bunch length with a Landau cavity giving optimum bunch lengthening. The remaining figures are for the natural bunch length which is not likely to be achieved in practice at high bunch currents.

**For BESSY these values have been assumed for the two operating radio frequencies. No Landau cavity has been proposed.

V. EFFECT OF INDUCTIVE WALLS

As mentioned above the low frequency impedance is predominantly inductive, giving a real shift to the coherent betatron frequency. For

this case Eq. 1 may be reduced to:

$$\Delta\omega = \frac{\beta c}{Q} \frac{I}{E} \frac{R^2}{b^2} \frac{(Z_{11}/n)}{ML} . \quad (6)$$

It will be noted that the frequency shift is higher for low energy, for small values of vertical aperture and for small numbers of bunches. It is in fact an effect dominated by the charge per bunch.

For the SRS parameters at injection, (600 MeV, 1A), with 160 buckets filled, even if (Z_{11}/n) could be reduced to as low as 1 ohm, the shift would be 2.5 kHz at the natural bunch length, or 750 Hz with a 3rd harmonic Landau cavity. The boundary of the SRS is far from smooth so that the impedance may be in the region of 4 ohms. Thus a big enough coherent shift will probably occur at injection to cause a loss of Landau damping.

Even the maximum use of octupoles will be insufficient to give adequate frequency spread. The spread achievable with 16-10 T/m³ octupoles over 2 σ is 620 Hz in the horizontal plane, 220 Hz in the vertical (at 10% coupling).

It turns out that this problem of loss of Landau damping is common to all electron rings with small emittance beams, even when injected at operating energies. In view of the small emittance, the use of octupoles to enhance the natural frequency spread is pointless. For the BNL x-ray using the use of 16 octupoles of 10 T/m², optimally positioned around the ring would give a frequency spread in the horizontal plane about 2 or 3 orders of magnitude below requirements.

Table II gives the inductive tune shift for the various machine on

TABLE II.

Machine	Energy (MeV)	Current (A)	Bunch Length (2 σ) (m)	M	R (m)	Q	$\Delta\omega$	ΔQ
SRS ⁶	600	1	0.023 0.076	160	15.28	3.25	15.6 k 4.7 k	0.79x10 ⁻³ 0.24x10 ⁻³
BNL X-ray ⁷	700	1	0.030 0.156	30	27	5.7 (v)	139.2 k 26.7 k	0.012 2.4x10 ⁻³
BNL ⁸ VUV	700	1	0.078 0.258	9	8.15	2.8	29.9 k 5.0 k	0.81x10 ⁻³ 0.13x10 ⁻³
BESSY ⁹	800	0.5	0.036 0.072	104 26	9.95	1.6 (v)	4.9 k 19.8 k	0.16x10 ⁻³ 0.66x10 ⁻³
ESRF ¹⁰	5000	0.565	0.020 0.0474	1008	96.19	13.7 (v)	5.0 k 2.1 k	1.6x10 ⁻³ 0.67x10 ⁻³

the assumption that the smooth wall induction impedance can be reduced to 1 ohm. As stated earlier the effect of this shift is to leave the ring vulnerable to instabilities caused by resistive or narrow band resonant impedances. The only effective natural damping against these is synchrotron radiation damping. The effective use of rf quadrupoles to counteract multi-bunch instabilities is considered later.

VI. RESISTIVE WALL EFFECTS

The resistance of the vacuum vessel wall is dominated by the skin effect. Thus the value of (Z/n) , resistive, varies at $\omega^{-1/2}$ and so becomes dominant at low frequencies. Thus a single line of the bunch spectrum at a low frequency can dominate the situation and cause instability. This is illustrated in Fig. 3. The condition is that $M - (n+Q)$ lies between 0 and -1.

Fig. 3. Possible resistive wall instability for SRS. The unstable line is at $f = 156 + Q = 160$.

In the SRS the transverse resistance is predominantly that derived from the strip lines formed by the clearing electrodes, which are nearly continuous throughout the bending magnets, quadrupoles and multipoles and are nearer to the beam than other parts of the vacuum vessels. With a conductivity 80% of copper, the resistance of the two strips in parallel is 2.5×10^{-3} ohm/m. The total resistance is 0.18 ohm at ω_0 and the transverse resistance is 8.8 K Ω /m.

The frequency shift is

$$\Delta\omega_0 = \frac{j\beta c}{4\pi Q} \frac{I}{E} R_T F'$$

where F' is 0.72 (Eq. 4).

At 600 MeV $\Delta\omega_0 = j86$ so the e-folding time is 12 ms and this is much smaller than the radiation damping time at injection. The corresponding figures for the various machines are given in Table III. The BNL and ESRF machines use extruded aluminum vessels, while BESSY uses stainless steel. It is apparent that all the rings are vulnerable to resistive wall instability at maximum current except the ESRF at 5 GeV. It is proposed to operate the latter energies down to 2.5 GeV where this ring too would be vulnerable.

TABLE III.

Machine	Energy	Current	Z/n	R	b	Z _T	$\Delta\omega_0$	τ	Trad.
SRS	600 MeV	1 A	0.18 ohm	15.18 m	0.025 m	8.8 K Ω /m	76	13 ms	186 ms(v) 312 ms(h)
BNL X-ray	700 MeV	1 A	0.38	28.4 m	0.022 m	48.9 K Ω /m	208	4.8 ms	21 ms
BNL VUV	700 MeV	1 A	0.195	8.15 m	0.022 m	7.2 K Ω /m	62	16 ms	66 ms
BESSY	800 MeV	0.5 A	0.74	9.95 m	0.025 m	23.5 K Ω /m	155	6.4 ms	165 ms
ESRF	5 GeV	0.565 A	0.675	96.19 m	0.015 m	577 K Ω /m	80	12.5 ms	8.3 ms

VII. CAVITY RESONANCES

Higher order cavity modes caused a severe current limitation on DORIS until they were suppressed by selective damping in the cavities. Kohaupt⁵ developed a theory showing how particular transverse modes, the

TM₁₁₀ and the TM₁₁₁, gave rise to multibunch instabilities. These modes are typified by a longitudinal electric field, zero on the axis of the cavity and rising linearly with radius (to a first order) and a transverse magnetic field, approximately uniform in the axial region, which deflects the beam.

The cavity can be excited if one of the lines of the bunch mode spectrum coincides with the resonant frequency.

$$\omega_p = (n + kM Q) \omega_o$$

It is clear that the smaller the machine the spacing between possible mode frequencies becomes larger and there is less chance of a coincidence. However it would not be safe to rely on this, since the resonant frequencies of cavity modes will be moved by operation of the tuning plunger. Also there is likely to be a spread of mode frequencies between cavities due to the effect of manufacturing tolerances. Further the bunch mode spectrum can be swept across a wide frequency range by change of chromaticity.

Kohaupt's expression for the frequency shift may after some manipulation be written as

$$\Delta\omega_o = \frac{\beta c}{4\pi Q_{v,r}} \frac{I}{E} \frac{2R}{b^2} \frac{ZT^2}{n} \eta_{v,r} F' \quad (7)$$

$\eta_{v,r} = \beta_{v,r} Q_{v,r}/R$ allows for the effect of the amplitude function, $\beta_{v,r}$ at the location of the cavity.

ZT^2 represents the transit time corrected shunt impedance at the resonant frequency, b is the radius of the cavity outer wall. Kohaupt does not have the factor F' but, following Sacherer, this should clearly be included to take account of the bunch mode spectrum. The parameters pertinent to DORIS using $Q_r = 4.37$, $b = 0.46$ m, $R = 45.8$ m, $\eta_v = 4$, $E = 2$ GeV are shown in Table IV. These predictions compare with the experimental findings in which instability arose with current about 1/3 to 1/2 of these values overcoming octupole Landau damping with a rate near to 1 ms.

TABLE IV.

Parameter	TM ₁₁₀	TM ₁₁₁
Resonant frequency	777 MHz	929 MHz
Shunt impedance Z	15 M Ω	24 M Ω
ZT^2	1.5 M Ω	4.3 M Ω
$\Delta\omega$	6750 I	15550 I
$T = 1$ ms for $I =$	146 mA	64 mA

The SRS cavities due to their optimised shape have their transverse modes pushed to higher frequencies, namely 819 MHz and 1392 MHz. However due to the smaller gap the transit time factor is much higher than in the DORIS cavities even at these higher frequencies. Putting in Eq. 7 the parameters for the SRS ($Q \simeq 3.25$, $b = 0.42$ m, $R = 15.28$ m, $\eta_v = 4.2$, $E = 600$ MeV at injection) $\Delta\omega_0$ is $0.024 ZT^2I$ and $0.014 ZT^2I$ for the 2 modes. Thus Z needs to be reduced by 3 orders of magnitude to avoid instabilities when accumulating 1 A. This degree of damping of particular modes has been shown to be possible both in Hamburg and Daresbury.

For the ESRF¹⁰, if the cavities are in long straight section, the horizontal β function (32) dominates the instability criteria. For $Q_r = 26.2$, $b = 0.45$ m, $R = 96.19$ m, $\eta_v \sim 9$, $E = 2.5$ GeV (the lowest operational energy) and assuming higher order modes similar to DORIS cavities then $\Delta\omega_0 = 0.0028 ZT^2I$ and $0.0023 ZT^2I$ for the TM_{110} and TM_{111} modes respectively. Thus the ESRF will be an order of magnitude less sensitive than the SRS to higher order modes. However damping of the cavities will still be necessary if 0.565 A is to be achieved. The radiation damping time at 2.5 GeV is 66 ms.

It is difficult to say anything meaningful about the Brookhaven rings in the absence of information on higher order modes in co-axial resonators. However if the rf cavity in the x-ray ring is placed in a long straight when the β function is low in both planes the η values are only about 0.5. This is not the case in the VUV ring where $\eta_r \sim 4$.

Beside the rf cavities, all other cavity-like insertions in a ring should be effectively damped. Since this damping is against all modes and not selective it is relatively easy to accomplish using resistors.

The BESSY 800 MeV VUV ring⁹ is interesting in that it is proposed to have two autonomous rf systems, one at 125 MHz and one at 500 MHz. This, at any rate, should allow interesting accelerators physics experiments to be done on beam limitations. For the 500 MHz system and assuming a cavity similar to DORIS, then for $Q_r = 1.6$, $b = 0.45$, $R = 9.95$ m, $\eta_r \sim 2.9$, $E = 800$ MeV, $I = 0.5$ A and higher modes as for DORIS then the frequency shifts are 3.8×10^3 and 8.7×10^3 for the two modes, TM_{110} and TM_{111} respectively, or double this is the operating energy is reduced to 400 MeV. Again considerable damping of higher order modes is required.

VIII. USE OF AN rf QUADRUPOLE

Since a reasonable arrangement of octupoles is ineffectual in giving adequate frequency spread to a beam of low emittance and therefore small dimensions, the only effective means of stabilising against multi-bunch instabilities is the rf quadrupole. This device can easily give a total betatron frequency spread about 10^4 . For optimum effect a quadrupole should be located at high β positions, so that it may be desirable to have two quadrupoles, one optimised for each plane. On the other hand since instability effects will in general be different for the two planes a position may be determined where the β values are appropriate to requirements in each plane.

The required frequency spread to overcome an instability growth rate of $-j\Delta\omega$ is usually taken to be that the full spread of $Q\omega_0$ should be greater than $\Delta\omega$. So $\Delta Q > 2\Delta\omega/\omega_0$. The focal length of the quadrupole to

produce a given Q shift is $f = \beta/4R \Delta Q$ and the corresponding quadrupole gradient $G = E(\text{MeV})/300 \ell f$ where ℓ is the effective length of the quadrupole.

To give an example, the rf quadrupole for the SRS¹¹ will employ single loop coils at a radius of 120 mm with a current of 90 A per coil. The length is 0.75 m. This will give a maximum gradient of 5 mT/m, sufficient with β values of 17 (radial) and 6 (vertical) to give $\Delta\omega$ values of $9.8 \cdot 10^4$ (radial) and $3.5 \cdot 10^4$ (vertical) at 600 MeV. A similar quadrupole in the BNL x-ray ring placed when $\beta = 20$ would give $\Delta\omega = 5.5 \cdot 10^4$.

It is interesting to discuss what should be the frequency for the quadrupole. This choice is in any case restricted by practical limitations in the design of rf quadrupoles. Assume one is counteracting long range wakes lasting more than one orbit. At one extreme, if it were practicable, one might run the quadrupole at half the radio frequency. This would divide the bunch population into two distinct families and only halve the growth rate. At the other extreme a quadrupole operating at the orbit frequency would not adequately uncouple bunches near the peaks of the quadrupole waveform where the tune change is very small. No theoretical treatment of stability in the presence of a sinusoidally varying Q around the ring has been presented.

The approach used at the SRS was to use a frequency 5 times the orbital frequency (~ 16 MHz) where the practical problems of making a quadrupole are easily solvable. The present proposal is to lock the quadrupole frequency to the orbital frequency. However it is possible that improvement could be obtained in its effectiveness if the frequency ratio was fractional, say 5.25. Clearly this is something that could easily be tried in practice. DORIS used a frequency 15 times the orbital frequency (also ~ 16 MHz) and the quadrupole was effective in reducing growth rates by a factor of 4-6 with $\Delta\omega = 2.5 \times 10^4$.

Another practical consideration in making a quadrupole is whether to put it inside or outside the vacuum system. The advantage of the former is that it can be made smaller e.g. only 22 A would be needed to give 5 mT/m if the conductors are on a radius of 60 mm. On the other hand there is the difficulty of feeding it and tuning it in a vacuum and of making a vacuum tight enclosure. The alternative solution, adopted at Daresbury, is to insert a ceramic tube through the quadrupole, when although the whole arrangement becomes larger the feeder and tuning arrangements are simpler.

The ceramic needs a thin resistive coating to prevent charge accumulation, but it cannot usually be thick enough to form an effective conductor for image currents because (a) at 100 ohms/square the heat dissipation becomes intolerable¹² and (b) there is not enough frequency difference between the quadrupole frequency and the storage ring rf to allow a thick enough coating without affecting the quadrupole field. A thin coating, > 1000 ohms/square, is therefore advocated. This is too thin to contribute to the resistive wall instability. The whole quadrupole enclosure could behave cavity-like and measures must be taken to suppress unwanted resonances by damping.

Finally, the quality of the quadrupolar field is clearly completely unimportant except that symmetry should be maintained.

IX. CONCLUSIONS

The main conclusions of the study may be summarised as follows:

- a) The inductive wall of a storage ring is likely to cause a real frequency shift sufficient to cause loss of Landau damping even if the walls are reasonably smooth in contour. The effect is worse the smaller the height of the vacuum vessel. Being a single bunch effect it is worse for fewer bunches or lower rf. Increase of bunch length by a higher harmonic cavity helps to reduce the effect.
- b) With low emittance beams octupoles have a negligible effect in increasing the betatron frequency spread. This is the case for nearly all modern rings designed as synchrotron frequency sources.
- c) Resistive wall instabilities can arise for one multi-bunch mode with an unstable mode line near the orbital frequency. High conductivity walls and larger vertical aperture reduce the growth rate.
- d) All transverse modes in rf cavities must be damped, especially the TM_{110} and TM_{111} modes in the case of cylindrical cavities. Damping of all modes in other cavity-like insertions by insertion of resistive material should also be carried out.
- e) Bunch lengthening by a higher harmonic cavity will help to reduce the possibility of unstable conditions by shortening the frequency range of the mode spectrum. However, this also theoretically increases the possibility of $m = 1$ modes.
- f) Growth rates may well exceed the radiation damping rates. This will apply where the injection energy is significantly below the operating energy or where the operating energy itself varies over an appreciable range.
- g) Finally, the only available additional means of effective damping of multi-bunch transverse modes is by the appropriate use of rf quadrupoles. Feedback systems would need to be very broadband and have not been considered.

Further theoretical studies are required on the effect on stability of a sinusoidal distribution of betatron frequency among the M bunches. Also the effect of unequal distributions of intensity between bunches should be considered.

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ON REDUCING THE COUPLING IMPEDANCE OF STEP DISCONTINUITIES
IN VACUUM CHAMBERS

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We consider the possibility that by filling in the corner of a step discontinuity in a vacuum chamber with conducting material, the coupling impedance of the step discontinuity can be substantially reduced.

Consider a step of the form

where b and d = the radii of the circular vacuum chamber on both sides of the discontinuity. We assume that the magnetic fields generated by the beam are allowed to penetrate the 45° shaded area at the discontinuity, but that the electric fields are prevented from entering the shaded area by the electrostatic shielding of the adjacent walls. We wish to calculate the effective impedance of the shaded area.

The inductance of the shaded area is given by

$$L = \frac{1}{I^2} \iiint \frac{B^2}{\mu_0} r dr d\theta dz \quad (1)$$

where

$$B = \frac{\mu_0 I}{2\pi r}$$

then

$$L = \frac{\mu_0}{(2\pi)^2} \int_0^{d-b} \int_{b+z}^d \int_0^{2\pi} d\theta \frac{dr}{r} dz \quad (2)$$

$$= \frac{\mu_0}{2\pi} \int_0^{d-b} [\ln(d) - \ln(b+z)] dz$$

$$= \frac{\mu_0}{2\pi} [(d-b)\ln(d) - d\ln(d) + d + b\ln(b) - b]$$

$$L = \frac{\mu_0}{2\pi} [(d-b) - b\ln(d/b)] \quad (3)$$

We tabulate the value of $(L/\mu_o)[b/(d-b)^2]$ for 3 models:
the 45° model; the Keil-Zotter¹ model; and the Markcuvitz² model:

$$L_{kz} = 0.241 \mu_o \frac{(d-b)^2}{d + 0.412b} \quad (4)$$

$$L_m = \frac{\mu_o}{(2\pi)^2} \frac{(d-b)^2}{b} \left[1 - 2 \ln \left(\frac{d-b}{2b} \right) \right] \quad (5)$$

Step size	$L \times \frac{b}{\mu_o (d-b)^2}$		
	$\frac{d-b}{b}$		
	45°	Keil-Zotter	Markcuvitz
0.1	0.074	0.150	0.177
0.2	0.070	0.150	0.142
0.5	0.060	0.126	0.096

which leads us to ask whether the coupling impedance of a step discontinuity can be reduced substantially by "filling-in" part of the step with conducting material-e.g. a taper. A limited number of measurements have been done³ which indicate that a taper can substantially reduce the coupling impedance. These measurements were done above cutoff frequency however, and it is not clear how valid they may be at lower frequencies where wave lengths are long with respect to the taper.

As there are roughly 750 clearing electrode-pumping station cavities in each ISABELLE ring, i.e. about 1500 step discontinuities, a substantial reduction in coupling impedance may result from tapering the ends of these cavities.

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ESTIMATE OF THE COUPLING IMPEDANCE
FOR
THE STORAGE RINGS OF THE NSLS

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I. INTRODUCTION

The most important ingredient to evaluate the stability of a particle beam in a storage ring is the longitudinal coupling impedance Z/n and the transverse impedance $Z_{\perp}$ which is usually associated to the former.

In this paper we calculate these impedances for the two storage rings which are part of the NSLS, namely the Ultra Violet Ring (UVR) and the x-Ray Ring (XRR). The parameters for these two rings which are used throughout the paper are shown in Table I.

TABLE I: MACHINE PARAMETERS

	<u>UVR</u>		<u>XRR</u>	
E, energy	0.7	GeV	2.0	GeV
I_{AV} , av. current	1	A	1	A
M, no. of bunches	9		30	
$2\pi R$, circumference	51	m	170	m
ω_0 , $2\pi \times$ revol. frequency	$2\pi \times 5.88$ MHz		$2\pi \times 1.76$ MHz	
α , mom. comp. factor	0.02345		0.0065	
σ_{ℓ} , rms bunch length	3.84	cm	4.0	cm
σ_E/E , rms energy spread	4.4×10^{-4}		6.6×10^{-4}	
I_p , peak current	49.2	A	47.2	A
ν , vert. betatr. tune	1.4		5.7	
a, av. beam radius	0.2	mm	0.25	mm
b, av. pipe radius (circ. approximation)	3	cm	3	cm
bending radius	1.91	m	6.875	m
rectan. vac. chamber: w x h	8 x 4	cm ²	8 x 4	cm ²

The method we used to estimate the impedance is the one outlined by Faltens and Laslett.¹ They emphasize that starting from the lowest cut-off frequency of the vacuum pipe, charged particles are actually radiating in the free space, and therefore for frequencies sufficiently large the impedance is mostly contributed by the radiation in the vacuum. For the

special case of the two rings under consideration, the length of the beam bunches is comparable to the lowest cut-off wavelength and a considerable fraction of the bunch current spectrum lies in the range of the cut-off frequencies. Moreover, the most serious stability considerations concern the motion within a bunch and henceforth involve the range of frequencies we have just mentioned.

But before we estimate quantitatively the impedances, we must first get an idea of how much of them can be reasonably allowed. For the longitudinal motion we can take the classical stability criterion which, derived for coasting,² has been proven to be valid also for a bunched beam³

$$\left| \frac{Z}{n} \right| \leq \frac{\alpha E}{e I_p} \left(\frac{\Delta E}{E} \right)^2$$

where α is the momentum compaction factor, E the particle total energy, I_p the peak current that we define as

$$I_p = I_{AV} \frac{2\pi R}{3M\sigma_\ell}$$

with I_{AV} the average current in M bunches, with rms length σ_ℓ and over a circumference $2\pi R$. Finally $\Delta E/E$ is the FWHM of the relative energy distribution. We shall take $\Delta E/E = 3\sigma_E/E$ where σ_E is the rms spread. With the numbers in Table I we obtain

$$\left| \frac{Z}{n} \right| \leq \begin{array}{l} 0.6 \, \Omega \text{ for UVR} \\ 1.1 \, \Omega \text{ for XRR} \end{array} \quad (1)$$

The meaning of n is the ratio of the storage ring circumference to the wavelength of the perturbation traveling around the bunch. It is also a harmonic number, the ratio of an actual frequency to the revolution frequency.

For the transverse motion we can take the following stability criterion as derived by Sacherer⁴ for the case of smooth transverse impedance

$$\Delta \nu \geq \frac{e |Z_\perp| I_{AV} R^2}{6 \nu \nu_{EM}}$$

where $\Delta \nu$ is the required betatron tune spread by other means than sextupole, for the lowest head-tail mode. Higher modes require smaller spreads. In this equation ν is the (vertical) betatron tune.

It is possible to relate the transverse impedance $Z_\perp$ to the longitudinal one⁴

$$Z = -i \frac{R Z_0}{\beta^2 \gamma^2} \left(\frac{1}{a^2} - \frac{1}{b^2} \right) + \frac{2R}{b^2} \frac{Z_n}{n}$$

where a is the average beam radius and b the average pipe radius, and $Z_0 = 377 \, \text{ohm}$. Also β and γ are the usual relativistic factors. In the

following, when possible, we shall always take $\beta = 1$. We obtain

$$Z_1 = \begin{cases} (-i \ 39 + 18 \frac{Z}{n}) k\Omega & \text{for UVR} \\ (-i \ 10 + 60 \frac{Z}{n}) k\Omega & \text{for XRR} \end{cases}$$

Therefore if the Z/n is kept within the limit (1), the transverse impedance for the UVR is essentially contributed by the space charge term, whereas in the XRR most of the contribution will come from the longitudinal term.

The required tune spread for individual bunch stability is

$$\Delta\nu = \begin{cases} 1.6 \times 10^{-10} & \text{for UVR} \\ 0.7 \times 10^{-3} & \text{for XRR} \end{cases}$$

There is a chance that such tune spread exists already without addition of machine nonlinearities, because of the ions produced by scattering on the residual gas.⁵

II. THE LONGITUDINAL COUPLING IMPEDANCE

A. Smooth Pipe

As we suggested by Faltens and Laslett¹ let us start by assuming that the beam is performing circular trajectories in the free space, far away from any environment, without a vacuum chamber. The free space is associated with an impedance¹ which for pure circular orbits we can write

$$\frac{Z}{n} = \frac{354}{n^{2/3}} \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right) \Omega$$

Such a formula nevertheless must be modified to include the straight sections of the storage rings where the particles do not radiate. For both rings we obtain

$$\frac{Z}{n} = \frac{220}{n^{2/3}} \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right) \Omega \quad (2)$$

The spectrum of this impedance extends up to some critical harmonic number $n_{\text{crit}} = (3/2)\gamma^3$, about 4×10^9 for the UVR, and 10^{11} for the XRR. For $n > n_{\text{crit}}$ the impedance is effectively reduced to zero. As we shall see, this is the largest frequency range we have to consider. All other items which can give a contribution to Z/n are effective over a much narrower range.

Now let us surround the beam with a vacuum chamber. But first, let us assume the pipe is perfectly conductive, perfectly smooth, circular and straight. Let b be the inner radius. With such a pipe, there are two contributions to Z/n . One is given by the static inductance and

capacitance of the beam

$$\frac{Z}{n} = i \frac{Z_0}{2\beta\gamma} \ln \frac{R}{nb} \quad (3)$$

which is very negligible since at most is 10^{-3} ohm for the UVR at $n=1$. This impedance has obviously a cut-off for $n = R/b$. The second contribution is the usual space charge term

$$\frac{Z}{n} = i \frac{Z_0}{2\beta\gamma} (1 + 2 \ln \frac{b}{a}) \quad (4)$$

which rolls off at $n = n_b \sim \gamma R/b$. We have

	<u>UVR</u>	<u>XRR</u>
Z/n	$1 \times 10^{-3} \Omega$	$1.2 \times 10^{-4} \Omega$
n_b	3.8×10^5	3.6×10^6

Also this is a very negligible contribution.

But actually the vacuum pipe, even when it is smooth, perfectly conductive and straight, contributes more than (3) and (4). It has the capability to shield the beam against the "free space" radiation. The "free space" impedance as given by (2) is quite large, 220 ohm for $n=1$, but it is reduced practically to zero with the presence of the pipe. The only contributions left would be those given by (3) and (4). The pipe shielding, though, is not effective at large frequency. It is well known, as one can derive by solving the Maxwell's equations for a circular pipe of radius b , that there are cut-off values of the wavelengths λ given by the equation

$$J_0 \left(2\pi \frac{b}{\lambda} \right) = 0$$

where J_0 is the Bessel function, for which the space occupied by the pipe wall would be at zero (the reference) voltage, whether there is the pipe or not. The frequencies associated to these wavelengths are therefore sort of windows to the "free space". At these frequencies the particles really radiate undisturbed. Therefore at the same frequencies the impedance would correspond as given by (2). In terms of harmonic numbers, those which correspond to a cut-off are

$$n_s = \frac{R}{b} j_s \quad s = 1, 2, 3, \dots \quad (5)$$

where j_s are the zeros of the Bessel function J_0

$$j_1 = 2.405$$

$$j_2 = 5.520$$

$$j_3 = 8.654$$

$$j_s \approx (s - \frac{1}{4}) \pi \quad \text{for } s \gg 1.$$

By inserting (5) in (2) we can estimate the impedance at the cut-off values. Between these values the impedance should go back to the pipe value, but actually this does not happen quite so; at higher harmonics it becomes more difficult to shield the beam from its own radiation, until when, after few resonating cut-off harmonics, the impedance of the "free space" predominates everywhere.

In reality the vacuum chamber in the UVR and in the XRR has a rectangular cross section with a width $w = 8$ cm and a height $b = 4$ cm. For this geometry the cut-off harmonic numbers are

$$n_s = \frac{2\pi R}{w} s, \quad s = 1, 2, 3, \dots$$

At the first cut-off, for instance, the "free space" impedance is reduced down to

	<u>UVR</u>	<u>XRR</u>
n_1	638	2125
Z/n (f.s.)	2.97 Ω	1.33 Ω

Anyway, for $n \gg n_b \sim \gamma n_1$ (9×10^5 for UVR, 8.5×10^6 for XRR) the pipe is completely ineffective and one is definitively in the free space region.

At this time it is interesting to compare the bunch spectrum to the impedance. In terms of harmonic number we can define $n_s = 2\pi R/2\sigma$ as the critical bunch number. We obtain $n_s = 665$ for UVR and $n_s = 2125$ for the XRR. Thus in both storage rings the bunch spectrum overlaps with the pipe-cut region and makes this a very sensitive area for beam stability.

B. Resistive And Bent Pipe

Still preserving the smoothness of the pipe, that is no discontinuities or transitions, but constant cross section, let us nevertheless consider a more realistic vacuum chamber with resistive wall and bent over a circumference $2\pi R$.

The vacuum chamber is made of aluminum (ρ , resistivity = $283 \mu \Omega \times \text{cm}$ at 20°C) except in a few long straight sections where the pipe is circular with 7.3 cm diameter and made of stainless steel ($\rho = 73 \mu \Omega \times \text{cm}$ at 20°C). The contribution to the impedance from the resistive wall is

$$\frac{Z}{n} = \frac{1-i}{\sqrt{n}} \sqrt{\frac{Z_0 \rho R}{2b^2}} \quad (6)$$

which is correct for a pipe with circular cross section and radius b .

Taking into account also the long straight sections (with stainless steel) we obtain

$$\begin{aligned} \frac{Z}{n} &= \frac{1-i}{\sqrt{n}} 0.40 \quad \Omega \quad \text{for} \quad \text{UVR} \\ &= \frac{1-i}{\sqrt{n}} 0.67 \quad \Omega \quad \text{for} \quad \text{XRR} \end{aligned}$$

Like most of any other pipe impedance this also rolls off at $n_b \sim \gamma n_1$. At the first cut-off harmonic number n_1 the contribution of the resistive wall is 0.015 ohm for both rings, quite negligible compared to the "free space" impedance.

Of quite different concern is the curvature of the pipe. If the pipe were straight, the electromagnetic waves were plane and propagating at the light velocity (assuming again there are no discontinuities or any sort of obstacle). The beam then can never catch up with the wave it radiates and resonances are not possible. But the picture is different for a bent pipe where for very short wavelengths the waves are not plane. A wave propagating at the outer side of a beam, though speeding at higher velocity, has nevertheless a longer trajectory than the one of the beam which is moving to the inside at slightly less velocity. Therefore, it is possible for the beam to catch up with the wave it radiated the revolution (or a few revolutions) before. This is the condition for the "pipe resonance" (or resonances). Faltens and Laslett¹ put the resonance condition as

$$\beta \left(1 + \frac{b}{R}\right) \simeq 1 + 0.80862 n_r^{-2/3} \quad (7)$$

which is valid for a toroidal vacuum chamber with major radius R and minor radius b . When (7) is applied with $b = w/2$ we obtain

$$n_r = \begin{cases} 2100 & \text{for} \quad \text{UVR} \\ 12,800 & \text{for} \quad \text{XRR} \end{cases}$$

Actually Laslett and Lenish⁶ have calculated the electromagnetic response of a toroidal vacuum chamber with a rectangular cross section with the inner wall at radius R_i and the outer wall at radius R_o . If the beam is located right in the middle of the vacuum chamber the radius of its orbit would be $R = (R_o + R_i)/2$ and the vacuum chamber width $w = R_o - R_i$. Define

$$q^2 = \beta^2 \frac{n^2}{R^2} - \frac{(2m+1)^2 \pi^2}{h^2}$$

when n can be any integer, then the resonance condition for the vacuum chamber is

$$J_n(qR_o)N_n(qR_i) - J_n(qR_i)N_n(qR_o) = 0 \quad (8)$$

Here J_n is the Bessel function of order n , and N_n is the Neumann function of order n . For a given vacuum chamber geometry and integer n , there is a value n_r which satisfies (8). This is the resonance harmonic number. The solution of (8) is not expected to be much different from the one that can be obtained from (7).

