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On The Existence of a Last Invariant of
Conservative Motion

Laurence S. Hall

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Lawrence
Livermore
Laboratory

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ON THE EXISTENCE OF A LAST INVARIANT OF CONSERVATIVE MOTION

I. INTRODUCTION

Recently, physicists have paid considerable new attention to the behavior of conservative dynamical systems under conditions in which a complete set of localizing invariants does not exist. There is now an extensive literature chronicling significant advances in the theory of non-integrable systems. Chirikov¹ has given us a useful review of this literature, which emphasizes the interaction of resonances and intrinsic stochasticity, KAM theory, Arnold diffusion, and the like.

On the other hand, there is an old and still largely unanswered question which lies at the separatrix between discussions of intrinsic stochasticity and the bulk of classical dynamics. It is a question which bothered physicists at the turn of this century: What is the nature of that class of conservative systems for which a complete set of isolating invariants does exist? (If we adopt the modern viewpoint that intrinsic stochasticity is "caused" by the overlapping of resonances, then we merely rephrase our question to ask: What is in the nature of that class of conservative systems for which there is no overlapping of resonances?) When, because of the existence of isolating invariants, is the motion of an object confined to a small volume of available space, or when is the motion such as to sample a much greater volume? One knows that isolating invariants exist for central forces or separable potentials, yet, what is it in the nature of forces that delimits the class? And of course, the practical physicist would like to lump adiabatic invariants together with strict invariants in setting down the class boundaries.

Beyond the invocation of simple symmetry principles, there is little in the literature which one can use to distinguish naturally between non-stochastic and intrinsically stochastic systems. We hope to reopen this problem here by developing, for two-dimensional motion, a theory for the existence of a last invariant in addition to the total energy.

The two-dimensional problem of classical Newtonian dynamics is of considerable practical importance in its own right. Indeed, the question of the existence or not of a certain class of axisymmetric magnetohydrodynamic reversed-field equilibria is what motivated the author's interest in the first place. As an illustration of the use to which our theory can be applied, it seems worthwhile to make the connection to reversed-field equilibrium² in this introduction.

Suppose that the presence of a small, self-consistent, self-generated toroidal magnetic field is necessary for the existence of a plasma in a reactor-suitable, stable, reversed-field magnetohydrodynamic equilibrium. Suppose further, perhaps just to isolate the physical questions involved, that there shall be no "force free" current along the magnetic field itself (i.e., an induced current of electrons tied to a line of force). There are some grounds for such speculation, certainly. Moreover, should these conjectures be necessary, they impose a significant constraint upon the internal structure of an acceptable equilibrium configuration.

In the axisymmetric system we have described, no particle can contribute to the poloidal current if that particle ergodically fills the volume of phase space accessible to it within its assigned angular momentum about the axis of symmetry, and within its assigned energy. Ergodicity requires that a particle reapproach a given spatial position at subsequent times with every orientation of poloidal velocity. The poloidal current which is required in order to

generate a toroidal magnetic field must be carried by particles which possess an additional invariant defining an average direction of movement past a given point.

for some particles, magnetic moment may be conserved. Such particles therefore possess the needed additional isolating invariant. Conceivably, these particles could be the source of an appropriate drift current to generate the needed toroidal field. But for the most energetic particles, and more importantly for all particles that can approach a magnetic field null (to which adiabatic particles are attracted), the magnetic moment fails as an invariant.

Absent the magnetic moment as an isolating invariant, our attention is directed to the fairly restricted class of field configurations for which an additional invariant naturally exists (for at least some significant particles). If there is such a class of equilibria which are appropriate for reversed-field confinement, and if some members of the class are indeed stably non-ergodic, then it is to just such a configuration that an experimental inquiry will lead us.

Here, therefore, we are directly confronted by our question as to the nature of candidate configurations. Moreover, the usual simplest cases (central potential, separable potential) are too restrictive for practical use. What are the full conditions for the existence of an additional invariant?

II. EARLY WORK

E. T. Whittaker³ considered the following equations of motion:

$$\ddot{x} = -\partial V / \partial x, \quad \ddot{y} = -\partial V / \partial y, \quad V = V(x, y) \quad (2.1)$$

He then asked under what conditions an invariant (other than the total energy) can exist such that it is at most of second order in the velocities; i.e., when is

$$\mathcal{J}_2(\dot{x}, \dot{y}; x, y) = P\dot{x}^2 + Q\dot{x}\dot{y} + R\dot{y}^2 + S\dot{x} + T\dot{y} + \mathcal{K} = \text{const} \quad ? \quad (2.2)$$

(Here P , Q , R , S , T , and $\mathcal{K}$ are each functions of x and y .) To answer this question, Whittaker differentiated $\mathcal{J}_2$ with respect to time, substituted for $\ddot{x}$ and $\ddot{y}$ from Eqs. (2.1), and then set the coefficients of the separate powers of $\dot{x}$ and $\dot{y}$ equal to zero. His conclusion was that when S and T are non-vanishing, V must be the potential of a central force, and if P , Q , and R all vanish, the invariant is the angular momentum. Otherwise, $S = T = 0$ and the equations are satisfied with a potential of the general (separable) form

$$V = [f(\xi) - g(\eta)]/(\xi^2 - \eta^2) \quad . \quad (2.3)$$

where $f(\xi)$ and $g(\eta)$ are arbitrary functions, and the new variables (ξ, η) result from a change to confocal coordinates:

$$(x/b)^2 = \xi^2 \eta^2 \quad , \quad (y/b)^2 = (\xi^2 - 1)(1 - \eta^2) \quad ,$$

where b is constant. (The transformation requires a specific initial location and orientation of x, y coordinates.)

Whittaker's work, however, was flawed by his failure to recognize that terms in $\dot{x}^2$ and $\dot{y}^2$ are not independent. They are related by conservation of energy. Therefore, his constraints were overrestrictive; the solutions he found are valid, but others may also exist.

On the other hand, Whittaker's basic approach is quite useful. It can be extended to rather general forms of the invariant, and magnetic fields may be included as well. Whittaker's method underlies the theory to be developed here.

