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DR# 0143-C

U UCID--20932

DE87 005280

Hydrodynamics of Exploding Foils: Progress on
Similarity Solution and a Stability Analysis for Early Time

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November 25, 1986

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National
Laboratory

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Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under Contract W-7405-Eng-48.

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In this report, we address two broad aspects of the behaviors of hydrodynamic flows, resulting from laser heated, exploding foils. In Part I, various aspects of the similarity solutions are considered, whereas in Part II we examine the stability of the flows at early times, before the similarity solutions are established.

Part I. The Similarity Solutions

Assuming that the temperature is spatially flat, two dimensional, Cartesian, similarity solutions are developed to represent the evolution of laser irradiated, exploding foils. If the heating occurs at a constant rate per gram, and has a duration of >10 ns, power law solutions are shown to be valid approximations of the flows. In cases of one dimensional expansion, an exact solution is derived for $L(t)$, which is valid for all times after the heating has ceased. The integrations of London and Rosen (1986) are found to be in excellent agreement with the analytical prediction. The corresponding solutions for two dimensional, adiabatic expansion can be integrated analytically only for very special sets of initial conditions. Notwithstanding, an exact integral of the motion, the energy integral, as well as the evolution of the temperature with time, is derived for more general adiabatic, two dimensional gas flows.

Consider an exploding foil, irradiated by a beam of laser light. Let the x axis be parallel to the laser beam, and let the y and z axes be perpendicular to the beam. We assume that the fluid is constrained not to move in the z direction, but that it expands freely in the x and y directions.

MASTER

Fig. 1. Schematic diagram of an expanding foil.

For this problem, the equation of continuity reads:

$$\frac{\partial \ln \rho}{\partial t} + v_x \frac{\partial \ln \rho}{\partial x} + v_y \frac{\partial \ln \rho}{\partial y} + \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \quad (1)$$

The component equations of motion are:

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{P}{\rho} \frac{\partial \ln \rho}{\partial x} = -\frac{T(t)}{K_1} \frac{\partial \ln \rho}{\partial x}, \quad (2)$$

and

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{P}{\rho} \frac{\partial \ln \rho}{\partial y} = -\frac{T(t)}{K_1} \frac{\partial \ln \rho}{\partial y}, \quad (3)$$

where we have used the perfect gas law, ρ , P , and T are, respectively, the gas density, pressure and temperature, and T is assumed to be spatially flat. The quantity K_1 is assumed constant and is defined by $K_1 = \rho T/P = A(Z+1)^{-1} m_a/k$, where k is Boltzmann's constant, A is the atomic number, Z the ionic charge, and m_a is the unit of atomic mass.

The first law of thermodynamics may be written in the form,

$$\frac{d \ln P}{dt} = \gamma \frac{d \ln \rho}{dt} + (\gamma-1) K_1 \frac{H(t)}{T(t)}, \quad (4)$$

where $\gamma = C_p/C_v$ ($= 5/3$ for a fully ionized gas), $H(t)$ is the heating rate/gm, and the convective derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}$.

Cooling, either by conduction or by atomic processes (e.g., x-ray line emission), has not been included explicitly in equation (4). (Thermal conduction has been included, implicitly, by constraining T to be spatially flat.) Equation (4) may be written as,

$$\frac{dT(t)}{dt} = (\gamma-1) T(t) \frac{d \ln \rho}{dt} + (\gamma-1) K_1 H(t). \quad (5)$$

OUTLINE

To construct the similarity solutions, let $V_x = [\dot{L}_x(t) x] / L_x(t)$, and $V_y = [\dot{L}_y(t) y] / L_y(t)$. Therefore,

$$\frac{\partial V_x}{\partial t} = \left[\frac{\ddot{L}_x(t)}{L_x(t)} - \frac{\dot{L}_x^2(t)}{L_x^2(t)} \right] x, \quad \frac{\partial V_y}{\partial t} = \left[\frac{\ddot{L}_y(t)}{L_y(t)} - \frac{\dot{L}_y^2(t)}{L_y^2(t)} \right] y,$$

$$V_x \frac{\partial V_x}{\partial x} = \frac{\dot{L}_x^2(t)}{L_x^2(t)}, \quad V_y \frac{\partial V_y}{\partial y} = \frac{\dot{L}_y^2(t)}{L_y^2(t)}, \quad \text{and} \quad V_y \frac{\partial V_x}{\partial y} = V_x \frac{\partial V_y}{\partial x} = 0.$$

Consequently, the equations of motion reduce to the forms,

$$L_x(t) \frac{\ddot{L}_x(t)}{L_x^2(t)} \frac{x}{L_x(t)} = - \frac{T(t)}{K_1} \frac{\partial \ln p}{\partial x}, \quad (6)$$

$$\text{and} \quad L_y(t) \frac{\ddot{L}_y(t)}{L_y^2(t)} \frac{y}{L_y(t)} = - \frac{T(t)}{K_1} \frac{\partial \ln p}{\partial y}. \quad (7)$$

Equations (6) and (7) may be separated into the following forms:

$$L_x(t) \frac{\ddot{L}_x(t)}{L_x^2(t)} = \frac{T(t)}{K_1}, \quad (8)$$

$$L_y(t) \frac{\ddot{L}_y(t)}{L_y^2(t)} = \frac{T(t)}{K_1}, \quad (9)$$

$$\frac{\partial \ln p}{\partial x} = - \frac{x}{L_x^2(t)}, \quad (10)$$

$$\text{and} \quad \frac{\partial \ln p}{\partial y} = - \frac{y}{L_y^2(t)}. \quad (11)$$

