AN ANALYTIC TREATMENT FOR A MODEL
OF SUPERCLUSTERING IN THE UNIVERSE

By

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AN ANALYTIC TREATMENT FOR A
MODEL OF SUPERCLUSTERING IN THE UNIVERSE

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Abstract

Recently, Weinberg, Ostriker, and Dekel (1989) (W.O.D.) have proposed a model of many spherical explosions which reproduces the large scale structure of the universe quite well. Because of the success of the model, we have developed an analytic treatment which gives the cluster-cluster correlation function and the cluster mass distribution in terms of the distribution of radii of the spherical voids. We also derive an expression for the distribution of radii in terms of the cluster mass distribution. Thus, the cluster-cluster correlation function is given in terms of the cluster mass distribution function. The numerical results of W.O.D. agree quite well with the predictions from the observed mass distribution. We give the cluster-cluster correlation function for several model distributions of radii. This formulation should be useful for future comparisons of observation and the model of W.O.D.

Section I: Introduction

It is a remarkable fact that the universe is inhomogeneous on a large scale at least up to 50 megaparsecs. This is manifested both by the presence of large voids in which galaxies are essentially absent, and by a correlation of the distance between clusters of galaxies. It seems likely that such a structure was produced sometime in the past by huge explosions. These explosions evacuated cosmological regions over such scales. Huge spherical shells of compressed matter were formed by the explosions, and this compressed matter cooled and formed into galaxies. Many such spherical shells were created and overlapping of them occurred. Arcs or points where the overlapping shells intersected would have a concentration of mass and hence gravitational forces that would draw in the galaxies in the shells. This would lead to a build up of clusters of galaxies at these loci.

The actual cluster producing the explosive energy that creates the spherical void is probably not critical to the model. It could be an explosion or alternatively an energetically vibrating superconducting cosmic string (Ostriker, Thompson, Witten 1986). The general inhomogeneity arising in the universe should be similar in either case.

Weinberg, Ostriker, and Dekel in a recent paper (1989) (referred to as W.O.D.) have constructed a simplified model based on these ideas to explain the statistics of clusters of galaxies. Their model consists of a large number of spherical shells of varying sizes whose centers are randomly disposed throughout the universe. The spherical shells are imagined to interpenetrate so that they are regarded as mathematical shells. In the W.O.D. model, the clusters are supposed to occur at the points of intersection of any three of these spherical shells. W.O.D. further give a physical argument that the mass of these clusters is proportional to the product of the

radii of the three shells. (Their argument actually involves the product of the surface densities of the three shells which is proportional to the product of the radii.) In fact, the mass they arrive at for the cluster is

$$M = R_1 R_2 R_3 \rho_c \alpha^2 [1 - (1 + Z_i)^{-1/2}]^2, \quad (1)$$

where the R_i^1 are the radii of the three spheres, ρ_c is the ratio of the baryonic mean density to the critical density ρ_c , α is the ratio of the spherical shell velocity to the Hubble velocity at the same radius, and Z_i is the initial red shift parameter at the time of the explosion. They treat α and Z_i as constants and simply assume $M = kR_1 R_2 R_3$.

On the basis of their model, W.O.D. numerically derive the mass function $\eta(M)$ for clusters of galaxies as well as the cluster-cluster correlation function $\xi_{cc}(R)$. $\eta(M)dM$ is the density of clusters whose mass is in dM . If there is a cluster at the origin, $\xi_{cc}(R)$ is the probability that there is a cluster in a unit volume a distance R away. They determine $\eta(M)$ and $\xi_{cc}(R)$ by a Monte Carlo calculation. Spheres are distributed randomly in a large box and their radii are selected randomly in accordance with an assumed distribution function for the radii of the spheres $n(R)$. By averaging these properties over an ensemble of such numerical realizations, they derive the mass function and correlation function for the clusters. Adjusting the distribution of radii $n(R)$, they find that a power law, $n(R) \propto R^{-q}$, $3.5 < q < 4.5$, approximately duplicates the observed mass distribution function of Schechter (Bahcall 1979)

$$\eta(M)dM = f^3 A \left(\frac{M}{M_*}\right)^{-\alpha} [\exp - (M/M_*)] M_*^{-1} R_*^{-3} dM. \quad (2)$$

Here $\alpha = -2$, f is the filling factor for the spherical shells, A is a dimensionless constant, and M_* and R_* are a reference mass and radius, respectively. We may take $M_* = kR_*^3$. They also determine the correlation function which they show is approximately an inverse power law

$$\xi_{cc} = \left(\frac{r}{r_0}\right)^{-\gamma} . \quad (3)$$

Remarkably, the observed distribution is also an inverse power law with $\gamma = 1.8$. Thus, the simple W.O.D. model duplicates an observational result. However, values for γ for various filling factors which they obtain are somewhat smaller than 1.8. Also the parameter r_0 in the correlation function can be compared to the mean intercluster spacing d . The results of their model for the ratio r_0/d are two or three times the observational value for this ratio. In other words their correlation function is too large.

Because of the partial success of their model, it should be taken seriously. It is the point of this note to show that the W.O.D. model can also be developed mathematically and most of the results of W.O.D. can be derived analytically. Perhaps the most important result of this paper is the relationship between $n(M)$ and ξ_{cc} : we show how ξ_{cc} can be derived from $n(M)$. This relationship which connects two observable quantities provides a sensitive test of the model. This mathematical development should make more feasible the further development of the model as well as confirmation of their present results. We first present the analytic results we have obtained for the model. The derivation of these results is given in the body of the paper.

Let the probability that a spherical shell has a radius R and has its center in the volume d^3r , be $n(R)dRd^3r$. Then we show in Sec. II that the mass function

$$\eta(M) = \frac{dN}{dM} = \frac{4\pi}{3k^3} M^2 F\left(\frac{M}{k}\right), \quad (4)$$

and

$$F(x) = \int_0^\infty dR_1 \frac{n(R_1)}{R_1} \int_0^\infty dR_2 \frac{n(R_2)}{R_2} n\left(\frac{x}{R_1 R_2}\right), \quad (5)$$

where $M = kR_1 R_2 R_3$, and R_1, R_2 and R_3 are the radii of the intersecting spherical shells. $\eta(M)dM$ is the probability per unit volume that there will be an intersection of "mass" M in dM .