The figure of quality Q of the vacuum chamber is estimated in Ref. (6). In the approximation that the wavelength which corresponds to n_r , namely $\lambda_r = 2\pi R/n_r$, is smaller than the two dimensions w and h of the pipe, one has the following formula

$$Q = \frac{4\pi^3 Z_o w}{\lambda_r^3 q^2 R_{\text{surf.}}}$$

where

$$R_{\text{surf}} = \frac{\text{resistivity}}{\text{skin depth}} = \sqrt{\frac{n_r Z_o \rho}{2R}}$$

Inserting number one obtains

$$Q = \begin{cases} 1.2 \times 10^5 & \text{for UVR} \\ 1.5 \times 10^5 & \text{for XRR} \end{cases}$$

In terms of the chamber Q the impedance at the resonance is given by

$$\frac{Z}{n} = \frac{2 Z_o Q \lambda_r}{n_r^2 \eta^2 h R w q^2}$$

where $\eta = w/2R$. For our rings this gives

$$\begin{aligned} Z/n &= 0.14 \quad \Omega \quad \text{for UVR} \\ &= 0.22 \quad \Omega \quad \text{for XRR} \end{aligned}$$

Though the resonance is very narrow, yet the amount involved is very large, a couple of kilo-ohm for the XRR, to cause enough worry about the beam stability. Moreover, as pointed out, there are several resonances, and we have looked only at the first one. From differentiation of (7), one can see that successive resonance harmonics are spaced by 0.0013 cm in the UVR and by 0.00021 cm in the XRR. As put by Faltens and Laslett this "may warrant reexamination of the usual relations between the peak value $|Z_n/n|$ and longitudinal stability".

Our advice is that the impedance from the curvature of the vacuum chamber deserves much more careful analysis than done here.

C. Discontinuities

A realistic pipe is made of discontinuities and obstacles. These will have the effect to scatter the radiated waves and brake the

electromagnetic shielding of the vacuum chamber, until the "free space" impedance is restored and even exceeded.

The major items that are included in storage rings like UVR and XRR are the following:

1. Pick-up plates
2. Clearing electrodes
3. Bellows
4. Steps and box-like objects, cavities
5. RF cavities
6. Vacuum ports,
Slits for synchrotron radiation extraction,
Holes for distributed pumping,
Slits for beam injection, extraction, dumping.

Each of these items requires a careful examination, and we have indeed estimated their contribution to the impedance as we shall see later. There are, of course, other elements, like ceramic pipes with kicker magnets, but they are not included in this analysis. Otherwise we believe that the above list is quite comprehensive.

3. Bellows

There are 8 bellows in the UVR and 16 in the XRR all with the same cross section of $8 \times 4 \text{ cm}^2$. They are made of copper ($\rho = 1.7 \mu\Omega \times \text{cm}$ at 20°C), each with 6 wiggles for a total length of 7.4 cm. The bellow height is 0.6 cm.

In the low frequency range, bellows cause just an inductance⁷

$$\frac{Z}{n} = -i Z_0 \left(\frac{M\ell}{2\pi R} \right) \ln \frac{\text{outer radius}}{\text{inner radius}}$$

where $M\ell$ is the total length. We obtain

$$\frac{Z}{n} = -i \begin{cases} 0.45 \Omega & \text{for UVR} \\ 0.26 \Omega & \text{for XRR} \end{cases}$$

A more serious problem nevertheless exists in the large frequency range when each convolution, thought as a cavity, starts to resonate. The calculation is done by Keil and Zotter⁸ but we also applied the technique of the radial transmission line model⁹ and we obtained about the same result. The lowest resonating frequency is around 4 GHz; for copper the corresponding Q is about 10^4 , and (unfortunately) the impedance Z about $10 \text{ k}\Omega$ for convolution. Since there are about 100 convolutions in the XRR this makes a total of $1 \text{ M}\Omega$, quite a large number that is balanced only by the narrow width of the resonance. Also, bellows are slightly different from each other and will resonate at different frequency. This will make a cumulative broader resonance with a lower impedance. The impedance for the higher order resonances decreases like $F^{-3/2}$.

Note that if the bellows were made of stainless steel, though Q could have been lowered by an order of magnitude making the resonances broader, the impedance is also reduced by the same amount.

1. Pick-Ups

There are 16 p.n. devices in the UVR and 32 in the XRR. Each device is made of four round buttons with a diameter of 1.8 cm, made of stainless steel and with a termination to the center. The capacitance of a button to ground is 6 pF which corresponds to a characteristic impedance of 10 Ω . Each button is terminated to a 50 Ω resistor.

Obviously one can apply the theory of clearing electrodes¹⁰ to the pick-up plates, but we have to approximate the circular geometry with a rectangular one, as it was formulated in the original calculations.¹⁰ The adjustment will cause a correction factor of 0.5 on the impedance. In the low frequency range the impedance is inductive again.

$$\frac{Z}{n} = -2\pi i Z_0 \left(\frac{4 M \ell}{2\pi R} \right) \left(\frac{\varphi_0}{\pi} \right)^2$$

where Z_0 is the characteristic impedance and φ_0 an effective semi-angular aperture. We obtain

$$Z/n = -i \begin{cases} 0.036 \quad \Omega & \text{for UVR} \\ 0.022 \quad \Omega & \text{for XRR} \end{cases}$$

In the large frequency range pick-up plates resonante. The technique used is the model of transmission lines.¹⁰ There are two kinds of resonances: the loaded resonances which depend on the ratio Z_T/Z_c of the termination impedance Z_T to the characteristic impedance Z_c , and the unloaded resonances with a wavelength that has a knot in correspondance of the termination and therefore makes the plate resonante as floating. The latter resonances are eventually damped by the resistivity of the plates. For the two storage rings UVR and XRR, because of the large value of γ and because of the material (stainless steel) involved, the two kinds of resonances merge to the same kind. They are rather broadband, resonate at frequencies

$$f_K = \frac{c}{\ell} \frac{2K-1}{2} \sim 8(2K-1) \text{ GHz}, \quad K=1,2,3,\dots$$

and have an impedance

$$Z_K = Z_c M \left(\frac{\varphi_0}{\pi} \right)^2 \left(\frac{Z_c}{Z_T} \right) = \begin{cases} 32 \quad \Omega & \text{for UVR} \\ 65 \quad \Omega & \text{for XRR} \end{cases}$$

Though they extend over a quite large frequency range, these resonances do not involve very large impedances.

2. Clearing Electrodes

There might be some need of cleaning excessive quantities of ions trapped by the electron beam after ionization of the residual gas.⁵ Then

we can conceive a situation where there are 16 pairs of clearing electrodes in the UVR and 32 in the XRR. Each electrode can be made round with 5 cm diameter. The expected characteristic impedance could be 5 ohm, whereas the termination can be a standard 50 ohm. The plates made of stainless steel will have the termination to the center. The same procedure outlined for the pick-ups can also apply here and all the considerations we made there still hold.

In the low frequency range we estimate the following inductance.

$$\frac{Z}{n} = -i \begin{matrix} 0.38 & \Omega & \text{for} & \text{UVR} \\ 0.23 & \Omega & \text{for} & \text{XRR} \end{matrix}$$

Again, the two kinds of resonances (loaded and unloaded) could not be distinguished from one another. The resonating frequencies are

$$f_K \approx 3 \times (2K-1) \text{ GHz} \quad K = 1, 2, 3, \dots$$

and the impedance is

$$Z_K = \begin{matrix} 65 & \Omega & \text{for} & \text{UVR} \\ 125 & \Omega & \text{for} & \text{XRR} \end{matrix} \quad \text{for any } K$$

quite a manageable amount.

The impedance from both pick-ups and clearing electrodes are eventually cut-off for $n \geq \gamma(2\pi R/w)$.

Observe that in these devices the termination impedance is not matched to the characteristic one. This will cause some loss of sensitivity in the pick-ups and has no effect in the clearing electrodes which are to be operated dc. But from the beam point of view there might be the inconvenience of the enhancement of a long range wake field.¹¹

4. Steps

In the UVR there are four long straight sections and in the XRR there are eight that have a different cross section from the rest of the ring, a circular pipe with radius of 7.3 cm. The transition is made with steps, which because of the considerable length of the straight sections (several meters) can be regarded as standing alone. We shall estimate the impedance in the worst case, the transition is sudden, whereas in practice it is tapered to a length of 8.8 cm.

Define S at the ratio of the outer dimension to the inner dimension b. We take S = 1.8. According to Hereward¹² the impedance per step is

$$\frac{Z}{n} = (1-i\pi) \frac{2Z_0}{\pi} (S-1)^2 \frac{b}{2\pi R} = \begin{matrix} 0.060 & \Omega & \text{for} & \text{UVR} \\ 0.018 & \Omega & \text{for} & \text{XRR} \end{matrix}$$

for $n \leq [4\pi R/4b(S-1)] = n_w$ where $n_w = 800$ for UVR and 2650 for XRR.
If $n \geq n_w$ then

$$\frac{Z}{n} = \frac{Z_o}{2\pi} \frac{(S-1)}{n} \sim \frac{50}{n} \Omega \text{ per step for both rings.}$$

Also these numbers are quite manageable.

The steps in the two rings are quite apart from each other and therefore we cannot identify any cavity or box-like object which could require a serious consideration.

5. RF Cavities

A major rf cavity can always be approximated with an RLC-resonating circuit. Usually a cavity is defined by a resonating frequency, which in our case is h times the revolution frequency, where h is the harmonic number, a figure of merit Q and a shunt impedance Z_s . The three impedance elements (R, L and C) can then be estimated from these. The impedance at the fundamental harmonic is compensated with proper phase and voltage loops. But for $n < h$ and $n > h$ there is an appreciable impedance which is respectively inductive and capacitive. One has

$$\frac{Z}{n} = \frac{Z_s}{Q^2 n} - i \frac{1}{h} \frac{Z_s}{Q} \text{ for } n < h$$

$$\frac{Z}{n} = i \frac{1}{n^2} h \frac{Z_s}{Q} \text{ for } n > h$$

The main parameters are

	<u>UVR</u>	<u>XRR</u>
h	9	30
Q	20,000	20,000
Z_s	2 M Ω	4 M Ω

and one has

$$\begin{aligned} \frac{Z}{n} &= -i 11 \Omega & \text{for} & \text{UVR} \\ &= -i 6.7 \Omega & \text{for} & \text{XRR} \end{aligned} \quad \text{for } n < h$$

and

$$\begin{aligned} \frac{Z}{n} &= i \frac{90}{n^2} \Omega & \text{for} & \text{UVR} \\ &= i \frac{6 \times 10^2}{n^2} \Omega & \text{for} & \text{XRR} \end{aligned} \quad \text{for } n > h$$

There is quite a large inductance for small harmonic number, left-over of the major rf system. Very likely our numbers are overestimated, but still considerably large to cause some concern.

On top of this, rf cavities are responsible for an all host of higher order modes. These are quite dangerous and one should keep control of them. We will not deal with them here.

6. Slits and Holes in the Vacuum Chamber

The impedance from a slit or hole is evidentially caused by the diffraction of the electromagnetic wave through them. Part of the wave is reflected back at the edge of the hole, and another part is lost beyond. Taking the same approach exposed by M. Sands,¹³ but working in the frequency domain instead of the time domain, we have estimated the impedance for a frequency $\omega = n\omega_0$. This is

$$\frac{Z}{n} = \frac{2Z_0 \alpha^2}{3\pi^3 R^4 b^2} \left[n^3 + i \frac{8}{\pi} (n^2 n_c + \frac{1}{3} n_c^3) \right] \quad (9)$$

where for a rectangular slit of dimensions $l \gg w$

$$\alpha = \frac{\pi}{16} l w^2$$

and b is the pipe radius. Also $n_c \sim 2\pi R/l$.

Equation (9) is valid only for $n \leq n_c$. For $n > n_c$, the wavelength is smaller. The dimension of the hole and the beam is practically radiating in the "free space", though for only the length of the slit.

We have estimated the impedance (9) for several hole geometrics in the two storage rings.

(a) Ports for synchrotron radiation extraction.

They are 40 cm long and 0.6 cm wide. There are 16 in the UVR and 32 in the XRR. For them we obtain

$$\frac{Z}{n} = \left[n^3 + i \frac{8}{\pi} (n^2 n_c + \frac{1}{3} n_c^3) \right] \times \begin{cases} 1.5 \times 10^{-10} \Omega & \text{for UVR} \\ 2.4 \times 10^{-12} \Omega & \text{for XRR} \end{cases}$$

with $n_c = 127$ for UVR and 425 for XRR.

(b) There are the major vacuum pump ports.

Each pump has seven rows of two slits each with $l = 5.2$ cm and $w = 0.6$ cm.

In this case we multiply the impedance also by a "coherence factor" of seven, because seven holes can resonate coherently at the same time.¹³ We obtain

$$\frac{Z}{n} = \left[n^3 + i \frac{8}{\pi} (n^2 n_c + \frac{1}{3} n_c^3) \right] \times \begin{cases} 5 \times 10^{-10} \Omega & \text{for UVR} \\ 8 \times 10^{-12} \Omega & \text{for XRR} \end{cases}$$

and $n_c = 784$ for UVR and 1568 for XRR.

(c) Holes for distributed pumping. For each magnet there are 43 pairs of holes $2.94 \times 0.67 \text{ cm}^2$. Each pair radiates coherently and therefore the impedance is multiplied by a factor of two.

The contribution from these holes is

$$\frac{Z}{n} = \left[n^3 + i \frac{8}{\pi} (n^2 n_c + \frac{1}{3} n_c^2) \right] \times \begin{cases} 1.1 \times 10^{-10} & \text{for UVR} \\ 1.8 \times 10^{-12} & \text{for XRR} \end{cases}$$

with $n_c \sim 1000$ for UVR and 2000 for XRR.

These holes, as the previous ones, can give a considerable contribution when n approaches the limit value n_c . Nevertheless the reactance is positive, which denotes a capacitance and can offset some of the large value of the inductance we have accumulated with the other vacuum chamber items.

III. CONCLUSIONS

The longitudinal impedance Z/n is plotted in Fig. 1 for the UVR and in Fig. 2 for the XRR. The solid line is our result. The dashed straight line is the impedance in the free space, and the dashed curve the contribution from the very dense resonances of the vacuum chamber curvature, assuming the same impedance for all of them. One can clearly notice the

FIG. 1.

FIG. 2.

contribution of the "wall resistivity" in the low frequency range, which is followed by a flat plateau which represents the region of the "inductance wall". The wiggles are caused by the lowest cut-off harmonic numbers.

The transition from the "pipe" impedance to the "free space" impedance is obvious.

All the contributions have been included in Figs. 1 and 2 except those from the main rf system and the high resonating modes of the below convolutions.

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PHENOMENOLOGICAL ANALYSIS OF CURRENT LIMITS IN STORAGE RINGS*

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SUMMARY

We study the limitations on the current which can be stored in a storage ring. A simple model is used for the impedance of the beam environment. We assume that feedback systems are used to their full capacity to provide beam stability and that in the frequency range where feedback can no longer be used Landau damping will determine how much current can be stored in the ring. Estimates are made for proton storage rings and for electron storage rings of interest as synchrotron radiation sources and as colliding beam systems for high energy physics.

I. INTRODUCTION

In this paper we discuss the limitations on the amount of current which can be stored in a ring. The effects limiting the current are assumed to be the collective instabilities due to the interaction of the beam with the vacuum chamber walls, rf cavities, pick-up electrodes and other devices constituting the environment in which the beam moves. We neglect other effects which, for very high currents, might introduce difficult technical problems, like, for instance, heating of the walls or radiation effects, since we want to study what is the maximum current obtainable under idealized conditions. We will consider mainly the limitations due to high frequency instability, assuming that all low frequency effects can be cured by feedback. We will only use a simplified and approximate description of the instability to obtain an order of magnitude estimate of the beam current and to see how it scales with different machine parameters.

In the next section we describe the model we assume for the beam environment, i.e. the impedance that we will use for the subsequent calculations. In section III we define our criterion for beam stability and in sections III and IV we apply it to the cases of coasting and bunched beams.

II. THE CHAMBER IMPEDANCE

To discuss the impedance seen by a beam in a storage ring, let us define first what are the different regions of interest in the frequency domain. Let ω_0 be the revolution frequency, $\omega_B = c/L$ the frequency associated with the bunch length L and $\omega_{co} \approx c/d$ the cut-off frequency of the vacuum pipe, whose transverse dimension is d . The beam environment

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is assumed to be a pipe with nearly uniform cross section, plus the rf cavities and beam measuring devices.

We assume that the total chamber impedance is the sum of three contributions:

- 1) the high-Q resonant contributions at low frequency ($\omega \ll \omega_B$);
- 2) a nonresonant broadband impedance extending to frequencies of the order of ω_{co} .
- 3) the resistive wall impedance.

The resonant part of the impedance gives rise to many bunch instabilities or to single bunch multiturn effects. We assume that all these effects can be cured with a feedback system, and in the following we will neglect this part of the impedance.

The broadband, nonresonant impedance is responsible for the high frequency instabilities involving the internal bunch modes, like the head-tail effect or the microwave instability. This is the part of the impedance that will be considered here.

The resistive wall impedance is usually small compared to the resonant impedance at low frequency and to the broadband impedance at high frequency and will also be neglected.

The broadband impedance is produced by discontinuities in the vacuum chamber, bellows, pick-up electrodes and other possible devices present in the beam environment. The characteristics of the longitudinal broadband impedance have been obtained at SPEAR from bunch length measurements.¹ The shape of this longitudinal impedance, is given in Fig. 1, where we have plotted the real and imaginary parts of $(Z_{||}/\omega)$ versus ω .

Fig. 1.

This impedance can be approximately described as that of a parallel R, C, L, resonant circuit, having the resonant frequency equal to ω_{co} and a Q of the order of 1. To completely define $Z_{||}$ we only need to know ω_{co}

and the shunt impedance, R_o . We then have

$$\frac{Z_{\parallel}(\omega)}{\omega} = \left(\frac{R_o}{\omega_{co}} \right) \frac{(\omega/\omega_{co}) + i Q \left[1 - \left(\frac{\omega}{\omega_{co}} \right)^2 \right]}{\left(\frac{\omega}{\omega_{co}} \right)^2 + Q^2 \left[1 - \left(\frac{\omega}{\omega_{co}} \right)^2 \right]^2}. \quad (2.1)$$

For our instability calculations we want to know the value of $Z_{\parallel}/\omega$ at $\omega = m\omega_o$ and in particular the peak value. For the new generation of storage rings one can assume that $(Z_{\parallel}/\omega)_{MAX} \approx 1\Omega$ and in the following we will use this value.

The chamber irregularities that give rise to $Z_{\parallel}$ will also generate fields which give the beam particles a transverse kick, and can cause unstable, coherent transverse oscillations. The transverse effect of the wall fields is most conveniently described in terms of a "Transverse impedance" Z_T --see W. Schnell²--which can, quite generally, be written in terms of $Z_{\parallel}$ by

$$Z_T(m\omega_o) \approx \frac{2R}{d^2} \frac{Z_{\parallel}(m\omega_o)}{m}, \quad (2.2)$$

where R is the ring radius.

Space-charge effects can be taken into account by adding an additional term to Z_T , namely,³

$$Z_T(\text{space charge}) = R \frac{Z_o}{\gamma^2} \left(\frac{1}{a^2} - \frac{1}{d^2} \right) \quad (2.3)$$

where γ is the beam energy factor, a is the beam radius and Z_o is the impedance of free space. We will take for Z_T the sum of the two contributions in (2.2) and (2.3).

III. CURRENT LIMITS FOR UNIFORM BEAMS

To determine the maximum current in a stable beam we use the following assumptions:

- 1) for $\omega < c/d$ we can use feedback systems to stabilize all transverse and longitudinal modes;
- 2) the effect of resonant impedances is also stabilized by feedback;
- 3) frequencies $\omega \gtrsim \omega_{co}$ are important, although $Z_{\parallel}$ starts to decrease at $\omega \approx \omega_{co}$;
- 4) for $\omega \gtrsim \omega_{co}$ the only cure for unstable modes is Landau damping;
- 5) we only consider unstable mode with a rise time, τ , longer than the revolution period, or

$$\frac{1}{\tau} < \frac{\omega_o}{2\pi}$$

thus excluding phenomena like the beam break-up in linear accelerators.

We now write down the conditions for beam stability considering the two cases of a coasting beam (that is, an unbunched beam) or of a bunched beam. For both the coasting and bunched beam case we consider separately the effect of longitudinal and transverse instabilities. For the longitudinal case we consider essentially the effect of the negative mass instability generalized to an arbitrary wall impedance⁵⁻⁷ and its extension to bunched beams, i.e. the microwave instability.^{8,9} For the transverse case the main effect comes from high frequency transverse coherent modes¹⁰ and the head-tail instability for bunched beams.^{11,12} In all cases stability is provided by Landau damping.

For a coasting beam we write the condition for longitudinal stability as⁷

$$e I_o \left(\frac{Z_{||}}{\omega} \right) \leq \alpha E \left(\frac{\Delta E}{E} \right)^2 \quad (3.1)$$

where I_o is the beam current, α the momentum compaction factor, E the beam energy and $(\Delta E/E)$ the beam energy spread.

The transverse stability condition is written as^{10,13}

$$\delta\nu \leq \Delta\nu \quad (3.2)$$

where $\delta\nu$ is the shift in the betatron tune, ν_B , due to the collective interaction with the environment

$$\delta\nu = \frac{e I_o Z_T R}{2\pi \nu_B E} \quad (3.3)$$

and $\Delta\nu$ is the tune spread producing the Landau damping. We assume Z_T to be given by (2.2), (2.3), or

$$Z_T = \frac{2R}{d^2} \frac{Z_{||}}{n} + R \frac{Z_o}{\gamma^2} \left(\frac{1}{a^2} - \frac{1}{d^2} \right). \quad (3.4)$$

The amount of tune spread, $\Delta\nu$, which can be allowed in a beam is limited by the distance between nonlinear resonances. In the following we will assume that $\Delta\nu \approx 10^{-2}$. If the machine chromaticity is of the order of one we also have $\Delta E/E \approx 10^{-2}$.

With these values of $\Delta\nu$ and $\Delta E/E$ and assuming $\alpha = 10^{-3}$, $E = 30$ GeV, $\gamma = 30$, $Z_{||}/n = 1\Omega$, we obtain from (3.1)

$$I_o \leq 3 \times 10^3 \text{ A}.$$

It is also interesting to rewrite (3.1) as a condition on the current density per unit energy spread, which is the quantity of interest during injection in a proton storage ring. For $\Delta E/E = 10^{-3}$ this gives

$$I_o / \frac{\Delta E}{E} \leq 3 \times 10^4 \text{ A}.$$

Assuming $E \sim RB$, $\alpha \approx v_\beta^2 \approx R^{-2}$ we also see that, for given $Z_\parallel$, I_0 scales as B/R so that, for a given energy, it is convenient to use high magnetic fields B .

To evaluate the transverse instability, we assume again $E = 30$ GeV, $\alpha = 10^{-3}$, $Z_\parallel/n = 1\Omega$ and $(R/d) = 10^4$, $v_\beta = 30$. The first and second term in (3.4) are of the same order for $(d/a) \approx 3$ and we assume $Z_T R = 5 \times 10^8 \Omega$. With these values we have from (3.3) (3.2)

$$I_0 \leq 10^4 \Delta v$$

or $I_0 \leq 10^2$ A for $\Delta v = 10^{-2}$. Notice that I_0 scales as $v_\beta E/R^2$ or as $(E/R) = B$ so that; as in the longitudinal case, it is convenient to work at high magnetic fields.

IV. BUNCHED BEAMS

We consider now the bunched beam case--of interest primarily for electron rings. The microwave stability condition for a bunched beam can be written as^{8,9}

$$e I_p (Z_\parallel/n)_{\text{eff}} < \alpha E \left(\frac{\Delta E}{E} \right)^2 \quad (4.1)$$

where I_p is the peak current, $\Delta E/E$ is the beam energy spread, E the beam energy, α the momentum compaction. The quantity $(Z_\parallel/n)_{\text{eff}}$ takes into account the fact that the frequency spectrum of a bunched beam contains many harmonics of the revolution frequency. It is defined as⁹

$$\left(\frac{Z_\parallel}{n\omega_o} \right)_{\text{eff}} = \sum_{p=-\infty}^{+\infty} \frac{Z_\parallel(p\omega_o)}{p\omega_o} \rho_m(p)$$

$\rho_m(w)$ being the bunch form factor for m -th mode of oscillation ($m = 1$ dipole, $m = 2$ quadrupole, etc.),

The condition for transverse stability can again be written in the form (3-2), with δv given by¹³

$$\delta v = \frac{e I_{\text{peak}} Z_{\text{Teff}} R}{2\pi v_\beta E} \quad (4.2)$$

where the effective transverse impedance is

$$Z_{\text{Teff}} = \frac{2R}{d^2} \left(\frac{Z_\parallel}{n\omega_o} \right)_{\text{eff}} + Z_o \frac{R}{\gamma^2} \left\{ \frac{1}{a^2} - \frac{1}{d^2} \right\} \quad (4.3)$$

Both (4.1) and (4.2) depend on the peak current. For a given peak current we can increase the average current, I_0 , by increasing the bunch length L . What limits the bunch length is the requirement of maintaining a long lifetime, determined mainly by quantum fluctuations for electrons and intrabeam scattering or other diffusion effects for proton beams. To keep down these effects we can use the condition that the

bunch should fill less than one tenth of the rf period or

$$\frac{L}{2\pi R} \leq \frac{1}{10h} \quad (4.4)$$

where h is the harmonic number. Assuming this maximum possible bunch length we then have for the average current per bunch

$$I_o = I_{\text{peak}} \frac{L}{2\pi R} = \frac{I_{\text{peak}}}{10h} \quad (4.5)$$

We can now evaluate the limits on I_o for two different cases: a low energy, $E = 1$ GeV, and a high energy, $E = 15$ GeV, electron storage ring. For the low energy ring we use the following parameters:

$$\begin{aligned} E &= 1 \text{ GeV} & (Z_{\parallel}/\omega)_{\text{eff}} &= 1\Omega \\ R &= 15 \text{ m} & Z_{\text{Teff}}^R &= 2 \times 10^4 \Omega \\ \alpha &= 10^{-3} & \Delta\nu_{\beta} &= 10^{-3} \\ \nu_{\beta} &= 5 \\ \nu_s &= 2 \times 10^{-2} \end{aligned}$$

The synchrotron tune, ν_s , is needed to relate $\Delta E/E$ and L , $\Delta E/E = (\nu_s/\alpha)(L/R)$. Using (4.1), (4.5) we obtain a threshold condition for the microwave instability

$$I_o \leq \frac{\nu_s^2 E}{8\pi \alpha h^3}$$

or

$$I_o \leq 10^7/h^3 \text{ Ampere}.$$

If we fill all the bunches we have a total current, $I_T = hI_o$, given by

$$I_T = \frac{10^7}{h^2} \text{ Ampere}.$$

Notice the very strong dependence of I_o on h , which makes it convenient to choose a low value of the harmonic number. For h of the order of 20, corresponding to an rf frequency of the order of 60 MHz, the limit on I_o is still very high, of the order of 10^3 A/bunch.

In the transverse case we obtain from (4.2), (4.5)

$$I_o \leq \frac{10^2}{h} \text{ Ampere}$$

or

$$I_T < 10^2 \text{ Ampere}.$$

In this case there is no dependence on h , although it is still convenient to use a small h if one wants a large current per bunch.

For the high energy ring we use the parameters

$$\begin{aligned} E &= 15 \text{ GeV} & (Z_{\parallel}/\omega)_{\text{eff}} &= 1 \Omega \\ R &= 300 \text{ m} & Z_{\text{Teff}} \times R &= 7 \times 10^7 \Omega \\ \alpha &= 3 \times 10^{-3} & \Delta v_{\beta} &= 10^{-3} \\ v_{\beta} &= 20 & \Delta v_s &= 10^{-3} \\ v_s &= 5 \times 10^{-2} \end{aligned}$$

For the microwave instability we have

$$I_0 \leq 2 \times 10^9 / h^3 \text{ Ampere.}$$

Choosing $h = 3 \times 10^3$, or a rf frequency $f_{\text{rf}} = 475 \text{ MHz}$, we have

$$I_0 \leq 7 \times 10^{-2} \text{ Ampere.}$$

For the transverse case we obtain

$$I_0 \leq \frac{1}{2h}$$

and for $h = 3 \times 10^3$

$$I_0 \leq 9.5 \times 10^{-4} \text{ A.}$$

The corresponding value of the peak current is rather large

$$I_p = 28 \text{ A}$$

and the bunch length L is about 7 cm.

Although the value of δv is rather large $\delta v = 10^{-3}$, causing the loss of Landau damping, it is still possible to use the radiation damping to cure instability if the rise time, τ , is longer than the radiation damping time. We can estimate this rise time for the case of a head-tail instability using the expression

$$\frac{1}{\tau} = \omega_0 \delta v \cdot 2\pi \frac{\xi v_{\beta}}{v_s} \left(\frac{\Delta E}{E} \right) \quad (4.6)$$

where ξ is the machine chromaticity, and the term $\xi v_{\beta}/v_s (\Delta E/E)$ measures the phase shift due to synchrotron oscillations. Ideally one can have $\xi = 0$ and thus an infinite rise time. However this condition can only be approximately satisfied because of machine nonlinearities making ξ a function of momentum. Assuming $\xi v_{\beta} \Delta E/E \approx 10^{-4}$ we obtain a rise time,

for the current corresponding to $\delta v = 10^{-3}$,

$$\frac{1}{\tau} \approx 10 \text{ s}^{-1}.$$

The order of magnitude of the radiation damping time is

$$\frac{1}{\tau_{\text{rad}}} = \frac{2}{3} \left(\frac{r_e}{\rho} \right) \gamma^3 \omega_0$$

and assuming the radius of curvature ρ to be 100 m, $\omega_0 = 10^6 \text{ s}^{-1}$ we have $1/\tau_{\text{rad}} = 20 \text{ s}^{-1}$ at 5 GeV and $1/\tau_{\text{rad}} \approx 5.4 \times 10^2 \text{ s}^{-1}$ at 15 GeV. Hence, at 5 GeV one can store twice the current corresponding to $\delta v = 10^{-3}$, or $0.66 \times 10^{-3} \text{ A}$, and at 15 GeV one can store about 50 mA.

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BENCH MEASUREMENT OF VACUUM CHAMBER IMPEDANCES WITH WIRES

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Particle beams travelling in an accelerator or storage ring vacuum chamber produce electric and magnetic fields which interact with the walls of the chamber. The induced wall currents in turn generate secondary fields which can interact with the beam, giving rise to both energy loss and other effects which can cause beam instability. In many simple geometries these effects can be calculated. In more complex geometries the calculations are difficult and very approximate. For this reason it is important that the effects be measured if possible prior to installation. As accelerators become larger and the number of vacuum chamber components more numerous, and as more severe beam intensity and stability requirements are placing lower limits on the longitudinal and transverse impedances, a less than adequate component will be more likely to limit machine performance.

In 1974 Sands and Rees¹ proposed a method of measuring longitudinal impedances of vacuum system components "on the bench" by emulating the beam with a current pulse down a coaxial system using the vacuum chamber as the outer conductor. Since then their basic ideas have been expanded upon by other groups.²⁻⁵ In all cases, the effect of the vacuum chamber components on the current pulse is measured by observing distortions in the transmitted (as opposed to the reflected) pulse. In the case of sinewave excitation, these distortions appear as phase shifts and attenuations.

The interaction of a beam with a vacuum chamber, and a current pulse in a coaxial system interacting with the same vacuum chamber are reality quite different, so some basic understanding of the differences is in order.

The Poynting vector accompanying the beam is dependent on the beam as its source of power. Any losses the Poynting vector encounters as a result of interacting with the vacuum components is replenished by the beam encountering a longitudinal voltage gradient. The vacuum component appears in effect like a series reactive or resistive component. The beam current is not affected. In the case of a current pulse in a coaxial system, the Poynting vector is the energy source, and interactions with the vacuum components cause changes in both the voltage and current of the pulse ($\vec{E}$ and $\vec{H}$ vectors).