III. THE WHITTAKER PROBLEM

As a first task, we will reexamine the problem which Whittaker posed, generalized to permit the introduction of a magnetic field perpendicular to the plane of two-dimensional motion. That is, we write

$$\ddot{x} = \Omega \dot{y} - \partial V / \partial x, \quad \ddot{y} = -\Omega \dot{x} - \partial V / \partial y \quad (3.1)$$

where both $\Omega(x,y)$ and $V(x,y)$ depend on position. For example, in terms of the three-dimensional axisymmetric equilibrium discussed in the introduction, we can let $V(r,z) = [(p_\theta + \psi)^2 / 2r^2] + \phi(r,z)$ represent the total potential energy (centrifugal plus electrostatic), where p_θ is the angular momentum about the axis of symmetry and $\psi(r,z)$ is the poloidal flux, and let $\Omega(r,z)$ represent the toroidal magnetic induction. If we allow for any convenient relocation and reorientation of coordinates by the linear transformation

$$x = (z - z_0) \cos \delta + (r - r_0) \sin \delta, \quad y = -(z - z_0) \sin \delta + (r - r_0) \cos \delta$$

(usually we would choose the origin at a maximum or minimum of V), then the equations of motion are just Eqs. (3.1).

Equations (3.1) naturally admit the total energy

$$\mathcal{E} = (1/2)(\dot{x}^2 + \dot{y}^2) + V(x,y), \quad (3.2)$$

as a constant of the motion. We now inquire as to the nature of $V(x,y)$ and $\Omega(x,y)$ such that $\mathcal{J}_2(\ddot{x}, \ddot{y}; x, y)$, Eq. (2.2), is also a constant of the motion.

A. The Linear Invariant

Suppose that P , Q , and R in Eq. (2.2) vanish identically. Then our invariant, which we now call $\mathcal{J}_1$, is only linearly dependent upon velocity:

$$\mathcal{J}_1(\dot{x}, \dot{y}; x, y) = S(x, y)\dot{x} + T(x, y)\dot{y} + X(x, y). \quad (3.3)$$

If $\mathcal{H}_1$ is constant, its time derivative must vanish so that from Eqs. (3.1),

$$0 = \dot{\mathcal{H}}_1 = -[S(\partial V/\partial x) + T(\partial V/\partial y)] + [(\partial \mathcal{H}/\partial x) - T\Omega]\dot{x} + [(\partial \mathcal{H}/\partial y) + S\Omega]\dot{y} \\ + (\partial S/\partial x)\dot{x}^2 + [(\partial S/\partial y) + (\partial T/\partial x)]\dot{x}\dot{y} + (\partial T/\partial y)\dot{y}^2.$$

Rewriting $\dot{x}^2$ and $\dot{y}^2$ in terms of $(\dot{x}^2 + \dot{y}^2)$ and $(\dot{x}^2 - \dot{y}^2)$ and using Eq. (3.2) to eliminate the former,

$$\dot{\mathcal{H}}_1 = [(\mathcal{E} - V)(\partial S/\partial x) - S(\partial V/\partial x) + (\mathcal{E} - V)(\partial T/\partial y) - T(\partial V/\partial y)] \\ + [(\partial \mathcal{H}/\partial x) - T\Omega]\dot{x} + [(\partial \mathcal{H}/\partial y) + S\Omega]\dot{y} \\ + (1/2)[(\partial S/\partial x) - (\partial T/\partial y)](\dot{x}^2 - \dot{y}^2) + [(\partial S/\partial y) + (\partial T/\partial x)]\dot{x}\dot{y}.$$

The five terms of Eq. (3.4) now involve linearly independent combinations of $\dot{x}$ and $\dot{y}$, and so each coefficient must separately vanish if $\dot{\mathcal{H}}_1 = 0$ identically.

Thus

$$\partial S/\partial x = \partial T/\partial y, \quad \partial S/\partial y = -\partial T/\partial x, \quad (3.5)$$

$$(\partial/\partial x)[S(\mathcal{E} - V)] + (\partial/\partial y)[T(\mathcal{E} - V)] = 0, \quad (3.6)$$

$$S(\partial \mathcal{H}/\partial x) + T(\partial \mathcal{H}/\partial y) = 0, \quad -T(\partial \mathcal{H}/\partial x) + S(\partial \mathcal{H}/\partial y) = -(S^2 + T^2)\Omega. \quad (3.7)$$

Suppose now that we introduce complex numbers

$$z = x + iy, \quad G = S + iT. \quad (3.8)$$

Then Eqs. (3.5) are just the Cauchy-Riemann conditions guaranteeing that G must be an analytic function of z . Hence, S and T are conjugate (orthogonal) solutions of Laplace's equation:

$$\partial^2 S/\partial x^2 + \partial^2 S/\partial y^2 = 0, \quad \partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 = 0. \quad (3.9)$$

Next, Eq. (3.6) is satisfied when

$$S(\mathcal{E} - V) = -\partial\phi/\partial y, \quad T(\mathcal{E} - V) = \partial\phi/\partial x. \quad (3.10)$$

Hence,

$$S(\partial\phi/\partial x) + T(\partial\phi/\partial y) = 0, \quad -T(\partial\phi/\partial x) + S(\partial\phi/\partial y) = -(S^2 + T^2)(\mathcal{E} - V), \quad (3.11)$$

which is a pair of equations of the same form as Eq. (3.7). Thus, it is convenient to introduce a transformation to coordinates along the characteristics (ξ, η) ,

$$\partial x = S\partial\xi - T\partial\eta, \quad \partial y = T\partial\xi + S\partial\eta. \quad (3.12)$$

That is,

$$dz = Gd\zeta, \quad d\zeta = d\xi + id\eta, \quad (3.13)$$

so that $z(z)$ is also analytic; the mapping is conformal.

