A LIMIT THEOREM FOR THE MAXIMUM OF AN
EXPONENTIAL AUTOREGRESSIVE PROCESS

by

MICHAEL CHERNICK
Stanford University

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Department of Energy, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

TECHNICAL REPORT NO. 14

December 16, 1977

STUDY ON STATISTICS AND ENVIRONMENTAL
FACTORS IN HEALTH

PREPARED UNDER SUPPORT TO SIMS FROM
ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION (ERDA)
ROCKEFELLER FOUNDATION
SLOAN FOUNDATION
ENVIRONMENTAL PROTECTION AGENCY (EPA)
NATIONAL SCIENCE FOUNDATION (NSF)

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED *EB*

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Table of Contents

Abstract

1. The Model

a) Definition

b) Existence and Uniqueness for the Distribution of the ϵ_n 's

2. The Exact Distribution for M_1 and M_2

3. Upper and Lower Bounds for the Limit Distribution

4. Loynes' Theorems

5. Verifying Loynes' Conditions

a) The Uniform Mixing Condition

b) The Sufficient Condition for M_n to Have the Same Limit as

$$M_n^*$$

References

A Limit Theorem for the Maximum of an
Exponential Autoregressive Process

Michael Chernick

Abstract

This paper deals with the asymptotic behavior of the maximum of a particular autoregressive process. The process was introduced by Gaver and Lewis in 1975 as a generalization of the Poisson process which allows for some dependence in the successive interarrival times.

The exact distribution of the maximum of the first two terms and the first three terms in the sequence (denoted by M_1 and M_2 respectively) is calculated and upper and lower bounds are obtained for M_n , the maximum of the first $n+1$ terms in the sequence.

Loynes' in 1965 gave conditions under which a stationary stochastic process has a maximum which behaves in the limit just as for independent identically distributed variables with the marginal distribution of the stationary process. Such conditions are often very difficult to verify in practice. However, for this particular example we are able to determine the joint distribution of the process at time n and $n+j+1$ and by using the Markov property we can verify the conditions to obtain the limit theorem.

1. The Model

a) Definition.

We define the process according to the following recursion

$$X_n = \rho X_{n-1} + \epsilon_n \quad \text{where } 0 \leq \rho < 1$$

and the ϵ_n 's are independent identically distributed random variables with ϵ_n independent of X_{n-1} . We require that X_n has an exponential distribution with parameter 1 (denoted $X_n \sim \exp(1)$) for each $n \geq 0$.

The process is initiated with $X_0 \sim \exp(1)$ and then $X_1 = \rho X_0 + \epsilon_1$, etc.

In requiring $X_n \sim \exp(1)$ we are naturally led to the question of

whether or not there exist a sequence of random variables ϵ_n giving

$X_n \sim \exp(1)$ for each n . Gaver and Lewis (1975) show that if we let

$\epsilon_n = 0$ with probability ρ and $\epsilon_n \sim \exp(1)$ with probability $1-\rho$,

then the resulting X_n 's have the required distribution. We note that

if we choose $\epsilon_n = 0$ with probability ρ and $\epsilon_n \sim \exp(\lambda)$ with prob-

ability $1-\rho$ then the X_n 's will be distributed $\exp(\lambda)$. With ρ and

the ϵ_n 's so specified, the process $\{X_n\}_{n=0}^{\infty}$ is a well defined stationary sequence.

In fact, given $X_n \sim \exp(1)$, the distribution $\epsilon_n = 0$ with probability ρ and $\epsilon_n \sim \exp(1)$ with probability $1-\rho$ is the only distribution that will work, as we shall see in the next subsection.

b) Existence and Uniqueness for the Distribution of the ϵ_n 's.

To prove the existence and uniqueness, we use a characteristic function argument. Suppose $X_{n-1} \sim \exp(1)$. Let $\phi_X(t)$ denote its characteristic function. It can easily be verified that $\phi_X(t) = \frac{1}{1-it}$, and

letting $\phi_{\rho X}(t)$ denote the characteristic function of ρX_{n-1} we know that $\phi_{\rho X}(t) = \phi_X(\rho t) = \frac{1}{1-i\rho t}$. Now since ϵ_n is independent of X_{n-1} , X_n is the convolution of ρX_{n-1} with ϵ_n . Therefore the characteristic function for X_n satisfies $\phi_{X_n}(t) = \phi_{\epsilon}(t) \phi_{\rho X}(t)$.