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Equations (10) and (11) may be integrated to give,

$$\ln \rho(x, y, t) = \frac{-x^2}{2L_x^2(t)} + \ln a(y, t) = \frac{-y^2}{2L_y^2(t)} + \ln b(x, t), \quad (12)$$

$$\text{or, } \rho(x, y, t) = a(y, t) e^{-\frac{x^2}{2L_x^2(t)}} = b(x, t) e^{-\frac{y^2}{2L_y^2(t)}}. \quad (13)$$

Therefore, $a(y, t) = C(t) \exp \{-y^2/[2L_y^2(t)]\}$, $b(x, t) = C(t) \exp \{-x^2/[2L_x^2(t)]\}$, and the density distribution follows the double Gaussian profile,

$$\rho(x, y, t) = C(t) e^{-\frac{x^2}{2L_x^2(t)}} e^{-\frac{y^2}{2L_y^2(t)}}. \quad (14)$$

The function $C(t)$ must be determined by normalization.

By definition, the constant foil mass,

$$\begin{aligned} M &= 2\bar{z}_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\rho(x, y, t) dx] dy = 8C(t)\bar{z}_0 \int_0^{\infty} e^{-\frac{y^2}{2L_y^2(t)}} dy \int_0^{\infty} e^{-\frac{x^2}{2L_x^2(t)}} dx \\ &= 4\pi C(t)\bar{z}_0 L_x(t) L_y(t). \end{aligned} \quad (15)$$

$$\text{Consequently, } C(t) = M/[4\pi\bar{z}_0 L_x(t) L_y(t)], \quad (16a)$$

$$\rho(x, y, t) = \left[\frac{M}{4\pi\bar{z}_0 L_x(t) L_y(t)} \right] e^{-\frac{x^2}{2L_x^2(t)}} e^{-\frac{y^2}{2L_y^2(t)}}, \quad (16b)$$

$$\begin{aligned} \text{and } \ln \rho(x, y, t) &= \ln [M/(4\pi\bar{z}_0)] - \ln L_x(t) \\ &\quad - \ln L_y(t) - x^2/[2L_x^2(t)] - y^2/[2L_y^2(t)]. \end{aligned} \quad (16c)$$

Using these results, continuity becomes,

$$\frac{d \ln \rho}{dt} = - \frac{\dot{L}_x(t)}{L_x(t)} - \frac{\dot{L}_y(t)}{L_y(t)}. \quad (17)$$

Hence, equation (5) may be written in the form,

$$\frac{dT(t)}{dt} = -(\gamma-1) \left[\frac{1}{L_x(t)} \frac{dL_x(t)}{dt} + \frac{1}{L_y(t)} \frac{dL_y(t)}{dt} \right] + (\gamma-1) K, H(t). \quad (18)$$

Equation (8), (9), and (18) are the system that must be solved for the time evolution of $L_x(t)$, $L_y(t)$, and $T(t)$. For computational purposes, it is best to reduce momentum equations to a first order system, viz,

$$\frac{dL_x(t)}{dt} = Q(t), \quad (19)$$

$$\frac{dL_y(t)}{dt} = R(t), \quad (20)$$

$$\frac{dQ(t)}{dt} = \frac{T(t)}{K, L_x(t)}, \quad (21)$$

and
$$\frac{dR(t)}{dt} = \frac{T(t)}{K, L_y(t)}. \quad (22)$$

Equations (18) through (22) are a set of five, first order differential equations which can be solved with high precision by any one of several techniques, such as a Runge-Kutta algorithm.

In special cases, equations (8), (9), and (18) may be solved exactly for the time evolution of the flow. We consider both one and two dimensional problems.

A. Asymptotic Solutions for One Dimensional Flows: ($L_y(t) = L_y(0) = \text{constant.}$)

Case 1. $H = \text{constant}$, valid for early times.

We let $\gamma = 5/3$, and assume power law solutions for $L_x(t)$ and $T(t)$. Thus, we find $T(t) = 1/3 K_1 H t$ and $L_x(t) = 2/3 H t^{3/2}$, in agreement with the one dimensional results of London and Rosen (1986).

Case 2. $H = 0$ (adiabatic expansion), valid for late times. With $\gamma = 5/3$, equation (17) integrates to $T(t) = T(0) [L_x(t)/L_x(0)]^{-2/3}$. For brevity, let $L_x(t)/L_x(0) \equiv l(t) = l$. The equation of motion becomes,

$$\ddot{l} = K l^{-5/3}. \quad (23)$$

In this expression, $K = T(0) / [K_1 L_x^2(0)] = k (Z+1) T(0) / [A m_a L_x^2(0)] = v_s^2(0) / L_x^2(0)$, where $v_s(0)$ is the isothermal sound speed at temperature $T(0)$. Multiply both sides of equation (23) by $2\dot{l}$. Then, the left hand side equals $d/dt \dot{l}^2$. The resulting expression can be integrated to give,

$$\dot{l} = \sqrt{a l^{2/3} - c} / l^{1/3}, \quad (24)$$

where $a = [L_x^2(0) + 3 v_s^2(0)] / L_x^2(0)$, and $C = 3 v_s^2(0) / L_x^2(0)$. Equation (24) may be integrated exactly to yield,

$$\frac{3l}{a} \left\{ \sqrt{1 - (c/a) l^{-2/3}} - \frac{2}{3} \left[1 - (c/a) l^{-2/3} \right]^{3/2} + l \left[\frac{2}{3} (1 - c/a)^{3/2} - \sqrt{1 - c/a} \right] \right\} = t. \quad (25)$$