From Eqs. (3.7) and (3.11),

$$\frac{\partial \mathcal{K}}{\partial \xi} = 0, \quad \frac{\partial \phi}{\partial \xi} = 0, \quad (3.14)$$

so that $\mathcal{K} = \mathcal{K}(\eta)$ and $\phi = \phi(\eta)$. Thus

$$\mathcal{E} - V = -(\partial\phi/\partial\eta) |\nabla\eta|^2, \quad \Omega = -(\partial\mathcal{K}/\partial\eta) |\nabla\eta|^2, \quad (3.15)$$

since

$$(S^2 + T^2)^{-1} = |d\zeta/dz|^2 = |\nabla\xi|^2 + |\nabla\eta|^2.$$

Therefore, if the potential and the induction can be expressed as in Eq. (3.15), where $\phi(\eta)$ and $\mathcal{K}(\eta)$ are any functions and η is any solution of Laplace's equation in the region of interest, then

$$\mathcal{E}_1 = \mathcal{K} + |\nabla\eta|^{-2} [x(\partial\eta/\partial y) - y(\partial\eta/\partial x)] \quad (3.16)$$

is invariant.

As an example of some importance, $\eta = \ln(a/r)$, where $r = (x^2 + y^2)^{1/2}$, satisfies Laplace's equation (except at $r = 0$). Setting $\phi = \int \phi(r) r dr = (1/2) \phi r^2$, Eqs. (3.15) become

$$V = \phi(r) \quad , \quad \Omega = (-1/r)(d\phi/dr) \quad . \quad (3.17)$$

Thus, a linear invariant (the angular momentum) can be constructed whenever $V = V(r)$ is a central potential and the induction satisfies $\Omega = \Omega(r)$.

B. The Quadratic Invariant

When P , Q , and R do not all vanish identically in Eq. (2.2), conservation of energy can still be used to eliminate one of the coefficients. Thus, the most general quadratic invariant may always be written

$$\mathcal{J}_2(\dot{x}, \dot{y}; x, y) = \mathcal{K} + A\dot{x} + B\dot{y} + C(\dot{x}^2 - \dot{y}^2) + 2D\dot{x}\dot{y} \quad . \quad (3.18)$$

Taking the time derivative of $\mathcal{J}_2$, we find

$$\partial C / \partial x - \partial D / \partial y = 0 \quad , \quad \partial C / \partial y + \partial D / \partial x = 0 \quad , \quad (3.19)$$

$$\partial A / \partial x - \partial B / \partial y = 4D\Omega \quad , \quad \partial A / \partial y + \partial B / \partial x = -4C\Omega \quad , \quad (3.20)$$

$$\partial \mathcal{K} / \partial x - 2C(\partial V / \partial x) + 2(\mathcal{E} - V)(\partial D / \partial y) - 2D(\partial V / \partial y) = B\Omega \quad , \quad (3.21a)$$

$$\partial \mathcal{K} / \partial y + 2C(\partial V / \partial y) + 2(\mathcal{E} - V)(\partial D / \partial x) - 2D(\partial V / \partial x) = -A\Omega \quad , \quad (3.21b)$$

$$(\partial / \partial x) [A(\mathcal{E} - V)] + (\partial / \partial y) [B(\mathcal{E} - V)] = 0 \quad . \quad (3.22)$$

Again the introduction of complex variables serves to satisfy two of the equations, Eqs. (3.19), so that

$$F = C + i D = F(z) \quad , \quad z = x + iy \quad .$$

Then, if we differentiate both Eq. (3.21a) and Eq. (3.21b) to obtain $\partial^2 V / \partial x \partial y$ in order to eliminate it from the results, we find

$$\begin{aligned} & -D(\partial^2 V / \partial x^2) + 2C(\partial^2 V / \partial x \partial y) + D(\partial^2 V / \partial y^2) + 3(\partial C / \partial y)(\partial V / \partial x) \\ & + 3(\partial C / \partial x)(\partial V / \partial y) - 2(\partial^2 C / \partial x \partial y)(\mathcal{E} - V) = \\ & - (1/2) [(\partial / \partial x)(\Omega A) + (\partial / \partial y)(\Omega B)] \quad . \end{aligned} \quad (3.23)$$

Likewise, eliminating Ω from Eqs. (3.20) yields

$$[C(\partial A / \partial x) + D(\partial A / \partial y)] - [-D(\partial B / \partial x) + C(\partial B / \partial y)] = 0 \quad . \quad (3.24)$$

These equations have different characteristic functions and so we introduce two new pairs of variables (μ, ν) and (ξ, η) , where

$$dx = C d\mu - D d\nu = P d\xi - Q d\eta, \quad dy = D d\mu + C d\nu = Q d\xi + P d\eta \quad , \quad (3.25)$$

such that

$$F(z) = (1/2)G^2 \quad , \quad G(z) = P + iQ \quad . \quad (3.26)$$

Thus, if

$$d\lambda = d\mu + i d\nu, \quad d\xi = d\xi + i d\eta \quad (3.27)$$

then we have

$$dz = F d\lambda = G d\xi \quad . \quad (3.28)$$

Specification of any one of the functions $F(z)$, $G(z)$, $\lambda(z)$, or $\xi(z)$, determines each of the others.

In terms of (μ, ν) variables, Eq. (3.24) is

$$\partial A / \partial \mu - \partial B / \partial \nu = 0 \quad ; \quad A = \partial \phi / \partial \nu \quad , \quad B = \partial \phi / \partial \mu \quad . \quad (3.29)$$

Hence, from Eq. (3.20),

$$4(p^2 + q^2)^{-1} (\partial^2 \phi / \partial \mu^2 + \partial^2 \phi / \partial \nu^2) = \partial^2 \phi / \partial \xi^2 + \partial^2 \phi / \partial \eta^2 = -4(p^2 + q^2)\Omega \quad (3.30)$$

Likewise, from Eq. (3.22),

$$(\mathcal{E} - V)(\partial \phi / \partial \mu) = \partial \Lambda / \partial x, \quad (\mathcal{E} - V)(\partial \phi / \partial \nu) = -\partial \Lambda / \partial y.$$

