STABILITY OF HIGHER-ORDER LONGITUDINAL MODES

IN A BUNCHED BEAM WITHOUT MODE COUPLING*

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Introduction

The theory of longitudinal instabilities of bunched beams was proposed by F. Sacherer.¹ Starting from the Vlasov equation, he derived the integral equation for the perturbed distribution function. While the general method to solve the integral equation was given by Sacherer, a number of other papers discussing longitudinal bunched beam instability have also been published.²⁻⁶ Here we want to propose another formalism with which we can treat the integral equation without mode coupling for the case of a Gaussian bunch. We then generalize the formalism for the other bunch distributions, and derive a practical method to analyze the instability for the case of a parabolic bunch. While the solution of the Sacherer equation that we find is not new, we present another approach to solve it. Since the integral equation for the transverse instability is similar to that for the longitudinal instability, this formalism is also useful for the transverse case.⁷⁻¹⁰

Single Gaussian Bunch

Until the Sacherer equation is derived, we will closely follow the derivation in Ref. 3. We consider a single-bunch beam with a longitudinal oscillation of mode m ($m = 1$ for dipole, $m = 2$ for quadrupole, etc.).

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We assume that the longitudinal distribution of this mode is given by

$$\rho_m(\theta) e^{-i\Omega_m t}, \quad (1)$$

where Ω_m is the mode frequency and $\theta = z/R$ is the angular coordinate whose origin moves with the center of the unperturbed bunch. The front of the bunch has $\theta > 0$. The normalization is such that $\rho_m(\theta)$ is in unit of charge per radian. Then the beam current is expressed in terms of the Fourier transform of $\rho_m(\theta)$:

$$i_m(t) = \omega_0 \sum_{p=-\infty}^{\infty} \tilde{\rho}_m(p) e^{-i(p\omega_0 + \Omega_m)t}, \quad (2)$$

where ω_0 is the angular revolution frequency, and

$$\tilde{\rho}_m(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho_m(\theta) e^{-ip\theta} d\theta. \quad (3)$$

A particle at longitudinal position θ will receive an increase in energy per revolution

$$eV(\theta, t) = -e\omega_0 e^{-i\Omega_m t} \sum_p \tilde{\rho}_m(p) Z(p\omega_0 + \Omega_m) e^{ip\theta} \quad (4)$$

where $Z(\omega)$ is the longitudinal impedance. We neglect the energy gain induced by the unperturbed bunch.

The Sacherer formalism is now applied. The Vlasov equation for the mode oscillation is

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \dot{\theta} \frac{\partial \psi}{\partial \theta} + \dot{\epsilon} \frac{\partial \psi}{\partial \epsilon} &= 0, \\ \dot{\theta} &= -k_0 \epsilon, \quad \dot{\epsilon} = \frac{\Omega_B^2}{k_0} \theta + \frac{eV(\theta, t)}{T}, \end{aligned} \quad (5)$$

where

ϵ = energy error

T = revolution period = $2\pi/\omega_0$

Ω_s = unperturbed synchrotron frequency

k_0 = $\alpha\omega_0/E_0$

α = momentum compaction factor

By the transformation of coordinates

$$r \cos \theta = \theta, \quad r \sin \theta = \frac{k_0}{\Omega_s} \epsilon, \quad (6)$$

Eq. (5) becomes

$$\frac{\partial \psi}{\partial t} + \Omega_s \frac{\partial \psi}{\partial \phi} + \frac{eV(\theta, t)}{T} \frac{k_0}{\Omega_s} \frac{\partial \psi}{\partial r} = 0. \quad (7)$$

To linearize the equation, we write the distribution function as a sum of an unperturbed term and a perturbation term

$$\psi(r, \phi, t) = \psi_0(r) + \psi_m(r) e^{im\phi} e^{-i\Omega_m t}. \quad (8)$$

We also assume that frequency shifts are small compared with Ω_s so that modes with different m 's do not mix.

