GAUGE AMBIGUITIES, ZERO MODES AND
CLASSIFICATION OF FIELD SIZE*

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ABSTRACT

The occurrence of zero modes for copies of the pure vacuum is studied in Coulomb and Landau gauges. We propose a precise criteria to distinguish small and large fields. An example of a zero mode that reflects a higher invariance of the generalized Gribov equation is noted.

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Two methods for quantization of Yang-Mills fields, the canonical formalism and the Faddeev-Popov formalism, although somewhat different in approach arrive at the same Feynman rules in the Coulomb gauge, for example. The gauge ambiguity⁽¹⁾ is seen in these two methods in a different, though related, way through the occurrence of zero modes. Here we exhaustively study the occurrence of zero modes for copies of pure vacuum in Coulomb and Landau gauges. Then we propose a precise criterion to distinguish "small" and "large" fields. Finally, an example of a zero mode which reflects a higher invariance of the generalised Gribov equation is noted.

In the Faddeev-Popov formalism⁽²⁾, one confronts the gauge ambiguity that two gauge fields A_μ, A'_μ connected by the gauge transformation U both satisfy, e.g. Coulomb gauge. That is,

$$\nabla_i \left[U^\dagger A_i U - \frac{1}{g} U^\dagger \nabla_i U \right] = 0 \quad (1)$$

$$\nabla_i \left[U A'_i U^\dagger - \frac{1}{g} U \nabla_i U^\dagger \right] = 0 \quad (2)$$

This implies that the functional gauge-fixing delta function occurring in the path integral may be satisfied more than once, thus invalidating one of the usual steps⁽²⁾. Indeed the Jacobian in Faddeev-Popov formalism should be read

$$\begin{aligned} \Delta_F^{-1}(A) &\equiv \int \prod_x dg(x) \prod_x \delta(\nabla_i A_i^g) \\ &= \int \prod_x dg \prod_n \sum (\det 0)_{g_n}^{-1} \delta(g_n - g) \\ &= \sum_n \left[\det 0^{ab}(A^{g_n}) \right]^{-1}, \end{aligned} \quad (3)$$

$$0^{ab}(A g_n) = \nabla_i D_i^{ab}(A g_n) \quad (4)$$

where the Coulomb gauge condition is satisfied more than once at $g = g_n$ due to the Gribov ambiguities, $\nabla_i A_i^{g_n} = 0$, and $D_i^{ab}(A)$ is the covariant derivative.

In the canonical formalism, in expressing dependent degrees of freedom in terms of independent ones, one is asked to invert a certain differential operator. For Coulomb gauge, this operator is

$$0^{ab}(A) = \nabla_i D_i^{ab}(A) \quad (5)$$

Since this operator is precisely the Faddeev-Popov determinant Eq. (4), the connection is made at once. It will be shown below the gauge ambiguities manifest themselves as zero modes of the operator $0^{ab}(A)$.

Coulomb Gauge Vacuum. Consider Gribov's original example⁽¹⁾.

$$U = \exp(i \sigma_a \hat{x}_a \alpha(r)) \quad (6)$$

$$\nabla_i A_i = \nabla_i (U^\dagger \nabla_i U) = 0 \quad (7)$$

giving the pendulum equation, with $t = \ln r$

$$\ddot{\alpha} + \dot{\alpha} = \sin 2\alpha \quad (8)$$

The zero mode equation

$$\nabla_i D_i^{ab}(A) h^b = 0 \quad (9)$$

with ansatz $h^a = \hat{x}^a H(r)$ becomes (for general A_μ)

$$\nabla^2 H = \frac{2H}{r^2} (1-F) \quad (10)$$

$$F = \frac{g}{2} e_{aij} x_i A_j^a \quad (11)$$

In the present case, $F = +2 \sin^2 \alpha$ so that

$$\ddot{H} + \dot{H} = 2H \cos 2\alpha \quad (12)$$

Comparison of Eqs. (8) and (10) then gives the zero mode

$$H(r) = \dot{\alpha} = r \frac{d}{dr} \alpha(r) \quad (13)$$

$$h^a = \hat{x}^a \dot{\alpha} \quad (14)$$

From the solution of Eq. (9) we may construct an infinitesimal gauge transformation