Equations (4) and (5) enable us to express the mass function η in terms of the distribution function $n(R)$. It is possible to invert this relation by means of the Mellin transform. Let

$$\hat{F}(s) = \int_0^\infty dx x^{s-1} F(x) \quad (6a)$$

and

$$\hat{n}(s) = \int_0^\infty dR R^{s-1} n(R). \quad (6b)$$

In terms of these transforms

$$\hat{F}(s) = [\hat{n}(s)]^3. \quad (7)$$

Then $n(R)$ can be found from the inversion of the Mellin transform (Courant and Hilbert 1953)

$$n(R) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} ds R^{-s} \hat{n}(s) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} ds R^{-s} [\hat{F}(s)]^{1/3} \quad (8)$$

(where σ is chosen in the usual way). For example, from the Schechter mass function, Eq. (2), we find that

$$n(R) = \left(\frac{3A}{4\pi}\right)^{1/3} \frac{f}{R_*^4} \left[\left(\frac{R}{R_*}\right)^{-(\alpha+2)} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{R}{R_*}\right)^{-p} [\Gamma(p)]^{1/3} dp\right], \quad (9)$$

where $M_* = kR_*^3$. The function of R/R_* with square bracket is derived in the appendix and a plot of it is given in Fig. 1 for $\alpha=2, 2.5$, and 3. The best power-law fit of W.O.D. matches this fairly well. The total number of clusters per unit volume obtained by integrating n over M is

$$\int_0^\infty n dM = \frac{4\pi}{3} \left[\int_0^\infty n(R) R^2 dR \right]^3. \quad (10)$$

In Sec. III we derive the cluster-cluster correlation function ξ_{cc} for a general distribution $n(R)$. To present it concisely, let us introduce some notation. We first disregard the mass of the cluster and consider all clusters. Let the one-body distribution function $f_1(\underline{r})$ give the density of clusters at any point $\underline{r}$. $f_1 d^3\underline{r}$ is the probability that some cluster is present in $d^3\underline{r}$. Let the two-body distribution function $f_2(\underline{r}_1, \underline{r}_2)$ give the probability that there is a cluster at $\underline{r}_1$ and one at $\underline{r}_2$. $f_2(\underline{r}_1, \underline{r}_2) d^3\underline{r}_1 d^3\underline{r}_2$ is the probability that there is a cluster in $d^3\underline{r}_1$ and one in $d^3\underline{r}_2$. By the homogeneity and isotopy of the model, f_1 is independent of $\underline{r}$ and f_2 is a function only of $|\underline{r}_1 - \underline{r}_2| \equiv 2\mathbf{l}$. We also define

$$n_{\mathbf{l}} \equiv \int_{\mathbf{l}}^\infty n(R) dR, \quad (11)$$

so that $n_{\mathbf{l}}$ is the mean density of spheres whose radii are bigger than $\mathbf{l}$. Let us further set

$$a_l^2 \equiv \frac{1}{n_l} \int_l^\infty n(R) R^2 dR \quad (12)$$

so that a_l^2 is the mean square radius of spheres whose radius is bigger than l .

Finally, let $a^2 = a_l^2$ for $l = 0$ and $\bar{n} = n_l$ for $l = 0$ and let $\phi = (4\pi/3)a^3\bar{n}$ be an effective filling factor. (The true filling factor involves a mean cube average.)

Then we show in Sec. III that the general correlation function, for clusters $2l$ apart, which is defined by

$$\xi_{cc}(l) \equiv (f_2(l) - f_1^2)/f_1^2, \quad (13)$$

is

$$\begin{aligned} \xi_{cc}(l) = & -\frac{1}{3\pi^2\phi^3} \left(\frac{n_l}{n}\right)^3 \frac{(a_l^2 - l^2)^2}{la^3} \\ & + \frac{2}{\pi^2\phi^2} \left(\frac{n_l}{n}\right)^2 \frac{(a_l^2 - l^2)(3l^2 + a_l^2)}{l^2a^2} \\ & + \frac{3}{4\phi} \left(\frac{n_l}{n}\right) \frac{a_l^2}{al}. \end{aligned} \quad (14)$$

Starting with an observed mass (luminosity) distribution $n(M)$, we can derive the distribution of void sizes $n(R)$ using Eqs. (4), (6a), and (8). Given $n(R)$ we can then derive ξ_{cc} using Eqs. (11), (12), and (14). Clearly, one test of this model is the agreement between ξ_{cc} calculated this way and the observed ξ_{cc} . In Sec. IV we give some applications of these results parallel to those of W.O.D.

Section II: The Mass Function $n(M)$

Consider a large number of spheres randomly positioned in a large box. By sphere we mean only the two-dimensional surface of the sphere. Let the distribution of radii of the spheres be such that in the mean there are $n(R)dR$ spheres per unit volume whose radii are in dR . We seek the probability that in a small cell of volume d^3r about the origin there will be an intersection point of the three spheres.

We first calculate the probability that for three chosen radii R_A , R_B , and R_C the three corresponding spheres intersect at a point in d^3r . Let the intersection point $\underline{r} = (x, y, z)$. (By numbers in parentheses, we denote a vector with these coordinates.) The centers of the three spheres are constrained to lie on three spherical shells centered at $\underline{r}$, of radii R_A , R_B , and R_C , respectively. Thus, it takes nine coordinates to specify these centers.

Introduce spherical polar coordinates about the origin and let the coordinates of the centers of the spheres be r_A, θ_A, ϕ_A ; r_B, θ_B, ϕ_B ; and r_C, θ_C, ϕ_C , respectively. Then for fixed angular coordinates one can determine the radial coordinates in terms of $\underline{r}$. Thus, we treat the coordinates x, y, z ; θ_A, ϕ_A ; θ_B, ϕ_B ; θ_C, ϕ_C as independent coordinates. The probability for the spheres to be in the nine-dimensional volume is

$$\begin{aligned}
 &= \frac{1}{\sqrt{3}} r_A^2 r_B^2 r_C^2 d\Omega_A d\Omega_B d\Omega_C dr_A dr_B dr_C \\
 &= \frac{1}{\sqrt{3}} r_A^2 r_B^2 r_C^2 d\Omega_A d\Omega_B d\Omega_C d^3r J, \quad (15)
 \end{aligned}$$

where

$$J = \frac{\partial(r_A, r_B, r_C)}{\partial(x, y, z)} \quad (16)$$

is the Jacobian of the radial coordinates with respect to the coordinates of the intersection point. $d\Omega$ is, of course, the element of solid angle and V the volume of the box.

Now $\nabla r_A = \hat{r} \cdot \hat{r}_A$ where $\hat{r}_A$ is the unit vector along r_A and similarly for B and C. Thus

$$J = \begin{vmatrix} \hat{x} \cdot \hat{r}_A & \hat{y} \cdot \hat{r}_A & \hat{z} \cdot \hat{r}_A \\ \hat{x} \cdot \hat{r}_B & \hat{y} \cdot \hat{r}_B & \hat{z} \cdot \hat{r}_B \\ \hat{x} \cdot \hat{r}_C & \hat{y} \cdot \hat{r}_C & \hat{z} \cdot \hat{r}_C \end{vmatrix} = \hat{r}_A \cdot \hat{r}_B \times \hat{r}_C \quad (17)$$

Integrating the expression in Eq. (15) over the solid angles, we get the probability Pd^3r , that the three spheres intersect in d^3r where

$$\begin{aligned} P &= \frac{R_A^2 R_B^2 R_C^2}{V^3} \int (\hat{r}_A \cdot \hat{r}_B \times \hat{r}_C) d\Omega_A d\Omega_B d\Omega_C \\ &= \frac{8\pi^4 R_A^2 R_B^2 R_C^2}{V^3} \end{aligned} \quad (18)$$

(Since r is small, $r_A = R_A, r_B = R_B$, and $r_C = R_C$.)