Specifically, a step in the diameter of the vacuum chamber appears to the beam as a series inductance, as a small volume in the corner "sees" the stored magnetic energy, but is shielded from the electric energy in the Poynting vector. In the coaxial case, the step appears as a shunt capacitance. This shunt capacitance may be interpreted as a series inductance however, if the effective impedance of the coaxial system is known; i.e. $L/C = Z^2$. For traveling waves this is straight forward. For standing waves, however (which may or may not be wanted) the effective impedance is position dependent. This is not a problem however when short pulses are used, rather than sinewave excitation, and the pulse lengths are short in relation to the length of components likely to cause standing waves. Due to the short bunch lengths in electron synchrotrons, bench measurements are carried out using short pulses.

In proton synchrotron applications, where rf frequencies normally are much lower, there is some interest in using sinewaves.

Typical bench systems using short pulses are described in the literature.²⁻⁵ In all cases the pulse length is in the range 100-300 picosec. The measurement consists of comparing the transmitted current pulse $i_m(t)$ through the component being tested with the current pulse $i_o(t)$ when the component is replaced with an equivalent length of standard vacuum pipe. The loss parameter k (having dimensions volts/coulomb) is then obtained from the relation

$$k = (2R_o/q^2) \int [i_o(t) - i_m(t)] i_o(t) dt \quad (1)$$

where R_o is the characteristic impedance of the wire in the vacuum pipe and $q = \int i_o(t) dt$ is the total charge in the pulse. The energy loss of one bunch of charge q passing through the structure is then $E_b = k^2 q$. Other relations are presented in Ref. 3.

Measurements with a total resolution of about $\Delta k = \pm 0.003$ volts/pico Coulomb have been obtained with a "single arm" setup (one transmission line where the component under test is alternated with a standard vacuum pipe), and resolutions of about $\Delta k \cong \pm 0.0005$ volts/pico Coulomb have been obtained with "double arm" setups (two transmission lines, which allow direct phase comparison). The latter setup,⁴ using 2.1 cm wide pulses, has yielded equivalent series resistance and inductance measurements on vacuum chamber components with a precision of ± 0.05 ohms and ± 0.05 nH respectively. The pulse width corresponds roughly to a frequency of 2 GHz.

In order to obtain the precision quoted above great care must be taken in stabilizing the temperature and minimizing resistive contacts. Systematic errors are introduced if the center wire is too large, and statistical errors dominate if the center wire is too small. Typical wire diameters used are in the range 1mm to 2mm.

A careful scattering matrix analysis of the coaxial structure has been carried out by Hahn and Peterson.⁶ They show that transmission type measurements are preferred over reflection type measurements for two reasons when sinewave excitation is used:

1. If two or more discontinuities are present, the transmission measurement is less sensitive to their exact location than is the reflection measurement.
2. As the "space charge" term in the longitudinal impedance is not diminished by the factor $\beta\gamma^2$ in the coax structure as it is in the case of a particle beam, the term appears in the bench measurements whenever there are two or more discontinuities present. This contribution is in addition to the shunt capacitance (series inductance) of the discontinuities themselves, and is proportional to their separation. This term is smaller in transmission type measurements than in reflection type.

For a simple two step discontinuity where the vacuum pipe is enlarged from a radius b to a radius d for a length ℓ , and a wire of

radius a is used, the voltage ratio is:

$$\frac{V_m(\omega t)}{V_o(\omega t)} = 1 - j \frac{2\omega L_s}{2R_o} + \left(\frac{\Delta R}{R_o}\right)^2 (1 - e^{-2ikl}) \quad (2)$$

where $V_m(\omega t)$ = voltage of perturbed transmitted sinewave
 $V_o(\omega t)$ = voltage of unperturbed transmitted sinewave
 $k = \omega/c$ is the wave number
 $R_o = 60/n(b/a)$
 $\Delta R = 60/n(d/a) - 60/n(b/a) = 60/n(d/b)$
 L_s = equivalent series inductance of each step.

Note that the "space charge" term varies as $1/R_o^2$ while the series step inductance term varies as $1/R_o$. Reducing the inner wire radius improves the rejection of the space charge term, but at the same time the skin effect attenuation in the wire must be considered. The attenuation is of the form $e^{-\alpha x}$ where

$$\alpha = \frac{1.38 \times 10^{-10} \sqrt{\omega}}{a \ln(b/a)} \text{ m}^{-1} \quad (3)$$

for a wire of radius a meters in a pipe of diameter b meters. Note that α is a function of pipe radius as well as of a and ω , meaning that under certain conditions variations in the pipe radius must be taken into account.

In order to use a thin wire, which results in $R_o \geq \sim 300$ ohms, it is important to use optimally tapered impedance matching sections. A good review of taper designs is given by R. E. Collin.⁷

As ISABELLE has a bunch length of about 50 nsec., frequencies of the order of a few MHz up to perhaps 1-2 GHz should be used to measure impedances. In addition, as the ring will be operated in the coasting beam mode, frequencies down to $(n-v)\omega_o/2\pi \cong 40$ kHz must be considered.

The bench measurement setup for ISABELLE components⁸ includes a "double arm" type arrangement with a 0.3 mm diameter wire in a 88 mm diameter beam pipe. A Hewlett Packard model 8505A sweep frequency network analyzer will allow measurements to over 1 GHz. Measurements of $\pm 0.5^\circ$ in phase and ± 0.1 db in voltage ratio represent accuracies of ± 1.0 nH and ± 8 ohms respectively in a $R_o = 340$ ohm structure of a 0.3 mm dia wire in a 88 mm dia beam pipe).

One nice feature of using sweep frequencies is to be able to identify parasitic resonances quickly. When a wire is used to excite the resonances however, the resonant frequency is shifted and the maximum observable Q is limited due to the wire inductance and the characteristic impedance R_o of the transmission line shorting the resonant cavity. Specifically the frequency shift Δf and the maximum observable Q_m are given by¹

$$\Delta f = \frac{1}{2\mu_o \ell} \left(\frac{R_{||}}{Q} \right) \frac{1}{\ln(b/a)} \text{ Hz.} \quad (4)$$

$$Q_m = \frac{120 \ell \ln(b/a)}{(R_{||}/Q)} \left[1 + \left(\frac{\pi f \mu_o \ell}{377} \right)^2 \right] \quad (5)$$

where $R_{||}$ and Q are the unperturbed shunt resistance and Q of the resonance, f is the resonant frequency, l is the length of the cavity gap, $\mu_0 = 4\pi \times 10^{-7}$ H/m, and $R_0 = 60 \ln(b/a)$ = transmission line impedance.

Measuring the natural $R_{||}$ and Q of the cavity seems to be beyond the capability of this method, however.

The major concern in proton rings is to minimize the imaginary component of the longitudinal impedance as the real components are normally zero except for the resistive wall heating term, which is calculable (in cold bore superconducting magnet designs this heating must be considered as a cryogenic load).

In ISABELLE, as $\omega_0 \approx 4.91 \times 10^5 \text{ sec}^{-1}$, the 1nH resolution at 1 GHz corresponds to $(Z/n) = j\omega_0 L = \pm j 4.9 \times 10^{-4}$ ohms. As there are of the order of 750 clearing electrode-pumping unit insertions in each ring, the measurement precision corresponds to about $\pm j 0.4$ ohms at 1 GHz. As $\text{Im}(Z/n)$ is nearly frequency independent (except for resistive wall) below 1 GHz, this is marginally adequate. The resolution could be improved by measuring several in series.

Hereward^{9,11} has proposed a method of measuring the non-resonant longitudinal impedance of a component by using standing waves in a section of vacuum chamber with standing waves. Specifically, the inductance of the component is

$$L = 2\mu_0 \delta l \frac{Z}{377} \text{ Henrys} \quad (6)$$

where δl is the apparent change in length of the resonant section, Z is the effective impedance of the vacuum chamber where the component is inserted, and $\mu_0 = 4\pi \times 10^{-7}$ Hy/m. Z becomes the characteristic impedance if the component is midway between a current mode and a voltage mode. The principle is essentially that the propagation velocity is proportional to $(1/LC)^{1/2}$ where L and C are the inductance and capacitance per unit length. If there is either excess capacitance or inductance at any point, the wave must slow down. For example, if one wanted to measure a 20 cm long component to $\pm 10\%$, the phase angle change over the unit should be about 6° or less, leading to a total length of about 300 cm for the test setup. Its quarter wave resonant frequency would be about 25 MHz. The impedance of a 200 ohm line open at both ends could be measured at the $\lambda/4$ resonance with a rf vector impedance meter. If the frequency of resonance could be determined for a phase angle of $0^\circ \pm 1^\circ$, and the Q were ~ 100 , then δf could be determined to about 1 part in $100 \times 57.3 \approx 1$ part in 6000. This leads to a δl of ± 0.5 mm, which corresponds to an inductance of $\delta L = \pm 0.7$ nH. The apparent improvement in resolution of this method over the method using the network analyzer is due entirely to the fact that one is measuring the phase angle of a high Q resonance rather than of a non resonant system, giving an improvement of the order of Q in the resolution. Although such a technique seems to have several advantages, it should only be used after a thorough review of the system using a scattering matrix analysis. Another possibility is to apply pulses to the line through a directional coupler, and observe the reflected pulses at the coupled part output. Measurement of time delay (a δl of ± 0.5 mm corresponds to a δt of ± 3.3 picosec) would yield the inductance.

Measurement of the transverse impedance of a beam line component is also important when its impedance is not easily calculable. Although

terms like resistive wall and space charge are easily calculable in circular components, other ones are not (e.g. a kicker magnet) and therefore need to be measured. Nassibian and Sacherer¹⁰ show that the transverse impedance is measurable by inserting a coil of length ℓ and width w into the component, and measuring the change in loop impedance $\delta Z = \delta R + i\omega \delta L$. Then

$$Z_T = \frac{c}{\omega} \frac{\delta Z}{2} \text{ ohms/meter} \quad (7)$$

for a length ℓ of the structure. As an example they use this method to measure a 84 mm diameter stainless pipe with a 30 cm long and 3.2 cm wide coil, using an excitation frequency of 32 MHz. They find $\delta A = (0.0154 \pm 0.003) - i(5.99 \pm 0.06)$ ohms which leads to $Z_T = (75 \pm 25\%) - i(29.1k \pm 2\%)$ ohms/m to be compared to a calculated value of $Z_T = 72 - i34k$ ohms/m for (resistive wall) $-i$ (space charge). Although the space charge term dominates $Im Z_T$, it might still be of interest to consider measuring longitudinal impedance by this method if the n dependence of $(Z/n)_\parallel$ were known. For example the $(Z/n)_\parallel$ for resistive wall is known to have a $(1+2\kappa^2/b^2)$ dependence,¹⁰ which leads to

$$\left(\frac{Z}{n}\right)_\parallel = \frac{\omega b^2}{c} \cdot Z_r \text{ ohms (resistive wall)} (\kappa=0). \quad (8)$$

In any case, the measurement would have to be corrected for the change in Z_T due to space charge ($Z_T = 377\ell/2\pi b^2$ ohms/m for a length ℓ of beam pipe of radius b). Also, certain components (e.g. kicker magnets) may have a large Z_T due to an off-axis $(Z/n)_\parallel$, but would not be seen in the on-axis $(Z/n)_\parallel$ measurements. Hence Z_T measurements would be useful in estimating off axis $(Z/n)_\parallel$. In addition, use of a sweep frequency generator in such a configuration would show up resonant Z_T and off-axis $(Z/n)_\parallel$ impedances which may not be observable in the on axis $(Z/n)_\parallel$ measurements.

Although a discussion of precise measurement of parameters of high Q resonant cavities is outside the scope of this paper, it is worthwhile to direct the reader to the paper by P. Bramham¹¹ which reviews this subject comprehensively.

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A CARBON FILAMENT BEAM PROFILE MONITOR
FOR HIGH ENERGY PROTON-ANTIPROTON STORAGE RINGS

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I. INTRODUCTION

The measurement of the evolution of the transverse profile of the stored beams in high energy proton storage rings such as the $p\text{-}\bar{p}$ colliders at CERN and at FNAL is of considerable importance. In the present note, a simple monitor is discussed which will allow almost non-destructive measurement of the profile of each individual proton and antiproton bunch separately. It is based on the "flying wire" technique first used at C.E.A.¹ and more recently at the C.P.S.². A fine carbon filament is passed quickly through the beam, acting as a target for secondary particle production. The flux of secondary particles is measured by two scintillator telescopes, one for protons and one for antiprotons, having an angular acceptance between 30 and 100 mrad. Measurements of secondary particle production performed at FNAL in this angular range show that a very respectable flux can be expected.

II. GENERAL CONSIDERATIONS

In the SPS proton-antiproton collider the protons and antiprotons will be stored in two counter-rotating beams at 270 GeV/c. Each beam consists of 6 bunches of 10^{11} particles per bunch. The wire diameter should be sufficient to give a reasonable flux in the detector in the tails of the beam. The displacement speed should be such that the wire moves at least by one diameter during one revolution period of the machine i.e. $v = d \cdot f_{\text{rev}}$, where d is the wire diameter. On average, each proton then traverses the material only once. The emittance blow-up and particle loss should be small and therefore a low Z material is required. In addition, the thermal properties should be compatible with the heating and thermal shock experienced during traversal of the beam. All of these conditions are satisfied by carbon filaments, and they have already been used in the more hostile environments of the CPS² and the FNAL main ring.³

The following parameters are assumed.

Beam ⁴	
Maximum Intensity	1.2×10^{12} particles
Energy	270 GeV/c
$E_{H\bar{p}}$ } proton emittances	$6.9 \times 10^{-8} \pi \text{ rad m}$
E_{vp} } at 270 GeV/c	$3.5 \times 10^{-8} \pi \text{ rad m}$
$E_{H\bar{p}}$ } antiproton emittances	$3.8 \times 10^{-8} \pi \text{ rad m}$
$E_{v\bar{p}}$ } at 270 GeV/c	$1.9 \times 10^{-8} \pi \text{ rad m}$
θ_H } betatron function	50 m
θ_v } at wire location	50 m

*CERN

†FNAL

3.6×10^{-5} grams. Assuming no conduction or radiation loss during the passage through the beam, the temperature rise is

$$\Delta T = \frac{6.3 \times 10^{-3} \text{ joules}}{3.6 \times 10^{-5} \text{ grams}} \times \frac{1 \text{ calorie}}{4.18 \text{ joules}} \times \frac{1 \text{ gram}}{0.24 \text{ calories}} = 174 \text{ }^{\circ}\text{C}.$$

V. PARTICLES LOST

The probability that a particle will be lost in traversing the filament is

$$40 \times 10^{-27} \frac{\text{cm}^2}{\text{nucleon}} \times \frac{\pi d}{4} \text{ cms} \times 2.3 \frac{\text{grams}}{\text{cm}^3} \times 6.02 \times 10^{23} \frac{\text{nucleons}}{\text{grams}}$$

$$\sim 4.4 \times 10^{-4}$$

i.e. 0.04% per scan.

VI. DETECTOR

Measurements at 400 GeV/c with a similar target at FNAL³ in the range 350 to 1700 mrad have shown that for a single counter (veto counter in Fig. 3 of the reference), the differential cross section per nucleon can be fitted by

$$\frac{d\sigma}{d\Omega} = \frac{67}{(0.1 + \theta)^2} \text{ millibarns per nucleon per steradian.}$$

Thus, if we have an annular detector around the beam pipe with an angular acceptance between 30 and 100 mrad, the cross section would be

$$2\pi \int_{0.03}^{0.1} \frac{67}{(0.1 + \theta)^2} \sin\theta \, d\theta \approx 67.2 \text{ mb/nucleon.}$$

For a gaussian beam distribution, the number of particles per bunch passing through the wire a distance x from the beam center is

$$N(x) = N_0 \frac{2\sqrt{2d}}{\sqrt{\epsilon\beta}} e^{-\frac{2\pi x^2}{\epsilon\beta}}.$$

where N_0 is the total number of particles per bunch. The number of counts in the detector would then be

$$N_c(x) = \frac{N_0}{\sqrt{\epsilon\beta}} \cdot \frac{\pi \sigma d^2}{\sqrt{2}} \cdot A \cdot e^{-\frac{2\pi x^2}{\epsilon\beta}}$$

where $A = 1.38 \times 10^{24} \text{ nucleons/cm}^3$

$\sigma = 67.2 \times 10^{-27} \text{ cm}^2/\text{nucleon.}$

Wire

Material	Carbon Filament
Diameter	0.1 mm
Density	2.3 grams/cc
Displacement Speed	4.3 m/sec
Radiation Length	42.7 grams/cm ²
dE/dx	1.78 MeV cm ² /gram
Total Cross Section	40 millibarns/nucleon
Heat Capacity	0.42 Cal/gram °C at 400 °K.

III. EMITTANCE BLOW-UP

The average thickness of the wire of diameter d is $\pi d/4$. The average rms. projected scattering angle due to multiple coulomb scattering is

$$\delta\theta = \frac{0.015}{270} \left(\frac{\pi d}{4} \times \frac{2.3 \text{ grams/cm}^3}{42.7 \text{ grams/cm}^2} \right)^{\frac{1}{2}}$$

$$= 1.1 \mu\text{rad for } d = 0.1 \text{ mm.}$$

The emittance blow-up is

$$\frac{\delta C}{\epsilon} = \frac{2\pi \beta \delta\theta^2}{\epsilon}$$

If the device is placed somewhere near the center of a lattice half-period where $\beta_H = \beta_V \sim 50$, then for the smallest foreseen emittance (antiprotons in the vertical plane) we have

$$\frac{\delta\theta}{\epsilon} = 0.6\%$$

IV. HEATING

The total energy deposited in the wire is

$$1.2 \times 10^{12} \text{ protons} \times 1.78 \frac{\text{MeV} \cdot \text{cm}^2}{\text{gram}} \times 2.3 \frac{\text{grams}}{\text{cm}^2} \times \frac{\pi d}{4} \text{ cm} \times 1.6 \times 10^{-13} \frac{\text{joules}}{\text{MeV}} = 6.3 \times 10^{-3} \text{ joules}$$

Assuming that all bunches have the vertical emittance of the antiprotons, the vertical beam size is about 2 mm at the position of the wire. The mass of material into which the heat is deposited is then

For $\sqrt{s} = 1.65$ cm (protons with a factor of 5 blow-up) this would give 1.3×10^6 particles through the detector in the beam center and 1.7×10^5 at 2 standard deviations for 10^{11} protons per bunch. This flux is too low for secondary emission detectors and marginal for an ion chamber, which also has clearing problems. Therefore we propose a simple scintillator telescope (Fig. 1.). Due to the strong forward peaking of particle production, two separate telescopes can be used (one for protons and one for antiprotons).

Fig. 1. Schematic diagram of double scintillator telescope.

Each bunch produces a burst of secondary particles of less than 5 nanoseconds duration. The integrated pulse height from the photo multiplier anode would be digitised and stored in a memory in the time interval between bunches ($\sim 3.8 \mu\text{sec}$). In this way the profile of each individual bunch can be built up.

VII. CONCLUSION

The carbon filament beam scanner linked to a scintillator telescope offers a simple, almost non-destructive method of measuring the profile of each individual bunch in the SPS $p-\bar{p}$ collider. The estimated particle flux in the telescope is sufficient to provide a spatial resolution of better than 0.1 mm well into the tails of the beam.

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LONGITUDINAL EFFECTS OF NEAR-UNIFORM BEAM-WALL WAKE FIELDS

A. G. Ruggiero* and Richard Talmann†

I. INTRODUCTION

Several theories have been developed in the past to explain longitudinal instabilities of individual bunches in particle accelerators and storage rings.¹ But they are less than satisfactory because in one way or another they rely on some mathematical approximation of doubtful physical meaning. Here we attempt to simplify the basic physical model in order to help to understand a little better the physics involved. Initially we shall investigate the motion of two particles executing phase oscillations in the same bunch under the influence of the wake field generated by the other. We shall show that this motion is stable and bounded even for the case of a slowly varying wake field. But the distortions of the trajectories can, nevertheless, be significant and can be interpreted as bunch lengthening and widening.

We then generalize to an N particle model. If the wake field is constant, the equations are simple enough that it is possible to solve exactly for the intensity-limited, self-consistent, longitudinal particle distribution. It is not necessary to solve, or even introduce, the Vlasov equation.

These models exhibit effects like dipole and quadrupole oscillations and bunch lengthening, in qualitative agreement with observed behavior, but no attempt has been made at quantitative comparison with observation.

II. THE TWO PARTICLE MODEL. UNIFORM WAKE

In this model we shall represent the bunch by two super-particles which would oscillate harmonically about a common center in the absence of interactions. Each particle produces a wake field which acts only to the back relative to its direction of motion. When a particle is leading it is not affected by the wake field of the other. After half a phase oscillation or so, the situation reverses: the two particles exchange position and coupling. This model is like the head-tail model of transverse beam-wall instabilities developed some time ago.² There is, though, a crucial difference. In the transverse case the wake field experienced by the trailing particle is proportional to two quantities: (i) a wake form which depends on their relative separation and (ii) the displacement of the leading particle; evaluated, though, at the earlier time when it was crossing the impedance region. It is the time lag in the second contribution which leads eventually to instability. In our case, only the first feature is present and hence there is no time lag and, as we shall see, no instability; that is, the system does not exhibit exponential growth of oscillations.

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Our system is described by the following equations of motion for particles "1" and "2".

$$\begin{aligned}\ddot{\varphi}_1 + \omega^2 \varphi_1 &= \lambda H(\varphi_2 - \varphi_1) \\ \ddot{\varphi}_2 + \omega^2 \varphi_2 &= \lambda H(\varphi_1 - \varphi_2)\end{aligned}\tag{1}$$

where λ is a coupling constant which depends on the impedance, the particle current, and other parameters. $H(\varphi)$ is the wake field form which depends on the frequency dependence of the wall impedance. Actually it is the associated Green's function. We shall make the reasonable assumption that the range of the wake field is short enough that "bunch-to-bunch" effects can be ignored. The other symbols in (1), φ and ω , have the meaning of phase displacement and unperturbed oscillation frequency due to "external" cause. Equation (1) should be interpreted as a "smooth approximation" where both external focussing forces and induced coupling forces are uniformly spread around the main closed orbit.

First consider the case $H(\varphi) = U(\varphi)$ where

$$\begin{aligned}U(\varphi) &= 0 \text{ for } \varphi < 0 \\ &= 1 \text{ for } \varphi > 0\end{aligned}\tag{2}$$

The wake field is constant and acts only to the back. Define new coordinates

$$S = \frac{\varphi_1 + \varphi_2}{2} \text{ and } \Delta = \frac{\varphi_1 - \varphi_2}{2}\tag{3}$$

in terms of which the equations of motion are

$$\ddot{S} + \omega^2 S = \frac{\lambda}{2}\tag{4a}$$

$$\ddot{\Delta} + \omega^2 \Delta = -\frac{\lambda}{2} \text{sign}(\Delta)\tag{4b}$$

The solution of the first equation can be written

$$S = \hat{S} \cos \omega t + S_0\tag{5}$$

where $S_0 = \lambda/(2\omega^2)$. Depending on the initial conditions, the "dipole" amplitude S can be as large as S_0 . This oscillation of the "center-of-mass" can be damped in an actual storage ring with an external phase loop. Similarly, the phase shift S_0 can be compensated by readjusting the synchronous phase of the rf cavities. These changes amount mathematically to adding the same correction factor to the rhs of both equations (1). This will result in the addition of the same factor to (4a), but will not alter (4b). In this way one achieves $S = 0$ and is left with Eq. (4b).

The motion of the relative phase coordinate Δ can be easily represented graphically in the normalized phase space with coordinates Δ and $\dot{\Delta}/\omega$, as is shown in Fig. 1 and Fig. 2. These figures illustrate the subsequent motion when the perturbation is turned on suddenly at $t = 0$.

Fig. 1. Trajectories of phase difference Δ for a two particle bunch $S_0 > 0$.

The two figures differ in the sign of S_0 . Let the initial unperturbed amplitude be

$$\Delta = \hat{\Delta} \cos(\omega t + \alpha).$$

With the interaction turned on, the trajectory is modified and consists of two circle arcs with centers respectively at $-S_0$ and $+S_0$. The construction is easy: one draws first the unperturbed circle (shown broken) with radius $\hat{\Delta}$. Point P represents the location at $t = 0$. The perturbed motion follows the heavy circular arcs. Also shown are the extreme trajectories for this value of $\hat{\Delta}$. One can observe in Fig. 1 that for $S_0 > 0$ the new trajectory has a smaller width $2\Delta_1$ and a larger height $2\Delta_2$ than the original diameter $2\hat{\Delta}$. With some simple geometrical relations one obtains (for $S_0 > 0$)

$$\Delta_1 = \sqrt{\hat{\Delta}^2 + S_0^2 + 2 S_0 \hat{\Delta} \cos \alpha} - S_0 \quad (6a)$$

$$\Delta_2 = \sqrt{\hat{\Delta}^2 + 2 S_0 \hat{\Delta} \cos \alpha} \quad (6b)$$

$2\Delta_1$ can be thought of as representing some kind of "bunch length" and $2\Delta_2$ the "bunch height" or "energy spread". For $S_0 > 0$ then, one observes "bunch shortening" accompanied by an equivalent "bunch widening"

Fig. 2. Trajectories of Δ for a two particle bunch $S_0 > 0$. Various relative values of $\hat{\Delta}$ and S_0 are shown.

with the new values given by (6a) and (6b).

This result is reminiscent of the theory of the negative mass instability³ which, above transition, predicts shortening of the bunch; though here no exponential growths are involved. For the negative mass instability, the condition $S_0 > 0$ means that the energy of the particles is above the transition energy provided that the impedance responsible for the coupling is "capacitive". In fact, λ is given by the product of the impedance and $\gamma^{-2} = \gamma_T^{-2}$. But the impedance can have either sign.

The case $S_0 < 0$ is illustrated in Fig. 2, for various values of $\hat{\Delta}$. Except in the top figure only the extreme trajectories are shown. There is now bunch lengthening. For $|S_0| \ll \hat{\Delta}$, Δ_1 and Δ_2 are given by

$$\Delta_2 = \sqrt{\hat{\Delta}^2 + S_0^2 - 2\hat{\Delta}|S_0|\cos\alpha} \quad (7a)$$

$$\Delta_1 = \Delta_2 + |S_0|. \quad (7b)$$

For larger values of $|S_0|$ a new type of behavior occurs in which the stable trajectories degenerate into circles around one or the other of the centers at $\pm S_0$. We will return to this later.

III. SLOWLY VARYING WAKE FIELDS

Several people have speculated⁴ that an instability could arise if the wake field is varying appreciably over the length of the bunch. To check this we choose the following wake form to substitute in the r.h. side of Eq. (1).

$$\begin{aligned} H(\varphi) &= e^{-\Gamma\varphi} U(\varphi) \\ &\approx (1 - \Gamma\varphi)U(\varphi) \end{aligned} \quad (8)$$

where the linear approximation will be valid if the variation over the bunch is modest. The form (8) could correspond to an oscillator impedance still with frequency small enough relative to the bunch length, but with a sufficiently low figure of merit Q that the decay can not be neglected. It represents well the response of conducting plates⁵ like clearing electrodes, pick-ups, etc.

In terms of the variables S and Δ , and in the linear approximation, the equations of motion become

$$\ddot{S} + \omega^2 S = \frac{\lambda}{2} - \frac{\lambda}{2} \Gamma |\Delta| \quad (9)$$

$$\ddot{\Delta} + \omega^2 \Delta = -\frac{\lambda}{2} \text{sign}(\Delta) + \frac{\lambda\Gamma\Delta}{2}. \quad (10)$$

The second equation is simpler and yields the result that the trajectories for Δ are as before except that the oscillation frequency around the temporary centers at $\pm S_0$ are altered to become

$$\omega_{\Delta} = \sqrt{\omega^2 - \lambda\Gamma}. \quad (11)$$

Let T be the period of oscillation of the Δ motion. The motion of Δ (which fortunately is independent of S) has the effect of applying a driving force $(-\lambda/2)\Gamma|\Delta|$ on the r.h. side of (9), the equation of motion for S . This threatens instability if the driving force has a Fourier component oscillating at frequency ω . But if Δ alternates in sign (as it does except in the cases at the bottom of Fig. 2) then the

Fourier frequencies of $|\Delta|$ are doubled. That is, they are multiples of $4\pi/T$ rather than $2\pi/T$. But T can never be greater than $4\pi/\omega$, which, for $S_0 < 0$, because of the negative sign in (11), assures that the fundamental is above ω . For $S_0 < 0$, it is possible for Δ to maintain the same sign. This avoids the frequency doubling but in this case, again by (11), the fundamental is above ω . The only remaining case where instability might occur is that of Fig. 1. But when the linear version of (8) is valid the decrease in T due to the alternating location of the center of rotation, outweighs the increase in T due to (11). Hence the motion is stable in all cases.

IV. SUMMARY OF QUALITATIVE FEATURES

We can now enumerate some features of these simple models. Most of these features will remain valid for many particles.

1. The "center-of-mass" coordinate S executes dipole oscillations at or near the natural frequency ω but relative to a displaced center.

2. There are "quadrupole" oscillations described by Eq. 4(b). It is possible to describe this motion in terms of an effective potential energy function $V(\Delta)$ which is distorted by the perturbation.⁶ $V(\Delta)$ is given by

$$V(\Delta) = \frac{1}{2} \omega^2 \Delta^2 + \frac{\lambda}{2} |\Delta| \quad (12)$$

and is shown in Fig. 3 for $\lambda < 0$.

Fig. 3. The effective potential energy function, $V_{\text{eff}}(\Delta)$ describing motion of a two particle bunch.

3. Because of the two minima there is the possibility of particles becoming trapped, so that, for example, particle 1 stays in the front half of the bunch.

4. For large amplitudes the oscillations of Δ are stable as is obvious from Fig. 3. The motion remains stable if the wake field varies slowly or if ω is allowed to depend on the oscillation

amplitude. If the machine exhibits longitudinal chromaticity (i.e., ω depends on $\dot{\phi}$) then instability can result. This has been called the longitudinal head-tail effect and analyzed by Hereward.⁷

5. A feature which may not have been emphasized before is that for motion described by a potential like (12) there is the possibility of irreversible increase of emittance but without exponential growth. That is, a particle injected near the origin in phase space will be accelerated irreversibly out to large radius in phase space and will then oscillate stably. This behavior results because of sufficiently small amplitude, the "defocussing" term on the right hand side of the equation dominates over the "focussing" term on the left side. This is the mechanism which lets a too dense bunch relax into a longer and less dense bunch within which the particles then oscillate stably.

V. MANY PARTICLE MODEL

To treat the bunch as N different particles redefine λ so that the acceleration of charge k due to charge j is

$$\lambda U_{jk} = C \left(\frac{N}{M} \right) \left(\frac{Q}{N} \right)^2 U_{jk} \quad j, k = 1, \dots, N.$$

Here C is a constant, Q is the total bunch charge and M is the total bunch mass. U_{jk} is an abbreviation for the step function $U(\phi_j - \phi_k)$ where ϕ_j is the phase coordinate of the j'th particle. For the case of greatest importance, λ is negative and that case will be discussed exclusively in what follows. The equations of motion are for $k = 1, 2, \dots, N$.

$$\ddot{\phi}_k + \omega^2 \phi_k = \lambda \sum_{j \neq k} U_{jk} \quad (13)$$

Define new coordinates

$$S = \frac{1}{N} \sum_{j=1}^N \phi_j \quad (14)$$

$$\Delta_i = \phi_i - \frac{1}{N-1} \sum_{j \neq i}^N \phi_j = \frac{N}{N-1} (\phi_i - S)$$

To simplify the discussion we will focus the discussion on a single particle, say the first, though any other is equivalent. In particular

$$\Delta_1 = \phi_1 - \frac{1}{N-1} \left(\sum_{j=2}^N \phi_j \right) \quad (15)$$

We will eventually let N be large so that $N-1$ could be replaced by N . But at least initially we will write equations valid for all N .