$$\hat{U} = 1 + \epsilon h^a_a \quad (15)$$

which maintains the field A'_μ , a copy of the vacuum, in the Coulomb gauge condition and produces from A' a field A'' related to A' by a dilation. The zero modes corresponding to translation and rotation may be found also, as follows

$$h^a_i = e^{aij} \hat{x}^j \frac{\sin^2 \alpha}{r} + (\delta^{ai} - \hat{x}^a \hat{x}^i) \frac{\sin 2\alpha}{2r} + \hat{x}^a \hat{x}^i \frac{\partial \alpha}{\partial r} \quad (16)$$

$$h^a_i = e^{aij} \hat{x}^j \frac{\sin 2\alpha}{2r} - (\delta^{ai} - \hat{x}^a \hat{x}^i) \frac{\sin^2 \alpha}{r} \quad (17)$$

The above examples are all associated with the "collective" invariances of the vacuum - dilation, translation and rotation.

We may find the general form of such zero modes directly in terms of the transformation U , without any assumption about spherical symmetry.

In the Coulomb gauge, the dilation mode may be established as follows. We have

$$\delta_{D_i} A_i' = \epsilon (x_j \nabla_j + 1) A_i' = \epsilon D_i(A') (U^\dagger x_j \nabla_j U) \quad (18)$$

$$\nabla_i (\delta_{D_i} A_i') = \epsilon (x_j \nabla_j + 2) \nabla_i A_i' = \epsilon \nabla_i D_i(A') (U^\dagger x_j \nabla_j U) = 0 \quad (19)$$

where we have used $\nabla_i (U^\dagger U) = 0$. Thus the infinitesimally dilated gauge field $A_i'' = A_i' + \delta_{D_i} A_i'$ also satisfy the Coulomb gauge condition and

$$\nabla_i D_i(A') (x_j U^\dagger \nabla_j U) = 0 \quad (20)$$

gives the general dilation zero mode.

The translation zero mode is given by

$$\nabla_i D_i(A') (U^\dagger \nabla_j U) = 0 \quad (21)$$

corresponding to

$$\delta_{T_i} A_i' = \epsilon_j \nabla_j A_i' = \epsilon_j D_i(A') U^\dagger \nabla_j U \quad (22)$$

Similarly the rotation zero mode is

$$\nabla_i D_i(A') (e_{jkl} x_k U^\dagger \nabla_l U) = 0, \quad (23)$$

corresponding to

$$\delta_{R_i} A_i' = \epsilon_j e_{jkl} x_k \nabla_l A_i' + \epsilon_j e_{jil} A_l' = \epsilon_j D_i(A') (e_{jkl} x_k U^\dagger \nabla_l U). \quad (24)$$

There is no straightforward zero mode corresponding to the conformal transformation,

$$\begin{aligned}\delta_C A'_i &= \epsilon_j (2x_j x_k - x^2 \delta_{jk}) \nabla_k A'_i + 2(\epsilon_i x_j - \epsilon_j x_i) A'_j + 4\epsilon_j x_j A'_i \\ &= D_i(A') \left[\epsilon_j (2x_j x_k - x^2 \delta_{jk}) U^+ \nabla_k U \right] + 2\epsilon_j x_j A'_i\end{aligned}\quad (25)$$

since the straightforward conformal transformation

$D_i(A') \epsilon_j (2x_j x_k - x^2 \delta_{jk}) U^+ \nabla_k U$ does not commute with the Coulomb gauge condition.

Landau Gauge Vacuum. Consider the gauge transformation⁽⁴⁾

$$U = \cos g(R, \alpha) + i \sigma_a \hat{x}_a \sin g(R, \alpha) \quad (26)$$

$$\nabla_\mu A'_\mu = \nabla_\mu \left[U^+ \nabla_\mu U \right] = 0 \quad (27)$$

The zero mode equation

$$\nabla_\mu D_\mu^{ab}(A') h^b(x) = 0 \quad (28)$$

gives, with ansatz $h^a(x) = \hat{x}^a H(R)$, (for general A'_μ)

$$\square H = \frac{2}{r^2} (1-F) H \quad (29)$$

and hence for a vacuum copy with $F = 2 \sin^2 g$

$$\square H = \frac{2H \cos 2g}{r^2} \quad (30)$$

On the other hand, Eq. (27) gives⁽⁴⁾

$$\square g = \frac{\sin 2g}{r^2} \quad (31)$$

Comparison of Eq. (30) and (31) then exhibits the zero mode.

$$H(R) = \dot{g} = R \frac{\partial}{\partial R} g(R, \alpha) \quad (32)$$

$$h^a = \hat{x}^a \cdot \dot{g} \quad (33)$$

Again this is the generator for dilation.