Now for the general problem with arbitrary distribution of radii, the probability that three spheres will have radii in dR_A, dR_B , and dR_C and further will intersect in d^3r is Pd^3r where

$$\begin{aligned} P &= 8\pi^4 n(R_A) n(R_B) n(R_C) R_A^2 R_B^2 R_C^2 dR_A dR_B dR_C \\ &= P' dR_A dR_B dR_C \end{aligned} \quad (19)$$

Let the intersection result in a cluster of galaxies of mass $M = kR_1R_2R_3$, where k is the constant in Eq. (1). The probability of a cluster of galaxies with mass $M < M_0$ in a unit volume is obtained by integrating P in Eq. (19) over the region $R_A R_B R_C < M_0/k$. The result is

$$\int_0^{M_0} n(M) dM = \frac{1}{6} \int_0^\infty dR_A \int_0^\infty dR_B \int_0^{\frac{M_0}{kR_A R_B}} dR_C P'$$

$$= \frac{8\pi^4}{6} \int_0^\infty R_A^2 dR_A n(R_A) \int_0^\infty n(R_B) R_B^2 dR_B \int_0^{\frac{M_0}{kR_A R_B}} n(R_C) R_C^2 dR_C \quad (20)$$

The factor of $1/6$ arises because in integrating over $R_A R_B R_C$ space the same situation is counted six times. Differentiating this equation with respect to $M_0 \equiv kR_0^3$ yields

$$n(M_0) = \frac{4\pi^4}{3} \frac{M_0^2}{k^3} \int_0^\infty n(R_A) \frac{dR_A}{R_A} \int_0^\infty n(R_B) \frac{dR_B}{R_B} n\left(\frac{R_0^3}{R_A R_B}\right) \quad (21)$$

an expression for the mass function identical with Eq. (4).

Multiply Eq. (5) by x^{s-1} and integrate from 0 to ∞ . There results

$$\hat{F}(s) = \int F(x) x^{s-1} dx$$

$$= \int n(R_A) R_A^{s-1} dR_A \int n(R_B) R_B^{s-1} dR_B \int n\left(\frac{x}{R_A R_B}\right) \left(\frac{x}{R_A R_B}\right)^{s-1} d\left(\frac{x}{R_A R_B}\right)$$

$$= [\hat{n}(s)]^3 \quad (21a)$$

or alternatively,

$$\hat{n}(s) = [\hat{F}(s)]^{1/3} . \quad (22)$$

Relation (21) enables us to determine the mass function $n(M)$ in terms of the distribution of radii of the spheres in the model $n(R)$. Equation (22) enables us to invert this relation and to find $n(R)$ in terms of $n(M)$. This also shows that $n(R)$ is uniquely determined - only one model can represent any observed mass function. Furthermore, once $n(R)$ is known, we show below that ξ_{cc} can be calculated and compared with observation.

As an example, consider the Schechter mass function Eq. (2). From Eq. (4) we have

$$F\left(\frac{M}{k}\right) = C \left(\frac{M}{M_*}\right)^{-\alpha-2} \exp\left(-\frac{M}{M_*}\right) ,$$

where $C = \frac{3}{4\pi} \frac{r^3}{X_*^3} \frac{A}{R_*^3}$ and $M_* = kX_* = kR_*^3$. Let $M = kX$. Then we may write

$$F(X) = C \left(\frac{X}{X_*}\right)^{-\alpha-2} \exp\left(-\frac{X}{X_*}\right) . \quad (23)$$

Taking the Mellin transform, we find

$$\begin{aligned} \hat{F}(s) &= CX_*^s \int_0^\infty dy y^{s-\alpha-3} \exp(-y) \\ &= CX_*^s \Gamma(s-\alpha-2) \end{aligned} \quad (24)$$

$$\hat{n}(s) = C^{1/3} X_*^{s/3} [\Gamma(s-\alpha-2)]^{1/3} . \quad (25)$$

Formula (8) for the inversion of n leads to

$$n(R) = \frac{C^{1/3}}{2\pi i} \left(\frac{R}{R_*}\right)^{-(\alpha+2)} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{R}{R_*}\right)^{-p} [\Gamma(p)]^{1/3} dp, \quad (26)$$

where $\sigma > 0$.

In the appendix it is shown that

$$n(R) \approx \frac{\sqrt{3}}{2\pi} C^{1/3} \Gamma\left(\frac{2}{3}\right) \frac{(R/R_*)^{-(\alpha+2)}}{|\ln(R/R_*)|^{2/3}}, \quad R \ll R_* \quad (27)$$

and

$$n(R) \approx \frac{\sqrt{3} C^{1/3}}{(2\pi)^{1/3}} \left(\frac{R}{R_*}\right)^{-(\alpha+1)} \exp(-R^3/3), \quad R \gg R_* \quad (28)$$

For $\alpha = 2$, W.O.D. find empirically that $n(R) \approx R^{-8}$ with two cutoffs duplicates the Schechter function if $+3.5 < s < 4.5$. We see that Eqs. (27) and (28) give about the same power but bring in the upper cutoff naturally. We evaluate $n/C^{1/3}$ in Eq. (26) numerically and plot it in Fig. 1.

If we integrate (19) over all radii R_A , R_B , and R_C without restrictions, we obtain the total density of clusters

$$\begin{aligned} f_1 = \bar{N} &= \int_0^\infty n(M) dM = \frac{4}{3} \pi^4 \left[\int_0^\infty n(R) R^2 dR \right]^3 \\ &= \frac{4}{3} \pi^4 \bar{n}^3 a^6 = \frac{9\pi}{16} \phi^3 a^{-3} \end{aligned} \quad (29)$$

in the notation of the introduction. Equation (29) is also identical to Eq. (10) and represents the one-body distribution for clusters, f_1 .

Section III: The Cluster-Cluster Correlation Function

Subsection III - 1

The Four Cases.

Consider as before a large number N of spheres randomly positioned in a large volume V and labeled 1 to N . (Again by a sphere we mean the two-dimensional surface of a solid sphere.) For the moment we leave their radii arbitrary. Any intersection point P must arise from three specific spheres with indices $i_1 < i_2 < i_3$. A second intersection point Q is associated with a further three spheres $i_4 < i_5 < i_6$ not necessarily distinct from the first three spheres. We seek the probability $f_2 d^3r_1 d^3r_2$ that P is in a box d^3r_1 at r_1 and Q is in a box d^3r_2 at r_2 .

One can distinguish four disjoint cases: For case I the three spheres i_1, i_2, i_3 are identical to the three spheres i_4, i_5, i_6 so that $i_1 = i_4$, $i_2 = i_5$, and $i_3 = i_6$. The contribution to the two-body distribution function f_2 from this case we denote as f_2^I .

For Case II only two of the spheres in the two triplets are identical and the third ones are distinct, e.g., $i_1 = i_4$, $i_2 = i_5$, $i_3 \neq i_6$. The contribution to f_2 for this case is f_2^{II} .

For Case III only one sphere is common to the two triplets and the other two spheres in each triplet are distinct, e.g., $i_1 = i_4$, $i_2 \neq i_5$, $i_3 \neq i_6$. The contribution to f_2 for this case is f_2^{III} . For Case IV all spheres are distinct and the contribution to f_2 from this case is $f_2^{IV} = f_1^2$, where f_1 is the one-body distribution function.

These cases are illustrated in Figs. 2a-2d. It should be noted that any three spheres, in general, intersect in two points or none. In case I these two points are the correlated points and the two clusters at these points have the same mass. These points are called intrinsically correlated. However, in

cases II-IV only one intersection point for each of the two triplets is considered, the others are irrelevant.