The Sacherer equation is then obtained:

$$i(m\Omega_s - \Omega_m)\psi_m(r) = -(i)^m \frac{2\pi m e k_0}{\Omega_s T^2} \frac{1}{r} \frac{\partial \psi_0}{\partial r} \sum_p \frac{1}{p} Z(p\omega_0 + \Omega_m) J_m(pr) \tilde{\rho}_m(p). \quad (9)$$

We can show that this equation is also valid for complex $\psi_m(r)$.

From the definition of

$$\tilde{\rho}_m(p) = \frac{1}{2\pi} \frac{\Omega_s}{k_0} \int_0^{2\pi} \psi_m(r) e^{im\phi} e^{-ipr \cos \phi} r dr d\phi, \quad (10)$$

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we have the equation in the frequency domain:

$$i(m\Omega_s - \Omega_m)\tilde{\rho}_m(q) = -\frac{2\pi me}{T^2} \sum_p \frac{1}{p} Z(p\omega_0 + \Omega_m)\tilde{\rho}_m(p) \int \frac{1}{r} \frac{\partial \psi_0}{\partial r} J_m(qr) J_m(pr) r dr \quad (11)$$

In what follows, we will treat a Gaussian bunch with an unperturbed distribution $\psi_0(r)$ given by

$$\psi_0(r) = \frac{Ne\alpha}{2\pi v_s E_0 \sigma^2} \exp\left\{-\frac{r^2}{2\sigma^2}\right\} \quad (12)$$

where

$$\iint \psi_0(r) dz d\theta = Ne$$

N = number of particles in one bunch

σ = r.m.s. bunch length in θ units = σ_z/R

$$v_s = \Omega_s/\omega_0$$

Then Eq. (11) becomes

$$i(m\Omega_s - \Omega_m)\tilde{\rho}_m(q) = \frac{mNe^2\alpha Z_0}{v_s E_0 T^2 \sigma^2} \sum_p g(p)\tilde{\rho}_m(p) \int_0^\infty e^{-\frac{t^2}{2}} J_m(\alpha qt) J_m(\alpha pt) t dt \quad (13)$$

where

$$Z_0 g(p) \equiv \frac{1}{p} Z(p\omega_0 + \Omega_m)$$

Z_0 = real impedance for normalization

After the integration we have the following eigenvalue problem:

$$\lambda \tilde{\rho}_m(q) = \sum_p K_m(q,p) \tilde{\rho}_m(p) \quad (14)$$

where

$$K_m(q, p) = g(p) \exp \left\{ -\frac{\sigma^2}{2} (q^2 + p^2) \right\} I_m(\sigma^2 qp) \quad (15)$$

$$m\Omega_s - \Omega_m = -i\lambda \frac{mN e^2 n Z_0}{v_s E_0 T_{\sigma}^2} \quad (16)$$

Since the modified Bessel function $I_m(x)$ can be expanded as

$$I_m(x) = \left(\frac{x}{2}\right)^m \sum_{k=0}^{\infty} \frac{1}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k} \quad (17)$$

the kernel $K_m(q, p)$ also can be expanded as

$$K_m(q, p) = g(p) \sum_{k=0}^{\infty} C_{mk}(\sigma q) C_{mk}(\sigma p) \quad (18)$$

where

$$C_{mk}(\sigma p) = \frac{1}{\sqrt{k!(m+k)!}} \exp \left\{ -\frac{\sigma^2 p^2}{2} \right\} \left(\frac{\sigma p}{\sqrt{2}} \right)^{m+2k} \quad (19)$$

and the vectors $C_{mk}(\sigma p)$, $k = 0, 1, 2, \dots$, are linearly independent.

