A LIMIT THEOREM FOR THE MAXIMUM OF AUTOREGRESSIVE
PROCESSES WITH UNIFORM MARGINAL DISTRIBUTIONS AND ITS
IMPLICATIONS TO THE ANALYSIS OF AIR POLLUTION DATA

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TECHNICAL REPORT NO. 17

JULY 1978

STUDY ON STATISTICS AND ENVIRONMENTAL
FACTORS IN HEALTH

PREPARED UNDER SUPPORT TO SIMS FROM

ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION (ERDA)

ROCKEFELLER FOUNDATION

SLOAN FOUNDATION

ENVIRONMENTAL PROTECTION AGENCY (EPA)

NATIONAL SCIENCE FOUNDATION (NSF)

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

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A Limit Theorem for the Maximum of Autoregressive Processes with Uniform Marginal Distributions and its Implications to the Analysis of Air Pollution Data

Michael R. Chernick

Abstract

Limit theorems for maxima of time series are often useful in making statistical inferences in practical problems. In the study of air pollution the peak levels over fixed time periods are of concern. This paper examines a class of models for which the classical theory does not apply. The results indicate that methods like those used in this paper might be necessary to obtain the appropriate limit in some simple stationary models (e.g., first order autoregressive models). Some of these stationary models may be adequate to describe the behavior of some air pollution time series, although the uniform model described in this paper may not.

Let $X_n^{(r)} = \frac{1}{r} X_{n-1}^{(r)} + \epsilon_n$ where $r \geq 2$ is an integer and ϵ_n are i.i.d. with $P(\epsilon_n = \frac{k}{r}) = \frac{1}{r}$ for $k = 0, 1, 2, \dots, r-1$ $n \geq 1$ and $X_0 \sim U[0,1]$ (i.e., X_0 is distributed as a uniform random variable on $[0,1]$) with X_0 independent of the ϵ_n 's. This defines a family of autoregressive processes with $X_n^{(r)} \sim U[0,1]$. For $\{X_i^*\}_{i=0}^\infty$ i.i.d. uniform random variables and $M_n^* = \max(X_0^*, X_1^*, \dots, X_n^*)$ it is well-known that $P(M_n^* \leq 1 - \frac{x}{n}) \rightarrow e^{-x}$ as $n \rightarrow \infty$ $\forall x > 0$. However in this paper we shall see that $P(M_n^{(r)} \leq 1 - \frac{x}{n}) \rightarrow e^{-[(r-1)/r]x}$ where $M_n^{(r)} = \max(X_0^{(r)}, X_1^{(r)}, \dots, X_n^{(r)})$ for $r \geq 2$ even though the processes satisfy Leadbetter's condition $D(u_n)$. Implications of this result on the analysis of pollution data are discussed.

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1. The Class of Models.

a) Definition.

We define the uniform AR(1) models as follows:

Let $X_n^{(r)} = \frac{1}{r} X_{n-1}^{(r)} + \epsilon_n$ for r an integer with $r \geq 2$ and the ϵ_n 's i.i.d. with $X_{n-1}^{(r)}$ independent of ϵ_n and the ϵ_n 's have their distribution given by $P(\epsilon_n = \frac{k}{r}) = \frac{1}{r}$ for $k = 0, 1, 2, \dots, r-1$ and $n \geq 1$. Let $X_0^{(r)} \sim U[0,1]$. We shall show in subsection b) that $\forall n \geq 1$ and $r \geq 2$ $X_n^{(r)} \sim U[0,1]$ and so the above definition gives a family of strictly stationary autoregressive processes of first order.

b) Proof that $X_n^{(r)} \sim U[0,1]$.

Let $\phi_\epsilon(t) = E(e^{it\epsilon_n})$ and $\phi_{X_n^{(r)}}(t) = E(e^{itX_n^{(r)}})$ the characteristic functions of ϵ_n and $X_n^{(r)}$ respectively. Define $\phi_U(t) = E(e^{itU})$ where $U \sim U[0,1]$. It is easy to show that $\phi_U(t) = \frac{e^{it}-1}{it}$ and $\phi_\epsilon(t) = \sum_{k=0}^{r-1} \frac{e^{itk/r}}{r}$.

$\phi_{X_0^{(r)}}(t) = \phi_U(t) = \frac{e^{it}-1}{it}$. The proof goes by induction.

Suppose $\phi_{X_{n-1}^{(r)}}(t) = \frac{e^{it}-1}{it}$. We must show that this implies

$$(1.1) \quad \phi_{X_n^{(r)}}(t) = \frac{e^{it}-1}{it}.$$

$\phi_{X_n^{(r)}}(t) = \phi_{X_{n-1}^{(r)}}\left(\frac{t}{r}\right) \phi_\epsilon(t)$ since $X_{n-1}^{(r)}$ and ϵ_n are independent.

$$\begin{aligned}\Phi_{X_n}^{(r)}(t) &= \frac{(e^{it/r} - 1)}{(it/r)} \frac{1}{r} \left(\sum_{k=0}^{r-1} e^{itk/r} \right) = \frac{e^{it/r} - 1}{it} \left(\sum_{k=0}^{r-1} e^{itk/r} \right) \\ &= \sum_{k=0}^{r-1} \left(\frac{e^{it(k+1)/r} - e^{itk/r}}{it} \right). \text{ This sum is telescoping, so after cancellation}\end{aligned}$$

only $e^{it} - 1$ remains. Hence (1.1) is satisfied and the proof is complete.

2. The Exact Distribution for M_1, M_2, M_3 When $r = 2$

a) Derivations.

We define $M_n = \max(X_0, X_1, \dots, X_n)$ where X_i is used to denote $X_i^{(2)}$.

$P(M_1 \leq 1 + y) = P(X_0 \leq 1 + y, X_1 \leq 1 + y)$ for $-1 \leq y \leq 0$. We recall that

$X_1 = X_0/2 + \varepsilon_1$. By conditioning on ε_1 we get $P(M_1 \leq 1 + y) = \frac{1}{2} P(X_0 \leq 1 + y) + \frac{1}{2} P(X_0 \leq 1 + 2y)$ and we observe that $P(X_0 \leq 1 + 2y) = 0$ if $y \leq -\frac{1}{2}$.

Hence $P(M_1 \leq 1 + y) = \frac{1+y}{2} + \frac{1+2y}{2} = 1 + \frac{3}{2}y$ for $-\frac{1}{2} \leq y \leq 0$ and $P(M_1 \leq 1 + y) = \frac{1}{2} + \frac{y}{2}$ for $-1 \leq y \leq -\frac{1}{2}$.

Now let us consider the case $n = 2$. The analysis is similar.

$P(M_2 \leq 1 + \frac{y}{2}) = P(X_0 \leq 1 + \frac{y}{2}, X_0/2 + \varepsilon_1 \leq 1 + \frac{y}{2}, X_0/4 + \frac{\varepsilon_1}{2} + \varepsilon_2 \leq 1 + \frac{y}{2})$

since $X_1 = X_0/2 + \varepsilon_1$ and $X_2 = X_0/4 + \varepsilon_1/2 + \varepsilon_2$. We now condition on ε_1 and ε_2 and arrive at the following result:

$$\begin{aligned}(2.1) \quad P(M_2 \leq 1 + \frac{y}{2}) &= \frac{1}{4} P(X_0 \leq 1 + \frac{y}{2}) + \frac{1}{4} P(X_0 \leq 1 + \frac{y}{2}, X_0 \leq 2 + 2y) \\ &\quad + \frac{1}{4} P(X_0 \leq 1 + \frac{y}{2}, X_0 \leq 1 + y) \\ &\quad + \frac{1}{4} P(X_0 \leq 1 + \frac{y}{2}, X_0 \leq 1 + y, X_0 \leq 1 + 2y)\end{aligned}$$

and this is defined for all $-2 \leq y \leq 0$. Using some algebra and taking care to notice at which values of y terms in (2.1) become zero we get the following:

$$\begin{aligned}
 P(M_2 \leq 1 + \frac{y}{2}) &= 1 + y & -\frac{1}{2} \leq y \leq 0 \\
 &= \frac{3}{4} + \frac{y}{2} & -\frac{2}{3} \leq y \leq -\frac{1}{2} \\
 &= 1 + \frac{7}{8}y & -1 \leq y \leq -\frac{2}{3} \\
 &= \frac{1}{4} + \frac{y}{8} & -2 \leq y \leq -1
 \end{aligned}$$

Similarly $P(M_3 \leq 1 + \frac{y}{3}) = P(X_0 \leq 1 + \frac{y}{3}, X_0/2 + \epsilon_1 \leq 1 + \frac{y}{3}, X_0/4 + \epsilon_1/2 + \epsilon_2 \leq 1 + \frac{y}{3}, X_0/8 + \epsilon_1/4 + \epsilon_2/2 + \epsilon_3 \leq 1 + \frac{y}{3})$ and when we condition on ϵ_1, ϵ_2 and ϵ_3 we get eight terms to evaluate with the final result as follows:

$$\begin{aligned}
 P(M_3 \leq 1 + \frac{y}{3}) &= 1 + \frac{5}{6}y & -\frac{3}{8} \leq y \leq 0 \\
 &= \frac{7}{8} + \frac{y}{2} & -\frac{3}{7} \leq y \leq -\frac{3}{8} \\
 &= 1 + \frac{19}{24}y & -\frac{3}{4} \leq y \leq -\frac{3}{7} \\
 &= \frac{5}{8} + \frac{7}{24}y & -1 \leq y \leq -\frac{3}{4} \\
 &= 1 + \frac{2}{3}y & -\frac{9}{8} \leq y \leq -1 \\
 &= \frac{5}{8} + \frac{y}{3} & -\frac{9}{7} \leq y \leq -\frac{9}{8} \\
 &= 1 + \frac{5}{8}y & -\frac{3}{2} \leq y \leq -\frac{9}{7} \\
 &= \frac{1}{8} + \frac{y}{24} & -3 \leq y \leq -\frac{3}{2}
 \end{aligned}$$

The important thing to observe from these calculations is that for every n $P(M_n \leq 1 + \frac{y}{n})$ is a piecewise linear function on $[-n, 0]$ increasing from 0 to 1.