In subsections III-2 to III-4 the separate contributions to f_2 ; f_2^I , f_2^{II} , f_2^{III} are evaluated. Their sum is

$$f_2 = f_2^I + f_2^{II} + f_2^{III} + f_2^{IV} = f_1^2 (1 + \xi_{cc}) \quad , \quad (30)$$

where ξ_{cc} is the two-body correlation function. Thus, $\xi(|r_2 - r_1|)$ can be derived. In this derivation we denote the intersection points by r_1 and r_2 while the location of the centers of the intersecting spheres will be r_A , r_B , r_C , etc. The distance between the centers of the two boxes $d^2_{r_1}$, and $d^3_{r_2}$ will be 2λ .

Subsection III - 2

Case I - Three Common Spheres.

Consider three spheres, A,B,C, of radii R_A , R_B , and R_C and let us calculate the probability that they intersect at two points P and Q in $d^3_{r_1}$, and $d^3_{r_2}$. By homogeneity and isotropy, we may take the centers of the boxes $d^3_{r_1}$ and $d^3_{r_2}$ at cartesian coordinates $(0,0,-\lambda)$ and $(0,0, \lambda)$. Introduce cylindrical coordinates r, θ, z about the z axis and let r_A , θ_A , z_A be the coordinates of the center of sphere A. Similarly, for spheres B and C. Let x_1, y_1, z_1 be cartesian coordinates of P and x_2, y_2, z_2 be cartesian coordinates of Q.

If the three spheres each pass through P and Q, they must satisfy the six conditions

$$|\underline{r}_A - \underline{r}_1| = |\underline{r}_A - \underline{r}_2| = R_A$$

$$|\underline{r}_B - \underline{r}_1| = |\underline{r}_B - \underline{r}_2| = R_B \quad (31)$$

$$|\underline{r}_C - \underline{r}_1| = |\underline{r}_C - \underline{r}_2| = R_C .$$

These six conditions do not uniquely determine $\underline{r}_A, \underline{r}_B$, and $\underline{r}_C$ uniquely in terms of $\underline{r}_1$ and $\underline{r}_2$. They do, however, restrict $\underline{r}_A, \underline{r}_B$, and $\underline{r}_C$ to lie on circles centered about $(\underline{r}_2 + \underline{r}_1)/2$ in a plane perpendicular to the vector $\underline{r}_2 - \underline{r}_1$. Therefore, if we specify θ_A, θ_B , and θ_C , then the other six cylindrical coordinates of $\underline{r}_A, \underline{r}_B$, and $\underline{r}_C$ are determined.

The probability that $\underline{r}_A$ is in the volume element $dV_A = r_A dr_A d\theta_A dz_A$ is dV_A/V . The probability that both R_A is in dR_A and the center of A is in dV_A is $n(R_A) dR_A dV_A$. The probability simultaneously for A, B, and C, that R_A, R_B , and R_C are in dR_A, dR_B , and dR_C , respectively, while their centers are in dV_A, dV_B, dV_C , respectively, is

$$dP = n(R_A)n(R_B)n(R_C) dR_A dR_B dR_C r_A dr_A d\theta_A dz_A \times r_B dr_B d\theta_B dz_B \times r_C dr_C d\theta_C dz_C . \quad (32)$$

To calculate the probability $\int_2^I d^3 \underline{r}_1 d^3 \underline{r}_2$, we must first express the element $dr_A dz_A dr_B dz_B dr_C dz_C$ in terms of $d^3 \underline{r}_1, d^3 \underline{r}_2$ by a Jacobian transformation.

$$dr_A dz_A dr_B dz_B dr_C dz_C = |J| dx_1 dy_1 dz_1 dx_2 dy_2 dz_2 , \quad (33)$$

where J is the Jacobian determinant

$$J = \frac{\partial(r_A, z_A, r_B, z_B, r_C, z_C)}{\partial(x_1, y_1, z_1, x_2, y_2, z_2)} \quad (34)$$

The elements of the determinant are derived by differentiating the relations Eq. (31) with respect to $x_1, \dots, z_2$ at fixed θ 's.

However, the determinant is more easily evaluated in terms of the sum and difference variables

$$\begin{aligned} X &= \frac{x_1 + x_2}{2}, & x &= \frac{x_2 - x_1}{2} \\ Y &= \frac{y_1 + y_2}{2}, & y &= \frac{y_2 - y_1}{2} \\ Z &= \frac{z_1 + z_2}{2}, & z &= \frac{z_2 - z_1}{2} \end{aligned} \quad (35)$$

and $J \approx 1/8 J'$, where

$$J' = \frac{\partial(r_A, \dots, z_C)}{\partial(X, x, Y, y, Z, z)} \quad (36)$$

$$= \begin{vmatrix} & x & y & z & X & Y & z \\ z_A & -r_A/l \cos \theta_A & -r_A/l \sin \theta_A & 1 & 0 & 0 & 0 \\ z_B & -r_B/l \cos \theta_B & -r_B/l \sin \theta_B & 1 & 0 & 0 & 0 \\ z_C & -r_C/l \cos \theta_C & -r_C/l \sin \theta_C & 1 & 0 & 0 & 0 \\ r_A & 0 & 0 & 0 & \cos \theta_A & \sin \theta_A & -l/r_A \\ r_B & 0 & 0 & 0 & \cos \theta_B & \sin \theta_B & -l/r_B \\ r_C & 0 & 0 & 0 & \cos \theta_C & \sin \theta_C & -l/r_C \end{vmatrix}$$

where J' is to be evaluated at the centers of the boxes $d_{r_1}^3$ and $d_{r_2}^3$. Hence,

$r_A^2 = R_A^2 - x^2$, etc. The symbols outside the determinant indicate the variables corresponding to the different rows and columns so, for example, the expression in the first column and second row is $\partial z_B / \partial x$. The choice of the sum and difference variables leads to 0's in the first three rows of the last three columns as well as the first three columns of the last three rows. This leads us to the product of two three-by-three determinants for the Jacobian, which is thus tractable. Let

$$\bar{p} = n(R_A) n(R_B) n(R_C) r_A r_B r_C \frac{J'}{8}.$$

We now obtain the probability $f_2 d^3 r_1 d^3 r_2$ by integrating $\bar{p}$ over $R_A, R_B, R_C, \theta_A, \theta_B$, and θ_C .

$$f_2^I = \frac{1}{6} \int \bar{p} dR_A dR_B dR_C d\theta_A d\theta_B d\theta_C \quad (37)$$

$$= \frac{1}{6} \int_{R_A, R_B, R_C > x} dR_A dR_B dR_C n(R_A) n(R_B) n(R_C) r_A r_B r_C \times \int d\theta_A d\theta_B d\theta_C \frac{J}{8}$$

The factor 1/6 arises because integration over all $R_A R_B R_C$ greater than x counts every triplet of spheres six times.