The equation of motion for S is

$$\ddot{S} + \omega^2 S = \frac{\lambda}{2} (N-1) \quad (16)$$

just like (4a). The perturbation term for Δ_1 depends on the magnitude of φ_1 relative to all the other φ_j . Suppose that at a certain instant the particles happen to have the order $\varphi_1 < \varphi_2 < \dots \varphi_N$. The equation for Δ_1 valid at that instant is

$$\ddot{\Delta}_1 + \omega^2 \Delta_1 = \lambda \left[N-1 - \frac{1}{N-1} \sum_{k=2}^N (U_{1k} + \sum_{\substack{j=2 \\ j \neq k}}^N U_{jk}) \right] = \frac{\lambda}{2} N.$$

Furthermore, permuting $\varphi_2, \varphi_3 \dots \varphi_N$ does not change the equation. But suppose that particle 1 is moved forward past J particles so that

$$\varphi_2 < \varphi_3 \dots \varphi_{J+1} < \varphi_1 < \dots \varphi_N.$$

The equation is then

$$\ddot{\Delta}_1 + \omega^2 \Delta_1 = \frac{\lambda N}{2} \left[1 - \frac{2J}{N-1} \right]. \quad (17)$$

Clearly all that matters is the number of particles in front of particle 1.

In passing, let us call attention to a surprising feature of this equation. Δ_1 suffers a space charge acceleration even when particle 1 is at the front of the bunch. In ordinary mechanics, due to the large inertia of particles 2 through N , Δ_1 would be about equal to $\dot{\varphi}_1$ which is zero when particle 1 is at the front of the bunch. But in this theory Newton's third law does not hold. Particle 1 is able to accelerate any number of trailing particles without being subject to a reaction force. The surprising consequence of this is that in the description of the motion relative to the "center-of-mass" the one-directional nature of the wake loses its importance. For example, a particle at the front is repelled forward away from the center of the bunch much as it would be by Coulomb repulsion at low velocities. Of course, this is just the reflection of the rest of the bunch being repelled backwards.

We will now switch to a statistical representation of the particle positions. Let $Np(\Delta)d\Delta$ be the number of particles in the range $d\Delta$ at Δ . $p(\Delta)$ is a normalized probability distribution. Define a cumulative probability distribution

$$P(\Delta) = \int_{-\infty}^{\Delta} p(\Delta) d\Delta. \quad (18)$$

If particle 1 is in front of J particles then

$$J = NP(\Delta_1) . \quad (19)$$

Substituting this in (17) we get

$$\ddot{\Delta} + \omega^2 \Delta = \frac{\lambda N}{2} [1 - 2P(\Delta)] \quad (20)$$

where we have assumed N large and have dropped the subscript 1 as this equation applies to any particle. If the bunch is not yet in equilibrium then $P(\Delta)$ evolves in time.

The simplest possible bunch shape is uniform with length $2L$.

$$p(\Delta) = \begin{cases} 0 & |\Delta| > L \\ \frac{1}{2L} & |\Delta| < L \end{cases} \quad (21)$$

$$P(\Delta) = \begin{cases} 0 & \Delta < -L \\ \frac{1}{2} + \frac{\Delta}{2L} & |\Delta| < L \\ 1 & L < \Delta \end{cases} \quad (22)$$

Substituting in (20) we get

$$\ddot{\Delta} + \omega^2 \Delta = \begin{cases} -\frac{\lambda N}{2} \frac{\Delta}{L} & |\Delta| < L \\ -\frac{\lambda N}{2} \frac{\Delta}{|\Delta|} & |\Delta| > L \end{cases} \quad (23)$$

The effective potential energy corresponding to this equation is

$$V(\Delta) = \begin{cases} \frac{1}{2} \left(\omega^2 + \frac{\lambda N}{2L} \right) \Delta^2 & |\Delta| < L \\ \frac{1}{2} \omega^2 \Delta^2 - \frac{1}{4} \lambda N (L - 2|\Delta|) & |\Delta| > L \end{cases} \quad (24)$$

For sufficiently small the motion is simple harmonic with frequency

$$\omega' = \left[\omega^2 + \frac{\lambda N}{2L} \right]^{1/2} \quad (25)$$

The effect of the wake field (still assuming $\lambda < 0$) is to reduce the frequency. But there is a maximum particle density beyond which ω' becomes complex and the motion becomes unstable. It is

$$\left(\frac{N}{L}\right)_{\text{Max.}} = \frac{2\omega^2}{-\lambda} \quad (26)$$

This could also be expressed as a minimum bunch length for N particles

$$L_{\text{Min}} = \frac{-N\lambda}{2\omega^2} = -\frac{C Q^2}{2\omega^2 M} \quad (27)$$

We have the good fortune that, if the bunch distribution is uniform and with the maximum particle density, then all oscillation frequencies are zero; all particles stay in fixed places in the bunch. Hence $P(\Delta)$ does not evolve in time and a self-consistent equilibrium distribution has been found. It is the intensity limited bunch shape. The effective potential $V_{\text{Lim}}(\Delta)$ for this limiting case is shown in Fig. 4.

Fig. 4. $V_{\text{eff}}(\Delta)$ for a current limited bunch.

SUMMARY

While the models which have been described are probably too oversimplified for direct comparison with observation, we hope and believe that the qualitative features are correct.

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A REQUIREMENT ON AVERAGE VALUES OF SPACE DOMAIN WAKE FIELDS
WITH AN EXAMPLE APPLICABLE TO THE CORNELL SYNCHROTRON

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INTRODUCTION

A point charge travelling off-center through a vacuum chamber leaves behind wake fields which give transverse forces on following particles. It is argued here that the magnetic portion of this force, when averaged over the spatial coordinate along the flight path, should give zero. A similar requirement applies to longitudinal electric forces but the emphasis in this paper is on transverse forces.

The argument about average magnetic fields is simple. An infinite line current passing through the same chamber can be thought of as the superposition of many moving point charges. Since the vacuum chamber is magnetically inert ($\mu_r = 1$) there is no "image" magnetic field due to the chamber. This implies that the magnetic field due to the presence of the chamber walls must average to zero along the flight path.

By a similar argument the space averaged longitudinal electric field can (in simple geometries) be argued to be zero. Note that neither of these principles applies to the average forces acting on the particle itself. For example, there is clearly a net longitudinal electric force slowing down the particle to compensate for radiated energy.

For realistic vacuum chambers, mathematical complications make it hard to calculate the wake field very close to the particle. Magnetic wake fields which have been calculated¹ are valid only well behind the particle and they do not satisfy the above averaging principle. This suggests there is a sign reversal in the magnetic field very close to the charge which allows the "average" requirement to be met.

To investigate these questions, a mathematically tractable (though physically somewhat unrealistic) example has been worked out and shown to exhibit the claimed behavior. The example consists of a point charge travelling in vacuum parallel to a plane vacuum chamber wall. The chamber wall can have complex dielectric constant (i.e., conductivity) but only the case of pure dielectric is worked out here in detail.

While the example is not directly applicable in most cases as it does not account for resistive wall impedance, the formulae can presumably be applied in general using numerical integration. But a real purpose of this paper is to call attention to the complicated situation close to the moving charge. Since, in modern storage rings, forces within a bunch are becoming increasingly important and bunches can have lengths comparable to the distance to the wall, it is important to solve this problem with more realistic geometry.

The other motivation for working out this example is that it is directly applicable to the Cornell synchrotron. In that machine the vacuum chamber wall is made of laminations as shown in Fig. 1(b). These laminations defeat longitudinal currents and hence make the wall behave as if it

had low conductivity but very high dielectric constant. This machine exhibits beam wall instability at very low currents, and this paper represents a first step at understanding this intensity limitation.

II. DESCRIPTION OF THE METHOD

The situation is illustrated in Fig. 1. The charge q is travelling in medium 1 in the x direction a distance a above medium 2. This problem has been worked out by numerous authors.² There is Cerenkov radiation emitted in medium 2, while in medium 1 there is no radiation but there is a superposition of surface waves decaying exponentially with distance z above the plane. I have used a different method which permits the solution to be worked out in closed form. This uses the plane wave spectrum of electromagnetic fields as described by Clemmow.³ This method closely resembles the more standard Fourier transform method, but the integral representations are somewhat different.

Fig. 1. Definition of coordinates.

The idea is that the free space fields of the moving charge are written as a superposition of plane waves (TM and TE). These waves are reflected at the medium with reflection coefficients none other than the Fresnel coefficients of physical optics. Of course these coefficients result from matching the boundary conditions for the incident, reflected, and refracted waves. For a pure dielectric (no dispersion), the Fresnel coefficients are independent of frequency and that simplifies

the situation enormously. In particular, it permits the inverse ω, t transform to be evaluated explicitly. This finally gives the spatial electric and magnetic fields as an angular superposition, where the angle is relative to the flight direction and in a plane parallel to the surface.

III. MATHEMATICAL SOLUTION

I follow Clemmow³ very closely and hence will give little description of the method and only very loose definition of the symbols. Also, we assume $\eta \gg 1, \gamma \gg 1$. (Defined below).

The current (actually, surface current density) is given by

$$\vec{j} = e \vec{v} \delta(x - vt) \delta(y). \quad (1)$$

Its Fourier transform is

$$j_x^\omega = \frac{e}{2\pi} e^{-i \frac{\omega}{v} x} \delta(y) \quad (2)$$

$$j_y^\omega = 0$$

The frequency components of the magnetic and electric fields of the point charge are given (for $z < a$) by

$$\vec{H}^\omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dl \, dm \left[0, -1, \frac{m}{n_1} \right] Q e^{i \frac{\omega}{c} [lx + my + n_1(z-a)]} \quad (3)$$

$$\vec{E}^\omega = Z_o \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dl \, dm \left[\frac{1 - \frac{c^2}{v^2}}{n_1}, \frac{mc}{vn_1} \frac{c}{v} \right] Q e^{i \frac{\omega}{c} [lx + my + n_1(z-a)]} \quad (4)$$

where M.K.S. units are in use.

$$Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \quad (5)$$

$$n_1 = \sqrt{1 - \frac{c^2}{v^2} - m^2} = -i |n_1| \quad (6)$$

$$Q = \frac{-\omega e}{8\pi^2 c} \delta\left(1 + \frac{c}{v}\right) \quad (7)$$

These are angular integrals with $-l$ and $-m$ being direction cosines of the propagation direction relative to the x and y axis respectively. Notice that n_1 is imaginary and the sign has been chosen to give attenuation as $z \rightarrow -\infty$.

The vectors in the square brackets must be decomposed into TM and TE components as these reflect differently. This decomposition is a function of m . (Notice that the l integration can be completed trivially). Define

$$\tilde{H} = \left(0, -1, \frac{m}{n_1} \right) \quad (8)$$

$$\tilde{E} = Z_0 \left(\frac{1 - \frac{c^2}{v^2}}{n_1}, \frac{c}{v} \frac{m}{n_1}, \frac{c}{v} \right).$$

After decomposition we get for the components of special interest to us

$$\begin{aligned} \tilde{H}_y^{TM} &= -\frac{1}{\beta^2} \frac{1}{K^2} \\ \tilde{H}_y^{TE} &= -\frac{m^2}{K^2} \\ \tilde{E}_z^{TM} &= \frac{Z_0}{\beta} \\ \tilde{E}_z^{TE} &= 0 \end{aligned} \quad (9)$$

where

$$K^2 = \frac{c^2}{v^2} + m^2, \quad \beta = \frac{v}{c}, \quad \gamma = (1 - \beta^2)^{-\frac{1}{2}}. \quad (10)$$

The Fourier transform of the reflected wave is now given by

$$(\vec{E}^{\omega}, \vec{H}^{\omega}) = -\frac{e\omega}{8\pi^2 c} \int_{-\infty}^{\infty} (\vec{E}, \vec{H})^{TM, TE} \rho_{TM, TE} e^{-\omega \left[i \frac{x}{v} + i \frac{my}{c} + \frac{|n_1|}{c} (-z-a) \right]} dm \quad (11)$$

where the Fresnel reflection coefficients are

$$\begin{aligned} \rho_{TM} &= \frac{\eta \sin \alpha - \sin \beta}{\eta \sin \alpha + \sin \beta} \\ \rho_{TE} &= \frac{\sin \alpha - \eta \sin \beta}{\sin \alpha + \eta \sin \beta} \end{aligned} \quad (12)$$

$$\eta = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{\frac{\epsilon_r - i\sigma/\omega}{\epsilon_0}} = \text{index of refraction} \quad (13)$$

$$\sin \alpha = n_1 \quad (14)$$

$$\cos \alpha = K$$

α is the angle of incidence (measured from the plane) and β is the angle of refraction, which, according to Snell's law, is given by

$$\cos \alpha = \eta \cos \beta.$$

To adequate accuracy, for $\eta \gg 1$, $\sin \beta \cong 1$.

To proceed we must integrate over m in (11) and then invert the ω transform. If $\rho_{TM,TE}$ are independent of ω , we can perform the ω transform first to give

$$\vec{E}, \vec{H} = - \frac{ec}{16\pi^2 a^2} \text{Re} \int_{-\infty}^{\infty} dm (\vec{E}^W, \vec{H}^W) \rho_{TM,TE} \frac{1}{(|\eta_1| + ix')^2} \quad (15)$$

where

$$x' = \frac{x}{2a\beta}, \quad y = 0, \quad z = a, \quad t = 0. \quad (16)$$

$$E_z^{TM} = \frac{ecZ_0}{16\pi^2 a^2} \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{(1 - \eta^2 |\eta_1|^2)(|\eta_1|^2 - x'^2) + 4x'\eta |\eta_1|^2}{(1 + \eta^2 |\eta_1|^2)(|\eta_1|^2 + x'^2)^2} dm \quad (17)$$

$$H_y^{TM} = - \frac{ec}{16\pi^2 a^2} \int_{-\infty}^{\infty} \frac{(1 - \eta^2 |\eta_1|^2)(|\eta_1|^2 - x'^2) + 4x'\eta |\eta_1|^2}{(1 + \eta^2 |\eta_1|^2)(|\eta_1|^2 + x'^2)^2} \frac{dm}{\frac{c^2}{v^2} + m^2} \quad (18)$$

$$H_y^{TE} = \frac{-ec}{16\pi^2 a^2} \int_{-\infty}^{\infty} \frac{(\eta^2 - |\eta_1|^2)(|\eta_1|^2 - x'^2) + 4\eta x' |\eta_1|^2}{(\eta^2 + |\eta_1|^2)(|\eta_1|^2 + x'^2)^2} \frac{m^2 dm}{\frac{c^2}{v^2} + m^2} \quad (19)$$

These integrands have been broken into partial fractions and integrated explicitly.

IV. RESULTS

To express the results it is convenient to define a constant E_0

which is the electric field at a distance $2a$ from a charge e at rest

$$E_0 = \frac{e}{4\pi \epsilon_0 (2a)^2} \quad (20)$$

The electric field due to an "image charge" $-e$ at $z = -a$ would be

$$E_z = -E_0 \frac{1/\gamma^2}{(1/\gamma^2 + x'^2)^{3/2}} \quad (21)$$

E_z is $-\gamma E_0$ at $x' = 0$ and falls off for $x' \sim 1/\gamma$ giving the well known foreshortened field. For large η , (17) reduces to exactly this. This is shown in Fig. 2.⁴ Notice also, for small x' , that the magnetic force approximately cancels the electric force. But, at large x' there is an opposite sign magnetic field which allows the magnetic field to average to zero.

A suspicious aspect of Fig. 2 is that the fields are symmetric for x' positive or negative. (The wake field precedes as much as it follows). This is an artifact of our assumption that $\eta \gg \gamma$ which allows the fields to precede the charge. For various values of η and γ the electric field is shown in Fig. 3. For the highly relativistic case, $\gamma = 1000$, $\eta = 100$ it can be seen that there is indeed a wake field. Note also the sign reversal very close to the charge.

Fig. 2. Electric and magnetic "wake" fields for $\eta \gg \gamma = 10$.

Fig. 3. Electric wave fields for various values of η and γ

V. TENTATIVE CONCLUSIONS

Some features of the calculation seem to me likely to be at least qualitatively valid in general as well as in the restricted case which has been worked out. Very close to the particle there is a Lorentz contracted image electric field of order γE_0 which falls off in a distance of order a/γ . This is approximately cancelled by a correspondingly Lorentz contracted magnetic field. Then follows a magnetic wake field stretching over a distance which depends on the chamber material. If the bunch length or structure of particles in the storage ring is long compared to the wake field, (in a coasting beam for example) then the curious situation holds that the force is dominated by the magnetic wake but is attractive towards the wall. If the bunch length is very short compared to the wake length, then the electric and magnetic forces cancel approximately. When the bunch length is comparable to the length of the wake there is a strong variation in the transverse force in going from the front to the back of the bunch. Such a force is known to lead to head-tail instability at sufficiently high current.

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LONGITUDINAL STABILITY OF A BUNCHED BEAM WITH RECTANGULAR SHAPE

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I. INTRODUCTION

The problem of the longitudinal instability of individual bunches is made difficult to investigate theoretically by the "incommensurability" between the electrodynamics aspects of the problem, which is "linear" involving only impedances and currents, and the beam dynamics side, which is "circular" because of the phase oscillations around the center of the bunch. As a consequence all the theories proposed¹ have to deal with some sort of compromise or approximations, some of them of quite unique physical meaning.

To go around the "incommensurability" aspect of the problem we discuss in this paper the model of "rectangular" bunches, where the particles are located in an external rectangular potential well² (instead of sinusoidal). Therefore the particles will execute "free" motion except at the edges of the bunch where they are rigidly reflected. The motion of the beam will look like a chopped coasting beam which will make the dynamics and the electrodynamics more "colinear" (or less "incommensurable"). Obviously though, one has to take into account very carefully the effect of the bunch edges.

This model has also some practical applications because it would describe the real case where the bunch motion is affected by some "lengthening cavity" (a sort of higher order Landau cavity) to actually diminish the amount of phase oscillations.³ This is believed to cause beam stability and has been successfully experimented in SPEAR.⁴

Our approach is the usual. We first determine a stationary distribution eventually taking into account the modification of the potential well induced by the beam itself. To test the stability of such distribution we then insert a perturbation on top. We chose this in the form of a standing wave oscillating at some frequency ω . The mode number can be defined as the ratio of the bunch length to the wavelength. We impose the condition that the perturbation exhibits knots at the edges of the bunch, because, since there is no charge there, the perturbation cannot be sustained. At closer look, this perturbation can be decomposed as two sets of travelling waves moving in opposite directions and cancelling outside the bunch. Finally assuming that the system is conservative, we write down the Vlasov equation including also the fields produced by the perturbation. We obtain a couple of dispersion relations by linearizing the Vlasov equation, since we are interested only in the stability condition. Our main result is that the motion can be unstable in a fashion very similar to the one peculiar of coasting beams. We believe this result is linked to our choice of perturbation which is really not propagating around the contour of the bunch. This model and result seems to conflict with some remarks by Hereward⁵ when he tried to apply coasting beam theory to bunches.

II. SEARCH FOR A STATIONARY DISTRIBUTION. THE POTENTIAL WELL

Let θ be angle around the closed orbit circumference, t the time and ω_0 the angular revolution frequency. We shall work with the

usual canonical variables θ, φ, w , where

$$\varphi = \theta - \omega_0 t \text{ and } w = E \frac{dE}{\omega(E)}.$$

Let $\varphi = \varphi(\theta, w, t)$ be the most general distribution function in the phase plane of coordinates θ and w . It will satisfy the Vlasov equation

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \dot{\theta} \frac{\partial \psi}{\partial \theta} + \dot{w} \frac{\partial \psi}{\partial w} = 0 \quad (1)$$

By definition a distribution φ_s is stationary when

$$\frac{\partial \psi_s}{\partial t} = 0, \quad \psi_s = \psi_s(\varphi, w)$$

and it satisfies

$$\dot{\varphi} \frac{\partial \psi_s}{\partial \varphi} + \dot{w} \frac{\partial \psi_s}{\partial w} = 0 \quad (2)$$

If the system is conservative there is an Hamiltonian function

$$H = \int \omega(w) dw + e U_{\text{ext}}(\varphi) + e U_{\text{s.c.}}(\varphi, I) \quad (3)$$

from which

$$\begin{aligned} \dot{\theta} &= \omega(w) \\ \dot{w} &= -e \frac{dU_{\text{ext}}}{d\varphi} - e \frac{dU_{\text{s.c.}}}{d\varphi} \end{aligned}$$

where $\omega(w)$ is the angular revolution frequency as function of the canonical momentum variable w .

$U_{\text{ext}}(\varphi)$ is the potential generated by external rf cavities and $U_{\text{s.c.}}(\varphi, I)$ is the potential induced by the beam, which therefore depends on the beam current. Introducing a Green function for the properties of the vacuum chamber and

$$g(\varphi) = \int \psi dw \quad (4)$$

as the bunch azimuthal distribution, we have quite generally

$$U_{\text{s.c.}} = I \int G(\varphi - \varphi') g(\varphi') d\varphi \quad (5)$$

where the Green function G will have therefore the dimension of an impedance and

$$\int g(\varphi) d\varphi = 1$$

If our distribution is stationary, then it must be matched to the induced potential $U_{s.c.}$ which will not change with time. We make here the reasonable assumption that it is possible to adjust the external potential so that the overall potential is "rectangular". Namely

$$U_{ext}(\varphi) + U_{s.c.}(\varphi, I) = U_0(I) R(\varphi) \quad (6)$$

where

$$\begin{aligned} R(\varphi) &= 1 & \text{for } |\varphi| < \Delta, \\ &= 0 & \text{for } |\varphi| > \Delta. \end{aligned} \quad (7)$$

The potential height U_0 and bunch half length Δ , which is also the potential half width, eventually will depend on the beam current I .

III. THE STATIONARY DISTRIBUTION

Let us arbitrarily set

$$\varphi_s = f_0(w) R(\varphi), \quad (8)$$

which is a uniform distribution over the range $-\Delta < \varphi < \Delta$ and some, unknown for the moment, distribution $f_0(w)$ in w . Let us insert (8) in (1) with

$$\dot{\theta} = \omega(w)$$

$$\dot{w} = -e U_0 dR/d\varphi,$$

where $dR/d\varphi$ is the difference of two delta functions

$$\frac{dR}{d\varphi} = \delta(\varphi + \Delta) - \delta(\varphi - \Delta).$$

Also

$$\frac{\partial \psi_0}{\partial t} = -f_0 \omega_0 \frac{dR}{d\varphi}$$

$$\frac{\partial \psi_0}{\partial \theta} = f_0 \frac{dR}{d\varphi}$$

$$\frac{\partial \psi_0}{\partial w} = R \frac{df_0}{dw}$$

Therefore

$$(w - \omega_0) f_0 \frac{dR}{d\varphi} - e U_0 \frac{dR}{d\varphi} R \frac{df_0}{dw} = 0. \quad (9)$$

We remark that $R \, dR/d\varphi = \frac{1}{2} \, dR/d\varphi$, therefore Eq. (9) is identically satisfied if

$$(\omega - \omega_0) f_0 - \frac{1}{2} e U_0 \frac{df_0}{d\omega} = 0,$$

which has solution

$$f_0 = C e^{-\frac{2}{eU_0} \int (\omega_0 - \omega) d\omega} \quad (10)$$

where C is a normalization constant. Below the transition energy $d\omega/d\varphi$ is negative and the distribution (10) corresponds to a Gaussian. But above the transition energy the same quantity is positive and the quantity at the shoulder of the exponent is also positive, and the distribution would increase away from the beam axis. To make this distribution physical one can put a cut on the distribution to the range $|\omega| < \delta$.

IV. THE PERTURBATION

The method to check the stability of the stationary distribution (8) and (10) is to insert a small perturbation and observe whether this is damped or not. Let us write for the total distribution

$$\psi = \psi_s + \psi_p \quad (11)$$

where ψ_s is given by (8) and (10) combined and ψ_p is chosen as a standing wave across the bunch length. We chose

$$\psi_p = [f_1(\omega) e^{in(\theta - \omega_1 t)} - f_2(\omega) e^{-in(\theta - \omega_2 t)}] R(\varphi). \quad (12)$$

The perturbation is zero outside the bunch. In the limit of zero current ($I=0$), the two frequency ω_1 and ω_2 are real and approach the value ω_0 . Also $f_1 = f_2$ in this case. In order for the perturbation to exhibit knots at the edge of the bunch, we can write from (12)

$$\psi_p = f_p(\omega) R(\varphi) \sin(n\varphi), \quad f_1 = f_2 = \frac{f_p}{2i}, \quad (12a)$$

where $n\Delta = \pi$.

To test the stability of the beam we let ω_1 and ω_2 to be different from ω_0 by an amount, the complex frequency shift, which will depend on the beam current and impedance function. In terms of coasting beam theory, the term at the rh side of (12) which travels with the beam is called the "fast" wave the other, moving in the opposite direction, the "slow" wave. In the coasting beam theory one of them would be growing and the other damped, therefore one needs $\omega_1 \neq \omega_2$.

In accordance to (4) and (5) one can also define a perturbed charge distribution

$$g_p = \int \psi_p dw = [\bar{f}_1 e^{in(\theta-\omega_1 t)} - \bar{f}_2 e^{-in(\theta-\omega_2 t)}] R(\varphi), \quad (13)$$

and the potential induced by the perturbation

$$U_p = I \int G(\varphi-\varphi') g_p(\varphi') d\varphi' = I \bar{f}_1 e^{-in\Delta\omega_1 t} \int G(\varphi-\varphi') e^{in\varphi'} R(\varphi') d\varphi' + \\ - I \bar{f}_2 e^{in\Delta\omega_2 t} \int G(\varphi-\varphi') e^{-in\varphi'} R(\varphi') d\varphi' \quad (14)$$

where $\bar{f}_i = \int f_i(w) dw$ and $\Delta\omega_i = \omega_i - \omega_0$.

V. THE VLASOV EQUATION

The total distribution function, Eq. (11), has to satisfy the Vlasov equation (1). We have

$$\frac{\partial \psi_p}{\partial t} + \theta \frac{\partial \psi_p}{\partial \theta} + \dot{w} \frac{\partial \psi_p}{\partial w} + \frac{\partial \psi_s}{\partial t} + \theta \frac{\partial \psi_s}{\partial \theta} + \dot{w} \frac{\partial \psi_s}{\partial w} = 0 \quad (15)$$

where now

$$\dot{w} = -e U_0 \frac{dR}{d\varphi} - e \frac{dU_p}{d\varphi}.$$

We have seen that

$$\frac{\partial \psi_s}{\partial t} + \theta \frac{\partial \psi_s}{\partial \theta} - e U_0 \frac{dR}{d\varphi} \frac{\partial \psi_s}{\partial w} = 0$$

We shall neglect the term $-e (dU_p/d\varphi)(\partial \psi_p/\partial w)$ because of higher order in the limit of small perturbation, and because we are only interested in checking the stability of the beam. Then (15) reduces to

$$\frac{\partial \psi_p}{\partial t} + \theta \frac{\partial \psi_p}{\partial \theta} - e U_0 \frac{dR}{d\varphi} \frac{\partial \psi_p}{\partial w} - e \frac{dU_p}{d\varphi} \frac{\partial \psi_s}{\partial w} = 0 \quad (17)$$

It is interesting to compare this equation to the corresponding one derived for coasting beams.⁵ The two new factors are the third term at the l.h. side of (17) and the last. The third term does not have an equivalent in the coasting beam theory: it is the product of the perturbation by the external potential. For "circular" bunches it is usually a troublesome factor which can be reduced only at cost of canonical transformation to angle-action variables, which will make the last term a more complicate expression. With our model of "rectangular"

bunches its impact is reduced to a minimum: it is zero everywhere except at the edges of the bunch where the beam (including the perturbation) experience a full reflection. The last term at the l.h. side of (17) has an equivalent in the coasting beam theory: it is the product of the unperturbed distribution by the potential induced by the perturbation. The difference is that now φ_s , rather than being constant everywhere, is zero everywhere except within the bunch. Moreover because the perturbation has to be confined within the bunch, the induced potential is the sum of the contributions of many travelling waves instead of only one as in the case of unbunched beam. Yet for the solution of (17) one needs only to know the perturbation potential U_p within the bunch. Then if $|\varphi| < \Delta$, Eq. (14) can be written as

$$U_p = I \bar{f}_1 e^{-in\Delta\omega_1 t} \int_0^\varphi G(\varphi-\varphi') e^{in\varphi'} d\varphi' - I \bar{f}_2 e^{in\Delta\omega_2 t} \int_0^\varphi G(\varphi-\varphi') e^{-in\varphi'} d\varphi'$$

Each of these two terms equals the contribution of a single travelling wave at one mode n and frequency ω by the corresponding impedance Z_n as in the case of unbunched beams. Therefore for $|\varphi| < \Delta$ we can also take

$$\frac{dU_p}{d\varphi} = I [\bar{f}_1 Z_n e^{in(\theta-\omega_1 t)} - \bar{f}_2 Z_{-n} e^{-in(\theta-\omega_2 t)}] \quad (18)$$

where Z_n is the complex impedance calculated at $Z_n = n\omega_0$ and Z_{-n} at $\omega = -n\omega_0$. Also

$$\text{Real } Z_n = \text{Real } Z_{-n}$$

$$\text{Imaginary } Z_n = - \text{Imaginary } Z_{-n}.$$

Equation (18) is the only contribution to the last term at the l.h. side of Eq. (17). Because U_p does not have discontinuities there are no delta functions at the edges of the bunch.

In conclusion the major difference between (17) and the equivalent for unbunched beams is that, aside from the contribution of the third term, two travelling waves with mode numbers n and m are to be taken simultaneously, because this is the only way a perturbation can be sustained on top of the bunch.

VI. THE DISPERSION RELATIONS

From (12) one derives

$$\begin{aligned} \frac{\partial \psi_p}{\partial t} = & [-in\omega_1 f_1 e^{in(\theta-\omega_1 t)} - in\omega_2 f_2 e^{-in(\theta-\omega_2 t)}] R(\varphi) + \\ & - \omega_0 \frac{dR}{d\varphi} [f_1 e^{in(\theta-\omega_1 t)} - f_2 e^{-in(\theta-\omega_2 t)}] \end{aligned} \quad (19)$$

and

$$\begin{aligned} \frac{\partial \psi}{\partial \theta} = & [f_1 e^{in(\theta-\omega_1 t)} - f_2 e^{-in(\theta-\omega_2 t)}] \frac{dR}{d\varphi} + \\ & + in[f_1 e^{in(\theta-\omega_1 t)} + f_2 e^{-in(\theta-\omega_2 t)}] R(\varphi) \end{aligned} \quad (20)$$

When (19) and (20) are inserted in the Vlasov equation (17) with (18), the left hand side of (17) splits in two parts: one is multiplied by R and the other by $dR/d\varphi$. Then in order the Vlasov equation is satisfied, each of the quantities multiplying these functions must be identically set equal to zero. Moreover for each of these two equations so obtained, the quantity in front of $\exp[in(\theta-\omega_1 t)]$ must vanish and so the quantity in front of $\exp[-in(\theta-\omega_2 t)]$. We obtain then the following four relations

$$\begin{aligned} (\omega-\omega_0)f_1(\omega) - \frac{1}{2} e U_0 \frac{df_1}{d\omega} &= 0 \\ (\omega-\omega_0)f_2(\omega) - \frac{1}{2} e U_0 \frac{df_2}{d\omega} &= 0 \\ in(\omega-\omega_1)f_1(\omega) &= eIZ_n \bar{f}_1 \frac{df_0}{d\omega} \\ - in(\omega-\omega_2)f_2(\omega) &= eIZ_{-n} \bar{f}_2 \frac{df_0}{d\omega} \end{aligned} \quad (21)$$

The last two equations are the dispersion relations we were looking for. These four equations match the number of unknown, f_1 , f_2 , ω_1 and ω_2 . From the dispersion relations we can determine ω_1 and ω_2 independently of f_1 and f_2 ; one has

$$1 = -i eIZ_n \int \frac{df_0/d\omega}{\omega-\omega_1} d\omega \quad (22a)$$

$$1 = i eIZ_{-n} \int \frac{df_0/d\omega}{\omega-\omega_2} d\omega \quad (22b)$$

The first two equations of (21) can be solved for $f_1(\omega)$ and $f_2(\omega)$. But they are of no relevance here.