Plotting the curve for $n = 3$ already shows us that the curve approximates the exponential shape. It is well known and easy to verify that if $M_n^* = \max(X_0^*, X_1^*, \dots, X_n^*)$ and X_i^* i.i.d. $U[0, 1]$ then $P[M_n^* \leq 1 + \frac{y}{n}] \rightarrow e^y$ as $n \rightarrow \infty$ for $-\infty < y \leq 0$. The limit distribution is an extreme value type II distribution with $\alpha = 1$. We shall see in section 8 that

$$(2.2) \quad P(M_n \leq 1 + \frac{y}{n}) \rightarrow e^{y/2} \quad \text{as } n \rightarrow \infty.$$

If we could calculate the exact distribution of M_n for "sufficiently large" n , we could see that (2.2) holds by plotting the curves. It may be feasible to do this with the aid of the computer. Feasibility depends on how large "sufficiently large" turns out to be.

It is apparent that $n = 3$ is not sufficiently large as can be seen by comparing e^y , $e^{y/2}$ and the values for $P(M_1 \leq 1 + y)$, $P(M_2 \leq 1 + \frac{y}{2})$ and $P(M_3 \leq 1 + \frac{y}{3})$. This is given in the table in the next subsection. The author was fooled into thinking that the limit was e^y . This illustrates the danger of guessing limits when n is small.

b) A Table Comparing the Distributions.

TABLE 1

y	$P(M_1 \leq 1 + y)$	$P(M_2 \leq 1 + \frac{y}{2})$	$P(M_3 \leq 1 + \frac{y}{3})$	e^y	$e^{y/2}$
- 3	0	0	0	.0498	.2231
- 2	0	0	.0417	.1353	.3679
$-\frac{3}{2}$	0	.0625	.0625	.2231	.4724
$-\frac{9}{7}$	0	.0893	.1964	.2765	.5258
$-\frac{9}{8}$	0	.1094	.25	.3247	.5698
- 1	0	.125	.3333	.3679	.6065
$-\frac{3}{4}$	.125	.3438	.4063	.4724	.6873
$-\frac{2}{3}$	.1667	.4167	.4722	.5134	.7165
$-\frac{1}{2}$	.25	.5	.6042	.6065	.7788
$-\frac{3}{7}$	.3571	.5714	.6607	.6514	.8071
$-\frac{3}{8}$	.4375	.625	.6875	.6873	.8290
0	1.0	1.0	1.0	1.0	1.0

3. Convergence for Fixed x Along a Subsequence: the Case $r = 2$.

We define $n_k = [2^k x]$ where $[\cdot]$ denotes the greatest integer function. M_n, X_n are as defined in section 2 and x is taken to be positive. The variable y in the previous section can be thought of as $-x$.

Lemma 3.1: a) $P(M_{n_k} \leq 1 - \frac{1}{2^k}) \rightarrow e^{-x/2}$ as $k \rightarrow \infty$

and hence also b) $P(M_{n_k} \leq 1 - \frac{x}{n_k}) \rightarrow e^{-x/2}$.

Proof: $P(M_{n_k} \leq 1 - \frac{1}{2^k}) = 1 - P(M_{n_k} > 1 - \frac{1}{2^k})$
 $= 1 - P(X_i > 1 - \frac{1}{2^k} \text{ for some } i \geq 0) .$ Now $X_i = \frac{X_1}{2^{i-1}} + \sum_{j=0}^{i-2} \frac{\epsilon_{i-j}}{2^j} .$

Suppose k consecutive ϵ_i 's $= \frac{1}{2}$. Call them $\epsilon_\ell, \epsilon_{\ell+1}, \dots, \epsilon_{\ell+k-1}$.

$$X_{k+\ell-1} = \frac{X_{\ell-1}}{2^k} + \sum_{j=0}^{k-1} \frac{\epsilon_{k+\ell-1-j}}{2^j} = 1 - \frac{1}{2^k} + \frac{X_{\ell-1}}{2^k} > 1 - \frac{1}{2^k} \Rightarrow M_{n_k} > 1 - \frac{1}{2^k} .$$

$$\therefore (k \text{ consecutive } \epsilon_i \text{'s} = \frac{1}{2} \text{ in } n_k \text{ trials}) \subseteq (M_{n_k} > 1 - \frac{1}{2^k})$$

and also we have (no run of length k in n_k trials) $\supseteq (M_{n_k} \leq 1 - \frac{1}{2^k})$.

Consequently $P(\text{no run of length } k \text{ in } n_k \text{ trials}) \geq P(M_{n_k} \leq 1 - \frac{1}{2^k})$.

$P(\text{no run of length } k \text{ in } n_k \text{ trials}) \rightarrow e^{-x/2}$ as $k, n_k \rightarrow \infty$.

This result comes from a Poisson limit law first proved by von Mises.

Feller (3) gives this result on p. 341. We identify $\frac{x}{2}$ with λ since

$$\frac{n_k}{2} \cdot \frac{1}{2^k} = \frac{n_k}{2^{k+1}} \rightarrow \frac{x}{2} \text{ by the definition of } n_k.$$

We can now conclude that $\lim_{k \rightarrow \infty} P(M_{n_k} \leq 1 - \frac{1}{2^k}) \leq e^{-x/2}$.

Let $B_k = (\text{no run of length } k \text{ in } n_k \text{ trials})$ and $A_k = (M_{n_k} \leq 1 - \frac{1}{2^k})$.

$B_k \supseteq A_k$ and $P(B_k) - P(A_k) = P(B_k \cap A_k^c) \leq P(X_0 > 1 - \frac{1}{2^k})$ or

$$1 - \frac{1}{2^j} < X_0 \leq 1 - \frac{1}{2^{j+1}} \text{ for some } 1 \leq j \leq k-1 \text{ and } \epsilon_1, \epsilon_2, \dots, \epsilon_{k-j} \neq 0$$

$$= P(X_0 > 1 - \frac{1}{2^k}) + \sum_{j=1}^{k-1} P(1 - \frac{1}{2^j} < X_0 \leq 1 - \frac{1}{2^{j+1}}) P(\epsilon_1 = \frac{1}{2}, \epsilon_2 = \frac{1}{2}, \dots, \epsilon_{k-j} = \frac{1}{2})$$

$$= \frac{1}{2^k} + \sum_{j=1}^{k-1} \frac{1}{2^{j+1}} \frac{1}{2^{k-j}} = \frac{1}{2^k} + \frac{k-1}{2^{k+1}} = \frac{k+1}{2^{k+1}}.$$

$$\text{So } P(B_k) - P(A_k) \leq \frac{k+1}{2^{k+1}} \rightarrow 0 \text{ as } k \rightarrow \infty \Rightarrow P(M_{n_k} \leq 1 - \frac{1}{2^k}) \rightarrow e^{-x/2}.$$

To show the second part of the lemma we have

$$|P(M_{n_k} \leq 1 - \frac{1}{2^k}) - P(M_{n_k} \leq 1 - \frac{x}{n_k})|$$

$$= P(m < M_{n_k} \leq M) \leq \sum_{i=0}^{n_k} P(m \leq X_i \leq M) = (n_k + 1) P(m \leq X_1 \leq M)$$

$$\text{where } M = \max(1 - \frac{1}{2^k}, 1 - \frac{x}{n_k}) \text{ and } m = \min(1 - \frac{1}{2^k}, 1 - \frac{x}{n_k}).$$

$$\text{Now } n_k P(m \leq X_1 \leq M) = n_k \left| \frac{x}{n_k} - \frac{1}{2^k} \right| = \left| x - \frac{n_k}{2^k} \right| \rightarrow 0 \text{ as } k \rightarrow \infty \text{ since}$$

$$\frac{n_k}{2^k} \rightarrow x \text{ as } k \rightarrow \infty. \text{ We conclude that } P(M_{n_k} \leq 1 - \frac{x}{n_k}) \rightarrow e^{-x/2} \text{ and this}$$

completes the proof of the lemma.

The author wishes to acknowledge that Professor Faith first conjectured this lemma. Professor Faith also provided the proof of part b) of the lemma.

Lemma 3.1 will not be used to prove the limit theorem but it is an important result as it tells us that we cannot expect $P(M_n \leq 1 - \frac{x}{n})$ to converge to e^{-x} as we originally expected it would. In section 4 we shall derive some bounds on $P(M_n \leq 1 - \frac{x}{n})$ with the aid of Lemma 3.1.