On converting to (ξ, η) coordinates, therefore,

$$(p^2 + q^2)(\mathcal{E} - V)(\partial \phi / \partial \xi) = 2(\partial \Lambda / \partial \xi), \quad (p^2 + q^2)(\mathcal{E} - V)(\partial \phi / \partial \eta) = -2(\partial \Lambda / \partial \eta)$$

or

$$(\partial / \partial \eta) [(p^2 + q^2)(\mathcal{E} - V)(\partial \phi / \partial \xi)] + (\partial / \partial \xi) [(p^2 + q^2)(\mathcal{E} - V)(\partial \phi / \partial \eta)] = 0 \quad (3.31)$$

Similarly, Eq. (3.23) is greatly simplified when it is expressed in terms of ξ and η as independent coordinates. We find

$$\begin{aligned} (\partial^2 / \partial \xi \partial \eta) [(p^2 + q^2)(\mathcal{E} - V)] &= (1/4) \{ (\partial / \partial \xi) [(p^2 + q^2)\Omega (\partial \phi / \partial \eta)] \\ &+ (\partial / \partial \eta) [(p^2 + q^2)\Omega (\partial \phi / \partial \xi)] \} \end{aligned} \quad (3.32)$$

Given $G(z)$, the three Eqs. (3.30) - (3.32) determine the three unknowns $\mathcal{E} - V, \Omega, \phi$. Moreover, the arrangement suggests a simple iterative solution for an adiabatic invariant when Ω is not too large. Thus, when Ω is small, we replace the right-hand side of Eq. (3.32) by zero, and

$$\mathcal{E} - V = f(\xi) |\nabla \xi|^2 + g(\eta) |\nabla \eta|^2 \quad (3.33)$$

Then, ϕ is determined by Eq. (3.31), whose solution contains an overall multiplicative constant which determines the magnitude of Ω . Thus, Eq. (3.31) permits

$$\phi = a \ln (f + g) + b(f - g) \quad (3.34)$$

where a and b are constants. Next, Ω itself is given by Eq. (3.30),

$$\begin{aligned} -4(P^2+Q^2)\Omega = a(f+g)^{-2} \{ [(d^2f/d\xi^2) + (d^2g/d\eta^2)] (f+g) - (df/d\xi)^2 \\ - (dg/d\eta)^2 \} + b [(d^2f/d\xi^2) - (d^2g/d\eta^2)] \end{aligned} \quad (3.35)$$

Finally, Eqs. (3.34) and (3.35) can be inserted on the right of Eq. (3.32), and the process repeated iteratively to the accuracy desired.

There is no difficulty in showing that the special case $z = a \cosh \eta$, for $\Omega = 0$, reduces to the result that Whittaker obtained. For this case $a^2 |\nabla \xi|^2 = a^2 |\nabla \eta|^2 = (\cosh^2 \eta - \cos^2 \xi)^{-1}$. Hence f and g may be chosen in Eq. (3.33) to eliminate the constant $\mathcal{C}$, as before. In this special limit, therefore, the form of Eq. (3.33) also holds for V alone.

IV. OTHER INVARIANTS

It is of some significance that Whittaker's procedure can be generalized to the case in which an invariant exists of the form

$$\mathcal{I} = (P/Q) \exp(-R/S) \quad , \quad (4.1)$$

where P , Q , R , and S are general series in $(\dot{x}, \dot{y})$:

$$P = \sum_{n=0}^{Np} \sum_{m=0}^n {}_n P_m(x, y) \dot{x}^m \dot{y}^{n-m} \quad , \quad \text{etc.} \quad (4.2)$$

If P is given by Eq. (4.2), then from the equations of motion, Eqs. (3.1),

$$\dot{P} = \sum_{n=0}^{Np+1} \sum_{m=0}^n {}_n \dot{P}_m(x, y) \dot{x}^m \dot{y}^{n-m} \quad . \quad (4.3)$$

That is, if P is a power series of order N_p in the velocities, then $\dot{P}$ is a power series of order $N_p + 1$. Hence, if $\mathcal{J}$ is given by Eq. (4.1), then

$$\dot{\mathcal{J}} = [Q^{-2}S^{-2} \exp(-R/S)] \tilde{\mathcal{J}} \quad (4.4)$$

where

$$\tilde{\mathcal{J}} = \dot{P}QS^2 - P\dot{Q}S^2 - PQ\dot{S}^2 - PQRS - PQR\dot{S} \quad (4.5)$$

If $\mathcal{J}$ is invariant, then $\tilde{\mathcal{J}} = 0$ is also a constant of the motion. The existence of $\mathcal{J}$ implies the existence of another invariant $\tilde{\mathcal{J}}$ which is a simple power series in the velocities, of order $N = N_p + N_q + N_0 + 1$, where N_0 is the greater of $N_p + N_s$ or $2N_s$. For a general invariant of the form of Eq. (4.1), it is therefore adequate to restrict our attention to invariants that can be expressed as a simple power series in the velocities.

Again, most of the $(N+1)(N+2)/2$ terms of the general form of the N th order invariant,

$$\mathcal{J}_N = \sum_{n=0}^N \sum_{m=0}^n n! I_m(x,y) \dot{x}^m \dot{y}^{n-m} \quad (4.6)$$

are not independent. They are related by conservation of energy. Thus, as suggested by Eq. (3.18) and upon repeated use of Eq. (3.2), Eq. (4.6) may be rewritten in what we shall take to be the standard form,

$$\mathcal{J}_N(\dot{x}, \dot{y}; x, y) = \frac{1}{2} C_0 + \sum_{n=1}^N [C_n \cos(n\gamma) + i_n \sin(n\gamma)] \quad (4.7)$$

where

$$\dot{x} = u \cos \gamma, \quad \dot{y} = u \sin \gamma, \quad \frac{1}{2} u^2 = \mathcal{E} - V \quad (4.8)$$

Then,

$$\ddot{\gamma} = u^{-1}(\ddot{y} \cos \gamma - \ddot{x} \sin \gamma) = -\Omega - u^{-1}\left(\frac{\partial V}{\partial y} \cos \gamma - \frac{\partial V}{\partial x} \sin \gamma\right) . \quad (4.9)$$