Now from Eq. (36)

$$|J'| = \frac{r_A r_B r_C}{x^3} \begin{vmatrix} \frac{x}{r_A} & \cos \theta_A & \sin \theta_A \\ \frac{x}{r_B} & \cos \theta_B & \sin \theta_B \\ \frac{x}{r_C} & \cos \theta_C & \sin \theta_C \end{vmatrix}^2 \quad (38)$$

so that a simple calculation shows that

$$\int_0^{2\pi} d\theta_A \int_0^{2\pi} d\theta_B \int_0^{2\pi} d\theta_C |J'| = \frac{r_A r_B r_C}{l^3} \int \int \int \left\{ \frac{1}{r_A} \sin(\theta_C - \theta_B) + \frac{1}{r_B} \sin(\theta_A - \theta_C) + \frac{1}{r_C} \sin(\theta_B - \theta_A) \right\}^2 d\theta_A d\theta_B d\theta_C = 4\pi^3 \frac{r_A r_B r_C}{l} \left(\frac{1}{r_A^2} + \frac{1}{r_B^2} + \frac{1}{r_C^2} \right) \quad (39)$$

Substituting this result in Eq. (37), we obtain the desired result

$$\begin{aligned} f_2^I &= \frac{\pi^3}{4l} \int_l^\infty n(r_A) (R_A^2 - l^2) dR_A \int_l^\infty n(R_B) (R_B^2 - l^2) dR_B \int_l^\infty n(R_C) dR_C \\ &= \frac{\pi^3}{4l} \left[\int_l^\infty n(R) (R^2 - l^2) dR \right]^2 \int_l^\infty n(R) dR \\ &= \frac{\pi^3}{4} \frac{n_l^3}{l} (a_l^2 - l^2)^2 \quad (40) \end{aligned}$$

Subsection III - 3

Case II - Two Common Spheres.

The logic in this section is parallel to the previous subsection. We now consider four spheres A, B, D, and E of radii R_A, R_B, R_D , and R_E and centers r_A, r_B, r_D , and r_E . Spheres A, B, and D intersect in point P while spheres A, B, and E intersect in point Q. Again we seek the probability $f_2^{II} d^3 r_1 d^3 r_2$ that P is in $d^3 r_1$ and Q in $d^3 r_2$. We describe r_A and r_B by cylindrical coordinates r_A, θ_A, z_A and r_B, θ_B, z_B as before. However, we introduce spherical polar coordinates r_D, θ_D, ϕ_D centered at $(0,0,-l)$ to describe r_D and similarly spherical polar coordinates r_E, θ_E, ϕ_E centered at $(0,0,l)$ to describe r_E .

r_A and r_B are constrained by Eq. (31), while r_D and r_E are constrained by the conditions

$$|\underline{r}_D - \underline{r}_1| = R_D \quad (41)$$

$$|\underline{r}_E - \underline{r}_2| = R_E$$

Now making use of the six conditions Eq. (31) and Eq. (41) we see that, for fixed angular coordinates, the six coordinates r_A, z_A, r_B, z_B, r_D and r_E are determined by $\underline{r}_1$ and $\underline{r}_2$.

Then arguing as before, we can easily see that

$$\begin{aligned} f_2^{II} = & \frac{1}{2} \int_{\substack{R_A > l \\ R_B > l}} n(R_A) n(R_B) n(R_D) n(R_E) dR_A dr_B dR_D dR_E \\ & \times r_A r_B R_D^2 R_E^2 \int d\theta_A d\theta_B d\theta_D d\phi_D d\theta_E d\phi_E \sin\theta_D \sin\theta_E \frac{1}{8} |J_2'|, \end{aligned} \quad (42)$$

where the factor 1/2 arises by symmetry between A and B and

$$J_2' = \frac{\partial(r_A, z_A, r_B, z_B, r_D, r_E)}{\partial(x, y, z, X, Y, Z)}. \quad (43)$$

The components of the Jacobian determinant are found by varying Eq. (35) and Eq. (41) at fixed angular coordinates. For example,

$$\frac{\partial \underline{r}_D}{\partial \underline{R}} = - \frac{\partial \underline{r}_D}{\partial \underline{r}} = \hat{\underline{r}}_D = (\cos\theta_D, \sin\theta_D \cos\phi_D, \sin\theta_D \sin\phi_D), \quad (44)$$

where $\hat{\underline{r}}_D$ is a unit vector in the $\underline{r}_D - \underline{r}_1$ direction and where $\underline{R}$ and $\underline{r}$ are the sum and difference vectors.

We thus obtain for J_2'

	z	x	y	z	x	y
r_A	$-l/r_A$	$\cos\theta_A$	$\sin\theta_A$	0	0	0
r_B	$-l/r_B$	$\cos\theta_B$	$\sin\theta_B$	0	0	0
r_D	$-\cos\theta_D$	$\sin\theta_D^x$ $\cos\phi_D$	$\sin\theta_D^x$ $\sin\phi_D$	$\cos\theta_D$	$-\sin\theta_D^x$ $\cos\phi_D$	$-\sin\theta_D^x$ $\sin\phi_D$
z_A	0	0	0	1	$-r_A/l\cos\theta_A$	$-r_A/l\sin\theta_A$
z_B	0	0	0	1	$-r_B/l\cos\theta_B$	$-r_B/l\sin\theta_B$
z_E	$\cos\theta_E$	$\sin\theta_E^x$ $\cos\phi_E$	$\sin\theta_E^x$ $\sin\phi_E$	$\cos\theta_E$	$\sin\theta_E^x$ $\cos\phi_E$	$\sin\theta_E^x$ $\sin\phi_E$

(45)

Unfortunately, J_2' does not factor into two three-by-three determinants as it stands since neither the upper right corner three-by-three minor nor the lower left corner minor are zero. However, it can be brought into such a form by first adding the first, second, and third columns to the fourth, fifth, and sixth columns, respectively. Next multiplying the fourth row by l/r_A and adding it to the first row and then multiplying the fifth row by l/r_B and adding to the second row, brings the determinant to a form with zeros in the upper right-hand corner. It can thus be evaluated as a product of two three-by-three determinants.

Define a vector

$$\underline{a} = -l\left(\frac{\sin\theta_A}{r_B} - \frac{\sin\theta_B}{r_A}\right), l\left(\frac{\cos\theta_A}{r_B} - \frac{\cos\theta_B}{r_A}\right), \sin\theta_A\cos\theta_B - \sin\theta_B\cos\theta_A. \quad (46)$$

Then the two three-by-three determinants can be written as $\underline{a} \cdot \hat{r}_D$ and $(2l^2/r_A r_B) \underline{a}^* \cdot \hat{r}_E$, respectively, $\hat{r}_D$ and $\hat{r}_E$ are unit vectors in the radial direction of the two spherical polar systems, while $\underline{a}^*$ is equal to the vector $\underline{a}$ with its z component reversed in sign. Consequently,

$$|J_2'| = \frac{2r_A r_B}{l^2} |\hat{a} \cdot \hat{r}_D| |\hat{a} \cdot \hat{r}_E| \quad (47)$$

Now,

$$\int |\hat{a} \cdot \hat{r}_D| \sin \theta_D d\theta_D d\phi_D = 2\pi |\hat{a}| \quad (48)$$

$$\int |\hat{a} \cdot \hat{r}_E| \sin \theta_E d\theta_E d\phi_E = 2\pi |\hat{a}| = 2\pi |\hat{a}|$$

while from Eq. (46)

$$\int d\theta_A d\theta_B a^2 = (2\pi)^2 \left(\frac{l^2}{r_B^2} + \frac{l^2}{r_A^2} + \frac{1}{2} \right) \quad (49)$$

Making note of the integrals Eq. (47) - (49) we may evaluate the angular integrals in Eq. (42). The result is

$$\begin{aligned} f_2^{II} &= \frac{2\pi^4}{l^2} \int_{R_A, R_B > l} n(R_A) n(R_B) n(R_D) n(R_E) R_D^2 R_E^2 (2l^2 r_A^2 + 2l^2 r_B^2 + r_A^2 r_B^2) dR_A dR_B dR_D dR_E \\ &= \frac{2\pi^2}{l^2} n^2 n_l^2 a^4 [4(a_l^2 - l^2) l^2 + (a_l^2 - l^2)^2] \\ &= \frac{2\pi^4}{l^2} n^2 n_l^2 a^4 (a_l^2 - l^2) (3l^2 + a_l^2) \end{aligned}$$

We have again made use of the results $r_A^2 = R_A^2 - l^2$, $r_B^2 = R_B^2 - l^2$.