When $l \gg 1$, $L(t) \rightarrow [L_x^2(0) + 3 v_s^2(0)]^{1/2} t$ in agreement with the linear behavior found by London and Rosen (1986). In the following table, the data in the first three columns are taken from a numerical calculation (London, 1986), while the times in the fourth column are calculated from equation (25).

$L_x(t) \times 10^2 (\text{cm})$	$l(t)$	(Numerical)	(Analytic)
		$t(\text{ns})$	$t(\text{ns})$
3.010	1.2557	0.5455	0.5455
4.037	2.6840	0.0733	0.7034
5.972	2.4915	0.9942	0.9944
13.40	5.5409	2.081	2.0815
26.59	11.093	3.972	3.9748
45.09	18.811	6.600	6.6035
70.82	29.545	10.23	10.240

The small differences between the numerical and analytical times are due to the fact that only four significant figures are given in the computer printout (which limited the accuracy of a and c). The main utility of equation (25) is that it provides an integral check upon the accuracy of the numerical integrations. No systematic differences were found between the numerical and the analytical predictions.

B. Asymptotic Solutions for Two Dimensional Flows

Case 1. $H = \text{constant}$. With $\gamma = 5/3$, we find the following power law solutions: $L_x(t) = L_y(t) = (2/3)^{3/2} H^{1/2} t^{3/2}$ and $T(t) = 2/9 K_1 H t$.

Figures 2 and 3 illustrate the rate of convergence of London's (1986) representative numerical integrations of the two dimensional, similarity equations, with constant heating, to the power law solution for L_x and L_y when $H = 2 \times 10^{25} \text{ erg g}^{-1} \text{ s}^{-1}$. In these models, $L_x(0) = 4.156 \times 10^{-6} \text{ cm}$. It is clear from Figure 3 that, when $L_y(0) = 10^{-1} \text{ cm}$ [$L_y(0)/L_x(0) = 2.406 \times 10^4$], more than 10 ns of heating is required before $L_y(t)$ lies within 10% of $L_x(t)$. Hence, the power law expression for L_x and L_y cannot be used in practical cases for which $L_y(0)/L_x(0) \simeq 10^3$, and the duration of the laser heating is $< 1 \text{ ns}$. The temperatures in these models converge more rapidly to the prediction of the power law. For $t > 10 \text{ ns}$, all of the model temperatures lie within $\sim 1\%$ of their corresponding power law values.

Case 2. $H = 0$ with $\gamma = 5/3$. As is the case in the one dimensional problem, the temperature evolution can be integrated exactly, $T(t) = T(0) \mathfrak{z}_1^{-2/3} \mathfrak{z}_2^{-2/3}$, where $\mathfrak{z}_1 = \mathfrak{z}_1(t) \equiv L_x(t)/L_x(0)$ and $\mathfrak{z}_2 = \mathfrak{z}_2(t) \equiv L_y(t)/L_y(0)$. We insert this expression into equations (8) and (9), and multiply both sides of these equations by $2\dot{L}_x(t)$ and $2\dot{L}_y(t)$ respectively. After adding the resulting equations, we arrive at the result,

$$\begin{aligned} 2\dot{L}_x(t)\ddot{L}_x(t) + 2\dot{L}_y(t)\ddot{L}_y(t) &= \frac{d}{dt} [\dot{L}_x^2(t) + \dot{L}_y^2(t)] = \\ 2V_s^2(0)\mathfrak{z}_1^{-5/3}\mathfrak{z}_2^{-2/3}\dot{\mathfrak{z}}_1 + 2V_s^2(0)\mathfrak{z}_1^{-2/3}\mathfrak{z}_2^{-5/3}\dot{\mathfrak{z}}_2 &= \\ -3V_s^2(0)\frac{d}{dt}(\mathfrak{z}_1^{-2/3}\mathfrak{z}_2^{-2/3}). \end{aligned} \quad (26)$$

Equation (26) can be solved for the energy integral of the motion,

$$\dot{L}_x^2(t) + \dot{L}_y^2(t) = \dot{L}_x^2(0) + \dot{L}_y^2(0) + 3V_s^2(0)(1 - \mathfrak{z}_1^{-2/3}\mathfrak{z}_2^{-2/3}). \quad (27)$$

This result was used to verify the integrity of the two dimensional, numerical integrations of the similarity equations (London, 1986).

Other integrals of the motion exist. Since the accelerations approach zero as the $\mathfrak{z}$'s become large, it follows that the velocity components must approach constant values as well. Therefore, as $t \rightarrow \infty$, $L_x(t)/L_y(t) \rightarrow \dot{L}_x(t)/\dot{L}_y(t) \rightarrow$ a particular constant for each case. These predictions are borne out by London's integrations with $H = 0$. What I have been unable to do, is to relate the asymptotic velocity components to the initial conditions, $L_{xy}(0)$, $\dot{L}_{xy}(0)$ and $V_s(0)$, excepting through the energy integral. At least one additional relation must exist, because London's integrations demonstrate empirically that $\dot{L}_x(t)/\dot{L}_y(t) \rightarrow \sqrt{L_y(0)/L_x(0)}$ as t becomes large if $L_x(0) = L_y(0) = 0$.