4. Upper and Lower Bounds for the Limit Distribution: the Case $r = 2$.

We are still using the notation of section 2. A lower bound for the limiting probability can now be obtained from the fact that $\left(X_n\right)_{n=0}^{\infty}$ is an associative process. All first order autoregressive processes with $\text{Cov}(X_n, X_{n+1}) \geq 0$ are associative. The definition of associative processes and the proof of this result are given in (2). In this case $p = \frac{1}{2}$ and associativity implies

$$(4.1) \quad P(M_n \leq t) \geq P(M_n^* \leq t)$$

and

$$(4.2) \quad P(m_n > t) \geq P(m_n^* > t) \quad \forall n \text{ and } t,$$

where

$$m_n = \min(X_0, X_1, \dots, X_n) \quad \text{and} \quad m_n^* = \min(X_0^*, X_1^*, \dots, X_n^*).$$

We therefore get that

$$P(M_n \leq 1 - \frac{x}{n}) \geq \left(1 - \frac{x}{n}\right)^n$$

and since $\left(1 - \frac{x}{n}\right)^n \rightarrow e^{-x}$

$$(4.3) \quad \lim_{n \rightarrow \infty} P(M_n \leq 1 - \frac{x}{n}) \geq e^{-x},$$

We shall see in section 8 that

$$\lim_{n \rightarrow \infty} P(M_n \leq 1 - \frac{x}{n}) \geq e^{-x/2}.$$

Using the results of the previous section we can show

$$(4.4) \quad \lim_{n \rightarrow \infty} P(M_n \leq 1 - \frac{x}{n}) \geq 2e^{-x/2} - e^{-x/4}$$

and

$$(4.5) \quad \overline{\lim}_{n \rightarrow \infty} P(M_n \leq 1 - \frac{x}{n}) \leq 2e^{-x/2} - e^{-x}.$$

Since

$$P(M_{n_k} \leq 1 - \frac{x}{n_k}) \rightarrow e^{-x/2}$$

we have

$$P(1 - \frac{x}{n_k} < M_{n_k} \leq 1 - \frac{x}{n_{k+1}}) \rightarrow e^{-x/4} - e^{-x/2}$$

and

$$P(1 - \frac{x}{n_k} < M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}}) \rightarrow e^{-x/2} - e^{-x} \text{ as } k \rightarrow \infty.$$

$$\forall n \exists k \ni n_k \leq n < n_{k+1}.$$

$$P(M_{n_k} \leq 1 - \frac{x}{n}) \geq P(M_n \leq 1 - \frac{x}{n}) \geq P(M_{n_{k+1}} \leq 1 - \frac{x}{n})$$

since M_n is increasing in n .

$$(4.6) \quad P(M_{n_{k+1}} \leq 1 - \frac{x}{n}) = P(M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}}) - P(1 - \frac{x}{n} < M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}}).$$

$$(4.7) \quad P(M_{n_k} \leq 1 - \frac{x}{n}) = P(M_{n_k} \leq 1 - \frac{x}{n_k}) + P(1 - \frac{x}{n_k} < M_{n_k} \leq 1 - \frac{x}{n}).$$

Combining (4.6) and (4.7) we get

$$(4.8) \quad P(M_{n_k} \leq 1 - \frac{x}{n}) + P(1 - \frac{x}{n_k} < M_{n_k} \leq 1 - \frac{x}{n}) \geq P(M_n \leq 1 - \frac{x}{n}) \\ \geq P(M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}}) - P(1 - \frac{x}{n} < M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}}).$$

We therefore have $\forall n \exists k \ni (4.8)$ holds so

$$\varliminf_{n \rightarrow \infty} P(M_n \leq 1 - \frac{x}{n}) \leq \lim_{k \rightarrow \infty} [P(M_{n_k} \leq 1 - \frac{x}{n_k}) + P(1 - \frac{x}{n_k} < M_{n_k} \leq 1 - \frac{x}{n})] \\ \leq e^{-x/2} + e^{-x/2} - e^{-x} = 2e^{-x/2} - e^{-x}$$

and

$$\lim_{n \rightarrow \infty} P(M_n \leq 1 - \frac{x}{n}) \geq \lim_{k \rightarrow \infty} [P(M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}}) - P(1 - \frac{x}{n} < M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}})]$$

$$\geq e^{-x/2} - (e^{-x/4} - e^{-x/2}) = 2 e^{-x/2} - e^{-x/4}.$$

So (4.4) and (4.5) hold. We have used the fact that

$$P(1 - \frac{x}{n_k} < M_{n_k} \leq 1 - \frac{x}{n}) \leq P(1 - \frac{x}{n_k} < M_{n_k} \leq 1 - \frac{x}{n_{k+1}})$$

and

$$P(1 - \frac{x}{n} < M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}}) \leq P(1 - \frac{x}{n_k} < M_{n_{k+1}} \leq 1 - \frac{x}{n_{k+1}}).$$

At this point (4.4) and (4.5) are the best bounds we can get.

5. Leadbetter's Theorems.

In 1965 Loynes (7) proved an important limit theorem for stationary sequences. He showed that if a strong mixing condition is satisfied M_n appropriately normalized can only converge to one of the three extreme value type distributions. Further he showed that if a sufficient condition is satisfied the norming constants and the limiting distribution are the same as if the sequence were i.i.d. . In 1974 Leadbetter (5) strengthened this result by weakening the mixing condition. Under the condition he calls $D(u_n)$ the first theorem of Loynes' holds and if in addition a sufficient condition $D'(u_n)$ holds ($D'(u_n)$ is similar to Loynes' condition but is easier to state) the second theorem also holds. Loynes refers to the mixing condition as uniform mixing. We shall use the term strong mixing as that appears to be more commonly used in the literature. Leadbetter's distributional mixing condition shall be referred to simply as $D(u_n)$ just as Leadbetter refers to it.

Since the conditions $D(u_n)$ and $D'(u_n)$ will be used in sections 6 and 7 respectively we shall give Leadbetter's definitions and state his theorems.

Definition: A stationary sequence (X_n) is said to satisfy the condition $D(u_n)$ if for any integers $1 \leq i_1 < i_2 < \dots < i_p < j_1 < \dots < j_q \leq n$ with $j_1 - i_p \geq \ell$, $|F_{i_1, i_2, \dots, i_p, j_1, \dots, j_q}(u_n) - F_{i_1, \dots, i_p}(u_n) F_{j_1, \dots, j_q}(u_n)| \leq \alpha_{n, \ell}$ where $\lim_{\ell \rightarrow \infty} \lim_{n \rightarrow \infty} \alpha_{n, \ell} = 0$ and u_n is some sequence of real numbers. $F_{i_1, i_2, \dots, i_p}(u_n)$ is notation for $P(X_{i_1} \leq u_n, X_{i_2} \leq u_n, \dots, X_{i_p} \leq u_n)$.

In section 6 we will verify $D(u_n)$ for the uniform AR(1) models. We will find a function $\phi(\ell) = \alpha_{n, \ell}$ which will not really depend on n .

Using a series of lemmas Leadbetter shows in (5) that Loynes' form of Gnedenko's theorem holds under the weaker condition of $D(u_n)$. The condition $D(u_n)$ was chosen so that the proof of Loynes' theorem would go through. Leadbetter has stripped out the properties of strong mixing which are not essential to the limit theorem.

The condition $D'(u_n)$ is similar to Loynes sufficient condition but is easier to state and may sometimes be easier to verify. $D'(u_n)$ is defined as follows: $D'(u_n)$ holds if

$$\overline{\lim}_{n \rightarrow \infty} \left[n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) \right] = o\left(\frac{1}{k}\right) \text{ as } k \rightarrow \infty.$$

When $D'(u_n)$ holds in addition to $D(u_n)$ Leadbetter gets $P(M_n \leq a_n x + b_n) \rightarrow G(x)$ if $P(M_n^* \leq a_n x + b_n) \rightarrow G(x)$ where $G(x)$ is one of the three extreme value distributions. This is done similarly to the way Loynes gets the result when he assumes strong mixing and his sufficient condition. In (2) Chernick gets a limit theorem for an exponential AR(1)

process by verifying Loynes' conditions. In that case Leadbetter's mixing condition $D(u_n)$ is not easier to verify than strong mixing. This is because for Markov processes a sufficient condition for strong mixing exists and is easy to check. This result with some examples will appear in Chernick's Ph.D. thesis.

For the uniform processes the sufficient condition for strong mixing is not easy to check but it is possible to check that $D(u_n)$ is satisfied. The condition $D'(u_n)$ is more convenient to check than Loynes' condition.

We shall see in section 7 that $D'(u_n)$ fails for the uniform process in the case $r = 2$. In fact, it just barely fails. That is to say

$$\overline{\lim}_{n \rightarrow \infty} \left[n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) \right] = O\left(\frac{1}{k}\right)$$

$$\left\{ \text{i.e., } x + \frac{x}{k} \geq \overline{\lim}_{n \rightarrow \infty} \left[nk \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) \right] \geq x > 0 \right\}$$

where

$$u_{nk} = 1 - \frac{x}{nk} = 1 + \frac{y}{nk}, \quad x > 0 \text{ fixed and } y = -x$$

It is worthwhile noting that another theorem of Loynes holds with $D(u_n)$ replacing strong mixing. We are referring to theorem 2, p. 995 in (7). The improved version of the theorem can be stated as follows:

Theorem 5.1. Let (X_n) be a strictly stationary sequence satisfying $D(u_n)$ and suppose $P[X_1 > u_n(x)] \sim \frac{\tau(x)}{n}$ as $n \rightarrow \infty$ then $P[M_n \leq u_n(x)] \rightarrow e^{-k\tau(x)}$ for some $0 < k \leq 1$ if it converges. [$u_n(x) = u_n$]

This theorem was not given by Leadbetter in (5) but it does follow quite easily. Leadbetter shows that $D(u_n)$ implies

$$(5.1) \quad \lim_{m \rightarrow \infty} \left[P(M_n \leq u_n) - P^{\ell}(M_m \leq u_n) \right] = 0$$

where $n = m\ell$. Loynes shows that if (5.1) is satisfied for $\ell = 2$ the solution to $\lim_{n \rightarrow \infty} P(M_n \leq u_n)$ can only have the form $e^{-k\tau(x)}$ for some $0 < k \leq 1$. He proves the result for strong mixing but he really only uses (5.1) so the result holds under $D(u_n)$.