Then Eq. (14) can be rewritten as

$$\lambda \tilde{\rho}_m(q) = \sum_{\nu=-\infty}^{\infty} \sum_{k=0}^{\infty} g(p) C_{mk}(\sigma q) C_{mk}(\sigma p) \tilde{\rho}_m(p) \quad (20)$$

We now look for an eigenmode $\tilde{\rho}_m(q)$ given by

$$\tilde{\rho}_m(q) = \sum_{k=0}^{\infty} a_k C_{mk}(\sigma q) \quad (21)$$

This equation indicates that $C_{mk}(\sigma p)$ corresponds to the spectrum of the k th radial mode. In particular, the lowest mode $C_{m0}(\sigma p)$ is just equal

to the spectrum of the Hermite mode in the Gaussian bunch. By this transformation, we can obtain the final eigenvalue problem

$$\lambda \alpha_k = \sum_{l=0}^{\infty} M_{kl} \alpha_l, \quad (22)$$

where

$$M_{kl} = \sum_p g(p) C_{mk}(op) C_{ml}(op). \quad (23)$$

Since M_{kl} is complex symmetric but not Hermitian, the eigenvalues and the eigenvectors are complex. For this reason, we have a complex eigenmode distribution. Let $\rho_m(\theta)$ be written as

$$\rho_m(\theta) = \rho_{mR}(\theta) + i\rho_{mI}(\theta). \quad (24)$$

Then the longitudinal distribution of this mode is seen to be

$$\rho_m(\theta, t) = \rho_{mR}(\theta) \cos \Omega_m t + \rho_{mI}(\theta) \sin \Omega_m t, \quad (25)$$

a sum of two modes which have a $\pi/2$ phase shift between them. The normalization of the eigenvector α_k is such that its length is unity:

$$\sum_{k=0}^{\infty} |\alpha_k|^2 = 1. \quad (26)$$

Let $\tilde{\rho}_{mn}(p)$ be an eigenmode for the n^{th} eigenvalue λ_n . We have the relation

$$\sum_p |\tilde{\rho}_{mn}(p)|^2 g(p) = \lambda_n \quad (27)$$

which connects the complex frequency shift with the impedance and the power spectrum of the eigenmode.

In the application of this formalism, we choose the number of modes for the summation in Eq. (18). The number depends upon the bunch length σ and the impedance function $g(p)$. If $g(p)$ is a slowly changing function with p , we may replace the summation in Eq. (23) with an integration.

Multibunch Gaussian Beam

Until the Sacherer equation is obtained, we will follow the derivation in Ref. 3. As in this reference, we take an example from PEP. We assume that the three bunches have the same intensity and are equally spaced. Then the perturbed beam current is given by

$$i_{m\mu}(t) = 3\omega_0 \sum_{p=-\infty}^{\infty} \tilde{\rho}_{m\mu}(3p+\mu) e^{-i(\Omega_m + 3p\omega_0 + \mu\omega_0)t} \quad (28)$$

The parameter μ is the coupled bunch mode index, which in this case can take three different integers, $\pm 1, 0$. The induced energy gain which is given to a particle in the reference bunch is

$$eV(\theta, t) = -3e\omega_0 e^{-i\Omega_m t} \sum_p \tilde{\rho}_{m\mu}(3p+\mu) Z((3p+\mu)\omega_0 + \Omega_m) e^{i(3p+\mu)\theta} \quad (29)$$

We now obtain the equation

$$i(m\Omega_s - \Omega_m)\psi_{m\mu}(r) = -3(i)^m \frac{2\pi me k_0}{\Omega_s T^2} \frac{1}{r} \frac{\partial \psi_0}{\partial r} \sum_p \frac{1}{3p+\mu} \times Z((3p+\mu)\omega_0 + \Omega_m) J_m((3p+\mu)r) \tilde{\rho}_{m\mu}(3p+\mu) \quad (30)$$

In the frequency domain we have the eigenvalue problem

$$\lambda \tilde{\rho}_{m\mu}(3q+\mu) = \sum_p K_m(3q+\mu, 3p+\mu) \tilde{\rho}_{m\mu}(3p+\mu) \quad (31)$$

where

$$K_m(3q+\mu, 3p+\mu) = g(3p+\mu) \sum_{k=0}^{\infty} C_{mk}((3q+\mu)\sigma) C_{mk}((3p+\mu)\sigma), \quad (32)$$

$$m\Omega_s - \Omega_{m\mu} = -1\lambda \frac{3mNe^2 \alpha_0^2}{v_s E_0 T^2 \sigma^2}, \quad (33)$$

and N is the number of particles in one bunch. We look for the eigenmode given by

$$\tilde{\rho}_{m\mu}(3q+\mu) = \sum_{k=0}^{\infty} \alpha_k C_{mk}((3q+\mu)\sigma). \quad (34)$$