¹ The author outlined this problem to a scientific collaborator, Professor George Contopoulos of the University of Athens. Working independently, Dr. Contopoulos was unable to find any integrals other than those described above.

In the special case when $l_1 = l_2 = l$, both equations of motion reduce to the form,

$$\ddot{l} = K l^{-7/3} \quad (28)$$

Hence,

$$d\dot{l}^2 = 2K l^{-7/3} dl, \quad (29)$$

which integrates to

$$\frac{d\dot{l}}{d\tau} = \frac{\sqrt{g l^{4/3} - cl}}{l^{2/3}}, \quad (30)$$

where $d = 3/2 V_S^2(0)/L^2(0)$ and $g = (L^2(0) + 3 V_S^2(0)/2)/L^2(0)$.

Letting $x = l^{1/3}$, equation (29) may be integrated, viz,

$$3 \int_0^{x^{4/3}} \frac{x^4 dx}{\sqrt{x^2 + b^2} \sqrt{x^2 - b^2}} = g^{1/2} t, \quad (31)$$

where $b = (d/g)^{1/4}$.

The integral in equation (31) may be written as a combination of algebraic terms and incomplete elliptic integrals of the first kind.

Defining $S = 1/\sqrt{2}$, $\epsilon_1 = \cos^{-1} b$ and $\epsilon_2 = \cos^{-1}(b/l^{1/3})$, equation (31) becomes,

$$\frac{b^3}{\sqrt{2}} \left[F(\epsilon_2, S) - F(\epsilon_1, S) \right] + l^{1/3} \sqrt{(l^{4/3} + b^2)(l^{2/3} - b^2)} - \sqrt{(1 + b^2)(1 - b^2)} = \sqrt{L^2(0) + 3 V_S^2(0)/2} t / L(0), \quad (32)$$

where

$$F(\epsilon, S) = \int_0^\epsilon \frac{d\phi}{\sqrt{1 - S^2 \sin^2 \phi}}, \quad (33)$$

the incomplete elliptic integral of the first kind. As $l(t)$ becomes large,

$$L_x(t) = L_y(t) \longrightarrow \sqrt{L^2(0) + 3 V_S^2(0)/2} t. \quad (34)$$

Part II. The Stability of the Flows

The one dimensional expansion of laser irradiated, exploding foils is examined at early times. The flows are shown to be Rayleigh-Taylor unstable for roughly the first 100 ps of expansion, and strongly stable thereafter. Transient, local Kelvin-Helmholtz instabilities may occur as well. However, it seems unlikely that initial fluctuations in such flows can grow by as much as an order of magnitude before they begin to damp. Notwithstanding, vorticity excited at early times may persist for all practical times of interest, owing the relatively low viscosity of the fluid. For this reason, it would be worthwhile to undertake a numerical study of the evolution of instabilities having a range of initial wavelengths. A possible approach to this problem is outlined.

In this section, we examine the stability of the hydrodynamic flows of laser heated, exploding foils. At early times, the flows are followed numerically, using 1-D LASNEX simulations. During the first few hundred picoseconds of heating, the flows essentially are one dimensional in the directions perpendicular to the foils. Defining x as the direction of expansion, and allowing for (small) y motions as well, the equations describing the problem are:

$$\frac{d\rho}{dt} + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = 0, \quad (1)$$

$$\frac{dv_x}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{T}{K, \rho} \frac{\partial \rho}{\partial x} - \frac{1}{K,} \frac{\partial T}{\partial x}, \quad (2)$$

$$\frac{dv_y}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} = -\frac{T}{K, \rho} \frac{\partial \rho}{\partial y} - \frac{1}{K,} \frac{\partial T}{\partial y}, \quad (3)$$

and

$$\frac{dp}{dt} = \gamma \frac{p}{\rho} \frac{d\rho}{dt} + (\gamma-1) H p + (\gamma-1) \vec{\nabla} \cdot v(\tau) \vec{\nabla} T, \quad (4a)$$

or

$$\frac{dT}{dt} = (\gamma-1) \frac{T}{\rho} \frac{d\rho}{dt} + (\gamma-1) K_1 H + (\gamma-1) \frac{K_1}{\rho} \vec{\nabla} \cdot \nu(T) \vec{\nabla} T, \quad (4b)$$

where $\nu(T)$ is the coefficient of thermal conduction. In the above equations, the effects of radiation pressure and viscosity were ignored, and the perfect gas law was used. These equations can be generalized to include more physics, such as an improved equation of state.