An interesting quotation from Loynes' paper is "The question of what values of k can actually arise suggests itself immediately; the author does not know, having been unable to find any process for which the limit exists with $k \neq 1$ ". We shall show for the process $\left(X_n^{(r)} \right)_{n=0}^{\infty}$ $P(M_n^{(r)} \leq u_n) \rightarrow e^{-[(r-1)/r]x}$ and hence we see that $(r-1)/r$ for $r = 2, 3, 4, \dots$ are possible values for k . Other examples were given by O'Brien (8).

6. Verifying $D(u_n)$.

In this section we will verify the condition $D(u_n)$ for the class of uniform AR(1) processes.

Theorem 6.1. The uniform AR(1) process defined by $X_n^{(r)} = \frac{1}{r} X_{n-1}^{(r)} + \epsilon_n$, ϵ_n i.i.d. with $P(\epsilon_n = \frac{k}{r}) = \frac{1}{r}$ for $k = 0, 1, 2, \dots, r-1$ satisfies

$$(6.1) \quad |F_{i_1, i_2, \dots, i_p, j_1, \dots, j_q}(u_n) - F_{i_1, \dots, i_p}(u_n) F_{j_1, \dots, j_q}(u_n)| \leq \phi(\ell)$$

where

$$\ell \leq j_1 - i_p \quad p + q \leq n, \quad \phi(\ell) = \left(\frac{1}{r} \right)^{\ell}$$

and

$$i_1 < i_2 < \dots < i_p < j_1 < \dots < j_q.$$

Proof: For convenience we shall suppress the superscript r . It is understood that $X_n = X_n^{(r)}$ and $M_n = M_n^{(r)}$.

$\frac{1}{r} > 0$ so by Theorem 1 in (2) $\left(X_n\right)_{n=1}^{\infty}$ is associative. Hence

$$\begin{aligned} 0 &\leq F_{i_1 \dots j_1 \dots j_q}(u_n) - F_{i_1 \dots i_p}(u_n) F_{j_1 \dots j_q}(u_n) \\ &= F_{i_1 \dots i_p}(u_n) \left[P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n) - F_{j_1 \dots j_q}(u_n) \right] \\ &\leq P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n) - F_{j_1 \dots j_q}(u_n) \end{aligned}$$

Now

$$\begin{aligned} &P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n) \\ &= \int_{v \leq u_n} P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{i_p} = v) d G_{i_p}(v) \end{aligned}$$

where

$$G_{i_p}(v) = P(X_{i_p} \leq v | X_{i_1} \leq u_n, \dots, X_{i_p} \leq u_n)$$

Let $\rho = \frac{1}{r}$ and define $W_{j_t} = X_{j_t} - \rho^{j_t - i_p} X_{i_p}$ for $t = 1, 2, \dots, q$

We consider

$$\int_{v \leq u_n} P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{i_p} = v) dG_{i_p}(v) .$$

Now

$$P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{i_p} = v) = \int_0^{u_n} P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{j_1} = u, X_{i_p} = v) dH_v(u)$$

where

$$H_v(u) = P(X_{j_1} \leq u | X_{i_p} = v) .$$

For each v

$$\begin{aligned} &P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{j_1} = u, X_{i_p} = v) \\ &= P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{j_1} = u) . \end{aligned}$$

Define

$$g(u) = \begin{cases} P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{j_1} = u) & \text{for } 0 \leq u \leq 1 \\ 1 & \text{for } u < 0 \end{cases}$$

We have

$$P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n | X_{i_p} = v) = \int_0^{u_n} g(u) dH_v(u).$$

It is important to note that $g(u)$ is a nonincreasing function of u .

We now consider the measure $H_v(u)$.

$$H_v(u) = P(X_{j_1} \leq u | X_{i_p} = v) = P(W_{j_1} + \rho^\ell v \leq u)$$

and

$$\forall 1 \leq t \leq q \quad X_{j_t} > W_{j_t} = X_{j_t} - \rho^{(j_t - i_p)} X_{i_p} > X_{j_t} - \rho^\ell.$$

Hence

$$\begin{aligned} P(W_{j_1} + \rho^\ell v \leq u) &\leq P(X_{j_1} - \rho^\ell + \rho^\ell v \leq u) \\ &\leq P(X_{j_1} \leq u + \rho^\ell) \quad \forall 0 \leq u, v \leq 1. \end{aligned}$$

Since g is a nonincreasing function of u

$$\begin{aligned} \int_0^{u_n} g(u) dH_v(u) &\leq \int_0^{u_n} g(u) dP(X_{j_1} - \rho^\ell \leq u) = \int_{-\rho^\ell}^{u_n - \rho^\ell} g(w + \rho^\ell) dw \\ &\leq \int_{-\rho^\ell}^{u_n - \rho^\ell} g(w) dw = \rho^\ell + \int_0^{u_n - \rho^\ell} g(w) dw \leq \rho^\ell + \int_0^{u_n} g(w) dw = \rho^\ell \\ &\quad + P(X_{j_1} \leq u_n, \dots, X_{j_q} \leq u_n) \end{aligned}$$

and from this we conclude that $D(u_n)$ holds with $\phi(\ell) = \rho^\ell = \left(\frac{1}{r}\right)^\ell$,
since $\phi(\ell) \rightarrow 0$ as $\ell \rightarrow \infty$.

7. Proof that $D'(u_n)$ Does Not Hold.

We consider

$$n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) .$$

Associativity implies

$$\begin{aligned} P(X_1 > u_{nk}, X_j > u_{nk}) &\geq P(X_1 > u_{nk}) P(X_j > u_{nk}) \quad \forall j \geq 2 \\ &= P^2(X_1 > u_{nk}) = \left[1 - u_{nk}\right]^2 = \left(\frac{x}{nk}\right)^2 \end{aligned}$$

So

$$n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) \geq n \sum_{j=2}^n \left(\frac{x}{nk}\right)^2 = \frac{n(n-1)x^2}{n^2 k^2} = \frac{(n-1)}{n} \frac{x^2}{k^2} .$$

Also

$$X_j = W_j + \rho^{j-1} X_1$$

where

$$W_j = \epsilon_j + \rho \epsilon_{j+1} + \dots + \rho^{j-2} \epsilon_2 . \quad 0 \leq X_1 \leq 1 \Rightarrow X_j < W_j + \rho^{j-1} .$$

So

$$P(X_j > u_{nk}) < P(W_j > u_{nk} - \rho^{j-1}) .$$

Now W_j is independent of X_1 so

$$n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) \leq n \sum_{j=2}^n P(X_1 > u_{nk}, W_j > u_{nk} - \rho^{j-1})$$

$$\begin{aligned}
&= n \sum_{j=2}^n P(X_1 > u_{nk}) P(W_j > u_{nk} - \rho^{j-1}) = n P(X_1 > u_{nk}) \sum_{j=2}^n P(W_j > u_{nk} - \rho^{j-1}) \\
&= \frac{x}{k} \sum_{j=2}^n P(W_j > u_{nk} - \rho^{j-1}) < \frac{x}{k} \sum_{j=2}^n P(X_j > u_{nk} - \rho^{j-1}) \\
&= \frac{x}{k} \sum_{j=2}^n (1 + \rho^{j-1} - 1 + \frac{x}{nk}) = \frac{(n-1)}{n} \frac{x^2}{k^2} + \frac{x}{k} \sum_{j=2}^n \rho^{j-1} < \frac{x^2}{k^2} + \frac{x}{k}.
\end{aligned}$$

We see from this that

$$\overline{\lim}_{n \rightarrow \infty} n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) \leq O(\frac{1}{k})$$

To actually show equality we must work a little harder.

Lemma 7.1: In the case $r = 2$

$$\overline{\lim}_{n \rightarrow \infty} n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) = O(\frac{1}{k})$$

$$\left[\text{i.e., } \lim_{k \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} nk \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) > 0 \right]$$

and hence $D'(u_n)$ fails.

Proof: For each j

$$P(X_1 > u_{nk}, X_j > u_{nk}) = \int_{u_{nk}}^1 P(W_j > u_{nk} - \frac{v}{2^{j-1}}) dv$$

$$P(W_j > u_{nk} - \frac{v}{2^{j-1}}) = 1 - \frac{(\ell_j + 1)}{2^{j-1}} \text{ if } u_{nk} - \frac{v}{2^{j-1}} \geq \frac{\ell_j}{2^{j-1}}$$

$$= 1 - \frac{\ell_j}{2^{j-1}} \text{ if } u_{nk} - \frac{v}{2^{j-1}} < \frac{\ell_j}{2^{j-1}}$$

where ℓ_j is defined by $\frac{\ell_j}{2^{j-1}} \leq u_{nk} < \frac{\ell_j + 1}{2^{j-1}}$ and ℓ_j is an integer.