Equation (22a) is the standard dispersion relation for unbunched beams. The second dispersion relation (22b) is exactly the same as the former with the only difference that one has "changed sign to the real part of the impedance".

Suppose the impedance is a pure reactance (no resistance) then the two dispersion relations are identical.

At this point in order to have an idea of how the dispersion relations work, we believe it is not necessary to be consistent with the previous choice of stationary distribution. For instance with rectangular distribution for $f_0(w)$ we obtain

$$\omega_{1,2} = \omega_0 \pm \sqrt{K_0^2 \delta^2 - i \frac{eIZ K_0}{2n\Delta}} \quad (23)$$

where $K_0 = d\omega/dw$ is a machine parameter.

Above the transition energy $K_0 < 0$ and for a perfectly conductive wall

$$Z_{||} = \frac{2\pi i n}{\beta c \gamma^2} (1 + 2 \log \frac{b}{a})$$

where b/a is the ratio of the vacuum chamber wall radius to beam radius, βc is the bunch velocity and γ its energy normalized to the rest energy. Then below the transition energy, the collective frequency (23) is real and the motion is stable. The perturbation is a stationary standing wave oscillating at the frequency $\Delta\omega_c = \omega_c - \omega_0$. Above the transition energy the motion is unstable; in fact for sufficiently small δ the quantity under the square root at the r.h. side of (23) is negative, this makes ω_c complex, with an imaginary component. The perturbation is still a standing wave, which is oscillating at the frequency $\text{Real}[\Delta\omega_c]$, but it is also growing at the rate $\text{Imag}[\Delta\omega_c]$. Observe that there are always two solutions, complex conjugated to each other, one representing a "fast" wave (sign +), unstable and the other a "slow" wave (sign -), stable. To preserve the shape of a standing wave for the perturbation, it has been necessary to associate the two frequencies ω_1 and ω_2 with the two values of ω_c .

The result we have just illustrated can be easily generalized to include any case where the impedance is a pure reactance (no resistance), for instance an inductance. Equation (23) would still hold.

If the impedance is resistive then the collective frequency ω_c [Eq. (23)] splits in four possible solutions

$$\omega_1 = \omega_0 \pm \sqrt{K_0^2 \delta^2 + \frac{eIK_0}{2n\Delta}} (X_n - iR_n)$$

$$\omega_2 = \omega_0 \pm \sqrt{K_0^2 \delta^2 + \frac{eIK_0}{2n\Delta}} (X_n + iR_n)$$

where $Z_n = R_n + iX_n$. For a uniform distribution in w , the bunch is always unstable (no threshold). But, in order to preserve again the shape of the standing wave, the solutions should be paired according to the sign in front of the squared root.

Like in the theory of coasting beam, the bunch can also be made

stable with a more reasonable w-distribution with tails and spread. Then one would derive again the usual stability criterion

$$\left| \frac{Z}{n} \right| \leq \frac{\eta E}{e I_p} \left(\frac{\Delta p}{p} \right)^2 \quad (24)$$

where $\eta = |1/\gamma^2 - 1/\gamma'^2|$ and $I_p = \pi/\Delta$ is the peak bunch current.

VII. CONCLUSION

With our model of bunch with rectangular shape we have seen that an individual bunch can indeed be unstable against longitudinal oscillations. This is in contrast with the normal statement⁵ that an individual bunch should be stable because a wave travelling along the contour is damped on one side and anti damped on the other. We believe that the mistake when the coasting beam theory is applied to bunches is that for an unbunched beam only one single traveling wave is required to investigate the stability of the motion: for a coasting beam there are boundary conditions to fulfill. The single wave usually presents two solutions, one stable and the other not. But for a bunched beam one requires at least two travelling waves, one in the opposite direction of the other, so that the boundary conditions at the edge of the bunch are satisfied. Each of these waves in turns has two possible solutions, namely two possible ways of propagating. It is this fact which led us to conclude that there is an unstable solution also for a bunch.

Curiously enough though, for the case of rectangular bunches, one could derive dispersion relations similar to the one for coasting beams. In particular we have derived a stability condition (by Landau damping) which is just the old formula⁵ with the average current replaced by the local current. We believe nevertheless that for a more "circular" bunch Eq. (29) could still apply but that the impedance at the l.h. side should be modified with a sum of contributions at more than one frequency.

Finally, as a particular case, we have proven that the "negative mass instability" is also possible for a bunched beam above the transition energy.

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A STUDY OF MICROWAVE INSTABILITIES BY MEANS OF A SQUARE-WELL POTENTIAL

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I. INTRODUCTION

The subject of microwave instabilities has attracted a lot of theoretical activity recently. A series of papers by Sacherer¹ has played the leading role in the field. Further development of his work is being actively pursued by several authors.² However, the mathematical complexity of the theory makes it very hard to grasp the essential physics underlying microwave instabilities. This is rather unfortunate since the qualitative features of microwave instabilities are easy to understand³ by applying coasting beam theory.⁴

In this paper, microwave instabilities are analyzed in a simple model, in which the usual synchrotron oscillation of a particle is replaced by particle motion in a square-well potential. The motivation for doing this was the following: In the usual synchrotron oscillation, a particle moves along an elliptic trajectory. The most natural coordinates for such a motion are the action and the angle variables. On the other hand, the distribution of the particles along the ring is most conveniently described by azimuthal variables. The complexity of the theory of microwave instabilities derives from the fact that the two sets of the variables are not simply related. The difficulty disappears if the synchrotron motion is approximated by the motion in a square-well potential.

The square-well potential may seem extremely unphysical. However, it should be remarked that the form of the potential with addition of a Landau cavity looks more or less like a square-well. At any rate, the main motivation of introducing the square-well here is to simplify the mathematics of and thereby gaining some insight into microwave instabilities.

The model is exactly soluble. The results are in general agreement with the conclusions obtained from qualitative arguments³ based on coasting beam theory. However, some of the detailed features of the solution, for example the behavior of ω^2 as a function of impedance, are surprising.

In section II, the model is defined precisely. In section III, the model is solved. The paper is concluded in section IV by discussing the properties of the solution.

II. THE MODEL

The canonically conjugate variables are:

σ : The azimuthal distance from the reference particle.

ϵ : The energy difference $E - E_s$, where E and E_s are the energy of the particle under consideration and the reference particle, respectively.

Let $\Psi(\sigma, \epsilon, t)$ be the distribution function in phase space. It satisfies the following Vlasov's equation:

$$\frac{\partial \Psi}{\partial t} + \Gamma \cdot \epsilon \frac{\partial \Psi}{\partial \sigma} + \left[F_{\text{ext}} + \frac{ev\Omega_{\text{rev}}}{2\pi} U(\sigma, t) \right] \frac{\partial \Psi}{\partial \epsilon} = 0. \quad (1)$$

Here, the constant Γ is defined in terms of R = machine radius, Ω_{rev} = revolution frequency, $\beta = v/c$ and $\eta = \gamma \epsilon^2 - \gamma^{-2}$ as follows:

$$\Gamma = \frac{R \Omega_{\text{rev}}}{\beta^2 E_s} . \quad (2)$$

$U(\sigma, t)$ is the collective potential given by

$$U(\sigma, t) = \int d\sigma' G(\sigma' - \sigma) \rho(\sigma', t) . \quad (3)$$

Here $\rho(\sigma, t)$ is the line density,

$$\rho(\sigma, t) = \int d\epsilon \Psi(\sigma, \epsilon, t) . \quad (4)$$

$G(\sigma)$ in Eq. (5) is the Fourier transform of the impedance function $Z(k)$;

$$G(\sigma) = - \int Z(k) e^{-ik\sigma} dk, \quad Z(k) = \frac{-1}{2\pi} \int e^{ik\sigma} G(\sigma) d\sigma . \quad (5)$$

So far, everything is quite general. The model enters in specifying the form of the external force F_{ext} . In a usual synchrotron oscillation, a particle moves in a harmonic potential as shown in Fig. 1a. Thus F_{ext} is proportional to σ and the motion in phase plane is elliptic as shown in Fig. 1b. This leads to the difficulties discussed in the Introduction. In this paper, I will replace the harmonic oscillator potential by the square-well potential shown in Fig. 2a. The corresponding trajectory in phase plane is shown in Fig. 2b. Here, the particle moves from the point A to the point B with a constant velocity, jumps to the point C, then moves again with a constant velocity to the point D, jumps to the point A, etc.

Fig. 1. Harmonic oscillator potential and the motion in phase plane.

One of the simplicities of the potential being a square-well is that F_{ext} vanishes inside the well, $0 < \sigma < L$. The sharp potential barrier at the edge $\sigma = 0$ and $\sigma = L$ could in principle be taken into account by introducing a certain δ -function type force. However, the use of such a singular function can become quite tricky. The difficulty is easily

Fig. 2. Square-well potential and the motion in phase plane.

avoided; the reflection at the barrier can be expressed mathematically by means of suitable boundary conditions on Ψ . Consider a particle moving toward the barrier at $(0, -\epsilon)$ in phase plane. As soon as it arrives at the point $(0, -\epsilon)$, it jumps to the point $(0, \epsilon)$ immediately. This means that the points $(0, \epsilon)$ and $(0, -\epsilon)$ should be identified. The same is true for the points (L, ϵ) and $(L, -\epsilon)$. Therefore, the proper boundary conditions are

$$\begin{aligned}\Psi(0, \epsilon, t) &= \Psi(0, -\epsilon, t) \\ \Psi(L, \epsilon, t) &= \Psi(L, -\epsilon, t) .\end{aligned}\quad (6)$$

The model is therefore defined by the Vlasov's equation (1) with $F_{\text{ext}} = 0$ together with the boundary conditions (6).

As is usual, one linearizes, the Vlasov's equation. Write

$$\Psi(\sigma, \epsilon, t) = \Psi_0(\epsilon) + \Psi_1(\sigma, \epsilon, t) . \quad (7)$$

Here $\Psi_0(\epsilon)$ is the static solution in the absence of the collective force, and Ψ_1 is the perturbation. The linearized equation is, for $0 < \sigma < L$,

$$\frac{\partial \Psi_1}{\partial t} + \Gamma \epsilon \frac{\partial \Psi_1}{\partial \sigma} + \frac{e v \Omega_{\text{rev}}}{2\pi} U_1(\sigma, t) \frac{\partial \Psi_0(\epsilon)}{\partial \epsilon} = 0 . \quad (8)$$

In the above,

$$U_1(\sigma, t) = \int_0^L d\sigma' G(\sigma' - \sigma) \rho_1(\sigma') d\sigma' , \quad (9)$$

where ρ_1 is the line density associated with Ψ_1 . The limits of the integration in (9) arise from the obvious fact that there are no particles outside the potential well. For a general impedance function $G(\sigma)$, the appearance of the finite integration limits in (9) makes the solution of Eq. (8) difficult. However, the difficulty disappears if the function $G(\sigma)$ is sharply peaked at $\sigma = 0$. In other words, the interaction is similar to the one induced by the space charge effect. Explicitly, $G(\sigma)$ will be taken to be of the following form:

$$G(\sigma) = G_1 \delta'(\sigma) + G_2 \delta(\sigma) . \quad (10)$$

Equation (10) represents the overall features of the longitudinal impedance correctly. The limits of the integration in Eq. (9) can now be replaced by $-\infty$ and $+\infty$, enabling one to solve Eq. (8) by a simple Fourier transformation.

As for $\Psi_0(\epsilon)$, I take the simplest choice

$$\Psi_0(\epsilon) = \frac{I}{2v\Delta} \theta(\Delta - |\epsilon|) , \quad (11)$$

Where I is the average current in the ring. Notice that $\Psi_0(\epsilon)$ is an even function of ϵ and therefore satisfies the boundary conditions (6). Equation (11) applies only inside the potential well, i.e., when $0 < \sigma < L$. It is understood that Ψ_0 vanishes outside the well. The same remark holds for the functions Ψ_1 , A and B in the equations below. By differentiation, one obtains

$$\frac{\partial \Psi_0}{\partial \epsilon} = \frac{1}{2v\Delta} [\delta(\epsilon + \Delta) - \delta(\epsilon - \Delta)] . \quad (12)$$

From Eqs. (8) and (12), one sees that Ψ_1 is of the following form:

$$\Psi_1(\sigma, \epsilon, t) = A(\sigma, t) \delta(\epsilon + \Delta) + B(\sigma, t) \delta(\epsilon - \Delta) . \quad (13)$$

The functions A and B satisfy the following equations:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Gamma \Delta \frac{\partial}{\partial \sigma} \right) A(\sigma, t) + \frac{e\Omega_{\text{rev}} I}{2\pi 2\Delta} \int d\sigma' G(\sigma' - \sigma) [A(\sigma', t) + B(\sigma', t)] &= 0 \\ \left(\frac{\partial}{\partial t} - \Gamma \Delta \frac{\partial}{\partial \sigma} \right) B(\sigma, t) - \frac{e\Omega_{\text{rev}} I}{2\pi 2\Delta} \int d\sigma' G(\sigma' - \sigma) [A(\sigma', t) + B(\sigma', t)] &= 0 \end{aligned} \quad (14)$$

The boundary condition becomes

$$A(0, t) = B(0, t), \quad A(L, t) = B(L, t) . \quad (15)$$

Equation (14) is applicable to a coasting beam as well if σ is interpreted as the distance from a fixed point on the ring, say at $\sigma = 0$. However, the boundary conditions are modified as follows: Let the circumference of the ring be C . Since the points $\sigma = 0$ and $\sigma = C$ are identical, the boundary conditions become

$$A(0, t) = A(C, t), \quad B(0, t) = B(C, t) . \quad (16)$$

III. THE SOLUTION

Let us forget about the boundary condition for the moment. Write

$$\begin{aligned} A(\sigma, t) &= \sum_k e^{i(\omega t + k\sigma)} a_k \\ B(\sigma, t) &= \sum_k e^{i(\omega t + k\sigma)} b_k \end{aligned} \quad (17)$$

Equation (14) becomes

$$\begin{aligned} [\omega - \gamma k + ig(k)]a_k + ig(k)b_k &= 0 \\ -ig(k)a_k + [\omega + \gamma k - ig(k)]b_k &= 0, \end{aligned} \quad (18)$$

where

$$\gamma = \Gamma \Delta \text{ and } g(k) = \frac{\Omega_{\text{rev}}}{2\pi} \cdot \frac{eZ(k)I}{2\Delta}. \quad (19)$$

γ in the above is the velocity of the particle relative to the reference particle at the top or the bottom of the stack. In view of Eq. (10), $g(k)$ is of the following form:

$$g(k) = i\alpha k + \beta, \quad (20)$$

where α and β are real constants. The solubility of Eq. (18) requires the following dispersion relation:

$$\omega = \pm \omega(k), \quad \omega(k) = \sqrt{(\gamma k)^2 - 2i\gamma k g(k)}. \quad (21)$$

In Eq. (21), the square root is defined so that $\omega(k)$ has a positive real part.

To complete the solution, one should take into account the boundary condition. For a coasting beam, Eq. (16) requires that k be real and discrete as follows:

$$k = k_n = \frac{2\pi n}{C} \quad n = 0, \pm 1, \pm 2, \dots \quad (22)$$

The corresponding ω^2 is

$$\omega^2 = \omega_n^2 = (\gamma k_n)^2 - 2i\gamma k_n g(k_n) = \gamma Y k_n^2 - 2i\beta \gamma k_n, \quad (23)$$

where

$$Y = \gamma + 2\alpha. \quad (24)$$

If $n > 0$ and $\omega = +\omega_n$, it follows from Eq. (18) that $b_k \gg a_k$ when g is small. Thus the disturbance runs mainly along the top of the stack. See Eq. (13). Analogously the case $\omega = -\omega_n$ corresponds to the bottom wave. If the impedance is purely resistive, $\alpha = 0$ and $\beta > 0$. It is then easy to show that the bottom wave grows and the top wave damps. All of these features are well known from coasting beam theory.⁴

For bunched beam, the relevant boundary condition is given by Eq. (15). The top and the bottom waves couple with each other in an essential way. To proceed, notice first that the boundary condition applies at all times, so that the contributions from different frequencies can be analyzed separately. Therefore, it is necessary to find the k 's

which correspond to the same ω^2 . Consider the equation

$$\omega^2 = (\gamma k)^2 - 2i\gamma k g(k) , \quad (25)$$

which is equivalent to Eq. (21). From Eqs. (20) and (25), one obtains

$$k = k_{\pm} = \frac{i\beta}{Y} \pm \sqrt{\frac{\omega^2}{Y\gamma} - \left(\frac{\beta}{Y}\right)^2} , \quad (26)$$

where Y is defined in Eq. (24). The functions A and B that behave as $e^{i\omega t}$ are, in view of Eqs. (17) and (18), as follows:

$$\begin{aligned} A(\sigma, t) &= e^{i\omega t} (e^{ik_+ \sigma} a_+ + e^{ik_- \sigma} a_-) \\ B(\sigma, t) &= e^{i\omega t} (e^{ik_+ \sigma} b_+ + e^{ik_- \sigma} b_-) \end{aligned} \quad (27)$$

where

$$b_{\pm} = D_{\pm} a_{\pm} , \quad D_{\pm} = \frac{i g(k_{\pm})}{\omega + k_{\pm} - i g(k_{\pm})} . \quad (28)$$

The boundary condition (15) becomes

$$(1 - D_+) a_+ + (1 - D_-) a_- = 0 \quad (29.a)$$

$$(1 - D_+) d^{ik_{\pm} L} a_+ + (1 - D_-) e^{ik_{\pm} L} a_- = 0 . \quad (29.b)$$

One way to satisfy Eq. (29) is to require

$$D_{\pm} = 1 \text{ and } a_{\pm} = 0 \quad (30)$$

After some algebra, one finds that Eqs. (25) and (30) imply

$$\omega = 0, \quad k = i\kappa = 2i \frac{\beta}{Y} . \quad (31)$$

For this value of ω and k , A is identical to B and given by

$$A(\sigma, t) = B(\sigma, t) = e^{\kappa \sigma} . \quad (32)$$

This solution is time independent and therefore stable.

Equation (29) can also be satisfied if the two amplitudes a_+ and a_- are related by (29.a) and, furthermore, k_+ and k_- are related as follows:

$$k_+ - k_- = 2K_n = \frac{2\pi n}{L} , \quad n = 0, 1, 2, \dots \quad (33)$$

From (26), one obtains

$$\omega^2 = \omega_n^2 = \gamma(YK_n^2 + \frac{\beta^2}{Y}) . \quad (34)$$

and

$$k_{\pm} = k_{\pm n} = \pm K_n + i \frac{\beta}{Y}. \quad (35)$$

The corresponding functions A and B are easily obtained. One gets

$$\begin{aligned} A(\sigma, t) &= e^{i\omega_n t} (e^{ik_{+n} \cdot \sigma} a_{+n} + e^{ik_{-n} \cdot \sigma} a_{-n}) C \\ B(\sigma, t) &= e^{i\omega_n t} (e^{ik_{+n} \cdot \sigma} b_{+n} + e^{ik_{-n} \cdot \sigma} b_{-n}) C. \end{aligned} \quad (36)$$

In the above ()_± means that k should be replaced by k_{±n}, and C is an arbitrary normalization constant. This completes the solution of the problem.

IV. DISCUSSION AND CONCLUSIONS

Let us now discuss the properties of the solutions obtained in the previous section. First, if the impedance is small, Eq. (34) can be approximated by

$$\omega_n \sim K_n = \gamma \frac{\pi n}{L}. \quad (38)$$

To understand this formula, recall that γ is the particle velocity at the top or the bottom of the stack. Therefore the quantity $2L/\gamma$ can be interpreted as the period of one "synchrotron" oscillation. Thus, Eq. (38) can be written in the following expected form:

$$\omega_n = n\Omega_s \quad (39)$$

where $\Omega_s = 2\pi\gamma/2L$ is the angular synchrotron frequency.

Next, it is interesting to compare the frequency spectra for coasting beam and bunched beam given by Eqs. (23) and (34), respectively. For coasting beam, the presence of a resistive part β in the impedance always implies an instability. The situation is quite different for the case of coasting beam; the resistive part enters as β^2 in Eq. (34), and a bunched beam can be stable even if $\beta \neq 0$. Instabilities occur if the quantity $Y = \gamma + 2\alpha$ becomes negative. Therefore, a bunched beam is always stable if the impedance is small and hence $Y > 0$. This conclusion is in accord with the one reached by intuitive arguments³ based on coasting beam theory.

For coasting beam, $|\omega^2|$ generally increases as the impedance (or current) increases. The situation is again quite different for bunched beam. Figure (3) shows the behavior of ω^2 as a function of Y for a fixed n . If $\beta = 0$, the curve is similar to the case of coasting beam. However, the curve for the case $\beta \neq 0$ is qualitatively different; it is singular at $Y = 0$ and has minima at $Y = \pm \beta/k_n$. The presence of a resistive term therefore has an important bearing on the behavior of ω^2 in bunched beam, although it does not directly influence the stability criteria. Whether this feature is due to the specific model discussed in this paper remains to be seen. However, I suspect that it is a

general phenomena arising from the strong interference of the top and bottom waves inevitable in bunched beams.

Fig. 3. ω^2 versus $Y = \gamma + 2\alpha$ for a fixed K_n . The solid curve is for the case $\beta \neq 0$, while the dashed one represents the case $\beta = 0$.

Finally, it is also interesting to compute the line density $\rho(\sigma, t) = A(\sigma, t) + B(\sigma, t)$. The result is complicated, but when $\beta = 0$ it becomes

$$\rho(\sigma, t) = e^{i\omega_n t} \cos(K_n \sigma), \quad \omega_n = \sqrt{\gamma Y} K_n. \quad (40)$$

That is, $\rho(\sigma, t)$ is a standing wave, as would be reasonable for a purely reactive impedance. However, the distributions $A(\sigma, t)$ and $B(\sigma, t)$ are not separately standing waves.

In this paper, the physics of microwave instabilities have been studied using a simple model for the synchrotron oscillation. The results verify to a certain extent the validity of intuitive arguments. However, a certain aspect of the results is unexpected. One hopes that the insight gained in this analysis will be helpful in attacking a more realistic theory of microwave instabilities.

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SINGLE BUNCH FAST LONGITUDINAL INSTABILITY

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I. INTRODUCTION

Single bunch longitudinal instability producing an increase of the bunch area have been observed in proton synchrotron and storage rings. Signals at microwave frequencies are observed during the bunch blow-up and because of this the effect has been called the microwave instability.

A similar increase in bunch area is observed also in electron storage rings, where it is usually referred to as the bunch lengthening effect.

It was suggested by Boussard¹ that this effect might be explained using the well known longitudinal coasting beam instability²⁻⁵ with the assumptions that the unstable modes have a wavelength shorter than the bunch length and that their rise time is shorter than the synchrotron oscillation period. In this approach one can introduce a threshold for the instability using the formulas given in references 4 or 5, with the local bunch value of the current and energy spread.

Following the work by Boussard several authors have worked on this problem.⁶⁻¹¹ All these theories use some approximations either in the formulation of the Vlasov equation, or in its solution, or in the impedance used to describe the beam self-force. This paper is an attempt to obtain a more general theory of this effect. Here we describe the model used and the method of calculation, together with some general results. More detailed results will be given in another paper.

The main result of this paper is the derivation of a condition for the existence of a fast longitudinal bunch blow-up. This condition is a generalized "threshold formula", showing explicitly the dependence on the bunch energy spread and length. This condition is qualitatively in agreement with Boussard's suggestion.¹

The term "fast blow-up" that we use here has the following meaning. The time scale involved in longitudinal instabilities can be characterized by the revolution frequency, ω_0 , the synchrotron frequency, ω_s , and the rise time τ_{in} . We assume that

$$\omega_s \ll \omega_0$$

which is satisfied in all storage rings. The regime of "fast blow-up" is then defined by

$$\omega_s < \frac{1}{\tau_{in}} < \omega_0.$$

Although we assume that this inequality is satisfied we will not neglect the effect of synchrotron oscillations, so that $\omega_s \neq 0$.

II. THE VLASOV EQUATION

The convenient tool for the investigation of the coherent motions of a beam is provided by the Vlasov equation. The Vlasov equation for

the longitudinal motion can be written, assuming the focusing force to be linear, as

$$\frac{\partial \psi}{\partial t} + \varphi \frac{\partial \psi}{\partial \varphi} - \omega_s^2 \varphi \frac{\partial \psi}{\partial \varphi} + \frac{\partial \psi}{\partial \varphi} \int_0^{2\pi} d\varphi' \int_{-\infty}^{\infty} dt' G(\varphi - \varphi', t - t') \rho(\varphi', t') = 0, \quad (1)$$

where ω_s is the synchrotron frequency, φ is the angular distance between a particle in the bunch and the synchronous particle, $\psi = \psi(\varphi, \dot{\varphi}, t)$ is the particle distribution function in phase space and $\rho(\varphi, t) \equiv \int_{-\infty}^{\infty} d\dot{\varphi} \psi(\varphi, \dot{\varphi}, t)$ is the particle line density (per radian). The Green's function $G(\varphi, t)$ describes the self interaction of the particles in the beam. $G(\varphi, t)$ is periodic in φ with period 2π , and causality demands G to vanish for $t < 0$.

Note that the self-force due to discontinuities around the ring can also be described in the average by a periodic Green's function, provided that the revolution frequency ω_0 is much greater than ω_s . For example, the Green's function for a single resonance with shunt impedance R_s , frequency ω_R and width Γ is given by

$$G(\varphi, t) = \frac{e^2 \Gamma R_s \omega_0}{\pi R} \delta_p(\varphi + \omega_0 t) e^{-\Gamma t} [\cos \omega_R'(t - t') - \frac{\Gamma}{\omega_R'} \sin \omega_R'(t - t')] ,$$

where R is the ring radius, $\omega_R' = [\omega_R^2 - \Gamma^2]^{\frac{1}{2}}$ and δ_p is the periodic delta function. The Fourier components of $G(\varphi, t)$ are related to the conventional impedance Z by

$$G_n(\Omega) \equiv \frac{1}{(2\pi)^2} \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dt e^{-in\varphi} e^{i\Omega t} G(\varphi, t) = (2\pi)^{-3} \frac{\eta \omega_0}{m_0 \gamma R^2} Z_n(\Omega + n\omega_0). \quad (2)$$

where η is the frequency slip factor $\gamma_L^{-2} - \gamma^{-2}$ and m_0 is the particle mass.

As usual, we try to solve (1) within a linear approximation. Define

$$\begin{aligned} \psi &= \psi_0(\varphi, \dot{\varphi}) + \psi_1(\varphi, \dot{\varphi}) e^{-i\Omega t}, \\ \rho(\varphi, t) &= \rho_0(\varphi) + \rho_1(\varphi) e^{-i\Omega t}, \end{aligned} \quad (3)$$

where Ω is the coherent frequency.

Note that the coherent frequency ω as seen by the pick up system is related to Ω by

$$\omega = \Omega + n\omega_0, \quad (4)$$

where n is any integer. We assume

$$|\Omega| \ll \omega_0. \quad (5)$$

Expanding (1) to first order in G , we obtain

$$\dot{\varphi} \frac{\partial \psi_0}{\partial \varphi} - \omega_s^2 \varphi \frac{\partial \psi_0}{\partial \dot{\varphi}} = 0 \quad (6)$$

and

$$-i\Omega\psi_1 + \dot{\varphi} \frac{\partial \psi_1}{\partial \varphi} - \omega_s^2 \varphi \frac{\partial \psi_1}{\partial \dot{\varphi}} + 2\pi \frac{\partial \psi_0}{\partial \dot{\varphi}} \sum_{m=-\infty}^{+\infty} G_m(\Omega) e^{im\varphi} \int_0^{2\pi} d\varphi' e^{-im\varphi'} \rho_1(\varphi') = 0 \quad (7)$$

In Eqs. (6) and (7) we have ignored a term which causes a shift in the synchrotron frequency ω_s , and makes it a function of φ . This change in ω_s is important to study Landau damping effects for slow blow-up effects [$(1/\tau_{in}) < \omega_s$]. However it can be neglected when studying fast blow-up effects.

Now introduce polar coordinates for the synchrotron motion,

$$\varphi = r \cos\theta, \quad \dot{\varphi} = r\omega_s \sin\theta. \quad (8)$$

In terms of these variables, the general solution of (6) is given by

$$\psi_0 = \psi_0(r). \quad (9)$$

In this paper, we shall treat only the case of a Gaussian bunch,

$$\psi_0(r) = \frac{N}{2\pi L^2 \omega_s} e^{-r^2/2L^2}. \quad (10)$$

L is half of the rms bunch length in units of radians and N is the total number of particles in the bunch. The normalization of ψ_0 is such that $\omega_s \int_0^{2\pi} d\theta \int_0^\infty r dr \psi_0(r) = N$.

For later use, let us introduce here the Hankel transform of $\psi_0(r)$:

$$\psi_0(r) = \int_0^\infty dk A(k) J_0(kr), \quad (11)$$

$$A(k) = \frac{Nk}{2\pi\omega_s} e^{-\frac{1}{2}k^2 L^2}. \quad (12)$$

Note that the half width (rms) of $A(k)$ in k is L^{-1} . The full rms momentum spread Δp of the bunch is related to L by,

$$\frac{\Delta p}{p} = \frac{2\omega_s L}{|\eta| \omega_0}. \quad (13)$$

In order to solve (7), set

$$\lambda_{n,\mu} = \frac{\omega_s}{2\pi} (-i)^\mu \int_0^{2\pi} d\theta e^{i\mu\theta} \int_0^\infty r dr J_\mu(nr) \psi_1(r, \theta) \quad (14)$$

and

$$\rho_1(\varphi) = \sum_{n=-\infty}^{\infty} \sigma_n e^{in\varphi}. \quad (15)$$

σ_n and $\lambda_{n,\mu}$ are related by

$$\sigma_n = \sum_{\mu=-\infty}^{\infty} \lambda_{n,\mu}. \quad (16)$$

We call the indices n and μ above the revolution mode number and the synchrotron mode number respectively.

In terms of σ and λ , Eq. (7) becomes

$$\lambda_{m,\mu} = -i4\pi^2 \frac{\mu}{\Omega - \mu\omega_s} \sum_{n=-\infty}^{\infty} \int_0^{\infty} dr \psi'_0(r) J_{\mu}(nr) J_{\mu}(nr) \frac{G}{n} \sigma_n. \quad (17)$$

This equation was first obtained by Sacherer⁶ and by Chao and Gareyte.⁸

In solving (17), Sacherer assumes $1/r \psi'_0(r)$ to be constant throughout the region $0 < r < \infty$, and as a consequence, in his approach different revolution modes decouple. His assumption amounts to assuming a very long bunch, and it is true in general that for a long bunch (not much shorter than the ring circumference), different revolution modes decouple. Sacherer further assumes that the eigenvectors $\lambda_{m,\mu}$ are known vectors, and uses them to deduce, phenomenologically, information about the eigenvalue Ω .