If we define

$$S_n = 0 \quad , \quad n \leq 0 \quad ; \quad C_n = 0 \quad , \quad n \leq -1 \quad ;$$

$$S_n = 0 \quad , \quad n \geq N+1 \quad ; \quad C_n = 0 \quad , \quad n \geq N+1 \quad ;$$

a bit of algebra shows that

$$\begin{aligned} \ddot{g}_N = \frac{1}{2} \sum_{n=0}^{N+1} \left\{ \left[u \frac{\partial C_{n-1}}{\partial x} - \frac{n-1}{u} \frac{\partial V}{\partial y} S_{n-1} - u \frac{\partial S_{n-1}}{\partial y} + \frac{n-1}{u} \frac{\partial V}{\partial x} C_{n-1} - 2n\Omega S_n \right. \right. \\ \left. \left. + u \frac{\partial C_{n+1}}{\partial x} - \frac{n+1}{u} \frac{\partial V}{\partial y} S_{n+1} + u \frac{\partial S_{n+1}}{\partial y} - \frac{n+1}{u} \frac{\partial V}{\partial x} C_{n+1} \right] \cos n\gamma \right. \\ \left. + \left[u \frac{\partial C_{n-1}}{\partial y} + \frac{n-1}{u} \frac{\partial V}{\partial x} S_{n-1} + u \frac{\partial S_{n-1}}{\partial x} + \frac{n-1}{u} \frac{\partial V}{\partial y} C_{n-1} + 2n\Omega C_n \right. \right. \\ \left. \left. - u \frac{\partial C_{n+1}}{\partial y} - \frac{n+1}{u} \frac{\partial V}{\partial x} S_{n+1} + u \frac{\partial S_{n+1}}{\partial x} + \frac{n+1}{u} \frac{\partial V}{\partial y} C_{n+1} \right] \sin n\gamma \right\} . \end{aligned}$$

We can now require that the coefficients of each $\cos(n\gamma)$ and of each $\sin(n\gamma)$, except $\sin(0\gamma)$, separately vanish. Remembering that

$$u \frac{\partial u}{\partial x} = - \frac{\partial V}{\partial x} \quad , \quad u \frac{\partial u}{\partial y} = - \frac{\partial V}{\partial y} \quad ,$$

we find

$$\begin{aligned} u^n \left[\frac{\partial}{\partial x} (u^{1-n} C_{n-1}) - \frac{\partial}{\partial y} (u^{1-n} S_{n-1}) \right] \\ + u^{-n} \left[\frac{\partial}{\partial x} (u^{1+n} C_{n+1}) + \frac{\partial}{\partial y} (u^{1+n} S_{n+1}) \right] = 2n\Omega S_n \quad , \quad 0 \leq n \leq N+1 \quad , \quad (4.10a) \end{aligned}$$

$$\begin{aligned} u^n \left[\frac{\partial}{\partial y} (u^{1-n} C_{n-1}) + \frac{\partial}{\partial x} (u^{1-n} S_{n-1}) \right] \\ + u^{-n} \left[- \frac{\partial}{\partial y} (u^{1+n} C_{n+1}) + \frac{\partial}{\partial x} (u^{1+n} S_{n+1}) \right] = - 2n\Omega C_n \quad , \quad 1 \leq n \leq N+1 \quad . \quad (4.10b) \end{aligned}$$

Equations (4.10) are complete, they represent $2n+3$ equations for the $2N+3$ unknowns: C_n , $0 \leq n \leq N$; S_n , $1 \leq n \leq N$; Ω and u .

Once again, it is useful to introduce complex functions. If

$$F_n = u^{-n}(C_n + iS_n) \quad (4.11)$$

and we define conjugate operators

$$\nabla = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}, \quad \nabla^* = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}, \quad (4.12)$$

so that

$$\nabla \nabla^* = \nabla^* \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad (4.13)$$

then i times Eq. (4.10b) added to Eq. (4.10a) gives

$$\nabla^* F_{n-1} + u^{-2n} \nabla(u^{2n+2} F_{n+1}) = -2n \Omega F_n, \quad 0 \leq n \leq N. \quad (4.14)$$

Equation (4.14) is augmented by the degenerate constraints at the ends of the chain:

$$\text{Im}(F_0) = 0, \quad \text{Re}[\nabla(u^2 F_1)] = 0, \quad (4.15)$$

and

$$\nabla^* F_N = 0. \quad (4.16)$$

To this point, nothing is said about the analytic properties of the functions F_n . Indeed, in general, F_n will not be an analytic function of $z = x + iy$. But suppose that $G = P + iQ$ is analytic, where P and Q are real, and that its complex conjugate is $G^* = P - iQ$. Then, both of the equations,

$$\nabla^* G = 0, \quad \nabla G^* = 0,$$

are satisfied; either one is the expression of the Cauchy-Riemann conditions.

Equation (4.16), therefore, states that F_N is an analytic function of z , just what we found earlier for $N = 1$ and $N = 2$.

The existence of a solution of Eqs. (4.14) - (4.16) is both necessary and sufficient for the existence of true invariant $\mathcal{Q}_N(\dot{x}, \dot{y}; x, y)$.

V. NO TOROIDAL INDUCTION

Suppose that $\Omega = 0$. Then our equations only interconnect just the even or just the odd orders n . The invariant is either even or odd in the velocities.