By this transformation we obtain the final eigenvalue problem:

$$\lambda \alpha_k = \sum_{l=0}^{\infty} M_{kl} \alpha_l$$

where

$$M_{kl} = \sum_p g(3p+\mu) C_{mk}((3p+\mu)\sigma) C_{ml}((3p+\mu)\sigma).$$

Generalization of the Formalism

We will try to generalize this formalism for other bunch distributions. We may write the general Sacherer equation in the frequency domain as

$$\lambda \tilde{\rho}_m(q) = \sum_p g(p) \tilde{\rho}_m(p) \frac{1}{\sigma^2} \int W\left(\frac{r}{\sigma}\right) J_m(qr) J_m(pr) r dr, \quad (35)$$

where $W(r/\sigma)$ is a weight function given by $(1/r) \psi'_0(r)$ or $\psi_0(r)$ for the longitudinal and the transverse cases respectively. For a Gaussian bunch we could easily obtain a separable kernel for the above integral, which

makes it easy to solve the equation. In order to derive a separable kernel for the integral in the general case, we want to find orthogonal polynomials $S_k(t)$ which satisfy

$$\int_0^{\infty} t W(t) S_k(t) S_l(t) dt = \delta_{kl} \quad (36)$$

or

$$\int_0^{\infty} W(t) t^{2m+1} e_k^{(m)}(t^2) e_l^{(m)}(t^2) dt = \delta_{kl} \quad (37)$$

where

$$S_k(t) = t^m e_k^{(m)}(t^2) \quad \text{with} \quad e_0^{(m)}(t^2) = \text{const.}$$

Using the polynomials $e_k^{(m)}(t^2)$, we can expand the Bessel function as

$$J_m(\sigma t) = \sum_{k=0}^{\infty} C_{mk}(\sigma) t^m e_k^{(m)}(t^2) \quad (38)$$

where

$$C_{mk}(\sigma) = \int_0^{\infty} W(t) t^{m+1} e_k^{(m)}(t^2) J_m(\sigma t) dt \quad (39)$$

For the integral in Eq. (35) we have the expansion formula

$$\int_0^{\infty} W(t) J_m(\sigma t) J_m(\rho t) t dt = \sum_{k=0}^{\infty} C_{mk}(\sigma) C_{mk}(\rho) \quad (40)$$

We then obtain the eigenvalue problem

$$\lambda \tilde{\rho}_m(q) = \sum_{p=-\infty}^{\infty} \sum_{k=0}^{\infty} g(p) C_{mk}(\sigma q) C_{mk}(\sigma p) \tilde{\rho}_m(p) \quad (41)$$

Since this is exactly equivalent to Eq. (20), we can solve the problem with the same formalism used for a Gaussian bunch. The vector $C_{mk}(\sigma)$ also corresponds to the frequency spectrum of the k^{th} radial mode. In the Appendix we will derive explicit expressions for $C_{mk}(\sigma)$ for Gaussian and parabolic bunches.

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Appendix

A) Gaussian Bunch

We will derive the spectrum $C_{km}(op)$ using a generalized formalism and show that the results are equal to those obtained previously. Let the weight function $W(t)$ be written as

$$W\left(\frac{r}{\sigma}\right) = e^{-\frac{r^2}{2\sigma^2}} \quad (A1)$$

We look for orthogonal polynomials $e_k^{(m)}(x)$ which satisfy

$$\int_0^\infty e^{-\frac{t^2}{2}} t^{2m+1} e_k^{(m)}(t^2) e_l^{(m)}(t^2) dt = \delta_{kl} \quad (A2)$$