We concur with the opinion that the coupling of various revolution modes is an intrinsic part of high frequency coherent motion of a bunched beam.⁷

III. FAST BLOW-UP REGIME

We wish to study (17) in the following microwave fast blow up limits

$$L \gg \frac{1}{|n|}, \quad (18)$$

$$\frac{\text{Im}\Omega}{\omega_s} = \frac{\text{Im}\omega}{\omega_s} \gg 1. \quad (19)$$

(18) implies that the coherent wavelength is much shorter than the bunch length, and (19) means that the rise time of the instability is much faster than a synchrotron period.

Notice the pole $(\Omega - \mu\omega_s)^{-1}$ in (17). Since $\text{Im}\Omega \gg \omega_s$, many such poles corresponding to different μ 's contribute with roughly the same order of magnitude. In other words, when fast blow up occurs, many synchrotron modes are excited simultaneously. It is therefore convenient to sum Eq. (17) over all μ 's.

Using (16), (17) becomes

$$\sigma_m = \sum_{n=-\infty}^{\infty} M_{m,n} \sigma_n, \quad (20)$$

where

$$M_{m,n} = i \frac{G_n(\Omega)}{n} T_{m,n}, \quad (21)$$

and

$$T_{m,n} = -4\pi^2 \int_0^\infty dr \psi'_0(r) \sum_{\mu=-\infty}^{\infty} \frac{\mu}{\Omega - \mu \omega_s} J_\mu(mr) J_\mu(nr). \quad (22)$$

The series in (22) can be summed conveniently if we go to Hankel transformed space (11). The details of this calculation are given in Appendix A. Here we only describe the main results. In terms of a function v ,

$$\cos v = 1 - \frac{k^2 - (|m| - |n|)^2}{2|mn|}, \quad (23)$$

(22) can be expressed as

$$T_{m,n} = 4\pi \frac{|n| + |m|}{||n| - |m||} \int dk A(k) S^\pm(v, \Omega), \quad (24)$$

where S^+ and S^- are two different functions. We should use S^+ or S^- depending on whether n and m have the same or the opposite signs.

It suffices for the purpose here to give the functions $S^\pm$ just for the values of v in the interval $(0, \pi)$.

$$S^+(v, \Omega) = \frac{\pi}{\omega_s} \left[\cot \frac{\Omega \pi}{\omega_s} \sin \frac{\Omega v}{\omega_s} - \cos \frac{\Omega v}{\omega_s} \right], \quad (25)$$

$$S^-(v, \Omega) = \frac{\pi}{\omega_s} \sin \frac{\Omega v}{\omega_s} \left(\sin \frac{\Omega \pi}{\omega_s} \right)^{-1}. \quad (26)$$

Substituting (12) into (24), we obtain

$$T_{m,n} = \frac{N}{\omega_s} e^{-\frac{1}{2}(|n| - |m|)^2 L^2} \int_0^\infty dk'^2 e^{-\frac{1}{2}k'^2 L^2} S^\pm(u, \Omega), \quad (27)$$

where k' is related to k in (24) by

$$k'^2 = k^2 - (|n| - |m|)^2,$$

and

$$\cos u = 1 - \frac{k'^2}{2|mn|}. \quad (28)$$

Because of the exponential factor in the integrand of (27), the main contribution to the integral comes from the region $k' \leq \sqrt{2}/L$. Applying (18), we see from (28) that, effectively,

$$0 \leq u \ll \pi, \quad (29)$$

$$u \cong k' |mn|^{-\frac{1}{2}}. \quad (30)$$

$S^\pm(u, \Omega)$ can now be greatly simplified in the limit (19). Using (29) we see immediately that

$$S^+(u, \Omega) \cong -\frac{\pi}{\omega_s} \exp[i\Omega k'/\omega_s \sqrt{mn}], \quad (31)$$

$$S^-(u, \Omega) \cong 0. \quad (32)$$

We conclude from (32) that m and n of opposite signs do not couple. In other words, the "fast wave" and the "slow wave" components within the bunch do not interfere in the microwave fast blow up limits.

From now on, we consider only the case $m, n > 0$. Due to the factor $e^{-\frac{1}{2}(n-m)^2 L^2}$ in (27), $T_{m,n}$ is small when $|n-m| \gg \sqrt{2}/L$. Again using (18), we obtain

$$S^+(u, \Omega) \cong -\frac{\pi}{\omega_s} \exp[i\Omega k'/\omega_s m].$$

So finally, by approximating the upper integration limit of (27) by ∞ , we have

$$T_{m,m} = -\frac{2N\pi}{\omega_s^2} \int_0^\infty k dk e^{-\frac{1}{2}k^2 L^2} \exp(i\Omega k/m\omega_s), \quad (33)$$

$$T_{m,n} = e^{-\frac{1}{2}(n-m)^2 L^2} T_{m,m} \quad (34)$$

(33) can be expressed in terms of a parabolic cylinder function:

$$T_{m,m} = -\frac{2N\pi}{\omega_s^2} f\left(\frac{1}{2} / L m \omega_s\right) \quad (35)$$

$$f(\alpha) = e^{-\frac{1}{2}\alpha^2} D_{-2}(-i\alpha). \quad (36)$$

The function $f(\alpha)$ has the following properties:

$$f(0) = 1, \quad (37)$$

$$f(\alpha) < 1, \text{ if } \text{Im } \alpha \geq 0, \text{ or if } \text{Im } \alpha = 0, \alpha \neq 0. \quad (38)$$

To conclude this section we summarize the above results that the coherent frequencies Ω are solutions of the secular equation

$$\sigma_m = \sum_n M_{m,n} \sigma_n. \quad (39)$$

In the microwave fast blow up limits,

$$M_{m,n} = -\frac{i}{2\pi} \frac{\eta}{m_o \gamma R^2} \frac{1}{(\omega_s L)^2} \frac{N \omega_o}{2\pi} \frac{Z(n\omega_o)}{n} f(\Omega/Lm\omega_s) e^{-\frac{1}{2}(n-m)^2 L^2}, \quad (40)$$

where $f(\alpha)$ is given by (36).

From (40), we see right away that a necessary and sufficient condition for different revolution modes in a bunched beam to approximately decouple is that the bunch is long. The same conclusion can be reached from the exact expression (24).

IV. CONDITIONS FOR FAST BLOW UP

In order to draw conclusions from (39) and (40), we have to have some idea about the longitudinal impedance $Z(n\omega_o)$.

It has been pointed out¹² that, in the high frequency domain of interest for the microwave instability, the impedance function $Z(\omega)$ for many existing machines including AGS, ISR, SPEAR etc. can be roughly approximated by a giant resonance with unity Q-factor. The resonance frequency is taken to be the cut off frequency ω_c of the ring,*

$$Z(\omega) \approx i \frac{\omega \omega_c R_s}{i \omega \omega_c + (\omega^2 - \omega_c^2)}. \quad (41)$$

The parameter R_s varies from one accelerator to another. We shall base our discussions on the approximate expression (41).

We take the microwave region to be that satisfying

$$\omega = \Omega + n\omega_o \approx n\omega_o \gg \omega_c$$

Suppose we wish to find a coherent frequency near $\omega = n_o \omega_o$, $n_o \omega_o \gg \omega_c$. We see that $Z(n\omega_o)/n$ as given by (41) is approximately constant in the interval $n_o - \sqrt{2}/L \leq n \leq n_o + \sqrt{2}/L$, provided $1/\sqrt{2}L \omega_o/\omega_c \ll 1$. We assume the last inequality to hold.

So we approximate (40) by

$$M_{m,n} = -\frac{i}{2\pi} \frac{\eta}{m_o \gamma R^2} \frac{1}{(\omega_s L)^2} \frac{N \omega_o}{2\pi} \frac{Z(n_o \omega_o)}{n_o} f\left(\frac{\Omega}{L n_o \omega_s}\right) e^{-\frac{1}{2}(n-m)^2 L^2}. \quad (42)$$

If the matrix M has an eigenvalue which is equal to unity for some Ω with $L n_o \Omega \gg \omega_s$, there will be a coherent instability with that

*Defined as $\omega_c \approx c/d$, d being the vacuum chamber dimension.

frequency.

As an approximation we shall extend the range of n and m in (42) to be $-\infty < n, m < +\infty$. The most likely fast instability corresponds to the biggest eigenvalue of the matrix $M_{m,n}$. It is easy to see that the maximum eigenvalue of a matrix with $e^{-\frac{1}{2}(n-m)L^2}$ as its elements is given by

$$g(L) = \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}n^2 L^2} = \frac{\sqrt{2\pi}}{L} \sum_{n=-\infty}^{\infty} e^{-2n^2 \pi^2 / L^2}. \quad (43)$$

Therefore the condition for fast blow up becomes

$$-1 \frac{2}{\pi m_o \gamma |\eta| \beta_c^2} \left(\frac{\Delta p}{p} \right)^{-2} \frac{Z(n_o \omega_o)}{n_o} f\left(\frac{\Omega}{\omega_s n_o L} \right) I_{av} g(L) = 1, \quad (44)$$

where (13) has been used, and the average current is $I_{av} = N\omega_o/2\pi$.

Let us define the threshold current I_{th} by

$$I_{th} = \left[\frac{2}{\pi m_o \gamma |\eta| \beta_c^2} \left(\frac{\Delta p}{p} \right)^{-2} \left| \frac{Z(n_o \omega_o)}{n_o} \right| \right]^{-1}. \quad (45)$$

If $I_{av} g(L) < I_{th}$, then we see from (38) that (44) does not have a solution with $\Omega_I > 0$. That means under this condition there is no microwave fast blow up. (We use the notation, $\Omega = \Omega_R + i\Omega_I$).

If $I_{av} g(L) = I_{th}$, then (44) has a solution with

$$\frac{\Omega_I}{\omega_s n_o L} \rightarrow 0^+, \quad (46)$$

when the function $f(\Omega/\omega_s n_o L) = 1$. (Remember that $|\Omega_R| \ll \Omega_I$.)

In this sense, (46) together with (18) and (19) define the threshold conditions for microwave fast blow up in analogous to the conventional slow blow up threshold condition for bunch beams,

$$\frac{\Omega_I}{\omega_s} \rightarrow 0^+. \quad (47)$$

Note that the wavelength ℓ (in radians) is related to n_o by $\ell = 2\pi/n_o$. The quantity $\alpha_I \equiv (\Omega_I/\omega_s) \cdot (\ell/L)$ determines the current needed to sustain a fast blow up with growth rate Ω_I and wavelength ℓ . The greater the quantity α_I , the bigger the current needed for that instability. I_{th} measures the minimum current needed for any microwave fast blow up to occur.

The quantity α_I is the product of Ω_I/ω_s , which gives the ratio of the synchrotron oscillation period to the instability rise time, and ℓ/L whose inverse gives the number of waves of the perturbing electromagnetic field in the bunch length. For microwave perturbation, ℓ/L is small, and it is possible to have small α_I even if Ω_I/ω_s is large.

Let us look into the properties of $I_{av} g(L)$. If L is large, we can just keep the leading term in the first equality of (43), and obtain

$$I_{av} g(L) \simeq I_{av}.$$

On the other hand, if L is small then from the second equality of (43) we have

$$I_{av} g(L) \simeq I_{peak}.$$

(Remember that for a Gaussian bunch the peak current I_{peak} and the average current I_{av} are related by $I_{peak} = \sqrt{2\pi}/L I_{av}$).

DISCUSSIONS

We have reached the conclusions that (46) together with (18) and (19) define the threshold conditions for microwave fast blow up with the corresponding threshold current given by (45).

It is worth emphasizing that even if the current is below the microwave fast blow up threshold, we still have to worry about the slow blow up as defined by $\Omega_I \leq \omega_s$.

It is well known that in the long wave length and long wake field limits, a single bunch is always stable. However, it can also be shown that in the short wave length limit, no matter how large $\Delta p/p$ is, half of the synchrotron modes of a single bunch beam are always unstable with possibly small but finite $I_m \Omega$. (Small as compared to ω_s). But the last statement is based on the assumption that the synchrotron frequency is independent of the synchrotron amplitude r . So we expect that the threshold condition corresponding to short wave length and (47), if it exists, should depend on $\Delta\omega_s$ as well as $\Delta p/p$.

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APPENDIX A

In this appendix, we show how the series in Eq. (22) can be summed and thus obtain Eqs. (24)-(26).

Substituting (11) into (22), we obtain

$$T_{m,n} = 4\pi^2 \int_0^\infty k dk A(k) \sum_{\mu=-\infty}^{\infty} \frac{\mu}{\Omega - \mu\omega_s} \int_0^\infty dr J_1(kr) J_\mu(mr) J_\mu(nr). \quad (A-1)$$

But

$$\int_0^\infty dr J_1(kr) J_\mu(mr) J_\mu(nr) = \begin{cases} \frac{1}{\pi k \mu} \sin \mu v, \\ \frac{(-1)^\mu}{\pi k \mu} \sin \mu v, \end{cases} \quad (A-2)$$

where v is given by Eq. (23). When m and n have the same (opposite) sign, we should take the first (second) expression in (A-2). (A-2) is valid when $||n| - |m|| \leq k \leq |n| + |m|$. For k outside of this region, the integral vanishes.

So, we need now to sum the following two series:

$$S^+(v, \Omega) = \sum_{\mu=-\infty}^{\infty} \frac{1}{\Omega - \mu\omega_s} \sin \mu v, \quad (A-3)$$

and

$$S^-(v, \Omega) = \sum_{\mu} \frac{(-1)^\mu}{\Omega - \mu\omega_s} \sin \mu v, \quad (A-4)$$

$$= S^+(v + \pi, \Omega). \quad (A-5)$$

Define,

$$U(v, \Omega) = \sum_{\mu} \frac{1}{\Omega - \mu\omega_s} e^{i\mu v}. \quad (A-6)$$

The function U will be calculated below. For a real value of Ω which is not an integral multiple of ω_s , we have

$$S^+(v, \Omega) = \text{Im } U(v, \Omega). \quad (A-7)$$

We shall use (A-7) to obtain $S^+(v, \Omega)$ for real Ω , and then analytically continue the function S^+ into the complex Ω region.

The function U satisfies the following two conditions:

$$U(v \pm 2\pi, \Omega) = U(v, \Omega), \quad (A-8)$$

$$i\omega_s \frac{\partial U}{\partial \nu} + \Omega U = 2\pi \sum_{n=-\infty}^{\infty} \delta(\nu - 2n\pi). \quad (\text{A-9})$$

From (A-9), we have for $2n\pi < \nu < 2(n+1)\pi$,

$$U = C_n \exp(i\Omega\nu/\omega_s). \quad (\text{A-10})$$

The coefficient C_n is to be determined from the condition (A-8). and (A-10) imply that

$$C_{n+1} = C_n \exp(-i2\pi\Omega/\omega_s). \quad (\text{A-11})$$

We now calculate C_0 . From (A-9), we have

$$i\omega_s [C_0 - C_{-1}] = 2\pi.$$

This together with

$$C_0 = C_{-1} \exp(-i2\pi\Omega/\omega_s)$$

give

$$C_0 = -i \frac{2\pi}{\omega_s} \frac{1}{1 - \exp(2\pi i\Omega/\omega_s)}. \quad (\text{A-12})$$

From (A-11) and (A-12), we obtain

$$C_n = -i \frac{2\pi}{\omega_s} \frac{1}{1 - \exp(2\pi i\Omega/\omega_s)} \exp(-2\pi i n\Omega/\omega_s). \quad (\text{A-13})$$

Thus for $2n\pi < \nu < 2(n+1)\pi$, we have

$$U(\nu, \Omega) = -i \frac{2\pi}{\omega_s} \frac{1}{1 - \exp(2\pi i\Omega/\omega_s)} \exp[i(\nu - 2n\pi)\Omega/\omega_s]. \quad (\text{A-14})$$

Using (A-14), we obtain, for $2n\pi < \nu < 2(n+1)\pi$:

$$S^+(\nu, \Omega) = \frac{\pi}{\omega_s} \frac{\sin\left\{\left[\nu - \pi(n+1)\right] \frac{\Omega}{\omega_s}\right\} \sin\left[\pi(n+1)\Omega/\omega_s\right]}{\sin^2(\pi\Omega/\omega_s)} \quad (\text{A-15})$$

$$S^-(\nu, \Omega) = \frac{\pi}{\omega_s} \frac{\sin\left\{\left[\nu - \pi n\right] \frac{\Omega}{\omega_s}\right\} \sin\left[\pi(n+1)\Omega/\omega_s\right]}{\sin^2(\pi\Omega/\omega_s)} \quad (\text{A-15})$$

BUNCH LENGTHENING DUE TO LARGE REAL FREQUENCY SHIFTS

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SACHERER'S THEORY OF SINGLE BUNCH INSTABILITY¹

Coherent longitudinal oscillations of a bunch of charged particles inside a vacuum chamber characterized by an impedance $Z(\omega)$ can be described by solving the Vlasov equation for the perturbed distribution function

$$\psi_m = R_m(r) \cdot e^{jm\theta} \cdot e^{j\omega_m r} \quad (1.1)$$

where r and θ are polar phase-space coordinates, and m is the mode-number characterizing the polarity of the oscillation ($m = 1$ dipole, $m = 2$ quadrupole, etc.). The frequency of oscillation is given by

$$\omega_m = m\omega_s - \Delta\omega_m \quad (1.2)$$

where ω_s is the (incoherent) synchrotron frequency, and ω_m the frequency shift of the m -th mode.

The radial functions $R_m(r)$ are given by an integral equation, and the mode-frequencies ω_m as its eigenvalues. Since there is an infinite number of solutions for each value of m , the modes are further characterized by a second mode-number which is usually taken to equal the number of zeros of the radial function ('radial mode number').

The radial functions can be developed into a series of (orthogonal) eigenmodes of the unperturbed distribution. Then the integral equation becomes an infinite matrix equation for the expansion coefficients. The mode frequencies are given by the condition that the determinant of the matrix vanishes

$$|(\omega_m - m\omega_s)\delta_{mk} - M_{mk}| = 0 \quad (1.3)$$

where

$$M_{mk} = jC \frac{\sum_p \frac{Z(\omega_p)}{p} h_{mk}(p)}{\sum_p h_{mm}(p)} \quad (1.4)$$

and C is a constant (depending on machine parameters and proportional to the beam current)

$$\omega_p = p\omega_0 + m\omega_s \text{ for } -\infty < p < +\infty \quad (1.5)$$

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$$h_{mk}(p) = \tilde{\lambda}_m^*(p) \tilde{\lambda}_k(p) \quad (1.6)$$

when $\tilde{\lambda}_m(p)$ is the p -th Fourier-component of the perturbed line-density for mode m .

For a parabolic distribution in the presence of a purely reactive impedance, the matrix is diagonal (for Legendre modes) and we get

$$\Delta w = M_{mm} = j \frac{32 m w_s I_0}{\pi^4 (m+1) B^3 h V_T \cos \phi_s} \frac{\sum_p \frac{Z(w_p)}{p} h_{mm}(p)}{\sum_p h_{mm}(p)} \quad (1.7)$$

Since we assumed a purely imaginary Z , the frequency shift is purely real and does not explain the instability leading to bunch-lengthening of a single bunch by "turbulence". Even adding a small resistive part leads only to a slow growth (except for very narrow band resonators where it yields the Robinson instability) rather than to the fast blow-up observed in most machines.

In the presence of a resonant impedance, the matrix is no longer diagonal even for parabolic distributions. Modes with different mode-numbers m will be coupled and can become unstable if their mode-frequencies become equal. However, this mechanism requires a very strong resonator at very high frequency (often above cut-off of the vacuum tube) which is not likely to be present in every - if any - machine where bunch-lengthening occurs.

GENERALIZATION TO NEGATIVE MODE-NUMBERS

Because the equation of an oscillator is of second degree, the frequency shift due to a perturbation is usually given by a quadratic formula

$$w^2 = w_0^2 \pm \delta^2 \quad (2.1)$$

(which is sometime linearized if $\delta \ll w_0$ to $w = w_0 \pm \delta^2/2w_0$). A formula of this type can be obtained in Sacherer's theory if we include negative mode-numbers m in Eq. (1.1). Again expressing the equation in terms of positive values m and k only, we get, instead of Eq. (1.3)

$$|(w_m^2 - m^2 w_s^2) \delta_{mk} - 2m w_s M_{mk}| = 0 \quad (2.2)$$

If the matrix M is diagonal, we obtain

$$\Delta\omega = \omega_m - m\omega_s = m\omega_s \left[\sqrt{1 + \frac{2M_{mm}}{m\omega_s}} - 1 \right] \quad (2.3)$$

For $M_{mm} \ll m\omega_s$, we obtain again Eq. (1.7). However, if $M_{mm} < -m\omega_s/2$, the frequency shift becomes complex even for a purely reactive impedance (inductive above transition).

In physical terms we can interpret this as a loss of focusing when the frequency shift is equal to the synchrotron frequency itself. As a result of this, the bunch will blow up until some focusing is restored.

For a resonant impedance, we have found some time ago,² that the real part of M_{mm} [see Eq. (1.7)], can become quite large - compared to the intuitive approximation obtained by including only the sidebands next to the resonance frequency of a high-Q resonator - when the sum is evaluated over all sidebands. This is due to the slow fall-off of $\text{Im}Z(\omega_p)/p$ with p (compared to $\text{Re}Z(\omega_p)/p$, for which furthermore terms with p and $-p$ almost cancel).

Resonant impedances are always present in vacuum chambers (cross-section variations, higher harmonics of rf etc.) and don't need to be very strong to yield a large frequency shift. Their resonant frequencies may be much lower than those required to couple higher modes - as discussed in the first section - and may certainly be below the cut-off frequency of the vacuum tube.

DISCUSSION

Measurements of the frequency of the quadrupole mode in SPEAR³ show a variation of less than 10% up to the threshold for turbulent bunch-lengthening. In the model discussed above the dipole mode should have a 100% shift at that current (and hence be invisible) but also the quadrupole mode should have a much larger shift than observed in SPEAR.

On the other hand, observation at DORIS with rf excitation indicate the existence of strongly shifted mode-frequencies in that machine.⁴

Observations are thus rather ambiguous, but there is strong doubt whether the whole theory may be applied to electron machines with (single) short bunches. The method used is based on a steady-state solution of the Vlasov equation (single frequency solution) while for short bunches there is not enough time to reach equilibrium between oscillating fields and particles and transient effects are dominant. For this situation, the time-domain approach (Wakefield) appears to be better adapted than the frequency domain solution discussed here.

For long proton bunches, however, the approach shown here should be applicable but experimental verification is still lacking.

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WAKE FIELD IN MATCHED KICKER MAGNET

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Previously we have reported about coherent transverse instability observed in KEK booster proton synchrotron.^{1,2} This instability is induced by the interaction of the beam with kicker magnet for the fast beam extraction. To understand the mechanism completely, it is necessary to know the wake field in detail. We consider here the wake field or induced current in the kicker magnet which is terminated with matched resistance.

Kicker magnet is represented by a LC-lamped ladder network. The beam passing through the gap of the magnet induces current in the coil of the magnet through magnetic interaction. The relation between the beam current and induced one is given by mutual inductance. The induced current travels away in both directions of the magnet and it is absorbed on one side into the matched resistance, and on the other side it goes to the power supply. Since the cable of the power feed is fairly long we neglect the reflection of the induced current. Thus we imagine an infinitely long ladder network, a part of which is connected with mutual inductances. (See Fig. 1). Here we neglect the loss in the network, although it is very important in connection with the beam instability because the loss brings about phase lag and phase relation is essential to the instability.

Fig. 1.

We now consider what field is induced and how it travels through the network when a single particle has passed through the gap of the magnet. The problem is decomposed into three parts. One is the induced current $i_c(t)$ in Fig. 2a., the second is the travel through symmetric network of Fig. 2b., and the third is the summation of the induced currents that originate from all mutual inductances. In the following we consider these problems by turns.

Fig. 2a.

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Fig. 2b.

In Fig. 2a. the induced currents $i_k(t)$ ($k = a, b$ and c) are given by the following relations

$$\left. \begin{aligned} Z_0 i_a + \frac{sL}{2} i_a + (i_a - i_b) \frac{1}{sC} &= 0 \\ Z_0 i_c + \frac{sL}{2} i_c + (i_c - i_b) \frac{1}{sC} &= 0 \\ \frac{1}{sC} (i_b - i_a) + \frac{1}{s} (i_b - i_c) + sLI_b &= sMI_p \end{aligned} \right\} \quad (1)$$

where we used the operator $s = j\omega$. From these equations we get for $i_c(t)$

$$I_c(s) = \frac{sMI_p(s)}{2Z_0 + Z_0 s^2 LC + ZsL + s^3 L^2 C/2} \quad (2)$$

where we have applied the Laplace transformation to $i_c(t)$ and $i_p(t)$ and represented them with $I_c(s)$ and $I_p(s)$.

Next we consider about the ladder network of Fig. 2b. It is well known that in a symmetric four port the transmission matrix is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cosh \xi & Z_0 \sinh \xi \\ \frac{1}{Z_0} \sinh \xi & \cosh \xi \end{pmatrix} \quad (3)$$

where $Z_0 = \sqrt{B/C}$ and $\sinh \xi = \sqrt{BC}$. From this matrix we get for the induced current at the n -th port

$$I_n(s) = I_0(s) e^{-n\xi} \quad (4)$$

where we have already made the Laplace transformation. $I_0(s)$ should be replaced by $I_c(s)$. The matrix of one section of Fig. 2b. is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 + s^2 LC/2 & sL(1 + s^2 LC/4) \\ sC & 1 + s^2 LC/2 \end{pmatrix} \quad (5)$$

Therefore we get

$$Z_o = \sqrt{\frac{L}{C}} \sqrt{1+s^2/\omega_c^2} \quad (6)$$

and

$$\sinh \xi = \frac{2s}{\omega_c^2} \sqrt{s^2 + \omega_c^2} \quad (7)$$

where ω_c is the cutoff frequency and given by $\omega_c = 2/\sqrt{LC}$. It should be noted that this network is low pass filter; for the frequency higher than ω_c , ξ becomes real and such a wave rapidly damps. Using the relation $\tanh \xi = (e^{2\xi}-1)/(e^{2\xi}+1)$, we can get

$$e^{-n\xi} = \frac{\omega_c^{2n}}{\left(\sqrt{s^2 + \omega_c^2} + s\right)^{2n}} \quad (8)$$

Next we consider the inverse transformation. It is difficult to obtain that of Eq. (2). Z_o depends on s as given by (6). So we make a crude approximation; we regard the ladder network of Fig. 2a. as continuous, so that $L \rightarrow 0$ and $C \rightarrow 0$ with $Z_o \rightarrow \sqrt{L/C}$ and $\omega_c \rightarrow \infty$. Then from Eq. (2) we get

$$I_c(s) \approx \frac{1}{2} \sqrt{\frac{C}{L}} M I_p \delta \quad (9)$$

where we have replaced $I_p(s)$ by $I_p(\text{constant})$ since we regard the single particle current as $I_p \delta(t)$, that is $I_p(s) = I_p$. Then we get from (4), (8) and (9)

$$I_n(s) \approx \frac{1}{2} \sqrt{\frac{C}{L}} M I_p s \frac{\omega_c^{2n}}{\left(\sqrt{s^2 + \omega_c^2} + s\right)^{2n}} \quad (10)$$

Denoting the inverse transformation by L^{-1} we get

$$L^{-1} \left(\frac{1}{2} \sqrt{\frac{C}{L}} M I_p s \right) = \frac{1}{2} \sqrt{\frac{C}{L}} M I_p \frac{d}{dt} \delta(t) \quad (11)$$

and

$$L^{-1} \left(\frac{\omega_c^{2n}}{\left(\sqrt{s^2 + \omega_c^2} + s\right)^{2n}} \right) = \frac{2n J_{2n}(\omega_c t)}{t} \quad (12)$$

where J_{2n} is Bessel function with $2n$ order. The inverse transformation of (10) is obtained by the convolution

$$L^{-1}(I_n(s)) \approx \frac{1}{2} \sqrt{\frac{C}{L}} MI_p \int_0^t \frac{J_{2n}(\omega_c t)}{t} \delta'(t-\tau) d\tau \quad (13)$$

Since $\delta'(t)$ is expressed as the limit of $a \rightarrow 0$

$$\delta'(t) = \lim_{a \rightarrow 0} \frac{1}{a} [\delta(a) - \delta(0)] \quad (14)$$

we get

$$L^{-1}(I_n(s)) \approx \sqrt{\frac{C}{L}} MI_p \frac{n\{\omega_c t J'_{2n}(\omega_c t) - J_{2n}(\omega_c t)\}}{t^2} \quad (15)$$

Using the relation

$$J'_m(x) = \frac{m}{x} J_m(x) - J_{m+1}(x)$$

and reminding $L^{-1}[I_n(s)] = i_n(t)$, we get

$$i_n(t) \approx \sqrt{\frac{C}{L}} MI_p \omega_c^2 f_n(x) \quad (16)$$

$$f_n(x) = \frac{n\{(2n-1)J_{2n}(x) - J_{2n+1}(x) \cdot x\}}{x^2}$$

where $\omega_c t = x$. Figure 3 shows $f_n(x)$ for some n . Since $J_{2n}(x)$ is almost zero for $x \leq 0.7 \cdot 2n$, we do not see any induced current in the n -th port before the time $t \leq 0.7 \cdot 2n/\omega_c$, which means the traveling time per one section is given by

$$\tau \approx 0.7 \times \frac{2}{\omega_c} = 0.7 \sqrt{LC} \quad (17)$$

After the arrival the induced current oscillates at the n -th mesh with an approximate frequency $\Omega \approx \omega_c/3.5$ and slowly damps. In a continuous medium, the induced current does not oscillate, but only pass through with the same wave form at the origin. The third step is obvious and depends on the length of the magnet.

Fig. 3.

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THE TWO STREAM INSTABILITY IN ELECTRON STORAGE RINGS

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Electron bunches in storage rings can ionize the background gas and depending on the bunch length and spacing between bunches the ions formed can be trapped by the electron space charge forces. It is the purpose of this note to show that the possibility of exciting the two stream instability then exists. This instability if excited would saturate by giving thermal velocity to the electrons i.e., by blowing up the emittance of the beam. Experiments carried out at ADONE¹ have observed an increase in the electron beam emittance in the presence of background ions.

The two stream instability has been studied extensively in the Plasma Physics Literature where one deals with high density beams. Energy is taken out of the relative beam motion and is deposited into the thermal energy of both electrons and the background ions. For an electron beam passing through a background ion plasma in a cylindrical conducting tube it has been shown² that the instability has a threshold. Figure 1 is a plot of the dispersion relation for the system under consideration.² Only if the beam intensity is high enough can the instability occur. In Fig. 1(a) and 1(b) for each value of k there are four real values of ω . However in Fig. 1(c) there are only two real values of ω for small k and hence two complex ω are possible for a range of k . One complex value³ of ω corresponds to growing waves (instability) and the other to damped waves. Computer simulations⁴ have been carried out to show that the excited instability saturates by giving thermal velocity to the electrons and ions. Figures 2(a) and 2(b) are the initial and final states of the distribution function of the unstable system.

Fig. 1. Dispersion curve for an electron beam propagating through background ions in a cylindrical tube. The beam intensity is progressively increased through the instability threshold.

Fig. 2. Initial and final distribution functions of an electron beam system propagating through background ions that is two stream unstable.

For Maxwellian type distributions

$$F_{eo} = \left(\frac{m_e}{2\pi\kappa T_e} \right)^{\frac{1}{2}} e^{-m_e(v-V_0)^2/2\kappa T_e} \text{ and } F_{io} = \left(\frac{m_i}{2\pi\kappa T_i} \right)^{\frac{1}{2}} e^{-m_i v^2/2\kappa T_i}$$

where V_0 is the velocity of the electron beam. It has been shown³ that for equal temperatures $T_e = T_i = T$,

$$V_0 < 1.3 \sqrt{\frac{\kappa T}{m_e}} \text{ for stability.}$$

In conclusion, we suggest that the two stream instability could account for emittance blow-up in high current electron storage rings and one may have to place clearing electrodes in the machine. In proton storage rings when one has a coasting beam and background electrons neutralizing the beam the two stream instability could also be excited. However, because of their heavier mass the emittance blow up of the proton beam should not be such a significant effect. Detailed theoretical calculations for both electron and proton storage rings need to be carried out before one can reach any definite conclusions.