Since F_N is analytic, we may rewrite it in terms of another analytic function $G(z)$,

$$F_N = (1/N)G^N, \quad (5.1)$$

$$\nabla^* F_{N-2} + G^{N-1} \nabla(u^2 G) = 0, \quad (5.2)$$

$$\nabla^* F_{N-4} + u^{6-2N} \nabla(u^{2N-4} F_{N-2}) = 0, \quad \text{etc.} \quad (5.3)$$

If we now make the same conformal transformation of coordinates as before, Eq. (3.13), then the operators ∇ and ∇^* undergo the simple transformations

$$\mathcal{Q} = G\nabla = \frac{\partial}{\partial \xi} - i \frac{\partial}{\partial \eta}, \quad \mathcal{Q}^* = G^* \nabla^* = \frac{\partial}{\partial \xi} + i \frac{\partial}{\partial \eta}, \quad (5.4)$$

$$\mathcal{Q}\mathcal{Q}^* = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} = GG^* \nabla \nabla^* = (P^2 + Q^2) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \quad (5.5)$$

It is convenient to define a solution, ψ , of Poisson's equation,

$$\nabla \nabla^* \psi = u^2, \quad (5.6)$$

so that Eq. (5.2) becomes

$$\mathcal{Q}^*[F_{N-2} + G^{N-2} \mathcal{Q}^2 \psi] = 0. \quad (5.7)$$

Integrating Eq. (5.7)

$$F_{N-2} = -G^{N-2} \partial^2 \psi + f, \quad ,$$

where $f(\zeta)$ is analytic, $\partial^* f = 0$.

Suppose now that instead of the given function ψ , we take another solution, $\tilde{\psi}$, of Poisson's equation. Then $\tilde{\psi} = \psi - \Psi$ satisfies Laplace's equation. Hence $\tilde{f} = \partial^2 \tilde{\psi}$ is analytic, since $\partial^* \tilde{f} = 0$. Moreover, $\partial \tilde{f} = 2d\tilde{f}/d\zeta$.

Thus

$$F_{N-2} = -G^{N-2} \partial^2 \psi + \{f - 2G^{N-2} d\tilde{f}/d\zeta\}.$$

Clearly, therefore, the arbitrariness of the added function $f(\zeta)$ is preserved by permitting ψ to satisfy an arbitrary boundary condition, i.e., to be any solution of Eq. (5.6). In general, then,

$$F_{N-2} = -G^{N-2} \partial^2 \psi, \quad (5.8)$$

$$\partial \partial^* \psi = T, \quad T = GG^* u^2. \quad (5.9)$$

Next, for $n = N-3$, Eq. (5.3) becomes

$$\partial^* F_{N-4} - G^{N-4} [T \partial^3 \psi + (N-2)(\partial T)(\partial^2 \psi)] = 0.$$

That is,

$$\partial^* [F_{N-4} - \frac{1}{2}(N-2)G^{N-4} (\partial^2 \psi)^2] = G^{N-4} T \partial^3 \psi. \quad (5.10)$$

Thus

$$F_{N-4} = G^{N-4} [\frac{1}{2}(N-2) (\partial^2 \psi)^2 + \Lambda], \quad (5.11)$$

where Λ is the general solution of

$$\partial^* \Lambda = T \partial^3 \psi. \quad (5.12)$$

Once again an additive analytic function in F_{N-4} can be suppressed in terms of the general solution of Eq. (5.12). Solution of the next highest order equation proceeds in the same fashion, and so on down the chain.

VI. THE THIRD-ORDER INVARIANT

As an example of some importance, consider $N = 3$. Our basic equations are

$$F_1 = -G\psi^2, \quad R_e[\nabla(u^2 F_1)] = 0.$$

From the second of these, $u^2 F_1 = i\nabla^* \phi$, where ϕ is any real function. Thus,

$$i\nabla^* \phi = -T G \psi^2.$$

In terms of real operators, therefore,

$$\begin{aligned} \frac{\partial \phi}{\partial \xi} &= 2T \frac{\partial^2 \psi}{\partial \xi \partial \eta}, \\ \frac{\partial \phi}{\partial \eta} &= T \left(\frac{\partial^2 \eta}{\partial \xi^2} - \frac{\partial^2 \psi}{\partial \eta^2} \right). \end{aligned} \quad (6.1)$$

With Eq. (5.9), upon taking appropriate combinations of the derivatives of Eqs. (6.1), we can eliminate ψ in terms of T , finding

$$\begin{aligned} \frac{\partial T}{\partial \xi} &= \frac{\partial}{\partial \eta} \left(\frac{1}{T} \frac{\partial \phi}{\partial \xi} \right) + \frac{\partial}{\partial \xi} \left(\frac{1}{T} \frac{\partial \phi}{\partial \eta} \right), \\ \frac{\partial T}{\partial \eta} &= \frac{\partial}{\partial \xi} \left(\frac{1}{T} \frac{\partial \phi}{\partial \xi} \right) - \frac{\partial}{\partial \eta} \left(\frac{1}{T} \frac{\partial \phi}{\partial \eta} \right). \end{aligned} \quad (6.2)$$

A. The Special Case $T = T(\eta)$

Suppose that T is independent of ξ . Then from the first Eq. (6.1),

$$\phi = \phi(\eta) + \Gamma(\xi) T^{1/2},$$

and from the second Eq. (6.1), if C is any constant,

$$\frac{d\phi}{d\eta} = (C-T)T \quad , \quad \Gamma(d^2/d\eta^2)T^{-1/2} + T^{-1/2}(d^2/d\xi^2)\Gamma = 0 \quad .$$

Thus, either $\Gamma(\xi) = 0$, or

$$d^2\Gamma/d\xi^2 + k^2\Gamma = 0 \quad , \quad d^2T^{-1/2}/d\eta^2 - k^2T^{-1/2} = 0 \quad ,$$

where $k^2 = \text{const}$ is also arbitrary. The more interesting case is $\Gamma = 0$, which we proceed to examine. We have

$$d\phi/d\eta = (C-T)T \quad , \quad \psi = (1/4)(\xi^2 - \eta^2) + \psi(\eta) \quad , \quad d^2\psi/d\eta^2 = T \quad . \quad (6.3)$$

Then,

$$d^2\psi = C - T \quad , \quad F_3 = (1/3)G^3 \quad , \quad F_1 = -(C-T)G \quad . \quad (6.4)$$