In terms of the transformation, the dilation zero mode is given by

$$h = U^\dagger x_\mu \nabla_\mu U. \quad (34)$$

Similarly for other zero modes corresponding to translation and 4-dimensional rotation,

$$h_\mu = U^\dagger \nabla_\mu U \quad (35)$$

$$h_{\mu\nu} = e_{\mu\nu\rho\sigma} U^\dagger x_\rho \nabla_\sigma U \quad (36)$$

As in the Coulomb gauge, there is no straightforward zero mode corresponding to the conformal transformation in the Landau gauge,

$$\begin{aligned} \delta_C A'_\mu &= \epsilon_\alpha (2x_\alpha x_\beta - x^2 g_{\alpha\beta}) \nabla_\beta A'_\mu + 2(\epsilon_\mu x_\alpha - \epsilon_\alpha x_\mu) A'_\alpha + 6\epsilon_\alpha x_\alpha A'_\mu \\ &= D_\mu(A') \epsilon_\alpha (2x_\alpha x_\beta - x^2 g_{\alpha\beta}) U^\dagger \nabla_\beta U + 4\epsilon_\alpha x_\alpha A'_\mu. \end{aligned} \quad (37)$$

Classification of Field Size. The zero mode equation in Coulomb gauge, for a general field reads

$$-\nabla^2 H + \frac{2(1-F)}{r^2} H = 0 \quad (38)$$

with an ansatz $h^a = \hat{x}^a H$.

A zero mode, for this Schrodinger equation can exist only if the potential term is somewhere attractive. We are therefore led to classify a field as "small" if

$$F = \frac{g}{2} e_{aij} x_i A_j^a \leq 1 \text{ for all } x \quad (39)$$

If Eq. (39) is violated (for any x) the field is "large". We note the potential term in Eq. (38) for a copy field

$$1 - F_{\text{copy}} = (1-F) \cos 2\alpha$$

becomes attractive for $r > r_0$ where $\alpha(r_0) = \pi/4$. Namely Gribov ambiguities imply that a copy of a "small" field is a "large" field, the latter carrying zero modes.

Let us give a non-vacuum example in the Coulomb gauge. Consider the field

$$A_i^a = \frac{1}{g} e_{aij} \hat{x}_j \frac{F(r)}{r} \quad (40)$$

with $F(r) = r^2/(1+r^2)$. With the ansatz, Eq. (6), maintenance of the Coulomb gauge, requires

$$\nabla^2 \alpha = \frac{(1-F) \sin 2\alpha}{r^2} \quad (41)$$

with a solution $\alpha = \tan^{-1} r$. The zero mode equation, Eq. (38), becomes

$$\ddot{H} + \dot{H} = 2H(1-F) \cos 2\alpha \quad (42)$$

Putting $H = \dot{\alpha}\beta$ (defines β) one finds

$$F(1-F) \frac{d^2}{dF^2} \beta + \frac{1}{2} (5 - 8F) \frac{d\beta}{dF} - \beta = 0 \quad (43)$$

with a solution

$$\beta(r) = {}_2F_1\left(\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}; \frac{5}{2}; \frac{r^2}{(1+r^2)}\right) \quad (44)$$

Thus, the "large" copy has a non-singular zero mode as expected, while considering the zero mode equation for the original small field, Eq. (40),

$$\ddot{H} + \dot{H} = 2H(1-F) \quad (45)$$

leads only to singular solutions such as $H(r) = (1 + 1/r^2)$.

Of particular interest is that the zero mode corresponding to Eq. (44) is not associated with the collective invariances of dilation, rotation and translation. Instead it is the signature of a symmetry of the generalized Gribov equation, Eq. (41). Thus a given small field gives rise to some densely populated set of large field copies, related to each other through such a zero mode.

Classification of the field size with more general ansatz for zero modes will be discussed elsewhere.

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