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THE TACOMA MODE

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The name "Tacoma" refers to the Tacoma Narrows Bridge which collapsed on November 8, 1940 due to massive oscillations caused by high winds. One of the destructive modes was a torsion mode which was excited by transverse wind, a dipole force, and continued until the bridge collapsed.

We use the name to refer to a coherent mode of oscillation of a spectrum of oscillators in which the amplitude vs frequency graph contains one node, i.e.

where the node occurs near the driving frequency and $a(\omega)$ is not symmetric about zero. When this result is applied to vertical instabilities in coasting beams, it implies the existence of a coherent skew quadrupole moment, Q_{xy} , whenever a coherent dipole oscillation exists.

$$Q_{xy} \equiv \iint xy \rho_1(x,y) dx dy, \quad (1)$$

ρ_1 is the coherent part of the beam density.

To see the above result in a simple way consider a spectrum of oscillation x_k driven with a force proportional to their average displacement $\langle x_k \rangle$:

$$x_k + \omega_k^2 x_k = Z \langle x_k \rangle. \quad (2)$$

To find a coherent solution we let $x_k = a_k e^{-i\omega t}$ and obtain

$$a_k = \frac{Z \langle a_k \rangle}{\omega_k^2 - \omega^2}, \quad (3)$$

where ω is the coherent frequency. Notice that for $\omega_k > \omega$, a_k is in phase with $\langle a_k \rangle$ and for $\omega_k < \omega$, a_k is exactly out of phase with $\langle a_k \rangle$. If ω_k is real, a_k diverges at $\omega_k = \omega$; however, in systems of interest ω is generally complex so we obtain the behavior illustrated in Fig. 1. Note that Fig. 1 is necessarily asymmetrical about $a_k = 0$ since otherwise $\langle a_k \rangle = 0$. a_k is plotted vs frequency in Fig. 1; however if we

consider the case of vertical instabilities, then ω_k is linearly correlated with position due to the chromaticity. This means that a coherent dipole oscillation will be accompanied by a coherent skew quadrupole oscillation. In other words the beam twists as it oscillates vertically, much like the Tacoma Narrows Bridge did before its collapse.

In order to obtain a quantitative estimate of the size of the "skewness" due to this effect consider the typical equation of motion for the coherent density, $D(u)$:

$$(u - u_c) D(u) = \rho(u) G \int_{-1}^1 D(u) du \quad (4)$$

where $\rho(u)$ = beam density, $\int \rho(u) du = 1$

G = impedance

u = frequency variation in some arbitrary units

u_c = coherent frequency in those units.

If we write (4)

$$D(u) = \frac{\rho(u) G}{u - u_c} \int D(u) du \quad (5)$$

it bears a clear resemblance to Eq. (3). One can now integrate (5) to obtain the usual dispersion relation for u_c ,

$$1 = G \int \frac{\rho(u)}{u - u_c} du, \quad (6)$$

from which, for a given $\rho(u)$ we can calculate u_c . However, we can also integrate Eq. (4) directly to obtain

$$\begin{aligned} M &= (G + u_c) D, \\ \text{where} \quad M &\equiv \int u D(u) du \\ D &\equiv \int D(u) du \end{aligned} \quad (7)$$

M is the moment of $D(u)$ which is a measure of the asymmetry of $D(u)$ with respect to u .

Now let us consider a specific example: coherent vertical instabilities in ISABELLE.¹ In this case we take u , u_c , G , $\rho(u)$ as defined in section H of Ref. 1. $D(u)$ is the coherent dipole density, i.e., $D(u) du = \int y \rho_1(y, u) dy du$ where ρ_1 is the coherent beam density. The transverse dimension $x = \Delta u$, where Δ is the $\frac{1}{2}$ width of the beam, and $x = 0$ is at the beam center, ($-1 < u < 1$).

The coherent quadrupole moment Q_{xy} is given by

$$\begin{aligned} Q_{xy} &\equiv \int xy \rho_1(x, y) dx dy \\ &= \Delta \int u D(u) du \end{aligned} \quad (8)$$

so

$$Q_{xy} = \Delta M, \quad (9)$$

and from (7) we find

$$Q_{xy} = (G + u_c) \Delta D, \quad (10)$$

where D is the total coherent dipole moment. So the quadrupole moment is simply proportional to the dipole moment times the $\frac{1}{2}$ width of the beam.

If we now consider the parameters of an ISA small beam in Ref. 1, then G is a function of the beam center, so we can evaluate at each G the u_c that corresponds to the threshold of instability.

In Fig. 2 we plot the $|G + u_c|$ vs position of the beam center. For a beam near the center of the chamber $|G + u_c| \leq 1.0$ while near the edge it grows rapidly; in fact it increases by a factor of 100 for a small beam located at 9/10 of the chamber radius. If $|G + u_c|$ is large, a very small dipole moment can produce a large skew quadrupole oscillation.

The point here is that in some instances, a small dipole signal can be caused by an oscillation whose amplitude is out of this world... or at least out of the vacuum chamber.

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CRITICAL REVIEW OF THE THEORIES ON THE LONGITUDINAL INSTABILITIES OF INDIVIDUAL PROTON BUNCHES

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I. INTRODUCTION

The following is a brief review of theories that have been proposed in the past to explain longitudinal instabilities of individual bunches. The review is by no means complete, and we apologize for those references which happen not to be included in the list. We wanted to give emphasis to the most relevant developments of the past theoretical effort. We hope that this unified presentation can be of some help for the theorist that wants to give a further look, more in detail, of the phenomena in question. We shall limit ourselves only to the case of proton bunches though we recognize that theories have been developed also for electron bunches that could apply as well in either case.

II. THE PROTON BEAM APPROACH

2.1 The Coasting Beam Theory

In the beginning there was the coasting beam theory. It was originally formulated in two steps: the negative mass instability¹ and the resistive wall instability.² The process is Liouvillian, namely phase space area conserving, therefore one makes use of the generalized Liouville equation, the Vlasov equation, which is then linearized. (In the case of electron bunches, one starts from the Fokker Plank. The solution of this is usually referred to as turbulence.) The main result is a dispersion relation that one resolves to obtain growth rates at the onset of the instabilities and thresholds, either in terms of wall impedance Z_n and beam current I or in terms of beam relative momentum spread, Δ_p/p (FWHM).

The dispersion relation was originally solved for rectangular and Lorentzian energy distribution and, tentatively for a Gaussian distribution. Later a systematic investigation of the dispersion relation³ was undertaken for several physical distributions. As a result one obtained a stability diagram as shown in Fig. 1 where the coordinates on the axis are

$$U' - i V' = - i \frac{2e I Z_n / n}{\pi |\eta| E \left(\frac{\Delta p}{p} \right)^2}$$

and E is the particle total energy, $\eta = \gamma_T^{-2} - \gamma^{-2}$ and k_0 is related to η ; for the case of above the transition energy ($\gamma > \gamma_T$) $k_0 < 0$. For a given distribution function one has stability provided the normalized impedance value falls inside the region bounded by the continuous curve. One can notice that for negative values of $\text{Sign}(k_0)$ there is no significant dependence on the shape of the energy distribution, whereas this is a crucial factor for the opposite case.

In the original papers^{1,2} the wall wake field effects were described in terms of the "energy shift" $U - iV$. Later⁴⁻⁶ it was recognized that this can be simply related to the wall impedance Z_n estimated

at the frequency $\omega = n\omega_0$, multiple of the revolution frequency, so that the longitudinal field acting back to the center of the beam can be written as

$$E_z = - \frac{I_n Z_n}{2\pi R} e^{i(n\phi - n\omega_0 t)}$$

where $2\pi R$ is the machine circumference and I_n is the n -th Fourier harmonics of the beam current.

Finally it was pointed out by Keil and Schnell⁷ that above the transition region the stability diagram can be reasonably well fit with a circle so to lead to the following stability condition criterion

$$\left| \frac{Z_n}{n} \right| \leq \frac{\eta E_0}{eI} \left(\frac{\Delta p}{p} \right)^2 \quad (1)$$

Since then Eq. (1) has been the reference equation to estimate a coasting beam stability.

III. APPLICATION OF COASTING BEAMS THEORY TO BUNCHED BEAMS

Equation (1) has also been used to estimate whether an accelerated beam would freely debunch or not.⁸ An experiment carried out in the CERN-PS has indeed shown⁹ that debunching was inhibited, at least unless at cost of increasing the momentum spread, by a wall impedance that was estimated of about four hundred ohms, when Eq. (1) was strictly applied by taking for I the average current. Boussard⁹ then noticed that the responsible impedance could be made as small as few tens of ohms if the average current I at the rf side of (1) was replaced by the peak current I_p within the bunch. Boussard went a little further and introduced the concept of "local" stability across the bunch length. He pointed out that if the bunch has parabolic charge as well as momentum distribution the "local" value of $(\Delta p/p)^2/I$ would remain an invariant and therefore (1) would apply identically at any location within the bunch. Unfortunately the parabolic distribution, as one can see from Fig. 1, does not yield a stability region too consistent with (1).

The same simple application of the coasting beam theory for bunched beams was later used also in a Fermilab experiment.¹⁰ Actually both experiments at Fermilab and at CERN (PS and ISR)^{9,16} have shown a strong coherent signal in the range of 1 to 2 GHz, corresponding to a wave length much shorter than the bunch length. This prompted the use of the term "microwave" instability for the effect under investigation.

In a second period Hereward¹¹ tried to justify the application of the coasting beam criteria (1) to bunched beams in spite of the fact that Sacherer in the mean time claimed to have proven that individual bunches cannot possibly be longitudinally unstable (see sec. 3-1). Hereward tried a compromise between Sacherer theory and Boussard approach also relying on some suggestions from Hardt, who has been trying to explain negative mass instabilities for bunched beams just after transition.¹² Hereward argues that, as for a coasting beam, also in a bunch there are two possible "waves", one stable the other unstable, one fast and the other slow, one propagating on top of the bunch the other on the bottom. Actually these two waves are the branches of one propagating mode around

The various curves correspond to different distributions:

1. Lorentzian
2. Gaussian
3. 5th-order parabola
4. 4th-order parabola
5. 3rd-order parabola
6. Squared cosine
7. 2nd-order parabola
8. Truncated cosine
9. 1st-order parabola

Fig. 1. Longitudinal stability diagram.

the contour of the bunch. An instability would grow when the wave travels on one side of the bunch, to be damped when the wave is on the other side. After one complete oscillation therefore one would remain with zero effect. All this agrees with Sacherer theory (again, see later), except that, according to Hereward, an instability would nevertheless be possible if two conditions are met: (i) the perturbation wave length is much smaller than the bunch length, and (ii) the instability is considerably fast compared to one phase oscillation period. In this case the main result of the coasting beam theory, the stability criterion (1) can again be applied "locally" to a bunch. Moreover, the definition of the mode number n is unique: it is the ratio of the machine circumference $2\pi R$ to the perturbation wave length. This theory has the drawback, nevertheless, that it is not capable to estimate the effect of the bunch edges.

3.1. Sacherer Theory¹³⁻¹⁵

This is probably the most remarkable theory that has been proposed because of the considerable amount of effort devoted to it. It is also one of the most complete theories. The original point of Sacherer was to unify bunch-to-bunch modes in individual-bunch modes of instability. But here we shall look only after the contribution to the individual bunch instability.

Sacherer recognized that an instability mode can be characterized by a wave travelling around the contour of the bunch. He carries as far as he can the analogy with the coasting beam for instance, the synchrotron frequency now takes place of the revolution frequency; moreover, whereas in the coasting beam the revolution frequency spread is crucial for the stability via Landau damping, in the bunched beam this role is played by the synchrotron frequency spread. In both cases nevertheless the involved spread can be measured by the momentum spread of the beam, bunched or not. Sacherer introduces also for the first time the so-called individual-bunch mode number m , which is the number of waves one counts around the contour of the bunch; for instance $m = 1$ is a dipole, $m = 2$ a quadrupole $m = 3$ a sextupole mode, and so on. Observe that mode number m cannot be simply related to the coasting beam mode number n appearing in (1). Moreover, Sacherer did not have to require, as in Hereward theory, for the perturbation to be short compared to the bunch length.

In the theory of the bunched beam two major problems arise that did not exist for the case of coasting beams: (i) If the beam is bunched, the current has a frequency spectrum also in absence of a perturbation. Moreover, because the perturbation is travelling along the contour of the bunch, it cannot be represented by a single frequency $\omega = n\omega_0$ as in the coasting beam theory, but it has also its own frequency spectrum. Therefore the longitudinal electric field acting back to the center of the bunch is a summation of products of current harmonics with the corresponding wall impedance. In the time domain representation, the summation can be replaced by the associated Green function. Because of all this, Sacherer introduced for the first time the concept of "effective" impedance Z_{eff} which will depend on the individual bunch mode number m . (ii) Whereas the electromagnetic part of the problem is "linear", in the sense that the longitudinal field is simply given by multiplying an impedance to a current,

both "linear" observables, and by summing over all possible frequencies, the beam dynamics is "circular", because of the motion of the particles around a common center.

These two elements are not "commensurable" and cause large mathematical difficulties when trying to solve the problem. Very recently the question of "incommensurability" has been raised again and the problem has been circumvented by employing a rectangular shaped bunch.^{17,18} With this approach one would corroborate the stability criterion (1) once more also for bunched beams.

Sacherer's approach was to search for a system of eigenfunctions that could be used to describe the bunch modes. Since the theory is linear, at the onset of the instability the modes were required not to interact with each other. Therefore the eigenfunctions are also required to form an orthogonal system. But even with a single parabolic charge distribution the bunch mode would be expressed by highly sophisticated functions (the Legendre functions). In conclusion one has a hard time to obtain a dispersion relation as in the coasting beam case, unless unrealistic distribution functions and impedance models are chosen. Sacherer then, as a way out, makes an approximation which is practically an average over one phase oscillation. He is then capable to obtain a dispersion relation, but his main result is that one individual bunch cannot exhibit instability if the range of the wake field is short enough. For a very high Q revolution he can though recover the Robinson individual bunch instability¹⁹ since now the interaction with the impedance propagates over several revolutions.

In a later paper²⁰ Sacherer proposes that an individual bunch can be unstable for practically any sort of wake field when the beam intensity is so large that two neighboring modes start to interfere with each other. He obtains there the following stability criterion

$$\frac{I_0}{2B_0 h V_T (\cos \varphi_s)} \left| \frac{Z_n}{n} \right| \leq 1.15 \quad (2)$$

where I_0 is the peak current, h the rf harmonic number B_0 is the ratio of the bunch length to the machine circumference, V_T the peak rf voltage including space charge modification and φ_s the synchrotron phase. The curious fact about Eq. (2) is that Sacherer devised it without invoking Landau damping.

3.2. Further Developments

At the same time Sacherer was producing his own work other theories were actually being proposed. Messerschmid and Month²² had still the intent to justify theoretically the application of the coasting beam theory to bunched beams. They start with the same approach as proposed by Sacherer, but to circumvent the "incommensurability" problem we mentioned above they make use of two important approximations. After linearizing the Vlasov equation they neglect the coupling of the synchrotron motion to the perturbation induced motion. By doing this they eliminate the contribution of the synchrotron frequency sidebands which were so crucial in the Sacherer theory, which actually with their mutual cancellation were inhibiting a bunch to be unstable.

When the synchrotron side lines are removed then it is possible to obtain a dispersion relation very similar to the one devised for a coasting beam. The synchrotron frequency never appears in the analysis, but only the revolution frequency spread happens to be important for the beam stability. This result is justifiable only in case of a fast instability as it was pointed out by Hereward. The second approximation is the usual average over the phase population. The stability criterion can again have the form of Eq. (1) where now Z_n is replaced by an effective impedance Z_{eff} which for a Gaussian bunch of angular spread Q_{rm} is given by

$$Z_{eff} = \sum_l Z_l e^{il^2 \theta_{rms}^2}$$

In summary this theory is not capable to explain the slow growth of bunched beams as it has been recently observed in CERN-SPS²³ and in the Fermilab Main Ring.²⁹

Finally one more theory was proposed by Ruggiero.²⁵ Once again the starting approach is similar to the one proposed by Sacherer. Again one makes phase average approximation and obtains linearized Vlasov equation which is not "constant", as Sacherer put it, namely is a function not only of the collective frequency but also of the radial variable (r is Sacherer notation and H is Ruggiero notation). As often pointed out the complexity is derived from the impedance function which must be integrated over the bunch spectrum with the perturbation. A way out, as noted also by Sacherer, is to simplify the self-force so that the Vlasov equation becomes independent of the radial variable. Sacherer chose two models: in one the self-force can excite only and only one mode, for example the dipole mode, in the other the interaction is local, like for instance space charge or an inductive wall. By doing this a dispersion relation is obtained very similar to the one for coasting beam. Unfortunately because of the model chosen (a force reactance) no instability is possible. Ruggiero points out that a local interaction is possible also with a pure constant resistance which actually can make the bunch unstable. This, moreover, would not require the two stringent conditions of short wave length and fast growth as proposed by Hereward.

As we said the dispersion relation for bunched beams is very similar to the one for coasting beams and can be solved with the same techniques. Nevertheless the variables involved are different.

The impedance Z_n for one coasting beam mode n is replaced by an effective impedance Z_{eff} for a particular bunch mode p . The revolution frequency is replaced by the synchrotron frequency, whereas the distribution over the radial variable, the Hamiltonian H , now takes the place of the distribution over the energy (w). Also for the bunched beam there are two waves solution, travelling around the contour of the bunch. One wave travels with the main rotation of the bunch, is "fast" and on the outer border; the other travels in the opposite direction, is "slower" and is inside, around the center of the bunch. Usually one of these waves is always unstable.

The stability criterion obtained by Ruggiero is very similar to (1) and can be written as

$$\left| \frac{Z_{\text{eff}}}{p} \right| < 0.861 \frac{\alpha E}{eI} \left(\frac{\Delta p}{p} \right)^2 B_0$$

where $Z_{\text{eff}} = Z/B_0 h$, Z is the resistance, I is the bunch average current and v the momentum compaction factor.

IV. CONCLUSION

It seems to us that the technique to evaluate the longitudinal stability of individual proton bunches is well established. The technique is essentially the one outlined by Sacherer with some modifications. Let us briefly review the entire method in steps and point out where further research is needed.

1. Assign an impedance model, that is either the complex impedance $Z(\omega)$ as function of frequency ω or the associated Green function $G(\varphi)$.

2. Define a particle distribution in the phase space of "rectilinear" coordinate φ and w

$$\psi = \psi(\varphi, w, t)$$

a linear charge distribution

$$g = \int \psi \, dw$$

and the self-force

$$F = \int_{-\varphi}^{\varphi} G(\varphi - \varphi') g(\varphi') \, d\varphi' \quad (3)$$

3. Introduce an external focusing potential function $U_{\text{ext}}(\varphi)$ and the current-impedance induced potential $U_{\text{sc}}(\varphi, t)$. The self-force (2) is derived from the latter.

Write the equation of motion

$$\begin{aligned} \dot{\varphi} &= kw \\ \dot{w} &= -e \frac{dU_{\text{ext}}}{d\varphi} - eF \end{aligned}$$

which can be derived from the Hamiltonian

$$H = \frac{1}{2} kw^2 + e U_{\text{ext}} + e U_{\text{sc}}$$

4. Write Vlasov equation from Liouville theorem

$$\frac{d\psi}{dt} \equiv \frac{\partial \psi}{\partial t} + \dot{\varphi} \frac{\partial \psi}{\partial \varphi} + \dot{w} \frac{\partial \psi}{\partial w} = 0 \quad (4)$$

and search for a stationary distribution in the presence of both external and induced forces. This is obtained by letting

$$\frac{\partial \psi_s}{\partial t} = 0$$

and imposing that U_{sc} does not depend explicitly on time.

The stationary distribution depends on $Z(w)$ and on the bunch current. The search of ψ_s includes the linear effects of the "modification of the potential well."

All possible solutions are

$$\psi_s = \psi_s(H)$$

and there are therefore infinite choices.

5. Investigate the stability of the stationary solution so found. One can do this by adding a perturbation ψ_p and check whether this is damped or not. The bunch is unstable if the perturbation grows. Therefore

$$\psi = \psi_s + \psi_p$$

6. It is convenient to transform from old "rectilinear" variables φ, w to a new pair of canonical angle-action variables, (r, θ) in Sacherer,¹ notation, (H, k) in Ruggiero's notation. The transformation can be accomplished with the side of the Hamiltonian-Jacobi method. One new variable is therefore the Hamiltonian H itself, for the other one can quite generally write

$$K = t - \frac{v}{\Omega_s(H)}$$

7. Define or search for perturbation modes. Quite generally

$$\psi_r = \psi_r(H, k) = \psi_r(H, t - \frac{v}{\Omega})$$

with a Fourier expansion

$$\psi_p = \sum_p A_p(H) \exp[i(p - \Omega t)]$$

At the onset of the instability one can assume that all modes (1) are normal with each other so one can investigate the stability of a one and disregard the others. Obviously this is a crucial step of the method and it is adopted because of some physical intuition but it is not otherwise corroborated.

8. Develop the Vlasov equation (3) to obtain

$$F_p \frac{d\psi_s}{dw} - i(\Omega - p\Omega_s)\psi_s = 0 \quad (5)$$

where

$$F_p = \int_0^\varphi G(\varphi - \varphi') g_p(\varphi') d\varphi'$$

$$g_p(\varphi) = \int \psi_r dw$$

and a second order term $F_p \partial \psi_p / \partial w$ has been neglected since we are only interested to investigate the bunch stability in proximity of the "threshold". Otherwise Eq. (4) would be exact.

4. Take a phase average of (4). Multiply both sides by $\exp[-i(pv - \Omega t)]$ and integrate over v from 0 to 2π . The integration is done over constant H . One obtains

$$Q_p \frac{d\psi_s}{dH} - i(\Omega - p\Omega_s)A_p = 0 \quad (6)$$

where

$$Q_p = \frac{k}{2\pi} \int_0^{2\pi} dv \{ w e^{-ipv} \int_0^\varphi G(\varphi - \varphi') [\int A_p(H) e^{ipv} dw] d\varphi' \} \quad (7)$$

Equation (5) depends on the radial variable H and therefore cannot be yet manipulated to obtain a dispersion relation similar to the one for coasting beam.

But what is the meaning of this average approximation? (The equivalent in Sacherer theory is the scalar multiplication of (4) with its "adjoint"). Do we miss some essential information in this way? Does it require that the instability has to be eventually slow compared to our phase oscillation?

10. Clearly the future effort should be concentrated on the integration of Eq. (5). Sacherer and Besnier²¹ have already tried in this sense but with no major success. One simple way is to replace the Kernel (6) with a "synthetic Kernel" or to choose unphysical impedance and distribution model.

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SYNCHROTRON RADIATION FROM A HELICAL WIGGLER

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I. INTRODUCTION

The use of wiggler magnets as an improved source of synchrotron radiation from high energy electron storage rings was proposed a few years ago.^{1,2} Since then it has also been suggested that synchrotron radiation from wiggler magnets placed in proton machines³ can be used to monitor energy, dimensions and position of the beam and that this effect is even more interesting in proton storage rings where the need to "see" the beam is greater. Most of the calculations carried out so far consider radiation from a single particle in a wiggler which is appropriate when the beam is radiating incoherently. In this paper we shall develop a general formalism for the case when the beam radiates coherently. We shall then apply these results to both electron and proton storage rings. For the electron case we shall derive an expression for the length of the bunch to use it as a more intense coherent radiation source. For proton machines we shall show how the radiation can be used to measure energy, current, transverse dimensions and longitudinal density variations in the beam.

II. THEORETICAL CALCULATIONS

We consider a helical wiggler of period λ_w with N periods and magnetic field $B = B_w \{ \hat{x} \cos(2\pi z/\lambda_w) + \hat{y} \sin(2\pi z/\lambda_w) \}$ as shown in Fig. 1. The wiggler parameter is given by $K = eB_w \lambda_w / 2\pi mc^2$. For electron machines the value of K can be made to vary, however because of the heavier mass of protons $K \ll 1$ is always valid for proton machines. The helical orbit of a particle in the helical magnetic field is shown in Fig. 2. The wavelength of radiation emitted by the particle is given by¹

$$\lambda = \frac{\lambda_w}{2\gamma^2} (1 + K^2 + \gamma^2 \theta^2)$$

where θ is the angle of observation ($\theta = 0$ on axis). The fractional line width $\Delta\lambda/\lambda \sim 1/N$. We shall assume, here, that other factors contributing to the line width are smaller than $1/N$. This implies that the energy spread of the beam must satisfy $\Delta E/E < 1/N$. Also, the angular size of the detector and the angular divergence of the beam $\sigma'_{x,y}$ in both transverse planes must satisfy $\sigma'_{x,y} < 1/\sqrt{N}$.

Now, the nature of the radiation emitted by a beam of N_p particles contained in a volume V depends on the interparticle distance $(V/N_p)^{1/3}$ and the wavelength emitted. If $\lambda < (V/N_p)^{1/3}$ one can calculate the energy radiated by a single particle and add up the contribution from all the particles. The beam is then radiating incoherently. However, if $\lambda > (V/N_p)^{1/3}$ one needs to consider the beam as a continuous distribution of charge and since waves emitted by the different particles interfere with each other the beam radiates coherently. The two cases are considered separately.

Fig. 1. A helical wiggler which is made up of two helical conductors of period λ_w in which the current is flowing in opposite directions. In the wiggler, near the axis

$$\vec{B} = B_w \left\{ \hat{x} \cos \frac{2\pi z}{\lambda_w} + \hat{y} \sin \frac{2\pi z}{\lambda_w} \right\}.$$

Fig. 2. Helical orbit of a charged particle in the helical magnetic field showing parameters used in the text.

A. Incoherent Radiation

$$\left[\lambda < \left(\frac{v}{N_p} \right)^{1/3} \right].$$

The energy radiated per unit solid angle per unit frequency interval by a single particle is⁴

$$\frac{d^2 I}{d\Omega d\omega} \Big|_{S.P.} = \frac{e^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} \hat{n} \times (\hat{n} \times \vec{\beta}) e^{i\omega[\tau - \hat{n} \cdot \vec{r}(\tau)]/c} d\tau \right|^2. \quad (1)$$

For N_p particles radiating incoherently the energy radiated by all the particles is

$$\left. \frac{d^2 I}{d\Omega d\omega} \right|_{S.P.} = N_p \left. \frac{d^2 I}{d\Omega d\omega} \right|_{S.P.} \quad (2)$$

For a single particle passing through a helical wiggler Eq. (1) has been evaluated by Kincaid¹ and is given by:

$$\left. \frac{d^2 I}{d\Omega d\omega} \right|_{S.P.} = \frac{e^2 \omega^2 K^2}{\pi^2 c \omega_0^2 \gamma^2} \sum_{n=1}^{\infty} \left[J_n'^2 \left(\frac{K \omega \theta}{\gamma \omega_0} \right) + \left(\frac{\gamma \theta}{K} - \frac{n \gamma \omega_0}{K \omega \theta} \right)^2 J_n^2 \left(\frac{K \omega \theta}{\gamma \omega_0} \right) \right] \frac{\sin^2 [N \pi (\omega/\omega_1 - n)]}{(\omega/\omega_1 - n)^2} \quad (3)$$

where

$$\omega_0 = \frac{2\pi \beta^* c}{\lambda_w}, \quad \beta^* = \beta \left[1 - \left(\frac{K}{\gamma} \right)^2 \right]^{\frac{1}{2}}, \quad \omega_1 = 2\gamma^2 \omega_0 / (1 + K^2 + \gamma^2 \theta^2)$$

and J_n , J_n' are the n th order Bessel function and its derivative. Thus the spectrum consists of a series of harmonics of the fundamental frequency ω_1 , with the intensity of the n th harmonic given by the n th term in the series. For N large i.e., as $N \rightarrow \infty$, $\sin^2 N \pi x / x^2 \rightarrow N \pi^2 \delta(x)$.

Using this approximation Eq. (3) becomes:

$$\left. \frac{d^2 I}{d\Omega d\omega} \right|_{S.P.} = \frac{N e^2 \omega^2 K^2 \omega_1}{c \omega_0^2 \gamma^2} \sum_{n=1}^{\infty} \left[J_n'^2 \left(\frac{K n \omega \theta}{\gamma \omega_0} \right) + \left(\frac{\gamma \theta}{K} - \frac{n \gamma \omega_0}{K \omega \theta} \right)^2 J_n^2 \left(\frac{K n \omega \theta}{\gamma \omega_0} \right) \right] \delta(\omega - n \omega_1). \quad (4)$$

The energy radiated per unit solid angle is given by¹:

$$\begin{aligned} \frac{dI}{d\Omega} \Big|_{S.P.} &= \int_0^{\omega} \left. \frac{d^2 I}{d\Omega d\omega} \right|_{S.P.} d\omega \\ &= \frac{N e^2 K^2 \gamma \omega_1^3}{c \omega_0^2} \sum_{n=1}^{\infty} n^2 \left[J_n'^2 \left(\frac{K n \omega_1 \theta}{\gamma \omega_0} \right) + \left(\frac{\gamma \theta}{K} - \frac{\gamma \omega_0}{K \omega_1 \theta} \right)^2 J_n^2 \left(\frac{K n \omega_1 \theta}{\gamma \omega_0} \right) \right] \end{aligned} \quad (5)$$

Since Eq. (4) can also be written in the form

$$\left. \frac{d^2 I}{d\Omega d\omega} \right|_{S.P.} = \frac{2 N e^2 K^2 \omega}{c \omega_0^2} \sum_{n=1}^{\infty} \left[J_n'^2 \left(\frac{K n \alpha}{\gamma^2 \omega_0} \right) + \left(\alpha - \frac{n \gamma^2 \omega_0}{K \omega \alpha} \right)^2 J_n^2 \left(\frac{K n \alpha}{\gamma^2 \omega_0} \right) \right] \delta(\alpha^2 - \alpha_n^2) \quad (6)$$

with

$$\alpha^2 = \gamma^2 \theta^2 \quad \text{and} \quad \alpha_n^2 = \frac{2 \gamma^2 \omega_0 n}{\omega} - 1 - K^2.$$

We get energy radiated per unit frequency interval as¹:

$$\begin{aligned} \left. \frac{dI}{d\omega} \right]_{S.P.} &= \int_0^{2\pi} d\phi \int_0^{\pi} \left. \frac{d^2 I}{d\Omega d\omega} \right]_{S.P.} \sin\theta d\theta = \frac{2\pi}{\gamma^2} \int_0^{\infty} \left. \frac{d^2 I}{d\Omega d\omega} \right]_{S.P.} \alpha d\alpha \\ &= \frac{2\pi N e^2 K^2 \omega}{\gamma^2 \omega_0^2 c} \sum_{n=1}^{\infty} \theta(\alpha_n^2) \left[J_n^2 \left(\frac{K \omega \alpha_n}{\gamma^2 \omega_0} \right) + \left(\frac{\alpha_n}{K} - \frac{n \gamma^2 \omega_0}{K \omega \alpha_n} \right)^2 J_n^2 \left(\frac{K \omega \alpha_n}{\gamma^2 \omega_0} \right) \right] \quad (7) \end{aligned}$$

where $\theta(\alpha_n^2)$ is the Heaviside Step Function.