Because the flows are time dependent, and most of the important timescales in this problem are of the same order of magnitude, it is not possible to attempt a full analytical treatment of the linear stability problem. However, the following numerical approach should work. Since accurate, one dimensional solutions, valid for early times, can be generated numerically, the stability of the flows may be studied by examining the development of small fluctuations imposed upon them. This is accomplished analytically by letting total density, temperature, x-velocity and y-velocity be $\rho + \rho'$, $T + T'$, $V_x + V_x'$, and V_y' , respectively where the unprimed variables denote the unperturbed flow, and primed quantities are the perturbations. Then, equations (1) through (4) may be written as

$$\frac{d\rho}{dt} + \rho \frac{\partial V_x}{\partial x} = 0, \quad (5)$$

$$\frac{dV_x}{dt} = -\frac{T}{K_1 \rho} \frac{\partial \rho}{\partial x} - \frac{1}{K_1} \frac{\partial T}{\partial x}, \quad (6)$$

$$\frac{dT}{dt} = (\gamma-1) \frac{T}{\rho} \frac{d\rho}{dt} + (\gamma-1) K_1 H + (\gamma-1) \frac{K_1}{\rho} \frac{\partial}{\partial x} \left[\nu(T) \frac{\partial T}{\partial x} \right], \quad (7)$$

$$\frac{d\rho'}{dt} + \frac{\partial V_x}{\partial x} \rho' = -\frac{\partial \rho}{\partial x} V_x' - \rho \left(\frac{\partial V_x'}{\partial x} + \frac{\partial V_y'}{\partial y} \right), \quad (8)$$

$$\frac{dV_x'}{dt} + \frac{\partial V_x'}{\partial x} v_x' = \frac{T}{K_1 \rho} \frac{\partial \rho'}{\partial x} - \frac{1}{K_1 \rho} \frac{\partial \rho}{\partial x} T' + \frac{T}{K_1 \rho^2} \frac{\partial \rho}{\partial x} \rho' - \frac{1}{K_1} \frac{\partial T'}{\partial x}, \quad (9)$$

$$\frac{dV_y'}{dt} = -\frac{T}{K_1 \rho} \frac{\partial \rho'}{\partial y} - \frac{1}{K_1} \frac{\partial T'}{\partial y}, \quad (10)$$

and

$$\begin{aligned} \frac{dT'}{dt} = & (\gamma-1) \frac{T}{\rho} \frac{d\rho'}{dt} + (\gamma-1) \frac{d\rho}{\rho} T' - (\gamma-1) \frac{T}{\rho^2} \frac{d\rho}{dt} \rho' \\ & + (\gamma-1) \frac{K_1 v(\tau)}{\rho} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T' + 2(\gamma-1) \frac{K_1}{\rho} \frac{dv(\tau)}{dT} \frac{\partial T}{\partial x} \frac{\partial T'}{\partial x} \\ & + (\gamma-1) \frac{K_1}{\rho} \frac{\partial}{\partial x} \left[\frac{dv(\tau)}{dT} \frac{\partial T}{\partial x} \right] T' - (\gamma-1) \frac{K_1}{\rho^2} \frac{\partial}{\partial x} \left[v(\tau) \frac{\partial T}{\partial x} \right] \rho'. \end{aligned} \quad (11)$$

In deriving equation (11), H was assumed constant - this constraint can be relaxed.

Perturbation equations (8) through (11) may be Fourier analyzed in y , viz,

$$\rho' = \rho'(x, t) \cos(ky) \equiv \rho' \cos(ky), \quad (12a)$$

$$T' = T'(x, t) \cos(ky) \equiv T' \cos(ky), \quad (12b)$$

$$V_x' = V_x'(x, t) \cos(ky) \equiv V_x' \cos(ky), \quad (12c)$$

and $V_y' = V_y'(x, t) \sin(ky) \equiv V_y' \sin(ky).$ (12d)

The (x, t) dependences of the perturbed variables appearing in equations (8) through (11) are described by the following system:

$$\frac{d\rho'}{dt} + \frac{\partial v_x}{\partial x} \rho' = -\frac{\partial \rho}{\partial x} v_x' - \rho \frac{\partial v_x'}{\partial x} - k \rho v_y', \quad (13)$$

$$\frac{dv_x'}{dt} + \frac{\partial v_x}{\partial x} v_x' = -\frac{T}{K_1 \rho} \frac{\partial \rho'}{\partial x} - \frac{1}{K_1 \rho} \frac{\partial \rho}{\partial x} T' + \frac{T}{K_1 \rho^2} \frac{\partial \rho}{\partial x} \rho' - \frac{1}{K_1} \frac{\partial T'}{\partial x}, \quad (14)$$

$$\frac{dv_y'}{dt} = -\frac{kT}{K_1 \rho} \rho' - \frac{kT'}{K_1}, \quad (15)$$

and

$$\begin{aligned} \frac{dT'}{dt} = & (\gamma-1) \frac{T}{\rho} \frac{d\rho'}{dt} + (\gamma-1) \frac{d\rho}{\rho} \frac{T'}{dt} - (\gamma-1) \frac{T}{\rho^2} \frac{d\rho}{dt} \rho' \\ & - (\gamma-1) k \frac{K_1 \nu(T)}{\rho} T' + (\gamma-1) \frac{K_1 \nu(T)}{\rho} \frac{\partial^2 T'}{\partial x^2} + 2(\gamma-1) \frac{K_1}{\rho} \frac{d\nu(T)}{dT} \frac{\partial T}{\partial x} \frac{\partial T'}{\partial x} \\ & + (\gamma-1) \frac{K_1}{\rho} \frac{\partial}{\partial x} \left[\frac{d\nu(T)}{dT} \frac{\partial T}{\partial x} \right] T' - (\gamma-1) \frac{K_1}{\rho^2} \frac{\partial}{\partial x} \left[\nu(T) \frac{\partial T}{\partial x} \right] \rho'. \end{aligned} \quad (16)$$

Equations (13) through (16) should be solved simultaneously with unperturbed flow equations (5) through (7), using the same algorithm. Instabilities, as well as the times when they occur, will be apparent in the numerical solutions.