Requiring $\phi_{X_n}(t) = \phi_X(t)$ gives

$$\begin{aligned}\phi_{\epsilon}(t) &= \frac{\phi_{X_n}(t)}{\phi_{\rho X}(t)} = \frac{1/(1-it)}{1/(1-i\rho t)} = \frac{1-i\rho t}{1-it} = \frac{1-\rho+\rho-i\rho t}{1-it} \\ &= \frac{\rho(1-it) + (1-\rho)}{1-it} = \rho+(1-\rho) \frac{1}{1-it}.\end{aligned}$$

We see that ϵ_n must have the characteristic function $\phi_{\epsilon}(t) = \rho+(1-\rho) \frac{1}{1-it}$, and hence if $\rho+(1-\rho) \frac{1}{1-it}$ is a characteristic function, we have demonstrated the existence part. It can easily be checked that $\phi_{\epsilon}(t) = \rho+(1-\rho) \frac{1}{1-it}$ is the characteristic function of a random variable that has probability ρ of being zero and is distributed $\exp(1)$ with probability $1-\rho$.

Now $\phi_{\epsilon}(t) = \rho+(1-\rho) \frac{1}{1-it}$ is the only characteristic function that gives $\phi_{X_n}(t) = \frac{1}{1-it}$ and so by the uniqueness theorem for characteristic functions, the ϵ_n 's must have the distribution corresponding to the characteristic function $\phi_{\epsilon}(t)$.

2. The Exact Distribution for M_1 and M_2

Let $M_1 = \max(X_0, X_1)$, $M_2 = \max(X_0, X_1, X_2)$, and in general we define $M_n = \max(X_0, X_1, \dots, X_n)$. By computing the distribution of M_1 and M_2 , we shall see the difficulties involved in obtaining the exact distribution for M_n .

$$\begin{aligned}
P(M_1 \leq t) &= P(X_0 \leq t \text{ and } X_1 \leq t) = P(X_0 \leq t, \rho X_0 + \varepsilon_1 \leq t) \\
&= P_t(X_0 \leq t, 0 < \varepsilon \leq t - \rho X_0) + P(X_0 \leq t, \varepsilon_1 = 0) \\
&= \int_0^t e^{-x} \left[\int_0^{t-\rho x} (1-\rho) e^{-u} du \right] dx + \rho(1-e^{-t}) \\
&= (1-\rho) \int_0^t e^{-x} (1-e^{-t} e^{\rho x}) dx + \rho(1-e^{-t}) \\
&= (1-\rho)(1-e^{-t}) - (1-\rho)e^{-t} \left(\frac{1 - e^{-(1-\rho)t}}{1-\rho} \right) + \rho(1-e^{-t}) \\
&= 1 - e^{-t} - e^{-t} + e^{-(2-\rho)t} = 1 - 2e^{-t} + e^{-(2-\rho)t}.
\end{aligned}$$

Let $M_n^* = \max(X_0^*, X_1^*, \dots, X_n^*)$ where X_i^* are i.i.d. $\exp(1)$. We note that

$$\begin{aligned}
P(M_1^* \leq t) &= P(X_0^* \leq t, X_1^* \leq t) = P(X_0^* \leq t) P(X_1^* \leq t) \\
&= (1-e^{-t})^2 = 1 - 2e^{-t} + e^{-2t}.
\end{aligned}$$

Now since $e^{-(2-\rho)t} > e^{-2t} \forall t > 0, \rho > 0$ we see that $P(M_1 \leq t) > P(M_1^* \leq t) \forall t > 0$ provided $\rho \neq 0$.

For stationary Gaussian processes, if

$$r_n = \frac{\text{cov}(X_0, X_n)}{\text{Var } X_0} \geq 0 \quad \forall n$$

then $P(M_n \leq t) \geq P(M_n^* \leq t) \forall t$ and $\forall n$.

This result is a consequence of Slepian's Lemma. Knowing this, one might conjecture that for the exponential first order autoregressive process $P(M_n \leq t) \geq P(M_n^* \leq t), \forall t > 0$ and for all n . In Section 3 we shall prove that this inequality is true for any process which satisfies an associativity property. Since the exponential process is associative,

the conjecture is correct and $P(M_n^* \leq t) = (1-e^{-t})^{n+1}$ provides a lower bound for $P(M_n \leq t)$.