Suppose $u_{nk} > \frac{2^{j-1}-1}{2^{j-1}}$ then $\frac{\ell_j}{2^{j-1}} < u_{nk}$ since

$$(7.1) \quad \frac{\ell_j}{2^{j-1}} \leq \frac{2^{j-1}-1}{2^{j-1}}.$$

(7.1) holds because if $\frac{\ell_j}{2^{j-1}} > \frac{2^{j-1}-1}{2^{j-1}}$ then $\frac{\ell_j}{2^{j-1}} = 1$ but $u_{nk} < 1$ and

$\frac{\ell_j}{2^{j-1}} \leq u_{nk}$ a contradiction.

So

$$u_{nk} = 1 - \frac{x}{nk} > \frac{2^{j-1}-1}{2^{j-1}} \Leftrightarrow 2^{j-1}(1 - \frac{x}{nk}) > 2^{j-1}-1 \Leftrightarrow 2^{j-1} < \frac{nk}{x}$$

$$\Leftrightarrow j < \log_2 n + \log_2 k - \log_2 x + 1.$$

When

$$j \leq [\log_2 n + \log_2 k - \log_2 x] + 1$$

we have

$$u_{nk} > \frac{2^{j-1}-1}{2^{j-1}}$$

and

$$P(W_j > u_{nk} - \frac{v}{2^{j-1}}) = 1 - \frac{\ell_j}{2^{j-1}} \quad \forall u_{nk} \leq v \leq 1.$$

Define

$$f(n) = [\log_2 n + \log_2 k - \log_2 x] + 1$$

then

$$\begin{aligned} n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) &= n \left[\sum_{j=2}^{f(n)} (1 - u_{nk}) \left(1 - \frac{\ell_j}{2^{j-1}}\right) \right] \\ &+ n \left[\sum_{f(n)+1}^n \int_{u_{nk}}^1 P(W_j > u_{nk} - v/2^{j-1}) dv \right] \geq n \sum_{j=2}^{f(n)} \left(\frac{x}{nk}\right) \left(1 - \frac{\ell_j}{2^{j-1}}\right). \end{aligned}$$

Now

$$n \sum_{j=2}^{f(n)} \left(\frac{x}{nk}\right) \left(1 - \frac{\ell_j}{2^{j-1}}\right) = \frac{x}{k} \sum_{j=2}^{f(n)} \frac{1}{2^{j-1}}$$

since $\ell_j = 2^{j-1} - 1$ for $j \leq f(n)$ and it follows that

$$\lim_{n \rightarrow \infty} n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) \geq \frac{x}{k}$$

since

$$\frac{x}{k} \sum_{j=2}^{f(n)} \frac{1}{2^{j-1}} \nearrow \frac{x}{k} \text{ as } n \rightarrow \infty.$$

So

$$\overline{\lim}_{n \rightarrow \infty} n \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) = O\left(\frac{1}{k}\right)$$

and $D'(u_n)$ fails.

A similar but perhaps slightly more complicated argument can be given to show $D'(u_n)$ fails for general r . This result indicates that there is a possibility that $P(M_n \leq u_n)$ does not converge to e^{-x} . This adds to the evidence we have shown in earlier sections. In the next section we shall prove that $P(M_n^{(r)} \leq u_n) \rightarrow e^{-[(r-1)/r]x}$ as $n \rightarrow \infty$.

8. The Limit Theorem for the Uniform AR(1) Models.

We shall prove the limit theorem for all $r \geq 2$ but for simplicity of notation we shall continue to suppress the superscript r .

Let $M_0 = X_0$. We have

$$P(M_0 \leq 1 - \frac{x}{n}) = 1 - \frac{x}{n}$$

$$P(M_1 \leq 1 - \frac{x}{n}) = P(X_0 \leq 1 - \frac{x}{n}, X_1 \leq 1 - \frac{x}{n}) = P(X_0 \leq 1 - \frac{x}{n}, \frac{X_0}{r} + \varepsilon_1 \leq 1 - \frac{x}{n}) .$$

Conditioning on ε_1 gives

$$P(M_1 \leq 1 - \frac{x}{n}) = \frac{1}{r} \sum_{k=0}^{r-1} P(X_0 \leq 1 - \frac{x}{n}, X_0 \leq r - k - \frac{rx}{n}) .$$

We see that $r - k - \frac{rx}{n} \geq 1 - \frac{x}{n}$ if $r - k - 1 \geq \frac{(r-1)x}{n}$.

For n sufficiently large $\frac{(r-1)x}{n} < 1 \leq r - k - 1$ if $k \leq r - 2$. So for large n we have

$$P[M_1 \leq 1 - \frac{x}{n}] = \left\{ \sum_{k=0}^{r-2} \frac{1}{r} P[X_0 \leq 1 - \frac{x}{n}] \right\} + \frac{1}{r} P[X_0 \leq 1 - \frac{rx}{n}]$$

$$= 1 - \frac{(2r-1)x}{rn}.$$

In general, if r is small enough relative to n (i.e., $\frac{(r-1)x}{n} < 1$) for each m

$$(8.1) \quad P[M_m \leq 1 - \frac{x}{n}] = \frac{(r-1)}{r} P[M_{m-1} \leq 1 - \frac{x}{n}] + \frac{1}{r} P[M_{m-1} \leq 1 - \frac{x}{n},$$

$$X_{m-1} \leq 1 - \frac{rx}{n}]$$

and

$$(8.2) \quad P[M_m \leq 1 - \frac{x}{n}, X_m \leq 1 - \frac{r^i x}{n}] = \frac{(r-1)}{r} P[M_{m-1} \leq 1 - \frac{x}{n}]$$

$$+ \frac{1}{r} P[M_{m-1} \leq 1 - \frac{x}{n}, X_{m-1} \leq 1 - \frac{r^{i+1} x}{n}]$$

where $i \leq j - 1$ and j will be defined later.

These recursive type formulae enable us to calculate $P[M_m \leq 1 - \frac{x}{n}]$ when m is small relative to n . This will be made explicit shortly.

We compute $P[M_2 \leq 1 - \frac{x}{n}]$ to see how the general formula arises.

$$\begin{aligned}
P[M_2 \leq 1 - \frac{x}{n}] &= \left(\frac{r-1}{r}\right) P[M_1 \leq 1 - \frac{x}{n}] + \frac{1}{r} P[M_1 \leq 1 - \frac{r}{n}, X_1 \leq 1 - \frac{rx}{n}] \\
&= \left(\frac{r-1}{r}\right)^2 P[M_0 \leq 1 - \frac{x}{n}] + \frac{(r-1)}{r^2} P[M_0 \leq 1 - \frac{rx}{n}]
\end{aligned}$$

$$+ \frac{1}{r^2} \sum_{k=0}^{r-1} P[M_0 \leq 1 - \frac{x}{n}, \frac{X_0 + k}{r} \leq 1 - \frac{rx}{n}]$$

$$= \left(\frac{r-1}{r}\right)^2 (1 - \frac{x}{n}) + \frac{(r-1)}{r^2} (1 - \frac{rx}{n}) + \frac{1}{r^2} \sum_{k=0}^{r-1} P[X_0 \leq 1 - \frac{x}{n}, X_0 \leq r - k - \frac{r^2 x}{n}] .$$

Now $r - k - \frac{r^2 x}{n} \geq 1 - \frac{x}{n}$ if $r - k - 1 \geq \frac{(r^2 - 1)x}{n}$ and for n sufficiently large $\frac{(r^2 - 1)x}{n} < \frac{r^2 x}{n} < 1$ and $r - k - 1 \geq 1$ when $k \leq r - 2$.

So

$$\begin{aligned}
P[M_2 \leq 1 - \frac{x}{n}] &= \left(\frac{r-1}{r}\right)^2 (1 - \frac{x}{n}) + \frac{(r-1)}{r^2} (1 - \frac{rx}{n}) \\
&\quad + \frac{(r-1)}{r^2} (1 - \frac{x}{n}) + \frac{1}{r^2} (1 - \frac{r^2 x}{n})
\end{aligned}$$

and after some algebra we get

$$P[M_2 \leq 1 - \frac{x}{n}] = 1 - \frac{(3r-2)x}{rn} .$$

Let j be the integer for which $1 - \frac{r^{j-1}x}{n} \geq 0 > 1 - \frac{r^j x}{n}$. We can use equations (8.1) and (8.2) to show for every $k \leq j - 1$

$$(8.3) \quad P[M_k \leq 1 - \frac{x}{n}] = 1 - \frac{((k+1)r-k)x}{rn}$$

and in particular we get

$$(8.4) \quad P[M_{j-1} \leq 1 - \frac{x}{n}] = 1 - \frac{jx}{n} + \frac{(j-1)x}{rn}.$$