In order to estimate the maximum unstable growth rates that might occur, we ignore thermal conduction and consider the asymptotic limit of arbitrarily short wavelength fluctuations. In real cases, such disturbances would be damped by conduction (although at early times the conduction coefficient would be small, due to the relatively low electron temperature). Notwithstanding, the limit of large wavenumbers is of importance in the present context, because these perturbations, if undamped by conduction and viscosity, would have the most rapid growth rates.

Defining the relative density perturbation $\eta' = \rho' / \rho$, the perturbation equations may be written in the forms,

$$\frac{\partial \eta'}{\partial t} + \left(\frac{D\rho}{\rho} \right) v_x' + \left(\frac{\partial v_x'}{\partial x} + \frac{\partial v_y'}{\partial y} \right) = 0, \quad (17)$$

$$\frac{\partial v_x'}{\partial t} + (Dv_x)v_x' = -\frac{1}{\rho} \frac{\partial p'}{\partial x} + \left(\frac{DP}{\rho}\right) \eta', \quad (18)$$

$$\frac{\partial v_y'}{\partial t} = -\frac{1}{\rho} \frac{\partial p'}{\partial y}, \quad (19)$$

$$\text{and} \quad \left[\frac{\partial}{\partial t} + \gamma(Dv_x) \right] p' + B_x v_x' = \gamma p \frac{\partial \eta'}{\partial t}, \quad (20)$$

where $\frac{\partial}{\partial t}$ is the convective time derivative following the unperturbed flow,

$D \equiv \frac{\partial}{\partial x}$, and Schwarzschild convective coefficient $B_x \equiv (DP) - \frac{\gamma p}{\rho} (D\rho)$.

For an approximate representation, let each perturbed quantity $\xi'(x, y, t) = \xi'(t) e^{i(k_1 x + k_2 y)}$. Clearly this procedure is not a proper normal mode analysis of the perturbations. However, it should provide an index of the stability of the flow for arbitrarily local disturbances (large k 's) because:

(1) Defining E as an unperturbed flow quantity and ξ' as any perturbed quantity, $\frac{\partial}{\partial x, y} E \xi' = E \frac{\partial \xi'}{\partial x, y}$, meaning that the medium appears locally flat on the length scale of the perturbations; and (2) the slowly propagating modes of interest are represented, mathematically, by the terms in the "dispersion relation" that are multiplied by the highest order coefficient in the wavenumbers ($k_{1,2}^2$) and, therefore, no essential information is lost due to the formal non-commutation of the differential operators.

Proceeding in this fashion, equations (17) through (19) may be combined to give,

$$k^2 \frac{\partial v_x'}{\partial t} + k_x^2 (Dv_x)v_x' \approx \frac{(DP)}{\rho} k_x^2 \eta', \quad (21)$$

where $k^2 = k_1^2 + k_2^2$.

Similarly, equations (18) and (20) may be combined to give the approximate result,

$$B_x V_x' \approx \gamma P \frac{\partial \eta'}{\partial t}. \quad (22)$$

Finally, η' may be eliminated between equations (21) and (22) to give,

$$\begin{aligned} \frac{\partial^2 V_x'}{\partial t^2} + \left[\frac{\partial}{\partial t} \left(\ln \frac{\rho}{|\mathbf{D}\mathbf{P}|} \right) + q^2 (DV_x) \right] \frac{\partial V_x'}{\partial t} \\ + q^2 \left[(DV_x) \frac{\partial}{\partial t} \left(\ln \frac{\rho}{|\mathbf{D}\mathbf{P}|} \right) - \frac{B_x(DP)}{\gamma P \rho} \right] V_x' \approx 0, \end{aligned} \quad (23)$$

where $q^2 = k_2^2/k^2$. In the present limit of large wavenumbers, it is a straightforward matter to show that the velocity perturbations are approximately solenoidal, even though the fluid is compressible. Hence, $V_y' \sim -(k_1/k_2) V_x'$, which shows that V_y' follows the same evolution as V_x' .

Essentially this same result may be arrived at in an intuitive fashion by the following reasoning. Since we are considering arbitrarily local disturbances, a displaced parcel of fluid must come to immediate pressure equilibrium with its surroundings, i.e., $p' = 0$ in a time average sense. Therefore, consider only equations (18) and (20) with $p' = 0$. The resulting expressions may be combined to give,

$$\begin{aligned} \frac{\partial^2 V_x'}{\partial t^2} + \left[\frac{\partial}{\partial t} \left(\ln \frac{\rho}{|\mathbf{D}\mathbf{P}|} \right) + (DV_x) \right] \frac{\partial V_x'}{\partial t} \\ + \left[(DV_x) \frac{\partial}{\partial t} \left(\ln \frac{\rho}{|\mathbf{D}\mathbf{P}|} \right) - \frac{B_x(DP)}{\gamma P \rho} \right] V_x' \approx 0, \end{aligned} \quad (24)$$

a result identical with equation (23), excepting for the missing factors of q^2 . The latter argument, which has been exploited in treating convection in stellar interiors (Tassoul, 1978), has the advantage of circumventing the (formally incorrect) Fourier "analysis" used in

arriving at equation (23). However, it suffers the shortcoming of not including the q^2 factor, which premultiplies the convective term in that equation. Clearly the q^2 factor must be present - no convection is possible if $k_2 = 0$, meaning that the disturbance in a plane wave in the x direction.