Subsection III-4

Case III - One Common Sphere

We next consider the case where only one sphere A is common to the points

P and Q. Let the other two spheres through P be D and F and the other two spheres through Q be E and G. Again, as in the previous subsection, we introduce two spherical polar coordinate systems for D,F, and E,G, respectively. We continue to describe A by cylindrical coordinates.

Now holding all angular coordinates fixed, the two cylindrical coordinates for A and the four radial coordinates for D,E,F, and G are determined by Eqs. (31), (41), and

$$|r_E - r_1| = R_F \quad (51)$$

$$|r_G - r_2| = R_G$$

The contribution to f_2 from this case is

$$\begin{aligned} f_2^{III} &= \frac{1}{4} \int dR_A dR_D dR_E dR_F dR_G n(R_A) n(R_D) n(R_E) n(R_F) n(R_G) \\ &\times r_A^2 R_D^2 R_E^2 R_F^2 R_G^2 \int d\theta_A d\Omega_D d\Omega_E d\Omega_F d\Omega_G \frac{1}{8} |J_3'|, \end{aligned} \quad (52)$$

where the factor $1/4$ comes from the symmetry between D and F and the symmetry between E and G.

J_3' denotes the Jacobian

$$J_3' = \frac{\partial(r_A, z_A, r_D, r_E, r_F, r_G)}{\partial(x, y, z, X, Y, Z)} \quad (53)$$

It can easily be written down in analogy with Eq. (45). However, it is also not immediately factorable in terms of two three-by-three determinants. But again, it can be brought into this form by manipulating the rows and

columns. In fact, we choose to employ the same order of columns as in Eq. (45) and choose the analogous order of rows (i.e., r_A , r_D , r_{LF} , z_A , r_E , r_g). Then we add the first three columns to the last three columns and finally add R/r_A times the fourth row to the first row. The result is

$$J' = \begin{pmatrix} r_A \\ z \\ z \end{pmatrix} \begin{vmatrix} -1/r_A & , & \cos\theta_A & , & \sin\theta_A & , & 0 & , & 0 & , & 0 \\ -\cos\theta_D & , & \sin\theta_D^* & , & \sin\theta_D^* & , & 0 & , & 0 & , & 0 \\ & & \cos\theta_D & & \sin\theta_D & & & & & & \\ -\cos\theta_F & , & \sin\theta_F^* & , & \sin\theta_F^* & , & 0 & , & 0 & , & 0 \\ & & \sin\theta_D & & \sin\theta_D & & & & & & \\ * & , & * & , & * & , & -1/r_A & , & \cos\theta_A & , & \sin\theta_A \\ * & , & * & , & * & , & 2\cos\theta_E & , & 2\sin\theta_E^* & , & 2\sin\theta_E^* \\ & & & & & & \cos\theta_E & & \sin\theta_E & & \\ * & , & * & , & * & , & 2\cos\theta_G & , & 2\sin\theta_G^* & , & 2\sin\theta_G^* \\ & & & & & & \cos\theta_G & & \sin\theta_G & & \end{vmatrix}$$

$$= 4 \frac{r_A}{z} (\hat{b} \cdot \hat{r}_D^* \times \hat{r}_F^*) (\hat{b} \cdot \hat{r}_E \times \hat{r}_G) , \quad (54)$$

where $\hat{b}$ is the vector

$$\hat{b} = \left(\frac{-1}{r_A}, \cos\theta_A, \sin\theta_A \right) \quad (55)$$

and $\hat{r}_B, \hat{r}_E, \hat{r}_F$, and $\hat{r}_G$ are the radial unit vectors in the spherical coordinate systems. $\hat{r}_D^*$ and $\hat{r}_F^*$ are $\hat{r}_D$ and $\hat{r}_F$ rotated by 180° about the polar axes.

The integration over the spherical solid angles can be carried out using

$$\int |\hat{b} \cdot (\hat{r}_E \times \hat{r}_G)| d\Omega_G = \int |(\hat{b} \times \hat{r}_E) \cdot \hat{r}_G| d\Omega_G = 2\pi |\hat{b} \times \hat{r}_E|$$

$$\int |\hat{b} \cdot (\hat{r}_D^* \times \hat{r}_F^*)| d\Omega_F = 2\pi |\hat{b} \times \hat{r}_D^*|$$

$$2\pi \int |(\underline{b} \times \hat{\underline{r}}_E)| d\Omega_E = 2\pi^3 |\underline{b}| \quad (56)$$

$$2\pi \int |\underline{b} \times \hat{\underline{r}}_D^*| d\Omega_D = 2\pi^3 |\underline{b}|$$

so that

$$\int |J_3'| d\Omega_D d\Omega_E d\Omega_F d\Omega_G = 16\pi^6 \frac{r_A}{2} |\underline{b}|^2 \quad (57)$$

Further, we have from Eq. (55)

$$\int b^2 d\theta_A = 2\pi \left(\frac{r_A^2}{2} + 1 \right) \quad (58)$$

so combining Eqs. (52), (54), (56), (57), and (58) we obtain

$$f_2^{III} = \frac{\pi^7}{2} \int_{R_A > 1} dR_A dR_D dR_E dR_F dR_G n(R_A) n(R_D) n(R_E) n(R_F) n(R_G) r_A^2 R_D^2 R_E^2 R_F^2 R_G^2 \left(\frac{r_A^2}{2} + 1 \right) = \frac{\pi^7 n_A^4 n_D^8 a_A^2}{2} \quad (59)$$

Subsection III - 5

Case IV No Common Spheres.