The polynomials are given by

$$e_k^{(m)}(x) = D_k^{(m)} \cdot L_k^m\left(\frac{x}{2}\right) \quad (A3)$$

where $L_k^m(x)$ is the generalized Laguerre polynomial defined by

$$L_k^m(x) = \sum_{q=0}^k (-1)^q \binom{m+k}{k-q} \frac{x^q}{q!} \quad (A4)$$

and

$$D_k^{(m)} = 2^{-\frac{m}{2}} \sqrt{\frac{k!}{(m+k)!}} \quad (A5)$$

We then have

$$C_{mk}(op) = D_k^{(m)} \int_0^\infty e^{-\frac{t^2}{2}} t^{2m+1} L_k^m\left(\frac{1}{2}t^2\right) J_m(op) dt \quad (A6)$$

Using the integration formula¹¹

$$\int_0^{\infty} t^{m+1} e^{-\beta t^2} L_k^m(\alpha t^2) J_m(yt) dt$$

$$= \frac{1}{2^{m+1} \beta^{m+k+1}} (\beta - \alpha)^k y^m e^{-\frac{y^2}{4\beta^2}} L_k^m\left(\frac{\alpha y^2}{4\beta(\alpha - \beta)}\right), \quad (A7)$$

Eq. (A6) becomes

$$C_{mk}(\sigma p) = \frac{1}{\sqrt{k! (m+k)!}} e^{-\frac{1}{2} \sigma^2 p^2} \left(\frac{\sigma p}{\sqrt{2}}\right)^{m+2k}, \quad (A8)$$

which is exactly equal to Eq. (19).

Next we will derive the longitudinal distribution of the mode, which is given neglecting a constant factor by

$$\rho_{mk}(\theta) = \sum_{p=-\infty}^{\infty} C_{mk}(\sigma p) e^{ip\theta} \quad (A9)$$

Since the function inside the summation is very slowly varying with p , we may replace the summation with an integration as

$$\rho_{mk}(\theta) \sim \int_{-\infty}^{\infty} C_m(\sigma p) e^{ip\theta} dp \quad (A10)$$

For a Gaussian bunch, we find

$$\rho_{mk}(\theta) \sim e^{-\frac{\theta^2}{2\sigma^2}} H_{m+2k}\left(\frac{\theta}{\sigma}\right), \quad (A11)$$

where $H_n(x)$ is the Hermite polynomial. This formalism extends the Hermite modes to include higher order radial modes.

Plots of $C_{m0}(\sigma p)$ and $\rho_{m0}(\theta)$ for a Gaussian bunch are given in Figs. 1 and 2 for $m = 0, 1, 2, 3, 4$. The magnitude of the curves is adjusted so that their shapes can be shown better.

B) Parabolic Bunch

Let the weight function be written as

$$w\left(\frac{r}{\sigma}\right) = \left(1 - \left(\frac{r}{\sigma}\right)^2\right)^\mu \quad \text{with } 0 \leq r \leq \sigma, \quad \mu > -1. \quad (\text{A12})$$

We want to find orthogonal polynomials $e_k^{(m)}(x)$ which satisfy

$$\int_0^1 (1-t^2)^\mu t^{2m+1} e_k^{(m)}(t^2) e_l^{(m)}(t^2) dt = \delta_{kl}. \quad (\text{A13})$$

These polynomials are given by

$$e_k^{(m)}(x) = D_k^{(m)} \cdot G_k(m+\mu+1, m+1, x), \quad (\text{A14})$$

where $G_k(\alpha, \gamma, x)$ is the Jacobi polynomial defined by

$$G_k(\alpha, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha+k)} \sum_{q=0}^k (-1)^q \binom{k}{q} \frac{\Gamma(\alpha+k+q)}{\Gamma(\gamma+q)} x^q, \quad (\text{A15})$$

and

$$\left\{D_k^{(m)}\right\}^2 = \frac{2(m+\mu+2k+1)\Gamma(m+\mu+k+1)(m+k)!}{k!(m!)^2 \Gamma(\mu+k+1)}. \quad (\text{A16})$$