We ignore the derivatives of the logarithmic terms, and fit e^{nt} solution for $V_x'(t)$, valid for small times about an initial time. Thus, we obtain the following expression for the instantaneous growth rates:

$$n^2 + g^2 (DV_x') n - g^2 \left[(DP) - \frac{\gamma P}{\rho} (DP) \right] \frac{(DP)}{\gamma P \rho} \approx 0. \quad (25)$$

If we define an acceleration $g = - (DP)/\rho$, then the constant term in equation (25) is $-q^2 \left[\frac{q^2}{c^2} + g D(\ln \rho) \right]$, where c is the sound speed.

For an incompressible fluid this becomes $-q^2 g D(\ln \rho)$, since $c = \infty$ in that limit. In the case of a static, incompressible atmosphere, with an exponential unperturbed density distribution of scale height B^{-1} , the asymptotic growth rate $\rightarrow \sqrt{gB}$ as $k \rightarrow \infty$ (Chandrasekhar, 1981), exactly the above result with $q = 1$.^{*} In the discussion to follow, we will use equation (25), to estimate upper bounds for the unstable growth rates at early times in exploding foils.

The constant term in equation (25) describes the growth of Rayleigh-Taylor, or convective, instabilities. Instability is guaranteed if $(DP)(D\rho) < 0$; if $(DP) < 0$, the instability criterion reduces to Schwarzschild's famous result for the onset of convection, $(DP) - \gamma P/\rho (D\rho) < 0$. We define the square of the Rayleigh-Taylor growth rate by,

$$n_{RT}^2 = \frac{P (D \ln P)}{\gamma \rho} \left[(D \ln P) - \gamma (D \ln P) \right]. \quad (26)$$

^{*} $q \equiv 1$ in the incompressible, exponential atmosphere, since, in that case, $k^2 \equiv k_1^2$, the square of the total horizontal wavenumber. For details, see Chandrasekhar (1981. pp 428-436).

The additional advection term, which is present in time dependent flows, has associated with it the dynamical growth rate,

$$\eta_p = - (DV_x). \quad (27)$$

Instability is enhanced if $(DV_x) < 0$. It should be emphasized that these expressions represent the maximum rates of unstable growth - longer wavelengths will grow more slowly, and dissipative effects (e.g. conduction) will diminish the growth.

The following values of η_{RT} and η_D were estimated from the output of a typical LASNEX simulation (London, 1986).

1. $t_1 = 4.0387 \times 10^{-11}$ s. Crossed gradients of P and ρ exist between mass shells 16 and 38. By finite differencing, $(D \ln P) [(D \ln P) - 5/3 (D \ln \rho)] = 0.785 \times 10^9 \text{ cm}^{-2}$. Moreover, the mean value of p/ρ , $P/\rho = 2.036 \times 10^{12} \text{ erg g}^{-1}$. Hence, $\eta_{RT} = 3.097 \times 10^{10} \text{ s}^{-1}$ and $t_{RT} \equiv \eta_{RT}^{-1} = 3.229 \times 10^{-11}$ s. $\eta_D > 0$ only between mass shells 13 and 16. In this local domain, $\eta_D = 7.46 \times 10^{10} \text{ s}^{-1}$.
2. $t_2 = 6.0447 \times 10^{-11}$ s. Crossed gradients occur between mass shells 12 and 32. $\eta_{RT} = 2.309 \times 10^{10} \text{ s}^{-1}$ and $t_{RT} = 4.311 \times 10^{-11}$ s. $\eta_D > 0$ only between mass shells 5 and 8; in that regime, $\eta_D = 7.15 \times 10^{10} \text{ s}^{-1}$.
3. $t_3 = 7.0067 \times 10^{-11}$ s. Crossed gradients exist between mass shells 6 and 32. $\eta_{RT} = 1.773 \times 10^{10} \text{ s}^{-1}$ and $t_{RT} = 5.641 \times 10^{-11}$ s. $\eta_D < 0$ everywhere.
4. $t_4 = 8.0456 \times 10^{-11}$ s. Crossed gradients exist between mass shells 11 and 26. $\eta_{RT} = 1.300 \times 10^{10} \text{ s}^{-1}$ and $t_{RT} = 7.690 \times 10^{-11}$ s. $\eta_D > 0$ between shells 3 and 8. $\eta_D = 7.52 \times 10^{10} \text{ s}^{-1}$.

For $t > 10^{-10}$ s, the flow is stable.