The case where the spheres through P and G are independent is clearly the same as the situation of Section II. Thus we have

$$f_2^{IV} = f_1^2 \quad (60)$$

Subsection III-6

Summary The Cluster-Cluster Correlation Function ξ_{cc}

The cluster-cluster function ξ_{cc} is defined by Eq. (13) where $z = 1/2|r_2 - r_1|$ is the half distance between the two points at which the correlation is taken. The two-body function is the sum of four terms f_2^I , f_2^{II} , f_2^{III} , and $f_2^{IV} = f_1^2$ corresponding to our four cases. The f_1^2 terms cancel in Eq. (13). f_2^I is given by Eq. (40), f_2^{II} is given by Eq. (50), and f_2^{III} is given by Eq. (59). From these results we have

$$\begin{aligned} f_1^2 \xi_{cc}(z) = & \frac{\pi^3}{4} \frac{n_z^3}{z} (a_z^2 - z^2)^2 \\ & + \frac{2\pi^4}{z^2} n^2 n_z^2 a_z^4 (a_z^2 - z^2)(3z^2 + a_z^2) + \frac{\pi^7}{z} n^4 n_z^4 a_z^8 a_z^2. \end{aligned} \quad (61)$$

We could express $\xi_{cc}(z)$ in terms of the filling factor ϕ_{true} , the total fraction of volume V occupied by all the solid spheres. To this purpose we need the mean cube radius so

$$\phi_{true} = \frac{4\pi}{3} \int_0^\infty R^3 n(R) dR. \quad (62)$$

However, an effective filling factor involving the mean square radius

$$\phi = \frac{4\pi}{3} n a^3 \quad (63)$$

is a more convenient quantity and in terms of ϕ one can eliminate n and write the cluster correlation in terms of ϕ . From Eq. (29), (61), and (63) we get the form for ξ_{cc} given in the introduction.

These expressions are, of course, the correlation function for all clusters of galaxies independent of mass. A more observationally relevant quantity would be the correlation function for all clusters larger than a given mass M since small clusters probably escape detection. For this restricted correlation function one must restrict the three integrals contributing to ξ by the condition $R_1 R_2 R_3 > M/k$ where R_1 , R_2 , and R_3 are the relevant radius of the spheres contributing to each point. That is, in Eq. (40) for f_2^I we demand $R_A R_B R_C > M/k$; in Eq. (50) for f_2^{II} we demand both $R_A R_B R_D$ and $R_A R_B R_E > M/k$ and in Eq. (59) for f_2^{III} we demand $R_A R_D R_F$ and $R_A R_E R_G > M/k$. The general result is quite complicated and we do not give it here. For any specific $n(R)$ it may be worked out.

Section IV: Applications

In this section we apply our results to several special cases. Once $n(R)$ is known it is possible in principle to evaluate both the mass function $n(M)$ and the cluster-cluster correlation function. We carry this out for the special cases of (1) spheres with all the same radius and (2) a power law distribution of spherical radii.

(1) For this case of equal spherical radii

$$n = \bar{n} \delta(r-a) \quad . \quad (64)$$

This leads to a delta function distribution for the mass function. From Eqs. (4) and (5)

$$\eta(M) = \frac{4\pi}{3} \frac{M^2}{k^2} \bar{n}^3 \delta(M-ka^3) \quad . \quad (65)$$

From Eq. (11) and (12) we have

$$\begin{aligned} n_l &= \bar{n} & l < a \\ n_l &= 0 & l > a \\ a_l &= a & l < a \\ a_l &= 0 & l > 0 \end{aligned} \quad (66)$$

so from Eq. (14)

$$\begin{aligned} \xi(l) &= \frac{1}{3\pi^2\phi^3} \frac{(a^2 - l^2)^2}{la^3} + \frac{2}{\pi^2\phi^2} \frac{(a^2 - l^2)(3l^2 + a^2)}{l^2a^2} \\ &\quad + \frac{3}{4\phi} \frac{a}{l} \quad l < a \\ &= 0 \quad l > a \end{aligned} \quad (67)$$

This expression for various filling factors was employed by W.O.D. and agrees very closely with their numerical results. They plot them in Fig. (3) of their paper. We here reproduce their figure as Fig. 3.

(2) We next consider a power law distribution with cutoffs R_1 and R_2 so that

$$\begin{aligned} n(R) &= \bar{n} (R_1/R)^\alpha & R_1 < R < R_2 \\ &= 0 & \text{otherwise} \end{aligned} \quad (68)$$

where $\alpha > 0$.

This function is used to model the observations by W.O.D., it gives a reasonable approximation to the Schechter function for the mass distribution

[compare Eq. (73) with Eq. (2)]. The major reason for using this function, as opposed to the $n(R)$ derived from the Schechter function, Eq. (9), is analytic convenience. The mass function $n(M)$ is most easily calculated by use of the Mellin transforms Eqs. (4) - (7). Thus, we easily obtain from Eq. (6b)

$$\hat{n}(s) = \frac{\bar{n}}{\alpha-s} [R_1^s - R_1^\alpha R_2^{s-\alpha}] \quad (69)$$

and from Eq. (7)

$$\hat{F}(s) = \frac{\bar{n}^3}{(\alpha-s)^3} [R_1^s - R_1^\alpha R_2^{s-\alpha}]^3 \quad (70)$$

an entire function of s . The inversion formula then gives

$$\begin{aligned} F(x) &= \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} \hat{F}(s) x^{-s} ds \\ &= \frac{\bar{n}^3}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{ds}{(\alpha-s)^3} \left[\left(\frac{R_1}{x}\right)^3 - 3\left(\frac{R_1}{R_2}\right)^\alpha \left(\frac{R_1^2 R_2}{x}\right)^s \right. \\ &\quad \left. + 3\left(\frac{R_1}{R_2}\right)^{2\alpha} \left(\frac{R_1 R_2^2}{x}\right)^s - \left(\frac{R_1}{R_2}\right)^{3\alpha} \left(\frac{R_2^3}{x}\right)^s \right] . \end{aligned} \quad (71)$$

This may be evaluated by making use of the result

$$\frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{t^s ds}{(\alpha-s)^3} = -\frac{t^\alpha (\ln t)^2}{2} \quad t > 1 \quad (72)$$

$$= 0 \quad t < 1$$

since the contour can be closed to the left in the first case and to the right in the second case. As a result we get zero terms from Eq. (71) if $x < R_1^3$,

one term if $R_1^3 < x < R_2 R_1^2$, two terms if $R_2 R_1^2 < x < R_2^2 R_1$, and three terms if $R_2^3 < x$. However, in the last case F vanishes.

Thus we may write compactly

$$\eta(M) = \frac{2\pi^4}{3} \frac{M^2}{k^3} \left(\frac{M}{M_1}\right)^{-\alpha} \bar{n}^3 \left[\ln^2\left(\frac{M}{M_1}\right) - \frac{1}{3} \ln^2\left(\frac{M^3}{M_1^2 M_2}\right) + \frac{1}{3} \ln^2\left(\frac{M^3}{M_1 M_2^2}\right) - \ln^2\left(\frac{M}{M_2}\right) \right] \quad (73)$$

where $M_1 = kR_1^3$, $M_2 = kR_2^3$ and we only keep those logarithmic terms in the brackets whose arguments are less than unity. Note if we keep all the terms the bracket vanishes. Thus, if $M < M_1$ or $M > M_2$, there are no clusters, as should be the case, and $\eta(M)$ vanishes. A plot of $\eta(M)$ is exhibited in Fig. 4 for $\alpha = 3.5$ and 4.5 where we assume $R_2 = 8R_1$ corresponding to the assumption in W.O.D. The upper part of the curve agrees with the numerical results of W.O.D.