One can show

$$\lim_{n \rightarrow \infty} P[M_n \leq 1 - \frac{x}{n}] \geq e^{-\frac{(r-1)}{r}x}$$

as follows:

$$\begin{aligned} P[M_n \leq 1 - \frac{x}{n}] &= P[X_0 \leq 1 - \frac{x}{n}, \dots, X_{j-1} \leq 1 - \frac{x}{n}, X_j \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}] \\ &\geq P[X_0 \leq 1 - \frac{x}{n}, \dots, X_{j-1} \leq 1 - \frac{x}{n}] P[X_j \leq 1 - \frac{x}{n}, \dots, X_{2j-1} \leq 1 - \frac{x}{n}] \\ &\dots P[X_{(k-1)j} \leq 1 - \frac{x}{n} \dots X_{kj-1} \leq 1 - \frac{x}{n}] P[X_{kj} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}] \end{aligned}$$

where $k = [\frac{n}{j}]$ and $[]$ denotes the greatest integer function. The inequality above is a consequence of associativity.

$$P[X_{rj} \leq 1 - \frac{x}{n}, \dots, X_{kj-1} \leq 1 - \frac{x}{n}] = P[X_0 \leq 1 - \frac{x}{n}, \dots, X_{j-1} \leq 1 - \frac{x}{n}]$$

by stationarity and hence

$$P[M_n \leq 1 - \frac{x}{n}] \geq P^k[M_{j-1} \leq 1 - \frac{x}{n}] P[X_{kj} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}].$$

$$P^k[M_{j-1} \leq 1 - \frac{x}{n}] \rightarrow e^{-\frac{(r-1)}{r}x}$$

as $j \rightarrow \infty$ since

$$\left(1 - \frac{jx}{n} + \frac{(j-1)x}{rn}\right)^k \sim \left(1 - \left(\frac{r-1}{r}\right) \frac{x}{k} - \frac{x}{n}\right)^k$$

and

$$\left(1 - \left(\frac{r-1}{r}\right) \frac{x}{k} - \frac{x}{n}\right)^k \rightarrow e^{-\left(\frac{r-1}{r}\right)x}$$

as $k \rightarrow \infty$ since

$$\frac{x}{n} = o\left(\frac{1}{k}\right).$$

We also need

$$(8.5) \quad P[X_{kj} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}] \rightarrow 1$$

as $m \rightarrow \infty$. (8.5) holds because there are at most $j-1$ X_i 's in the expression and (8.4) gives us that

$$P[M_{j-1} \leq 1 - \frac{x}{n}] \rightarrow 1$$

as $j \rightarrow \infty$. Since

$$P[X_{kj} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}] \geq P[M_{j-1} \leq 1 - \frac{x}{n}]$$

we get (8.5).

Lemma 8.1: Let $k = \lfloor \frac{n}{j} \rfloor$. Define

$$I_1 = \{0, 1, 2, \dots, j-1-m\} \quad I_1^* = \{j-m, \dots, j-1\}$$

$$I_2 = \{j, j+1, \dots, 2j-1-m\} \quad I_2^* = \{2j-m, \dots, 2j-1\}$$

⋮

⋮

$$I_k = \{(k-1)j, \dots, kj-1-m\} \quad I_k^* = \{kj-m, \dots, kj-1\}$$

We have for the uniform process

$$|P[M_n \leq 1 - \frac{x}{n}] - P^k[M_{j-1} \leq 1 - \frac{x}{n}]| \rightarrow 0$$

as n, m , and $j \rightarrow \infty$ provided $\frac{m}{j} \rightarrow 0$.

The proof of this lemma is similar to the proof of lemma 2.4, Leadbetter (5) pp. 293-294. We denote by $M(I_i)$ the maximum of the X_j 's for $j \in I_i$.

$$\begin{aligned} & |P[M_n \leq 1 - \frac{x}{n}] - P^k[M_{j-1} \leq 1 - \frac{x}{n}]| \leq |P(M_n \leq 1 - \frac{x}{n}) \\ & - P[M_{jk} \leq 1 - \frac{x}{n}]| + |P\left(\bigcap_{i=1}^k (M(I_i) \leq 1 - \frac{x}{n})\right) - P(M_{jk} \leq 1 - \frac{x}{n})| \\ & + |P\left(\bigcap_{i=1}^k (M(I_i) \leq 1 - \frac{x}{n})\right) - P^k(M(I_1) \leq 1 - \frac{x}{n})| \\ & + |P^k(M(I_1) \leq 1 - \frac{x}{n}) - P^k(M_{j-1} \leq 1 - \frac{x}{n})|. \end{aligned}$$

We must show that all four terms tend to zero as j, m and $n \rightarrow \infty$ with $\frac{m}{j} \rightarrow 0$.

First

$$|P^k(M(I_1) \leq 1 - \frac{x}{n}) - P^k(M_{j-1} \leq 1 - \frac{x}{n})| \rightarrow 0$$

since

$$P^k(M_{j-1} \leq 1 - \frac{x}{n}) \rightarrow e^{-[(r-1)/r]x}$$

as $j, n \rightarrow \infty$ and

$$P[M(I_1) \leq 1 - \frac{x}{n}] = 1 - (j-1-m) \left(\frac{r-1}{r} \right) \frac{x}{n} - \frac{x}{n}.$$

If

$$\frac{m}{j} \rightarrow 0 \quad [1 - (j-1-m) \frac{r-1}{r} \frac{x}{n} - \frac{x}{n}]^k \rightarrow e^{-[(r-1)/r]x}.$$

$$0 \leq P \left[\bigcap_{i=1}^k (M(I_i) \leq 1 - \frac{x}{n}) \right] - P(M_{jk} \leq 1 - \frac{x}{n}) \leq k P[M(I_1) \leq 1 - \frac{x}{n} \leq M(I_1^*)].$$

This follows just as in lemma 2.4 Leadbetter [8]. Now

$$kP[M(I_1) \leq 1 - \frac{x}{n} \leq M(I_1^*)] \leq kP(1 - \frac{x}{n} \leq M(I_1^*))$$

$$< \frac{kx}{n} + \frac{kmx}{n} \left(\frac{r-1}{r} \right), \quad \frac{kx}{n} < \frac{x}{j}$$

and

$$\frac{kmx}{n} \left(\frac{r-1}{r} \right) < \frac{mx}{j} \left(\frac{r-1}{r} \right).$$

So

$$\frac{kx}{n} + \frac{kmx}{n} \left(\frac{r-1}{r} \right) \rightarrow 0$$

as m, n and $j \rightarrow \infty$. Here we need the fact that $\frac{m}{j} \rightarrow 0$ as m and $j \rightarrow \infty$.

It remains for us to show that

$$|P[M_n \leq 1 - \frac{x}{n}] - P[M_{jk} \leq 1 - \frac{x}{n}]| \rightarrow 0$$

and

$$|P(\bigcap_{i=1}^k (M(I_i) \leq 1 - \frac{x}{n})) - P^k(M(I_1) \leq 1 - \frac{x}{n})| \rightarrow 0.$$

Using the argument which led to (8.5) we have the following:

$$\begin{aligned} P[M_{jk} \leq 1 - \frac{x}{n}] &\geq P[M_n \leq 1 - \frac{x}{n}] \\ &\geq P[M_{jk} \leq 1 - \frac{x}{n}] P[X_{jk+1} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}] \end{aligned}$$

and by (8.5)

$$P[M_{jk} \leq 1 - \frac{x}{n}] \{1 - P[X_{jk+1} \leq 1 - \frac{x}{n}, \dots, X_n \leq 1 - \frac{x}{n}]\} \rightarrow 0 \text{ as } j, n \rightarrow \infty.$$

and hence

$$P[M_{jk} \leq 1 - \frac{x}{n}] - P[M_n \leq 1 - \frac{x}{n}] \rightarrow 0.$$

Let $A_i = \{M(I_i) \leq 1 - \frac{x}{n}\}$ for $i = 1, 2, \dots, k$. We then have that

$$\left| P\left(\bigcap_{i=1}^k (M(I_i) \leq 1 - \frac{x}{n})\right) - P^k(M(I_i) \leq 1 - \frac{x}{n}) \right| = \left| P\left(\bigcap_{i=1}^k A_i\right) - \prod_{i=1}^k P(A_i) \right|$$

and we shall see that this can be bounded by $2k \frac{x}{n}$. Since $2k \frac{x}{n} < 2 \frac{x}{j}$

and $2 \frac{x}{j} \rightarrow 0$ as $j \rightarrow \infty$ this bound will be good enough to get the result of the lemma.

$$\left| P\left(\bigcap_{i=1}^k A_i\right) - \prod_{i=1}^k P(A_i) \right| = P\left(\bigcap_{i=1}^k A_i\right) - \prod_{i=1}^k P(A_i)$$

$$= P(A_k | \bigcap_{i=1}^{k-1} A_i) P\left(\bigcap_{i=1}^{k-1} A_i\right) - P(A_k) P\left(\bigcap_{i=1}^{k-1} A_i\right)$$

$$+ P(A_k) P\left(\bigcap_{i=1}^{k-2} A_i\right) P(A_{k-1} | \bigcap_{i=1}^{k-2} A_i)$$

$$- P(A_k) P(A_{k-1}) P\left(\bigcap_{i=1}^{k-2} A_i\right) + \dots$$

$$+ \left\{ \prod_{j=3}^k P(A_j) \right\} P(A_2 | A_1) - \prod_{j=1}^k P(A_j)$$

$$\leq P(A_k | \bigcap_{i=1}^{k-1} A_i) - P(A_k) + P(A_{k-1} | \bigcap_{i=1}^{k-2} A_i) - P(A_{k-1})$$

$$+ \dots + P(A_2 | A_1) - P(A_2) .$$

For a typical term

$$P(A_s | \bigcap_{i=1}^{s-1} A_i) - P(A_s)$$

we shall show that

$$0 \leq P(A_s | \bigcap_{i=1}^{s-1} A_i) - P(A_s) < \frac{2x}{n}$$

and hence

$$|P(\bigcap_{i=1}^k A_i) - \prod_{i=1}^k P(A_i)|$$

will be less than $\frac{2kx}{n} = \frac{2x}{j}$.