For $K \ll 1$, all the higher harmonics become negligible and the particle emits radiation only at $\omega = \omega_1$. Then,

$$\left. \frac{d^2 I}{d\Omega d\omega} \right]_{S.P.} = \frac{N e^2 \omega^2 K^2 \omega_1}{4 c \omega_0^2 \gamma^2} \left[1 + \left(\frac{\omega \theta^2}{\omega_0} - 1 \right)^2 \right] \delta(\omega - \omega_1) \quad (8)$$

Also,

$$\left. \frac{dI}{d\Omega} \right]_{S.P.} = \frac{4 N e^2 \omega_0 K^2 \gamma^4 (1 + \gamma^4 \theta^4)}{c (1 + \gamma^2 \theta^2)^5} \quad (9)$$

and,

$$\left. \frac{dI}{d\omega} \right]_{S.P.} = \frac{\pi N e^2 K^2 \omega}{2 c \omega_0 \gamma^2} \left[1 + \left(1 - \frac{\omega}{\gamma^2 \omega_0} \right)^2 \right] \theta(\alpha_1^2) \quad (10)$$

where

$$\alpha_1^2 = \frac{2 \gamma^2 \omega_0}{\omega} - 1$$

The total energy radiated by a single particle is:

$$I_{S.P.} = \frac{8 \pi^2 N e^2 K^2 \gamma^2}{3 \lambda_w} \quad (11)$$

and $I = N_p I_{S.P.}$ if N_p particles radiate incoherently. The total energy radiated in the forward direction in a cone of angle θ is $\epsilon = I_{S.P.} f(\gamma^2 \theta^2)$ where

$$f(\gamma^2 \theta^2) = 1 - \frac{1}{2(1 + \gamma^2 \theta^2)^4} \left\{ 3 - 4(1 + \gamma^2 \theta^2) + 3(1 + \gamma^2 \theta^2)^2 \right\}.$$

Since the angular size of the detector must be smaller than $1/\sqrt{N}$ we calculate the radiation emitted in $\theta = 1/\sqrt{N}$ for $N = 50$, $\gamma^2 \theta^2 = 0.02$, $f(\gamma^2 \theta^2) = 0.057$ and $\epsilon = 0.057 I_{S.P.}$.

The total number of photons/proton that the detector collects is $n_{ph} = \epsilon/(hc/\lambda)$. Since $\gamma^2\theta^2 \ll 1$ we neglect the change in λ with θ and we get

$$n_{ph} = \frac{2}{3} \pi \bar{\alpha} N K^2 \times f(\gamma^2\theta^2) \quad (12)$$

where the finestructure constant $\bar{\alpha} = e^2/hc = 1/137$. The total number of photons per second that are detected is given by:

$$\dot{n}_{ph} = \frac{2}{3} \pi \bar{\alpha} N K^2 \frac{i_{av}}{e} \times f(\gamma^2\theta^2) \quad (13)$$

where i_{av} is the average current of the beam.

B. Coherent Radiation

$$\left[\lambda > \left(\frac{v}{N_p} \right)^{1/3} \right]$$

For a system of N_p particles radiating coherently we can consider the beam as a continuous distribution of charge of current density $\underline{J}(\underline{x}, t)$ in motion. Then,⁴

$$\frac{d^2 I}{d\Omega d\omega} = \frac{\omega^2}{4\pi^2 c^3} \left| \int dt \int d^3 x \hat{n} \times \hat{n} \times \underline{J}(\underline{x}, t) e^{i\omega[t - (\hat{n} \cdot \underline{x})/c]} \right|^2 \quad (14)$$

We consider a constant density charge distribution of length L , width d and thickness b propagating through the wiggler as shown in Fig. 3. In Eq. (14)

$$\begin{aligned} \hat{n} &= \sin\theta \cos\varphi \hat{x} + \sin\theta \sin\varphi \hat{y} + \cos\theta \hat{z} \\ \underline{J}(\underline{x}, t) &= \frac{N_p}{bdL} e c \underline{\beta} \theta(z - \beta^* ct) \theta(\beta^* ct + L - z) \theta(y - a \sin\omega_0 t) \theta(a \sin\omega_0 t + b - y) \\ &\quad \times \theta(x - a \cos\omega_0 t) \theta(a \cos\omega_0 t + d - x) \end{aligned}$$

and a is the radius of the helical orbit of a single particle. Evaluating Eq. (14) gives:

$$\begin{aligned} \frac{d^2 I}{d\Omega d\omega} &= \frac{2N_p^2 e^2 c^5}{\pi^2 b^2 d^2 L^2 \omega^4 \sin^4 \theta \cos^2 \theta \sin^2 \varphi \cos^2 \varphi} \left[1 - \cos \left(\frac{\omega d \sin\theta \cos\varphi}{c} \right) \right] \times \\ &\quad \left[1 - \cos \left(\frac{\omega b \sin\theta \sin\varphi}{c} \right) \right] \left[1 - \cos \left(\frac{\omega L \cos\theta}{c} \right) \right] \times \\ &\quad \left| \int dt \hat{n} \times [\hat{n} \times \underline{\beta}] e^{i\omega[t - (a/c)\sin\theta\cos\varphi\cos\omega_0 t - (a/c)\sin\theta\sin\varphi\sin\omega_0 t - \beta^* t\cos\theta]} \right|^2 \end{aligned}$$

Since θ is small ($\theta < 1/\gamma$)

$$\frac{d^2 I}{d\Omega d\omega} = \frac{8 N_p^2 c^6}{b^2 d^2 L^2 \omega^6 \theta^4 \sin^2 \varphi \cos^2 \varphi} \left[1 - \cos \left(\frac{\omega d \theta \cos \varphi}{c} \right) \right] \left[1 - \cos \left(\frac{\omega b \theta \sin \varphi}{c} \right) \right] \times$$

$$\left[1 - \cos \left(\frac{\omega L}{c} \right) \right] \frac{d^2 I}{d\Omega d\omega} \Big|_{S.P.} \quad (15)$$

We note that as $b, d, L \rightarrow 0$

$$\frac{d^2 I}{d\Omega d\omega} \rightarrow N_p^2 \frac{d^2 I}{d\Omega d\omega} \Big|_{S.P.} \quad (16)$$

This is just the collective effect one would expect when the dimensions of the bunch are smaller than the wavelength emitted.

Fig. 3. A constant density charge distribution propagating through the wiggler and the co-ordinate system used in the text.

Consider $\hat{n}$ (the unit vector to the observation point) in the xz plane. Then from Eq. (15)

$$\frac{d^2 I}{d\Omega d\omega} = \frac{4 N_p^2 c^4}{d^2 L^2 \omega^4 \theta^2} \left[1 - \cos \left(\frac{\omega d \theta}{c} \right) \right] \left[1 - \cos \left(\frac{\omega L}{c} \right) \right] \cdot \frac{d^2 I}{d\Omega d\omega} \Big|_{S.P.} \quad (17)$$

Note that $d^2 I / d\Omega d\omega$ does not depend on b , the transverse dimension of the beam in the yz plane.

Consider $\hat{n}$ in the yz plane. Then from Eq. (15)

$$\frac{d^2 I}{d\Omega d\omega} = \frac{4 N_p^2 c^4}{b^2 L^2 \omega^4 \theta^2} \left[1 - \cos \left(\frac{\omega b \theta}{c} \right) \right] \left[1 - \cos \left(\frac{\omega L}{c} \right) \right] \cdot \frac{d^2 I}{d\Omega d\omega} \Big|_{S.P.} \quad (18)$$

Now, $d^2 I / d\Omega d\omega$ does not depend on d , the transverse dimension of the beam in the xz plane.

Hence, sampling the radiation in the xz and yz planes gives a measure of the transverse dimensions d and b respectively of the beam.

For $\theta = 0$, $\hat{n}$ is on the z axis (valid if $\theta \ll c/\omega d \cos\phi$, $c/\omega b \sin\phi$) and we get:

$$\left. \frac{d^2 I}{d\Omega d\omega} \right]_{\theta=0} = \frac{2 N_p^2 c^2}{\omega^2 L^2} \left[1 - \cos \left(\frac{\omega L}{c} \right) \right] \left. \frac{d^2 I}{d\Omega d\omega} \right]_{S.P.} \quad (19)$$

$\theta=0$

Setting $\theta = 0$ in Eq. (4) gives

$$\left. \frac{d^2 I}{d\Omega d\omega} \right]_{S.P.} = \frac{N_e^2 K^2 \omega^2 \omega_1}{2 c \omega_0^2 \gamma^2} \delta(\omega - \omega_1) \quad (20)$$

$\theta=0$

Hence on axis all the higher harmonics vanish and we are left with a single narrow peak at

$$\omega = \omega_1 = \frac{2\gamma^2 \omega_0}{(1+K^2)}$$

Using Eqs. (19) and (20) and $\lambda = \lambda_w (1+K^2)/2\gamma^2$ we get:

$$\left. \frac{dI}{d\Omega} \right]_{\theta=0} = \left(\frac{N_p \lambda}{\pi L} \right)^2 \sin^2 \left(\frac{\pi L}{\lambda} \right) \times \left. \frac{dI}{d\Omega} \right]_{S.P.} \quad (21)$$

$\theta=0$

Figure 4 is a plot of Eq. (21). We see that radiation from particles separated by a distance $\lambda/2$ reach the observation point with phases such as to cancel out each others effect. If the length of the beam is written as $L = m\lambda + \ell$ ($0 < \ell < \lambda$ and m is an integer) then particles within each wavelength and hence within $m\lambda$ act destructively to cancel out each others effect.

For the case $L > \lambda$, the maximum value of

$$\left. \frac{dI}{d\Omega} \right]_{\theta=0} = \left(\frac{N_p \lambda}{L\pi} \right)^2 \left. \frac{dI}{d\Omega} \right]_{S.P.} \quad (22)$$

$\theta=0$

Now, if we had N_p particles radiating incoherently we know that

$$\frac{dI}{d\Omega} = N_p \left. \frac{dI}{d\Omega} \right]_{S.P.} \quad (23)$$

If we want the coherent effect to give us more intense radiation on axis then from Eqs. (22) and (23)

$$\left(\frac{N_p \lambda}{L\pi} \right)^2 > N_p$$

or

$$L < \frac{\sqrt{N_p} \lambda_w (1+K^2)}{2\pi\gamma^2} \quad (24)$$

Fig. 4. A plot of $dI/d\Omega]_{\theta=0}$, the energy radiated per unit solid angle on the axis of the wiggler against the length L of the bunch.

Next, using Eq. (15) we calculate

$$\frac{dI}{d\Omega} = \frac{8NN_p^2 e^2 K^2 c^5}{b^2 d^2 L^2 \omega_1^3 \omega_0^2 \gamma^2 \theta^4 \sin^2 \varphi \cos^2 \varphi} \sum_{n=1} \frac{1}{n^4} \left[1 - \cos\left(\frac{n\omega_1 d \theta \cos \varphi}{c}\right) \right] \times$$

$$\left[1 - \cos\left(\frac{n\omega_1 b \theta \sin \varphi}{c}\right) \right] \left[1 - \cos\left(\frac{n\omega_1 L}{c}\right) \right] \times \left[J_n^2\left(\frac{Kn\omega_1 \theta}{\gamma\omega_0}\right) + \left(\frac{\gamma\theta}{K} - \frac{\gamma\omega_0}{Kn\omega_1 \theta}\right)^2 \times \right.$$

$$\left. J_n^2\left(\frac{Kn\omega_1 \theta}{\gamma\omega_0}\right) \right] . \quad (25)$$

For $K \ll 1$ all higher harmonics become negligible and

$$\frac{dI}{d\Omega} = \frac{NN_p^2 e^2 K^2 c^5}{2b^2 d^2 L^2 \omega_1^5 \omega_0^8 \gamma^2 \theta^4 \sin^2 \varphi \cos^2 \varphi} \times (1+\gamma^2 \theta^2)(1+\gamma^4 \theta^4) \left[1 - \cos\left(\frac{\omega_1 d \theta \cos \varphi}{c}\right) \right] \times$$

$$\left[1 - \cos\left(\frac{\omega_1 b \theta \sin \varphi}{c}\right) \right] \left[1 - \cos\left(\frac{\omega_1 L}{c}\right) \right] . \quad (26)$$

In order to calculate $dI/d\omega$ we consider 2 cases: (a) $\hat{n}$ is near the xz plane and (b) $\hat{n}$ is near the yz plane. For case (a) since $\varphi = 0 \pm \varphi$,

with $\varphi' \ll 1$, using Eq. (15) we get:

$$\frac{d^2 I}{d\Omega d\omega} = \frac{8 N_p^2 c^6}{b^2 d^2 L^2 \omega_0^6 \varphi'^2} \left[1 - \cos\left(\frac{\omega d \theta}{c}\right) \right] \left[1 - \cos\left(\frac{\omega b \theta \varphi'}{c}\right) \right] \left[1 - \cos\left(\frac{\omega L}{c}\right) \right] \times \frac{d^2 I}{d\Omega d\omega} \Big|_{S.P.} \quad (27)$$

As long as $\Phi \ll 1$ we can calculate

$$\begin{aligned} \frac{dI(\Phi)}{d\omega} &= \int_{-\Phi}^{\Phi} d\varphi' \int_0^{\pi} \frac{d^2 I}{d\Omega d\omega} \sin\theta d\theta \\ &= \frac{16 N_p^2 e^2 K^2 \gamma^2 c^5}{b^2 d^2 L^2 \omega_0^5} \left[1 - \cos\left(\frac{\omega L}{c}\right) \right] \sum_{n=1}^{\infty} \theta(\alpha_n^2) \frac{1}{\alpha_n^4} \times \left[1 - \cos\left(\frac{\omega d \alpha_n}{\gamma c}\right) \right] \times \\ &\quad \left[J_n^2\left(\frac{K \omega \alpha_n}{\gamma^2 \omega_0}\right) + \left(\frac{\alpha_n}{K} - \frac{n \gamma^2 \omega_0}{K \omega \alpha_n}\right)^2 J_n^2\left(\frac{K \omega \alpha_n}{\gamma^2 \omega_0}\right) \right] \cdot \left\{ \frac{\cos \frac{\omega \alpha_n b \Phi}{\gamma c} - 1}{\Phi} + \right. \\ &\quad \left. \frac{\omega \alpha_n b}{\gamma c} \operatorname{Si}\left(\frac{\omega \alpha_n b \Phi}{\gamma c}\right) \right\} \end{aligned} \quad (28)$$

where $\operatorname{Si}(x) = x - x^3/3.3! + x^5/5.5! - x^7/7.7! \dots$

For $\Phi \ll \gamma c / \omega \alpha_n b$

$$\begin{aligned} \frac{dI(\Phi)}{d\omega} &= \frac{8 N_p^2 e^2 K^2 c^3}{d^2 L^2 \omega_0^3} \Phi \left[1 - \cos\left(\frac{\omega L}{c}\right) \right] \sum_{n=1}^{\infty} \theta(\alpha_n^2) \times \frac{1}{\alpha_n^2} \left[1 - \cos\left(\frac{\omega d \alpha_n}{\gamma c}\right) \right] \times \\ &\quad \left[J_n^2\left(\frac{K \omega \alpha_n}{\gamma^2 \omega_0}\right) + \left(\frac{\alpha_n}{K} - \frac{n \gamma^2 \omega_0}{K \omega \alpha_n}\right)^2 J_n^2\left(\frac{K \omega \alpha_n}{\gamma^2 \omega_0}\right) \right] \end{aligned} \quad (29)$$

As we would expect, the above expression does not depend on b , the transverse dimension of the beam in the yz plane.

For $K \ll 1$ from Eq. (28)

$$\begin{aligned} \frac{dI(\Phi)}{d\omega} &= \frac{4 N_p^2 e^2 K^2 \gamma^2 c^5}{b^2 d^2 L^2 \omega_0^3} \times \frac{1}{(2 \gamma^2 \omega_0 - \omega)^2} \times \left[1 + \left(1 - \frac{\omega}{\gamma^2 \omega_0} \right)^2 \right] \theta(\alpha_1^2) \times \\ &\quad \left[1 - \cos\left(\frac{\omega L}{c}\right) \right] \left[1 - \cos\left(\frac{\omega d \alpha_1}{\gamma c}\right) \right] \left\{ \frac{\cos\left(\frac{\omega \alpha_1 b \Phi}{\gamma c}\right) - 1}{\Phi} + \frac{\omega \alpha_1 b}{\gamma c} \operatorname{Si}\left(\frac{\omega \alpha_1 b \Phi}{\gamma c}\right) \right\}. \end{aligned} \quad (30)$$

For case (b) i.e., $\hat{n}$ near the yz plane, $d^2 I / d\Omega d\omega$ and $dI(\Phi) / d\omega$ are obtained by interchanging b and d in the above equations.

III. APPLICATIONS TO STORAGE RINGS

We shall consider numerical examples for a helical wiggler placed in ISABELLE and the National Synchrotron Light Source. It should be pointed out that no attempt to optimize parameters has been made in either case.

A. Proton Storage Rings-ISABELLE Case

Consider a helical wiggler with $\lambda_w = 15$ cm, $N = 50$, $L_w = N\lambda_w = 750$ cm, $B_w = 20$ kG and $K = 1.5 \times 10^{-2}$. ISABELLE has a circumference of 3834 m and a beam current of 8 amperes. Other relevant parameters are:

	30 GeV	400 GeV
$\Delta E/E$	1.9×10^{-2}	10^{-3}
Normalized Emittance		
$\epsilon_x = \epsilon_y$	$15\pi \times 10^{-6}$ rad -m	$15\pi \times 10^{-6}$ rad -m
Minimum β in cells β_m	10 m	10 m
$\sigma'_{x,y} = \sqrt{\epsilon_{x,y}/\pi\beta_m(B\gamma)_m}$	2.2×10^{-4} rad	0.6×10^{-4} rad
b	0.37 cm	0.1 cm
d	0.47 cm	0.13 cm
$(V/N_p)^{1/3}$	4.7×10^{-4} cm	2×10^{-4} cm
$1/2\sqrt{N}$	4.7×10^{-3} rad	3.5×10^{-4} rad
$\lambda = \lambda_w/2\gamma^2$	8.3×10^{-3} cm	4.7×10^{-5} cm
Resolution $\Delta = \lambda\gamma$	0.25 cm	0.02 cm

We see that the conditions $\Delta E/E < 1/N$, $\sigma'_{x,y} < 1/\sqrt{N}$ and $\Delta < b, d$ are satisfied. Hence the line width should not be broadened beyond $\Delta\lambda/\lambda = 1/N$ and the system should offer good resolution, especially at higher energies.

We note that at 30 GeV the beam radiates coherently since $\lambda > (V/N_p)^{1/3}$ whereas at 400 GeV it radiates incoherently since $\lambda < (V/N_p)^{1/3}$. The characteristics of the radiation emitted for both cases have been calculated. However, an interesting experimental observation would be in the intermediate energy range where $\lambda \sim (V/N_p)^{1/3}$ and the radiation character changes from coherent to incoherent.

Since for a coasting beam the total number of particles in the wiggler $N_w = 1.3 \times 10^{12}$ we get from Eq. (22) that at 30 GeV the maximum value of

$$\left. \frac{dI}{d\Omega} \right|_{\theta=0} / \left. \frac{dI}{d\Omega} \right|_{S.P.} = \left(\frac{N_w \lambda}{\pi L_w} \right)^2 = 2.1 \times 10^{13}$$

At 400 GeV from Eq. (23)

$$\left. \frac{dI}{d\Omega} \right|_{\theta=0} / \left. \frac{dI}{d\Omega} \right|_{S.P.} = N_w = 1.3 \times 10^{12}$$

Hence, the coherent radiation can be more intense than the incoherent radiation and should be observable.

Using Eq. (12), when the beam radiates incoherently the total number of photons emitted per proton in a cone of angle $1/\sqrt{N}$ is $n_{ph} = 10^{-5}$. Also, from Eq. (13)

$$\dot{n}_{ph} = 0.6 \times 10^{14} \times i_{av} \text{ photons/s} \quad (31)$$

where i_{av} is the average beam current in amperes.

Using Eq. (11) the total energy radiated in the wiggler for $\gamma = 400$ is $I_{S.P.} = 4.5 \times 10^{-4}$ eV. The radiative energy loss per revolution in a machine of radius ρ is given by⁴:

$$I_{Rev.} = \frac{4\pi}{3} \frac{e^2}{\rho} \beta^3 \gamma^4$$

For ISABELLE $I_{Rev.} = 0.25$ eV and hence the energy radiated in the wiggler at 400 GeV is small compared to that radiated in the rest of the machine.

Finally, we point out how the radiation from the helical wiggler can be used to measure beam energy, current, transverse dimensions and longitudinal density variations.

Beam Energy: Since the wavelength emitted on the axis of the wiggler is $\lambda = \lambda_w / 2\gamma^2(1+K^2)$ and λ_w and K are known parameters, by measuring λ one can calculate γ and hence the energy of the beam.

Current: When the beam radiates incoherently Eq. (31) can be used to calculate the average beam current in amperes by measuring the number of photons per second that are emitted in a cone of angle $1/\sqrt{N}$. From the formulas derived in Section II.B we see that for the coherent radiation case a measurement of the beam current should also be possible. However, it is not as straightforward since a knowledge of other beam parameters becomes necessary.

Transverse Dimension: When the beam radiates coherently we have shown that a measure of the transverse dimensions d and b can be obtained by sampling the radiation in the xz and yz planes respectively. Another method to measure the size of the beam would be to use mirrors to reflect the radiation onto a receiver which is either a vidicon followed by a TV set or a photomultiplier.⁶ The size of the light spot detected then gives a measure of the proton beam size. A rotating mirror that can be used to carry out a detailed spatial study of the proton beam is also planned for the Synchrotron Light Detector at CERN.

Longitudinal Density Variations: When the beam radiates incoherently a direct observation of the radiation should give a measure of the longitudinal beam density variations. When the beam radiates coherently one should also be able to get this information. However, in our analysis in Section II.B we have considered only a constant density beam and theoretical calculations of radiation from different longitudinal density distributions need to be carried out.

B. Electron Storage Rings-NSLS-VUV Ring Case

Consider a helical wiggler with $\lambda_w = 15$ cm, $N = 50$, $B_w = 2$ KG and $K = 3$.

The VUV Ring⁷ has a circumference of 51 m, a beam energy of 700 MeV, an energy spread $\Delta E/E = 0.00044$, a beam current of 1 ampere and 9 bunches

in the ring. Also, $1/\gamma\sqrt{N} = 10^{-4}$ rad and $\lambda = \lambda_w/2\gamma^2 (1+K^2) = 3.8 \times 10^{-5}$ cm.

Other relevant parameters are:

Unnormalized Emittance (rad-m)	$\epsilon = 9 \times 10^{-8}$	$\epsilon = 9 \times 10^{-10}$
β at position of wiggler (m)	$\beta^x = 11.6$	$\beta^y = 5.5$
Beam Size = $\sqrt{\epsilon\beta}$ (cm)	$\frac{x}{d} = 0.1$	$b = 0.007$
Angular Spread = $\sqrt{\epsilon/\beta}$ (rad)	$\sigma'_x = 0.88 \times 10^{-4}$	$\sigma'_y = 1.3 \times 10^{-5}$

We see that $\Delta E/E < 1/N$ and $\sigma'_{x,y} < 1/\gamma\sqrt{N}$ are satisfied.

Since the total number of electrons in each bunch $N_p = 1.2 \times 10^{11}$, $\lambda > (V/N_p)^{1/3}$ and the beam radiates coherently for $L < 9.4$ cm. We observe from Eq. (24) that if the coherent radiation is to be more intense than the incoherent radiation the length of the bunch should satisfy $L < 4.2$ cm. By making the length of the bunch much smaller than this limiting value one can greatly enhance the magnitude of the coherent radiation. Also, by making λ larger (radiation in the far infrared) one should be able to observe coherent radiation even without bunching the beam strongly.

Finally, it has been shown⁸ that a helical wiggler inserted in the machine should not drive the electron beam unstable.

IV. CONCLUSIONS

We have carried out analytical studies for incoherent and coherent radiation from a helical wiggler for electron and proton beams. The calculations can be repeated if one wishes to use, instead, a transverse wiggler. Wigglers are planned for the National Synchrotron Light Source to be built at BNL. Our calculations show that by bunching the beam strongly one can obtain a more intense coherent source of radiation. For ISABELLE, also being built at BNL, since the beam energy varies from 30 to 400 GeV wigglers are especially useful both as a diagnostic technique and for experimental studies of the character of the radiation emitted. Further theoretical work on the optimization of wiggler parameters for a given application and the consideration of different density distributions in a coasting beam needs to be carried out.

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TOUSCHEK LIFETIME FOR THE NSLS
FREE ELECTRON LASER EXPERIMENT*

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The VUV-Ring¹ of the National Synchrotron Light Source is designed to operate at an electron energy of 700 MeV with an average current of 1 A distributed in nine bunches. High brightness sources of synchrotron radiation are obtained by a lattice design yielding a small electron beam emittance, the uncoupled value of 700 MeV being 9×10^{-8} m-rad. Recently there has been interest in using the VUV-Ring for an experiment to study the operation of a free electron laser.² For this purpose it is necessary to operate at an energy lower than 700 MeV, say 400 or 500 MeV. A consequence of the lower energy is that the emittance will be reduced by a factor of 1/3 or 1/2, respectively, relative to the value at 700 MeV. Since it is the peak current which determines the laser gain, it is hoped to achieve an average current of 1 A distributed in three bunches, the length of a bunch being kept as short as possible. The combination of high peak current, small transverse dimensions, and low energy lead one to be concerned about the Touschek lifetime.

In this paper the Touschek lifetime for the free electron laser experiment is considered. The conclusion is reached that although minimizing the electron emittance is appropriate to optimize the performance of the ring as a synchrotron light source, for the free electron laser it may be desirable to operate at a lower tune having a larger emittance. This increases the Touschek lifetime and does not result in a major change of the laser gain. For normal operation of the VUV-Ring, the nominal tune values are $\nu_x = 3.32$ and $\nu_z = 1.32$. Using the SYNCH program we have found a satisfactory solution with $\nu_x = 2.85$ and $\nu_z = 0.85$, having an emittance twice that of the original tune. In considering even lower tunes, we encountered difficulty with the chromaticity correction, since the β -functions at the sextupole locations became unfavorable.

The circumference of the VUV-Ring is 51 m, corresponding to a harmonic number of 9. The distance between three equally spaced bunches is 17 m. For synchronization of the radiation pulse in the optical cavity of the laser with the electron bunches, we can take the length of the cavity L_{op} to be one-half the distance between bunches, i.e. $L_{op} = 8.5$ m. Diffraction sets a lower limit on the cross-sectional area of the standing wave in the cavity given by

$$\Sigma \sim \lambda L_{op} \quad (1)$$

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where λ is the wavelength of the amplified radiation.

If we consider

$$\lambda = 0.25 \mu\text{m},$$

then

$$\Sigma \sim 2 \text{ mm}^2. \quad (2)$$

For a transverse wiggler, the radiation wavelength λ is related to the wiggler wavelength λ_w and the wiggler magnetic field B_w by³

$$\lambda = \frac{\lambda_w}{2\gamma^2} \left(1 + \frac{K^2}{2}\right), \quad (3)$$

$$K \approx B_w (\text{KG}) \lambda_w (\text{cm}) / 10, \quad (4)$$

and $\gamma = E/mc^2$. If we take $\lambda_w = 5 \text{ cm}$, then the wiggler field corresponding to $\lambda = 0.25 \mu\text{m}$ is

$$B_w = 6.4 \text{ KG} \quad (E = 400 \text{ MeV}), \quad (5)$$

$$B_w = 8.2 \text{ KG} \quad (E = 500 \text{ MeV}).$$

These fields may be achievable with a full gap of about 2 cm and superconducting coils.

Let us consider the wiggler magnet to have $N = 50$ periods. For the operation of a free electron laser⁴ we desire:

1. the energy spread $\sigma_e/E < 1/2N$,
2. the transverse cross section of the electron beam is approximately equal to that of the radiation beam,
3. the angular divergence of the electron beam $\theta_{\text{rms}} < 1/\sqrt{2N}$.

When these conditions are satisfied the laser gain $G = \Delta I/I$ is approximately given by

$$G = 8\pi^2 N^3 \left(\frac{\lambda}{\ell_c}\right) \frac{K^2/2}{(1 + K^2/2)^2} \frac{\gamma I_{\text{av}}}{I_A}, \quad (6)$$

where $I_{\text{av}} = 1 \text{ A}$ is the average current, $\ell_c = 2\sigma_\perp \approx 14 \text{ cm}$ is the bunch length, $I_A = ec/r_e = 17,000 \text{ A}$, and $\lambda = 0.25 \mu\text{m}$ is the radiation wavelength. The gain corresponding to the magnetic fields given in (5) is:

$$G = 0.11, \quad (E = 400 \text{ MeV})$$

$$G = 0.095. \quad (E = 500 \text{ MeV}) \quad (7)$$

As a rough estimate of the Touschek lifetime τ (half-life, we take

$$\frac{1}{\tau} = \frac{\sqrt{\pi} r_e^2 c n C(\xi)}{\sigma' \gamma^3 (\Delta E/E)^2 V},$$

where $r_e = 2.8 \times 10^{-15} \text{ m}$, $n = 3.6 \times 10^{11}$ electrons per bunch, $\Delta E/E$ is the maximum energy deviation which is accepted by the ring,

$$V = 8\pi^{3/2} \sigma_x \sigma_z \sigma_\ell,$$

$$\sigma_x = \sqrt{\epsilon_x \beta_x}, \quad \sigma_z = \sqrt{\epsilon_z \beta_z}, \quad \sigma_\ell \cong 7 \text{ cm},$$

$$\sigma' = \left(\frac{\epsilon_x}{\beta_x} + \frac{\epsilon_z}{\beta_z} \right)^{1/2},$$

$$\beta_{x,z} \cong R/\nu_{x,z}, \quad R = 8.1 \text{ m},$$

$$\xi = \frac{\Delta E/E}{\gamma \sigma'}^2,$$

and $C(\xi)$ is a function defined in Bruck.⁵

We shall consider full coupling $\epsilon_x = \epsilon_z = \epsilon_x^0/2$, where ϵ_x^0 is the emittance in the absence of coupling. For the original tune $\nu_x = 3.32$ and $\nu_z = 1.32$, $\epsilon_x^0 = 18 \times 10^{-8} E_{\text{GeV}}^2$ (m-rad), and we find the Touschek lifetimes:

	E = 400 MeV (min)	E = 500 MeV (min)
$\Delta E/E = 1.0\%$	4	11
$\Delta E/E = 1.5\%$	13	33

For the alternate tune $\nu_x = 2.85$, $\nu_z = 0.85$, $\epsilon_x^0 = 38 \times 10^{-8} E_{\text{GeV}}^2$ (m-rad), and we obtain:

	E = 400 MeV (min)	E = 500 MeV (min)
$\Delta E/E = 1\%$	12	36
$\Delta E/E = 1.5\%$	38	100

We see that the Touschek lifetimes have been increased by a factor of three.

For the original tune the values of the amplitude functions at the wiggler locations are $\beta_x^w = 11.6$ m, $\beta_z^w = 5.5$ m, so

$$4 \sigma_x \sigma_z = 0.5 \text{ mm}^2, \quad (E = 400 \text{ MeV})$$

$$4 \sigma_x \sigma_z = 0.8 \text{ mm}^2. \quad (E = 500 \text{ MeV})$$

The corresponding values for the lower tune are $\beta_x^w = 11.9$ m, $\beta_z^w = 9.0$ m, and

$$4 \sigma_x \sigma_z = 1.2 \text{ mm}^2 \quad (E = 400 \text{ MeV})$$

$$4 \sigma_x \sigma_z = 1.9 \text{ mm}^2 \quad (E = 500 \text{ MeV})$$

These should be compared to the cross-section $\Sigma \sim 2 \text{ mm}^2$ of the standing wave in the cavity. The condition on the angular spread of the electron beam is always satisfied for both tunes since $\theta \sim 0.1$ mrad in all cases.

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