We calculate the cluster-cluster correlation function without any restriction on the masses. For this we need a_k^2 and n_k for substitution in Eq. (14); thus

$$\begin{aligned} n_k &= \frac{\bar{n} R_2}{\alpha-1} \left(\frac{R_1}{R_2}\right)^\alpha \left[\left(\frac{R_2}{R_1}\right)^{\alpha-1} - 1 \right] & k < R_1 \\ &= \frac{\bar{n} R_2}{\alpha-1} \left(\frac{R_1}{R_2}\right)^\alpha \left[\left(\frac{R_2}{k}\right)^{\alpha-1} - 1 \right] & k > R_1 \\ &= 0 & k > R_2 \end{aligned} \quad (74)$$

and

$$a_k^2 = \frac{\alpha-3}{\alpha-1} \frac{1 - (R_1/R_2)^{\alpha-2}}{1 - (R_1/R_2)^\alpha} \quad k < R_1 \quad (75)$$

$$= R_2^2 \frac{\alpha-1}{\alpha-3} \frac{[(R_2/l)^{\alpha-3}-1]}{[(R_2/l)^{\alpha-1}-1]} \quad l > R_1$$

$$= 0 \quad l_2 > R$$

Making use of these results it is easy to evaluate Eq. (14) for the correlation function. We plot it in Fig. 5 for the two cases $\alpha = 3.5$ and $\alpha = 4.5$ where again we choose $R_2 = 8R_1$. W.O.D. have compared the correlation function (obtained numerically) for this $n(R)$ with the observed correlation function. The agreement is moderate, the reader is referred to their paper for details.

These cases illustrate the method by which we obtain η and ξ for any choice of n .

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Appendix I. The Schechter Function

In Eq. (26) of Section II the density function $n(R)$ corresponding to the Schechter function Eq. (2) was given as

$$n(R) = C^{1/3} \left(\frac{R}{R_*} \right)^{-(\alpha+2)} I\left(\frac{R}{R_*}\right), \quad (A1)$$

where

$$I(r) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} r^{1/3}(p) r^{-p} dp \quad (A2)$$

($\sigma > 0$).

For $R \gg R_A$ we may evaluate $I(r)$ by steepest descent. As will appear immediately, the saddle point is on the real p axis and large compared with unity. We thus employ Stirlings formula valid for large p

$$r(p) \approx \sqrt{2\pi} p^{p-1/2} e^{-p} \quad (A3)$$

$$I(r) = \frac{1}{2\pi i} (2\pi)^{1/6} \int \frac{dp}{p^{1/6}} \exp\left(\frac{-p}{3} + \frac{p}{3} \ln p - p \ln r\right). \quad (A4)$$

The saddle point occurs at $p = r^3$. Let $p = r^3 + it$. Then expanding for small t

$$\begin{aligned} I(r) &= (2\pi)^{1/6} \frac{e^{-r^{3/3}}}{\sqrt{r}} \frac{i}{2\pi i} \int_{-\infty}^{\infty} \exp(-t^2/6r^3) dt \\ &= \frac{\sqrt{3}}{(2\pi)^{1/3}} r e^{-(r^3/3)} \quad r \gg 1. \end{aligned} \quad (A5)$$

For arbitrary r , $I(r)$ may be reduced to a real integral. The original contour can be distorted to a contour which extends from $-\infty$ to 0 along the

lower real axis surrounding the origin in the positive sense and return to $-\infty$ along the upper real axis. Then using

$$\Gamma(p) = \frac{\pi}{\sin \pi p} \frac{1}{\Gamma(-p)p} \quad (A6)$$

we can write

$$\begin{aligned} I(r) &= -\frac{1}{\pi} \sum_{n=1}^{\infty} \sin \frac{n\pi}{3} \int_{-(n-1)}^{-n} r^{-p} \left(\frac{\pi}{\sin \pi p} \right)^{1/3} \frac{1}{(-p \Gamma(-p))^{1/3}} dp \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \sin \frac{n\pi}{3} \int_{n-1}^n r^s \frac{\pi}{\sin \pi s} \frac{ds}{[s \Gamma(s)]^{1/3}} \end{aligned}$$

where $s = -p$. Finally setting $s = y + n - 1$ and using the recursion formula $\Gamma(s) = (s-1)\Gamma(s-1)$ $n-1$ times on the n^{th} term we have, (with $y = s - n + 1$)

$$I(r) = \frac{1}{\pi} \int_0^1 \frac{r^y}{[y \Gamma(y)]^{1/3}} \frac{\pi^{1/3}}{(\sin \pi y)^{1/3}} \sum_{n=1}^{\infty} \frac{r^{n-1} \sin n\pi/3}{[(y+1)\dots(y+n-1)]^{1/3}} dy \quad (A8)$$

For r small only the $n=1$ term survives and we may approximate $1/\Gamma(y)$ by unity near $y = 0$. Thus

$$\begin{aligned} I &= \frac{1}{\pi} \sin \pi/3 \int_0^{\infty} \frac{e^{-| \ln r | y}}{y^{1/3}} dy = \frac{0.0383}{(\ln r)^{2/3}} \\ &= \frac{\sqrt{3}}{2\pi} \frac{\Gamma(2/3)}{|\ln r|^{2/3}} \quad (A9) \end{aligned}$$

Equation (A8) may be evaluated numerically and we plot $I(r)$ and $r^{-(\alpha+2)} I(r)$ in Fig. 1 for $\alpha=2, 2.5$, and 3.

From Eqs. (A1) and (A5) we see that $n(R)$ varies as

$$n(R) = \frac{R^{-(\alpha + 2)}}{|\ln(R_1/R_*)|^{2/3}} \quad R \ll R_* \quad (A10)$$

for $R \ll R_*$ and from Eq. (A1) and (A9) we see $n(R)$ varies as

$$n(R) = \left(\frac{R}{R_*}\right)^{-(\alpha+1)} \exp - (R/R_*)^3 \quad R \gg R_* \quad (A11)$$

for large R . Since observationally $\alpha \approx 2$, n varies as R^{-3} for small R , and as R^{-4} with an exponential cutoff for large R . Hence, the choice by W.O.D. of a $R^{-3.5}$ dependence with an abrupt cutoff is a reasonable fit to $n(R)$ deduced here from the Schechter function.

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FIGURE CAPTIONS

1. The distribution of spherical radii $n(R)$ in the model necessary to reproduce the Schechter distribution of cluster masses $n(M)$ given in Eq. (2). α is the exponent in Eq. (2). C is defined before Eq. (23).
2. The configurations of the two triplets of spheres producing the two intersection points are illustrated for each of the cases treated in Sec. III. 2a. The case in which the two triplets are identical. 2b. The case in which the two triplets share two common spheres. 2c. The case in which the two triplets share only one common sphere. 2d. The case in which two triplets share no common sphere.
3. The correlation function $\xi(R)$ for spheres all of the same radii a , for various filling factors f . The solid lines are given by the analytic expression, Eq. 61. The symbols are from the Monte Carlo results of W.O.D.
4. The mass function for clusters of galaxies $n(M)$ derived from the power-law distribution for the radii of the spheres $n(R)$ given by Eq. (68).
5. The correlation function $\xi(l)$ for the same case as in Fig. 4.

Fig. 1

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Fig. 2a

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Fig. 2b

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Fig. 2c

Fig. 2d

FIG. 3

Fig. 4

Fig. 5

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