First we will show this for $P(A_2|A_1) - P(A_2)$. Let $q = j - 1 - m$.

$$\begin{aligned} P(A_2|A_1) - P(A_2) &= P[X_{q+m+1} \leq 1 - \frac{x}{n}, \dots, X_{2q+m} \leq 1 - \frac{x}{n} | A_1] \\ &\quad - P[X_{q+m+1} \leq 1 - \frac{x}{n}, \dots, X_{2q+m} \leq 1 - \frac{x}{n}] \end{aligned}$$

and we know

$$P(A_2) = 1 - \frac{qx}{n} + \frac{(q-1)x}{rn}.$$

$$P(A_2|A_1) = \int_0^{1-x/n} P[X_{q+m+1} \leq 1 - \frac{x}{n}, \dots,$$

$$X_{2q+m} \leq 1 - \frac{x}{n} | X_q = v] dF_{x_q}(v)$$

where

$$F_{x_q}(v) = P[X_q \leq v | X_0 \leq 1 - \frac{x}{n}, \dots, X_q \leq 1 - \frac{x}{n}] .$$

For each ℓ , $X_{q+\ell} = \rho^\ell X_q + W_\ell$ where W_ℓ is a discrete uniform random variable with $P[W_\ell = \frac{h}{r^\ell}] = \frac{1}{r^\ell}$ for $h = 0, 1, \dots, \frac{r^\ell - 1}{r}$.

$$\begin{aligned} & P[X_{q+m+1} \leq 1 - \frac{x}{n}, \dots, X_{2q+m} \leq 1 - \frac{x}{n} | X_q = v] \\ &= P[W_{m+1} \leq 1 - \frac{x}{n} - \rho^{m+1} v, \dots, W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v] . \end{aligned}$$

We observe that for $m+1 \leq \ell \leq m+q$

$$\begin{aligned} P[W_\ell \leq 1 - \frac{x}{n} \rho^\ell v] &= 1 & \text{if } v \leq 1 - \frac{r^\ell x}{n} \\ &= \frac{r^\ell - 1}{r^\ell} & \text{if } v > 1 - \frac{r^\ell x}{n} . \end{aligned}$$

This follows since

$$v \leq 1 - \frac{r^\ell x}{n} \Rightarrow 1 - \frac{x}{n} - \rho^\ell v \geq 1 - \frac{x}{n} - \rho^\ell + \frac{x}{n} = 1 - \rho^\ell$$

and

$$P[W_\ell \leq 1 - \rho^\ell] = 1 .$$

For $v > 1 - \frac{r^\ell x}{n}$ we have

$$1 - \rho^\ell > 1 - \frac{x}{n} - \rho^\ell v > 1 - \frac{x}{n} - \rho^\ell > 1 - 2\rho^\ell$$

and

$$P[W_\ell \leq 1 - 2\rho^\ell] = \frac{r^\ell - 1}{r^\ell}.$$

(Note that $1 - 2\rho^\ell < 1 - \frac{x}{n} - \rho^\ell \Leftrightarrow \frac{x}{n} < \rho^\ell \Leftrightarrow \ell \leq j - 1$.)

From this we see that

$$\begin{aligned} P(A_2 | A_1) &= \int_0^{1-(r^{j-1}x)/n} dF_{x_q}(v) \\ &\quad + \int_{1-(r^{j-1}x)/n}^{1-(r^{j-2}x)/n} P[W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v] dF_{x_q}(v) \\ &\quad + \int_{1-(r^{j-2}x)/n}^{1-(r^{j-3}x)/n} P[W_{m+q-1} \leq 1 - \frac{x}{n} - \rho^{m+q-1} v, W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v] dF_{x_q}(v) \\ &\quad + \dots + \int_{1-(r^{m+1}x)/n}^{1-x/n} P[W_{m+1} \leq 1 - \frac{x}{n} - \rho^{m+1} v, \dots, \\ &\quad \quad \quad W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v] dF_{x_q}(v). \end{aligned}$$

The integrands

$$P[W_\ell \leq 1 - \frac{x}{n} - \rho^\ell v, \dots, W_{m+q} \leq 1 - \frac{x}{n} - \rho^{m+q} v]$$

are constant over the interval of integration with the constant being $1 - \rho^\ell$.

The integrals simplify as follows:

$$\begin{aligned}
(8.6) \quad P(A_2|A_1) &= F_{x_q} \left(1 - \frac{r^{j-1}x}{n}\right) \\
&+ \sum_{\ell=1}^{q-1} \left(1 - \frac{1}{r^{j-\ell}}\right) \left[F_{x_q} \left(1 - \frac{r^{j-\ell-1}x}{n}\right) - F_{x_q} \left(1 - \frac{r^{j-1}x}{n}\right)\right] \\
&+ \left[1 - F_{x_q} \left(1 - \frac{r^{m+1}x}{n}\right)\right] \left(1 - \frac{1}{r^{m+1}}\right).
\end{aligned}$$

Claim: For each $0 \leq v \leq 1$ $F_{x_q}(v) \leq v + \rho^q$.

Proof: $F_{x_q}(v) = P[X_q \leq v | X_0 \leq 1 - \frac{x}{n}, \dots, X_q \leq 1 - \frac{x}{n}]$

$$\leq P[X_q \leq v | X_0 = 0, X_1 \leq 1 - \frac{x}{n}, \dots, X_q \leq 1 - \frac{x}{n}].$$

$X_0 = 0 \Rightarrow X_1, X_2, \dots, X_q$ are all less than or equal to $1 - \frac{x}{n}$ and hence we get

$$\begin{aligned}
P[X_q \leq v | X_0 = 0, X_1 \leq 1 - \frac{x}{n}, \dots, X_q \leq 1 - \frac{x}{n}] &= P[X_q \leq v | X_0 = 0] \\
&= P[W_q \leq v].
\end{aligned}$$

Now $W_q = X_q - \rho^q X_0 \geq X_q - \rho^q$ with prob. 1.

Therefore

$$P[W_q \leq v] \leq P[X_q - \rho^q \leq v] = P[X_q \leq v + \rho^q] = v + \rho^q.$$

After some algebra (8.6) simplifies to

$$(8.7) \quad P(A_2|A_1) = 1 - \sum_{\ell=1}^{q-1} \left\{ \frac{1}{r^{j-1}} \left[F_{x_q} \left(1 - \frac{r^{j-\ell-1}x}{n} \right) - F_{x_q} \left(1 - \frac{r^{j-\ell}x}{n} \right) \right] \right\} \\ + \frac{1}{r^{m+1}} F_{x_q} \left(1 - \frac{r^{m+1}x}{n} \right) - \frac{1}{r^{m+1}} .$$

$$F_{x_q} \left(1 - \frac{r^{j-\ell-1}x}{n} \right) - F_{x_q} \left(1 - \frac{r^{j-\ell}x}{n} \right) \geq 1 - \frac{r^{j-\ell-1}x}{n} - \left(1 - \frac{r^{j-\ell}x}{n} + \rho^q \right)$$

and

$$F_{x_q} \left(1 - \frac{r^{m+1}x}{n} \right) \leq 1 - \frac{r^{m+1}x}{n} + \rho^q .$$

Using these two inequalities with (8.7) we get

$$P(A_2|A_1) \leq 1 - \frac{qx}{n} + \frac{(q-1)x}{rn} + \frac{\rho^q}{r^{m+1}} + \rho^q \sum_{\ell=1}^{q-1} \frac{1}{r^{j-\ell}} \\ = 1 - \frac{qx}{n} + \frac{(q-1)x}{rn} + \frac{1}{r^j} + \frac{1}{r^j} \left(1 - \frac{1}{r^{j-n-2}} \right) .$$

We get

$$P(A_2|A_1) - P(A_2) \leq \frac{1}{r^j} \left(2 - \frac{1}{r^{j-m-2}} \right) < \frac{2}{r^j} \leq \frac{2x}{n} .$$

Now consider $P(A_s | \bigcap_{i=1}^{s-1} A_i)$ for $s \geq 3$.