For M_2 ,

$$\begin{aligned}
P(M_2 \leq t) &= P(X_0 \leq t, X_1 \leq t, X_2 \leq t) \\
&= P(X_0 \leq t, \rho X_0 + \epsilon_1 \leq t, \rho^2 X_0 + \rho \epsilon_1 + \epsilon_2 \leq t) \\
&= P(X_0 \leq t, 0 < \epsilon_1 \leq t - \rho X_0, 0 < \epsilon_2 \leq t - \rho \epsilon_1 - \rho^2 X_0) \\
&\quad + P(X_0 \leq t, 0 < \epsilon_1 \leq t - \rho X_0, \epsilon_2 = 0) \\
&\quad + P(X_0 \leq t, \epsilon_1 = 0, 0 < \epsilon_2 \leq t - \rho^2 X_0) \\
&\quad + P(X_0 \leq t, \epsilon_1 = 0, \epsilon_2 = 0) \\
&= \int_0^t e^{-x} \int_0^{t-\rho x} (1-\rho) e^{-u} \int_0^{t-\rho u-\rho^2 x} (1-\rho) e^{-v} dv du dx \\
&\quad + \rho \int_0^t e^{-x} \int_0^{t-\rho^2 x} (1-\rho) e^{-v} dv dx \\
&\quad + \rho \int_0^t e^{-x} \int_0^{t-\rho x} (1-\rho) e^{-u} du dx + \rho^2 (1-e^{-t}) .
\end{aligned}$$

After tedious computations, one gets

$$\begin{aligned}
P(M_2 \leq t) &= (1-\rho)^2 (1-e^{-t} (2 - e^{-(1-\rho)t})) - (1-\rho) \\
&\quad e^{-t} \left(\frac{1 - e^{-(1-\rho^2)t}}{(1-\rho^2)} - \frac{e^{-(1-\rho)t}}{(1-\rho)} (1 - e^{-(1-\rho)t}) \right)
\end{aligned}$$

$$\begin{aligned}
& + \rho(1-\rho) \left[1 - e^{-t} \left(\frac{2-\rho^2 - e^{-(1-\rho^2)t}}{1-\rho^2} \right) \right] \\
& + \rho(1-\rho) \left(1 - e^{-t} \left[\frac{1 + (1-e^{-(1-\rho)t})}{(1-\rho)} \right] \right) + \rho^2(1-e^{-t}) .
\end{aligned}$$

One can do some algebra and simplify to get $P[M_2 \leq t] = 1 - 3e^{-t} + 2e^{-(2-\rho)t} + e^{-(2-\rho^2)t} - e^{-(3-2\rho)t}$. We notice that since $P[M_2^* \leq t] = 1 - 3e^{-t} + 3e^{-2t} + e^{-3t}$, $P[M_2 \leq t] - P[M_2^* \leq t] = 2e^{-(2-\rho)t} + e^{-(2-\rho^2)t} - 3e^{-2t} + e^{-3t} - e^{-(3-2\rho)t}$. A little calculus shows that $P[M_2 \leq t] - P[M_2^* \leq t] > 0$, $\forall t > 0$.

Having done the computation for M_2 , we see that the computation for general M_n leads to complicated forms that are difficult to simplify.

Although it is possible to do the integrations and algebra for any n , it appears to be difficult to simplify even for $n=3$. There may be ways to simplify by approximating some of the integrals by upper bounds or by neglecting some of the integrals which would be negligible in the total. Such methods might be useful in determining rates of convergence for the maximum. For rates of convergence, the parameter ρ would play a role, whereas it does not appear in the limit theorem.

In general, we have the following:

$$\begin{aligned}
P(M_{n-} \leq t) &= (1-\rho)^n \int_0^t e^{-x} \int_0^{t-\rho x} e^{-u_1} \int_0^{t-\rho u_1-\rho^2 x} e^{-u_2} \\
&\quad \dots \int_0^{t-\rho u_{n-1}-\rho^2 u_{n-2}-\dots-\rho^n x} e^{-u_n} du_n du_{n-1} \dots du_1 dx \\
&+ \rho(1-\rho)^{n-1} \int_0^t e^{-x} \int_0^{t-\rho^2 x} e^{-u_2} \int_0^{t-\rho u_2-\rho^3 x} e^{-u_3} \\
&\quad \dots \int_0^{t-\rho u_{n-1}-\rho^2 u_{n-2}-\dots-\rho^{n-2} u_2-\rho^n x} e^{-u_n} du_n du_{n-1} \dots du_2 dx \\
&+ (1-\rho)^{n-1} \int_0^t e^{-x} \int_0^{t-\rho x} e^{-u_1} \int_0^{t-\rho u_1-\rho^3 x} e^{-u_3} \\
&\quad \dots \int_0^{t-\rho u_{n-1}-\rho^2 u_{n-3}-\dots-\rho^{n-3} u_3-\rho^{n-1} u_1-\rho^n x} e^{-u_n} du_n \dots du_3 du_1 dx \\
&+ \dots + \rho^n (1-e^{-t}) .
\end{aligned}$$

Note that there are 2^n terms in the sum. Some simplification is possible. For example, one can show that

$$P(M_n \leq t, \epsilon_1=0, \epsilon_i \neq 0, \text{ for } i \neq 1) > P(M_n \leq t, \epsilon_j=0, \epsilon_i \neq 0, \text{ for } i \neq j)$$

where $j \neq 1$. This might be useful in getting upper bounds for $P(M_n \leq t)$.

We also observe that $P(M_1 \leq t) - P(M_1^* \leq t) = e^{-2t}(e^{\rho t} - 1)$ is an increasing function of ρ for each fixed t . Similarly, $P(M_2 \leq t) - P(M_2^* \leq t) = 2e^{-2t}e^{\rho t} + e^{-2t}e^{\rho^2 t} - 3e^{-2t} + e^{-3t} - e^{-3t}e^{2\rho t}$ is an increasing function of ρ for each fixed t . It seems reasonable to expect $P(M_n \leq t) - P(M_n^* \leq t)$ to be an increasing function of ρ when n and t are held fixed.

3. Upper and Lower Bounds for the Limit Distribution

For M_n^* it is easy to show that $P[M_n^* \leq x + \ln n] \rightarrow e^{-e^{-x}} \forall x$ as $n \rightarrow \infty$. So $a_n \equiv 1$, $b_n = \ln n$ is the appropriate choice for $P[M_n^* \leq a_n x + b_n] \rightarrow e^{-e^{-x}}$. We see that $P[M_n^* \leq x + \ln n] = P[X_0^* \leq x + \ln n, \dots, X_n^* \leq x + \ln n]$
 $= P[X_0^* \leq x + \ln n]^{n+1} = [1 - e^{-(x + \ln n)}]^{n+1} = (1 - \frac{e^{-x}}{n})^{n+1}$ and
 $\lim_{n \rightarrow \infty} (1 - \frac{e^{-x}}{n})^{n+1} = e^{-e^{-x}}$. We shall soon see that $P[M_n \leq a_n x + b_n] \geq P[M_n^* \leq a_n x + b_n]$ so that at least $\lim_{n \rightarrow \infty} P[M_n \leq x + \ln n] \geq e^{-e^{-x}}$.

The following observation provides us with an upper bound for the limiting distribution:

$$P(M_n \leq t) = P(X_0 \leq t, \rho X_0 + \epsilon_1 \leq t, \rho^2 X_0 + \rho \epsilon_1 + \epsilon_2 \leq t, \dots, \rho^n X_0 + \rho^{n-1} \epsilon_1 + \dots + \rho \epsilon_{n-1} + \epsilon_n \leq t)$$

and the event $\{X_0 \leq t, \rho X_0 + \epsilon_1 \leq t, \dots, \rho^n X_0 + \rho^{n-1} \epsilon_1 + \dots + \rho \epsilon_{n-1} + \epsilon_n \leq t\} \subseteq \{X_0 \leq t, \epsilon_1 \leq t, \epsilon_2 \leq t, \dots, \epsilon_n \leq t\}$ since all ϵ_i 's ≥ 0 and $X_0 > 0$ with probability one. So $P(M_n \leq t) \leq P(X_0 \leq t, \epsilon_1 \leq t, \epsilon_2 \leq t, \dots, \epsilon_n \leq t) = P(X_0 \leq t) [P(\epsilon_1 \leq t)]^n$
 $= (1 - e^{-t}) (\rho + (1-\rho)(1 - e^{-t}))^n = (1 - e^{-t}) (1 - (1-\rho)e^{-t})^n$. We see therefore that $P[M_n \leq x + \ln n] \leq (1 - \frac{e^{-x}}{n}) (1 - (1-\rho) \frac{e^{-x}}{n})^n$, and since

$$\lim_{n \rightarrow \infty} (1 - \frac{e^{-x}}{n}) (1 - (1-\rho) \frac{e^{-x}}{n})^n = e^{-(1-\rho)e^{-x}} \forall x,$$

$$\lim_{n \rightarrow \infty} P[M_n \leq x + \ln n] \leq e^{-(1-\rho)e^{-x}} \forall x.$$