Then $C_{mk}(\sigma p)$ is given by

$$C_{mk}(\sigma p) = D_k^{(m)} \cdot \int_0^1 (1-t^2)^\mu t^{m+1} G_k(m+\mu+1, m+1, t^2) J_m(\sigma p t) dt. \quad (\text{A17})$$

Using the transformation formula

$$G_k(\alpha, \gamma, x) = (-1)^k \frac{\Gamma(\gamma) \cdot \Gamma(\alpha-\gamma+k+1)}{\Gamma(\gamma+k) \cdot \Gamma(\alpha-\gamma+1)} G_k(\alpha, \alpha-\gamma+1, 1-x), \quad (\text{A18})$$

the integral formula

$$\int_0^1 x^{m+1} (1-x^2)^\mu J_m(yx) dx = 2^\mu \Gamma(\mu+1) \frac{1}{y^{\mu+1}} J_{m+\mu+1}(y) \quad (\text{A19})$$

and the recurrence relation

$$\sum_{q=0}^k (-1)^q \binom{k}{q} \Gamma(\nu+k+q) \left(\frac{2}{x}\right)^q J_{\nu+q}(x) = (-1)^k \Gamma(\nu+k) J_{\nu+2k}(x), \quad (A20)$$

we finally have

$$C_{mk}(\sigma p) = \left[\frac{(m+\mu+2k+1)\Gamma(\mu+k+1)\Gamma(m+\mu+k+1)}{2 \cdot k! (m+k)!} \right]^{1/2} \cdot \left(\frac{2}{\sigma p}\right)^{\mu+1} J_{m+\mu+2k+1}(\sigma p) \quad (A21)$$

with which we can expand the following integral as

$$\int_0^1 (1-t^2)^\mu J_m(\sigma q t) J_m(\sigma p t) t dt = \sum_{k=0}^{\infty} C_{mk}(\sigma q) C_{mk}(\sigma p). \quad (A22)$$

We now consider another weight function which is given by

$$W\left(\frac{r}{\sigma}\right) = \delta\left(\frac{r}{\sigma} - 1\right). \quad (A23)$$

Since in Eq. (40) for this case

$$\int_0^1 \delta(t-1) J_m(\sigma q t) J_m(\sigma p t) t dt = J_m(\sigma q) J_m(\sigma p), \quad (A24)$$

we expect that there is only one radial mode for the m^{th} mode,

$$C_{m0}(\sigma p) = J_m(\sigma p). \quad (A25)$$

This result can be obtained in Eq. (A21) for $\mu = -1$ neglecting a divergent factor. Therefore the mode spectrum of this weight function may be added to Eq. (A21) with $\mu = -1$ and $k = 0$.

Next we derive the longitudinal distribution given by

$$\rho_{mk}(\theta) \sim \int_{-\infty}^{\infty} \frac{1}{(\sigma p)^{\mu+1}} J_{m+\mu+2k+1}(\sigma p) e^{ip\theta} dp. \quad (A26)$$

Using the two integral formulae,¹²

$$\begin{aligned} \int_0^{\infty} x^{-\nu} J_{\nu+2n}(x) \cos(yx) dx \\ = \frac{(-1)^n \cdot 2^{\nu-1} \cdot (2n)! \Gamma(\nu)}{\Gamma(2n+2\nu)} (1-y^2)^{\nu-\frac{1}{2}} C_{2n}^{\nu}(y) \quad \text{for } 0 < y < 1 \\ = 0 \quad \text{for } y > 1 \end{aligned} \quad (A27)$$

and

$$\begin{aligned} \int_0^{\infty} x^{-\nu} J_{\nu+2n+1}(x) \sin(yx) dx \\ = \frac{(-1)^n \cdot 2^{\nu-1} \cdot (2n+1)! \Gamma(\nu)}{\Gamma(2n+2\nu+1)} (1-y^2)^{\nu-\frac{1}{2}} C_{2n+1}^{\nu}(y) \quad \text{for } 0 < y < 1 \\ = 0 \quad \text{for } y > 1 \end{aligned} \quad (A28)$$

we obtain

$$\rho_{mk}(\theta) \sim \Gamma(\mu+1) \left(1 - \left(\frac{\theta}{\sigma}\right)^2\right)^{\mu+\frac{1}{2}} C_{m+2k}^{\mu+1}\left(\frac{\theta}{\sigma}\right) \quad (A29)$$

where $C_n^{\nu}(x)$ is the Gegenbauer polynomial. Therefore for the weight function given by Eq. (A12), we obtain the "Gegenbauer" mode. Later we can see that this mode includes the Legendre mode and the Tchebycheff mode.