Let

$$B_{s-2} = \bigcap_{i=1}^{s-2} A_i$$

so that

$$P(A_s | \bigcap_{i=1}^{s-1} A_i) = P(A_s | A_{s-1} \cap B_{s-2}) .$$

$$A_{s-1} = \{X_{(s-2)j} \leq 1 - \frac{x}{n}, \dots, X_{(s-1)j-1-m} \leq 1 - \frac{x}{n}\}$$

and

$$P(A_s | A_{s-1} \cap B_{s-2}) = \int_0^{1-x/n} P(X_{(s-1)j} \leq 1 - \frac{x}{n}, \dots, X_{sj-1-m} \leq 1 - \frac{x}{n}) dF_s(v)$$

where

$$F_s(v) = P(X_{(s-1)j-1-m} \leq v | A_{s-1} \cap B_{s-2}) .$$

Now

$$F_s(v) \leq P(X_{(s-1)j-1-m} \leq v | X_{(s-2)j} = 0, X_{(s-2)j+1} \leq 1 - \frac{x}{n} \dots)$$

$$X_{(s-1)j-1-m} \leq 1 - \frac{x}{n} \cap B_{s-2}) = P(X_{(s-1)j-1-m} \leq v | X_{(s-2)j} = 0) \leq v + \rho^q .$$

Applying stationarity we see that $P(A_s | A_{s-1} \cap B_{s-2})$ has the same upper bound as $P(A_2 | A_1)$ since the integrands are equal and the distribution functions have the same upper bound. Therefore we conclude that

$$P(A_s | A_{s-1} \cap B_{s-2}) - P(A_s) \leq \frac{2}{rj} < \frac{2x}{n} .$$

This completes the proof of the lemma. As a corollary we get the following limit theorem:

Theorem 8.2: For the processes

$$\left\{ X_n^{(r)} \right\}_{n=0}^{\infty} \quad r \geq 2$$

as defined in section 1

$$P(M_n^{(r)} \leq 1 - \frac{x}{n}) \rightarrow e^{-((r-1)/r)x} \quad \text{for all } x \geq 0$$

or equivalently

$$P(M_n^{(r)} \leq 1 - \frac{y}{n}) \rightarrow e^{((r-1)/r)y} \quad \text{for all } y \leq 0.$$

For this theorem it was not necessary for us to verify the strong mixing condition. However, it would be nice to know whether or not the strong mixing condition is satisfied since the condition is useful in proving other types of limit theorems such as the central limit theorem for the normalized sums. It seems likely that the strong mixing condition would hold for this process since X_{n+s} depends on X_n in only a minor way when s is large.

$$X_{n+s} = \left(\frac{1}{r}\right)^s X_n + W_s$$

and since

$$0 \leq X_n \leq 1 \quad W_s - \left(\frac{1}{r}\right)^s \leq X_{n+s} \leq W_s + \left(\frac{1}{r}\right)^s .$$

So for s large X_{n+s} is almost equal to W_s and remember W_s is independent of X_n .

The author can demonstrate that if

$$A = \left\{ \bigcap_{i \in J_1} (y_i \leq X_i \leq x_i) \right\}$$

and

$$B = \left\{ \bigcap_{i \in J_2} (y_i \leq X_i \leq x_i) \right\}$$

where $\forall i \ 0 \leq y_i \leq x_i \leq 1$, J_1 is a subset of $\{0, 1, 2, \dots, m\}$ and J_2 a subset of $\{m+k+1, \dots\}$ then

$$|P(A \cap B) - P(A)P(B)| \leq 2\left(\frac{1}{r}\right)^k .$$

This is close to what we need and is much stronger than $D(u_n)$. However, the author does not see a way to extend the result to arbitrary sets in the appropriate σ -fields. The class of cylinder sets does not satisfy the conditions needed to apply a monotone class argument.

9. Simulation Results, $x = 1$ and $x = \frac{1}{2}$ with $r = 2$

A simulation of the process was carried out for the case $r = 2$.

The results are presented here for two reasons. First, it gives independent

confirmation that $P(M_n \leq 1 - \frac{x}{n})$ appears to be converging to $e^{-x/2}$ and the second reason is that it may indicate something about the rate of convergence.

The simulation was done on the LOTS system using the library random number generator. In a more detailed analysis it would perhaps be a good idea to use a random number generator which is nearer to the state of the art. Such random number generators are readily available.

The following table summarizes the results. n refers to the number of X_i 's over which the maximum is taken and m is the number of replications. $nP(1)$ is the number of X_i 's $\leq 1 - \frac{1}{n}$ and $nP(\frac{1}{2})$ is the number of X_i 's $\leq 1 - \frac{1}{2n}$.

TABLE 2

m = 500			m = 1000		
<u>n</u>	<u>P(1)</u>	<u>$P(\frac{1}{2})$</u>	<u>n</u>	<u>P(1)</u>	<u>$P(\frac{1}{2})$</u>
50	.606	.790	50	.583	.765
75	.550	.744	75	.586	.765
100	.600	.756	100	.616	.782
125	.612	.774	125	.577	.775
150	.582	.780	150	.608	.767
175	.600	.796	175	.584	.753
200	.584	.752	200	.583	.773
			225	.607	.795
			250	.592	.771
			275	.591	.762
			300	.593	.768
			325	.602	.779
			350	.610	.777
			375	.593	.778
			400	.622	.792
			425	.590	.781
			450	.602	.780
			475	.604	.775
			500	.613	.770
			525	.621	.776
			550	.593	.775
			575	.609	.792
			600	.593	.787
			625	.608	.783
			650	.608	.782
			675	.605	.794
			700	.588	.778
			725	.603	.771

Since the limit value for $x = 1$ is $e^{-1/2}$ and for $x = \frac{1}{2}$ is $e^{-1/4}$, $P(1)$ and $P(\frac{1}{2})$ should be compared to the values .6065 and .7788 respectively. We see that the Monte Carlo approximation is good to within $\pm .02$ for n ranging from 50 to 725 and $x = 1$. For $x = \frac{1}{2}$ the approximation is even better with differences only in the third decimal place when $n \geq 500$. It is apparent that $P(1)$ is not converging to $e^{-1} \approx .3679$ and $P(\frac{1}{2})$ is not converging to $e^{-1/2} \approx .6065$.

Table 3 gives some idea of the variability due to random sampling. M_{500} was generated 1000 times and $P(1)$ and $P(\frac{1}{2})$ were computed. This was done five times in such a way that a different sequence of random numbers was used each time.

TABLE 3

m = 1000		n = 500	
replication	$P(1)$		$P(\frac{1}{2})$
1	.625		.789
2	.585		.778
3	.588		.774
4	.614		.780
5	.626		.773

We see that nearly all the variation between $P(1)$ and $e^{-1/2}$ and $P(\frac{1}{2})$ and $e^{-1/4}$ can be explained by sampling variation. This means that although these results are useful in confirming the convergence to $e^{-x/2}$ there is too much sampling variation to say much about the rate of convergence.

10. Implications on the Analysis of Air Pollution Data and Discussion

References.

Data provided by the Bay Area air monitoring stations has been studied with the hope of finding ways to predict the quality of the air in the future and to validate models which would be used to determine effects of changes in emissions. Time series analysis is a very important statistical tool that can be applied to the data. Attempts to apply Box-Jenkins type methodology have usually failed to show promise. It is the author's opinion that this failure is due to the difficulty in finding the right transformation to apply to the data to make it approximately normal.

An alternative approach would be to look at nonnormal ARMA-type models. The uniform processes studied in this paper are a special case of such models. Other authors have done work on exponential and, more generally, gamma-type ARMA models. One particular class of exponential models was studied by Jacobs and Lewis [4] and a special case of those models by Chernick [2]. Lewis has developed a class of gamma distributed ARMA models in a forthcoming paper [6]. This author believes that these gamma models might be applicable to pollution data and serve as an alternative to the lognormal models which seem to be the current practice.

The distribution of the maximum of the process over a time period is important because high levels of pollutants are of concern and one might want to know how likely it is that over a period of fixed length the level of a particular pollutant will exceed a known danger level. If the fixed length of time is large relative to the sampling interval, the limit distribution can provide a good approximation to the answer.

One of the currently adopted approaches (justified by the lognormal model) is to first apply a log transformation to the data. This will make the transformed data nearly normal if the data were originally approximately lognormally distributed over time. After this is done, time series methods for removing trends or seasonal effects are applied. These are generally linear filters of some kind and hence normality would be preserved. Hopefully one then has a stationary Gaussian process. One could then look at the sample autocorrelations to satisfy himself that $r_n \ln n \rightarrow 0$ and if the correlations appear to decay fast would use the norming constants for the maximum of i.i.d. normal random variables. Theoretical justification for this is given by Berman's theorem [1].

One could question how reasonable this approach would be. The theory could break down in several places. In the opinion of this author the log transformation may be the problem.

One approach which avoids the transformation would be to apply linear filters to make the process stationary and then fit a distribution to the filtered series. One would then like to invoke Leadbetter's theorem to justify norming the maximum as in the i.i.d. case. Unfortunately, it is much more difficult to convince oneself that the conditions hold. If we could fit an ARMA model to the series we might be satisfied that $D(u_n)$ would hold since recent papers indicate that this is often the case for ARMA processes. However, it is hard to know whether or not $D'(u_n)$ would hold. We have seen that $D'(u_n)$ can fail and even if it only barely fails the limit can be affected. The uniform examples have implications on data analysis not because they are reasonable models to apply

to the data, but rather because they suggest that even for reasonable models the condition $D'(u_n)$ might fail and the norming constant could be affected.

The methodology just described appears reasonable but the uniform example indicates that problems could arise. What approach could we take to solve this problem? Suppose a gamma type ARMA model gives a reasonable fit to the data after filters have been applied to get stationarity. Such a model could be used for prediction and in the context of maxima, if we assume the model describes the process, a limit theorem could be obtained using the model. For the model we could check Leadbetter's conditions. If they are satisfied we would proceed as previously described. If $D(u_n)$ is satisfied and $D'(u_n)$ fails there is a good chance that a limit theorem for the model could be obtained. It may be possible to apply methods similar to the ones in this paper to get the limit theorem for the model.

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