Let's consider the particular mapping $z = ae^{i\xi}$, where a is a constant. Then

$$x = r \cos \xi \quad , \quad y = r \sin \xi \quad , \quad (x^2 + y^2)^{1/2} = r = ae^{-\eta} \quad , \quad (6.5)$$

and

$$G = dz/d\xi = ire^{i\xi} \quad , \quad GG^* = r^2 \quad . \quad (6.6)$$

Therefore

$$\begin{aligned} F_3 &= (1/3)r^3(\sin 3\xi - i \cos 3\xi) \quad , \\ F_1 &= r(C-T)(\sin \xi - i \cos \xi) \quad . \end{aligned} \quad (6.7)$$

From the definition of F_n , Eq. (4.11), and from Eqs. (4.7) and (5.9),

$$\begin{aligned} \mathcal{A}_3 &= (r^3 u^3/3)(\sin 3\xi \cos 3\gamma - \cos 3\xi \sin 3\gamma) \\ &+ (C - r^2 u^2)r u(\sin \xi \cos \gamma - \cos \xi \sin \gamma) \quad , \end{aligned}$$

or

$$\mathcal{J}_3 = C r u \sin (\xi - \gamma) - (4/3) [r u \sin (\xi - \gamma)]^3 \quad (6.8)$$

In this case, the third order invariant is nothing more than an arbitrary combination of the angular momentum added to its cube. We should have expected as much, since we showed in Section III that if $T = T(\eta)$, a linear invariant exists.

B. A Perturbed System

Equations (6.2) are difficult to solve in the general case. On the other hand, it is quite natural to ask as to the nature of systems that are only slightly perturbed from the condition $T = T(\eta)$ which we just considered. Suppose, therefore, that

$$T = T_0(\eta) + T_1 + \dots,$$

$$\phi = \phi_0(\eta) + \phi_1 + \dots,$$

$$\psi = \psi_0 + \psi_1 + \dots,$$

where T_0 is assumed given, and ϕ_0 and ψ_0 are the solutions just found for $T = T_0(\eta)$.

We obtain

$$\begin{aligned} \frac{C}{T_0} T_1 &= \frac{2}{T_0} \frac{\partial \phi_1}{\partial \eta} - \frac{\phi_1}{T_0^2} \frac{dT_0}{d\eta}, \\ \frac{\partial}{\partial \eta} \left\{ \left(\frac{1}{T_0} - \frac{4}{C} \right) \frac{\partial \phi_1}{\partial \eta} + \phi_1 \frac{d}{d\eta} \left[\frac{1}{T_0} + \frac{2}{C} \ln \left(\frac{T_0}{C} \right) \right] \right\} + \frac{1}{T_0} \frac{\partial^2 \phi_1}{\partial \xi^2} &= 0, \end{aligned} \quad (6.9)$$

where we have taken the arbitrary function of η due to partial integration, which

appears in T_1 and ϕ_1 , into T_0 and ϕ_0 . Likewise, if ψ_1 is an arbitrary solution of

$$\frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial \eta^2} = T_1 \quad , \quad (6.10)$$

Then

$$\psi_1 = \psi_1 + \tilde{\psi}_1 \quad , \quad (6.11)$$

where $\tilde{\psi}_1$ satisfies Laplace's equation, and as well,

$$2 \frac{\partial^2 \tilde{\psi}_1}{\partial \xi \partial \eta} = -2 \frac{\partial^2 \psi_1}{\partial \xi \partial \eta} + \frac{1}{T_0} \frac{\partial \phi_1}{\partial \xi} \quad , \quad (6.12)$$

$$\frac{\partial^2 \tilde{\psi}_1}{\partial \xi^2} - \frac{\partial^2 \tilde{\psi}_1}{\partial \eta^2} = - \left(\frac{\partial^2 \psi_1}{\partial \xi^2} - \frac{\partial^2 \psi_1}{\partial \eta^2} \right) + \frac{1}{T_0} \left[\frac{\partial \phi_1}{\partial \eta} - (C - T_0) T_0 \right] \quad .$$

Presumably, only one of Eqs. (6.12) need be satisfied, since both Eqs. (6.12) taken together imply (the linearized) Eqs. (6.2), which we have already solved. We must pay attention to our ordering, however, since we are dealing only with the dominant terms of an expansion.

Next, let

$$\tau = CT_1/T_0 \quad , \quad \lambda = \phi_1/T_0 \quad , \quad (6.13)$$

and let us assume that $C \gg 4T_0$. Then

$$\tau = 2 \frac{\partial \lambda}{\partial \eta} + (\lambda/T_0) dT_0/d\eta \quad . \quad (6.14)$$

$$\frac{\partial^2 \lambda}{\partial \xi^2} + \frac{\partial^2 \lambda}{\partial \eta^2} = 0 \quad .$$

Suppose, therefore, we look for solutions

$$\lambda = \sum_k A_k e^{-k\eta} \cos(k\xi + \theta_k) \quad , \quad (6.15)$$

where k , A_k , and θ_k are constants, and the sum is over appropriate values of k . Then

$$\tau = \sum_k [(1/T_0)(dT_0/d\eta) - 2k] A_k e^{-k\eta} \cos(k\xi + \theta_k) \quad (6.16)$$

Hence,

$$\begin{aligned} T_1 &= \sum_k (A_k/C) [(dT_0/d\eta) - 2kT_0] e^{-k\eta} \cos(k\xi + \theta_k) , \\ \phi_1 &= \sum_k T_0 A_k e^{-k\eta} \cos(k\xi + \theta_k) , \\ \psi_1 &= \sum_k \alpha_k(\eta) \cos(k\xi + \theta_k) , \\ d^2\alpha_k/d\eta^2 - k^2\alpha_k &= (A_k/C) [(dT_0/d\eta) - 2kT_0] e^{-k\eta} . \end{aligned} \quad (6.17)$$