1) $\mu = -1$

This weight function is used for the water-bag model in the longitudinal problem and for the hollow bunch model in the transverse problem. The longitudinal distribution is given by

$$\begin{aligned} \rho_{m0}(\theta) &\sim \left(1 - \left(\frac{\theta}{\sigma}\right)^2\right)^{-\frac{1}{2}} \lim_{\mu \rightarrow -1} \Gamma(\mu+1) C_m^{\mu+1}\left(\frac{\theta}{\sigma}\right) \\ &\sim \left(1 - \left(\frac{\theta}{\sigma}\right)^2\right)^{-\frac{1}{2}} T_m\left(\frac{\theta}{\sigma}\right), \end{aligned} \quad (A30)$$

where $T_n(x)$ is the Tchebycheff polynomial. The above result is known as the Tchebycheff mode.

ii) $\mu = -\frac{1}{2}$

The longitudinal distribution is given by

$$\begin{aligned} \rho_{mk}(\theta) &\sim C_{m+2k}^{\frac{1}{2}}\left(\frac{\theta}{\sigma}\right) \\ &\sim P_{m+2k}\left(\frac{\theta}{\sigma}\right) \end{aligned} \quad (A31)$$

where $P_n(x)$ is the Legendre polynomial. This result is known as the Legendre mode for the lowest radial mode $k = 0$. Here we have extended the Legendre mode for the higher radial mode $k \neq 0$.

iii) $\mu = 0$

$$\begin{aligned} \rho_{mk}(\theta) &\sim \left(1 - \left(\frac{\theta}{\sigma}\right)^2\right)^{\frac{1}{2}} C_{m+2k}^1\left(\frac{\theta}{\sigma}\right) \\ &\sim \left(1 - \left(\frac{\theta}{\sigma}\right)^2\right)^{\frac{1}{2}} U_{m+2k}\left(\frac{\theta}{\sigma}\right) \end{aligned} \quad (A32)$$

where $U_n(x)$ is the Tchebycheff polynomial of the second kind. While the sinusoidal mode for this weight function is already known, with this formalism we obtain another mode which may be more convenient for treating the problem.

We have seen that the Gegenbauer mode is considered to be the generalization of the Legendre and the Tchebycheff modes. With this

mode we can treat various bunch distributions by changing μ . Especially for $\mu \geq 1$, the unperturbed bunch distribution,

$$\rho_{00}(\theta) \sim \left(1 - \left(\frac{\theta}{\sigma}\right)^2\right)^{\mu + \frac{1}{2}}, \quad (A33)$$

looks like a real bunch shape.

Plots of $C_{m0}(\sigma p)$ and $\rho_{m0}(\theta)$ are given in Figs. 3 and 4 for $\mu = 0, 1, 2, 3$, for the first four longitudinal modes.

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Fig. 1. $C_{m0}(\sigma\rho)$ for a Gaussian bunch, $m=0, 1, 2, 3, 4$.

Fig. 2. $\rho_{m0}(\theta)$ for a Gaussian bunch, $m=0, 1, 2, 3, 4$.

Fig. 3. $C_{m0}(\sigma p)$ for a parabolic bunch for $\mu = 0, 1, 2, 3$ and $m = 0, 1, 2, 3, 4$.

Fig. 4. $\rho_{m0}(\theta)$ for a parabolic bunch for $\mu = 0, 1, 2, 3$ and $m = 0, 1, 2, 3, 4$.