Consequently, a velocity perturbation of initial amplitude V^1 could grow to the (approximate) maximum value,

$$V^1(t) \sim V^1 \exp [(\omega_{RT_1} - \omega_{RT_2}) t_1 + (\omega_{RT_2} - \omega_{RT_3}) t_2 + (\omega_{RT_3} - \omega_{RT_4}) t_3 + \omega_{RT_4} t_4] \\ = V^1 e^{2.0195} \approx 7.53 V^1.$$

At this juncture, it is instructive to consider other timescales that are important in the present problem. These are:

A. Thermal conduction timescale, t_c . Using the results from Spitzer (1962),

$$t_c \approx 2.6 \times 10^{-11} \frac{L^2}{Z} \frac{\Lambda}{n_e} T_{kev}^{-5/2} s, \quad (28)$$

where L is in units of 2.1×10^{-2} cm, $Z = Z/25$, $\Lambda = \ln \Lambda / 5$, and $n_e = 1.78 \times 10^{20} \frac{n_e}{n_e}$. $t_c < t_{RT}$ when $T \sim 1$ keV, meaning that conduction will cause temperature fluctuations to decay for $t > 10^{-10}$ s.

B. Shear viscous timescale, t_μ . Using Spitzer's (1962) expression for the coefficient of viscosity,

$$t_\mu \approx 9.0 \times 10^{-5} \frac{Z^4}{A} \frac{L^2}{n_e} A^{-1/2} T_{kev}^{-5/2} s, \quad (29)$$

where $A = 80A$. This (classical) shear viscous timescale is relatively long because the momentum exchange occurs via the heavy ions. Consequently, while temperature (and density) fluctuations will be damped by electron conduction, velocity perturbations, and vorticity, may persist for all times of practical interest.

C. Kelvin-Helmholtz (K-H) timescales, t_{KH} . $t_{KH} \sim \lambda / (\pi \Delta V)$,

where λ is the perturbation wavelength and ΔV is the velocity difference across the shearing layer (Hunter, 1985). But $\lambda \sim d$, the thickness of the layer (ibid, p.17). Numerically,

$$t_{KH} \approx 6.7 \times 10^{-11} \frac{d}{V_e} s, \quad (30)$$

where d is in units of 2.1×10^{-2} cm and V_B is the velocity difference in units of 10^8 cm s $^{-1}$. An examination of a 2-D LASNEX output indicates that localized, K-H instabilities may develop in the (primarily diverging) flows, having $t_{KH} \sim 10^{-10}$ s. Again, the major concern is that velocity fluctuations associated with this instability may persist.

Before concluding this section, is it illuminating to consider the growth of perturbations in a one dimensional, homologously expanding, pressureless flow. Define Lagrangian variables (a,b) as the values of (x,y) at $t = 0$. Let the density be spatially flat, and let the unperturbed x velocity be given by $V_x(x,t) = \mathcal{H}(t) x = V_x(a,t) = \alpha a$. The undisturbed density $\rho(t) = \rho_0 / (1 + \alpha t)$, where ρ_0 is the (constant) density at $t = 0$, and $\mathcal{H}(t) = \alpha / (1 + \alpha t)$. At $t = 0$, let the perturbations have the forms,

$$\rho'(a,b,0) = \rho'_0 \cos(k_1 a) \sin(k_2 b), \quad (31a)$$

$$V_x'(a,b,0) = V_{x0}' \sin(k_1 a) \sin(k_2 b), \quad (31b)$$

$$\text{and } V_y'(a,b,0) = -V_{y0}' \cos(k_1 a) \cos(k_2 b). \quad (31c)$$

With respect to an observer, co-moving with the background flow, the evolution of the fluctuations is described by.

$$\rho'(a,b,t) = \left[\frac{\rho'_0}{(1+\alpha t)} - \frac{k_1 \rho'_0 V_{x0}' t}{(1+\alpha t)^2} - \frac{k_2 \rho'_0 V_{y0}' t}{(1+\alpha t)} \right] \cos(k_1 a) \sin(k_2 b), \quad (32a)$$

$$V_x'(a,b,t) = \frac{V_{x0}'}{(1+\alpha t)} \sin(k_1 a) \sin(k_2 b), \quad (32b)$$

$$\text{and } v_y'(a, b, t) = -v_{y0}' \cos(k_1 a) \cos(k_2 b). \quad (32c)$$

The relative density perturbation,

$$\eta' = \rho'/\rho = \left[\frac{\rho_0'}{\rho_0} - \frac{k_1 v_{x0}' t}{(1+\alpha t)} - k_2 v_{y0}' t \right] \cos(k_1 a) \sin(k_2 b). \quad (32d)$$

Consequently, in this example the influence of ρ_0' and v_{x0}' diminish with time, whereas the v_{y0}' influence persists. Real flows may exhibit similar behaviors.

In summary, it seems unlikely that initial disturbances can grow by even an order of magnitude during the early evolution of laser heated, exploding foils. For times $> 10^{-10}$ s, the expanding flows are strongly stable to Rayleigh-Taylor instabilities. The role of Kelvin-Helmholtz instabilities is more difficult to assess, but their temperature and density fluctuations also should damp after the temperature becomes high enough that electron conduction dominates. Notwithstanding, vorticity generated at early times may persist, because of the relatively low viscosity of the gas. Therefore, it would be worthwhile to follow the development of representative, unstable modes by numerical calculation.

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Fig. 2. The convergence of L_x and L_y to the power law predictions. All L_x behaviors are roughly the same; subscripts 1, 3 and 5 refer to the initial conditions $L_y(0) = 10^{-1}$, 10^{-3} and 10^{-5} cm, respectively.

Fig. 3. The convergence of L_y/L_x to unity in the cases of Fig. 2.