For example, if

$$\tau = \sum_m t_m e^{-2m\eta} ,$$

then

$$\alpha_k = \sum_m (A_k/2C) [(m-k)/m(m+k)] t_m e^{-(2m+k)\eta} .$$

In any event, $\psi_1 = O(A T_0/C)$, whereas both $T_0^{-1} \partial\phi_1/\partial\xi$ and $T_0^{-1} [(\partial\phi_1/\partial\eta) - C T_1]$ are $O(A)$, where A is a typical value of A_k . Hence,

$$\psi_1 \approx \tilde{\psi}_1 = - \sum_k (k^{-1} A_k) e^{-k\eta} \cos(k\xi + \theta_k) \quad (6.18)$$

Setting $A_k = C\epsilon_k/2k$, we find

$$\begin{aligned} T &= T_0 + \sum_k \epsilon_k [(1/2k) (dT_0/d\eta) - T_0] e^{-k\eta} \cos(k\xi + \theta_k) \\ \phi &= C \left[\int T_0 d\eta + T_0 \sum_k (\epsilon_k/2k) e^{-k\eta} \cos(k\xi + \theta_k) \right] , \\ \psi &= (C/4) \left\{ (\xi^2 - \eta^2) - \sum_k (2/k^2) \epsilon_k e^{-k\eta} \cos(k\xi + \theta_k) \right\} , \\ \phi_3 &= C u e^{-\eta} \left\{ \sin(\xi - \gamma) + \sum_k \epsilon_k e^{-k\eta} \sin[k\xi + \theta_k + (\xi - \gamma)] \right\} . \end{aligned} \quad (6.19)$$

The actual value of $C \neq 0$, now, is immaterial.

Again, consider the mapping $z = ae^{i\xi}$. Then $T_0 = 2(v/a)^2(\mathcal{E} - V_0)$ and

(6.20a)

$$V = V_0 + \sum_k (\epsilon_k/2k) \left\{ r \frac{d}{dr} \left[\left(\frac{r}{a} \right)^2 (\mathcal{E} - V_0) \right] + 2k \left(\frac{r}{a} \right)^2 (\mathcal{E} - V_0) \right\} \left(\frac{r}{a} \right)^{k-2} \cos(k\xi + \theta_k) ,$$

$$\mathcal{J}_3 = ru \left\{ \sin(\xi - \gamma) + \sum_k \epsilon_k (r/a)^k \sin[k\xi + \theta_k + (\xi - \gamma)] \right\} . \quad (6.20b)$$

If we choose $\epsilon_k = 0$ except for $k = 3$, $\epsilon_3 = \epsilon$, $\theta_3 = -\pi/2$, and if $V_0 = \epsilon_0 r^2/a_0^2$, then Eqs. (6.20) correspond to a potential very much like that considered by Heñon and Heiles⁴ in their computational study of representative galactic motion. (The difference lies in the higher-order-in- r terms from the appearance of V_0 in the summation.)

Notice that the first-order (in ϵ_k) expression for $\mathcal{J}_3$ is only first order in the velocities $\dot{x} = u \cos \gamma$ and $\dot{y} = u \sin \gamma$. But it was necessary to consider $\mathcal{J}_3$ in general (not $\mathcal{J}_1$) to obtain this result. The necessary and sufficient condition for the existence of a true linear invariant, as we found in Section 3, does not admit Eqs. (6.20), even as an approximation. Our result, Eq. (6.20b), is an adiabatic invariant.

Notice also that in the present case, particles of all energies may not possess the last invariant, even though particles of some energies might. The potential V , once fixed, is the same for all particles. Thus, suppose V is of the form of Eq. (6.20a) for particles of energy $\mathcal{E}_0$:

$$\begin{aligned} V &= V_0(r) + \sum_k (\epsilon_k^0/2k) \left\{ r \frac{d}{dr} \left[\left(\frac{r}{a_0} \right)^2 (\mathcal{E}_0 - V_0) \right] + 2k \left(\frac{r}{a_0} \right)^2 (\mathcal{E}_0 - V_0) \right\} \left(\frac{r}{a_0} \right)^{k-2} \cos(k\xi + \theta_k) \\ &= V_0 + \sum_k \left[(\epsilon_k^0 \mathcal{E}_0^k / a_0^k) / 2k \right] \left\{ r \frac{d}{dr} \left[\left(\frac{r}{a} \right)^2 (\mathcal{E} - \mathcal{E}_0 V_0 / \mathcal{E}_0) \right] + 2k (r/a)^2 (\mathcal{E} - \mathcal{E}_0 V_0 / \mathcal{E}_0) \right\} \\ &\quad \times (r/a)^{k-2} \cos(k\xi + \theta_k) . \end{aligned} \quad (6.21)$$

We may set $(\epsilon_k^0 \mathcal{E}_0^k / \mathcal{E}_0^k) = \epsilon_k$, recovering Eq. (6.20a) except that V_0 is replaced by $\mathcal{E}V_0/\mathcal{E}_0$ in the summation. Accordingly, there are in general two circumstances when Eq. (6.20b) gives an invariant:

- I. When the orbits (due to the existence of the invariant) are constrained so that both $V_0 \ll \mathcal{E}_0$ and $V_0 \ll \mathcal{E}$, then only $\mathcal{E}$ rather than $\mathcal{E} - \mathcal{E}V_0/\mathcal{E}_0$ or $\mathcal{E} - V_0$ appears in braces in the sums over k in the equations for V . Equations (6.20a) and (6.21) become the same.
- II. When $\mathcal{E}$ approaches $\mathcal{E}_0$, Eqs. (6.20a) and (6.21) become the same.

In the Hénon and Heiles case, the potential which they consider is constructed for $\epsilon_k^0 \rightarrow 0$, $\mathcal{E}_0 \rightarrow \infty$, so that ϵ is finite and $\mathcal{E}V_0/\mathcal{E}_0 \rightarrow 0$. Thus, they discover invariant behavior for motion confined to the region $V_0 \ll \mathcal{E} \ll \mathcal{E}_0$. Case 1. On the other hand, because $\mathcal{E}_0 \rightarrow \infty$, their computations would not reveal any additional invariant behavior for $\mathcal{E} \approx \mathcal{E}_0$. The existence of a last invariant for a band at finite energy as a consequence of the present theory is a matter of some interest.

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