This upper bound appears to be quite crude, but it is not trivial. For $\rho < 1$, $e^{-(1-\rho)e^{-x}} < 1$, $\forall x < \infty$. If $\lim_{n \rightarrow \infty} P[M_n \leq a_n x + b_n]$ exists for some $\{a_n\}$ and $\{b_n\}$ then $a_n \equiv 1$, $b_n = \ln n$ is an appropriate choice since $0 < e^{-e^{-x}} \leq \lim_{n \rightarrow \infty} P[M_n \leq x + \ln n] \leq e^{-(1-\rho)e^{-x}} < 1 \quad \forall -\infty < x < \infty$.

We shall now prove that $P[M_n \leq t] \geq P[M_n^* \leq t] \forall n$ and each fixed t . In order to accomplish this, we introduce the notion of associative random variables.

Def: Random variables $X_1, X_2, \dots, X_n$ are associated if $\text{Cov}(\Gamma(X_1, X_2, \dots, X_n), \Delta(X_1, X_2, \dots, X_n)) \geq 0$ for all pairs of increasing¹ binary functions Γ and Δ .

We shall consider the pair (M_{n-1}, X_n) and show that it is an associated pair of random variables $\forall n$.

In order to show (M_{n-1}, X_n) associated it will be sufficient to show that $(X_0, X_1, \dots, X_n)$ is associated for each n . M_{n-1} is an increasing function of $X_0, X_1, \dots, X_{n-1}, X_n$ and trivially so is X_n . Increasing functions of associated random variables are associated (see (P₃), Barlow and Proschan, p. 30). So we need only show that $(X_0, X_1, \dots, X_n)$ are associated.

By Theorem 2.2, p. 31 of Barlow and Proschan, any set of independent random variables is associated. So $(X_0, \epsilon_1, \epsilon_2, \dots, \epsilon_n)$ is associated. For each j , $X_j = \epsilon_j + \rho\epsilon_{j-1} + \dots + \rho^{j-1}\epsilon_1 + \rho^j X_0$ is an increasing function of $X_0, \epsilon_1, \epsilon_2, \dots, \epsilon_j$ since $\rho \geq 0$. $X_0, X_1, X_2, \dots, X_n$ are all increasing functions of the associated random variables $X_0, \epsilon_1, \epsilon_2, \dots, \epsilon_n$ and hence are associated by (P₃). Hence (M_{n-1}, X_n) is an associated pair of random variables.

(M_{n-1}, X_n) associated implies $\text{Cov}(f(M_{n-1}), g(X_n)) \geq 0$ for all non-decreasing f, g . Esary, Proschan and Walkup give comparisons of concepts of bivariate dependence and give applications and examples of associated random variables (Ann. Math. Statist., 1967). In particular, if we take

¹Increasing means the function does not decrease when any variable is increased and the others are held fixed.

$f(M_{n-1}) = 1_{\{M_{n-1} \leq s\}}$, $g(X_n) = 1_{\{X_n \leq t\}}$ we get $P[M_{n-1} \leq s, X_n \leq t] \geq P[M_{n-1} \leq s]$

$P[X_n \leq t] \forall s, t$. This property is called positive quadrant dependence.

Taking the special case $s=t$ gives the following lemma.

Lemma 1: $P(X_n \leq t, M_{n-1} \leq t) \geq P(X_n \leq t) P(M_{n-1} \leq t) \forall n \geq 1$.

Using this lemma we can now apply induction to get $P[M_n \leq t] \geq P[M_n^* \leq t]$.

It is worthwhile noticing that the argument showing (M_{n-1}, X_n) associated depended only on our model in a weak sense. We really only needed the condition $0 \leq \rho$. So for any first order autoregressive process with $0 \leq \rho < 1$, M_{n-1}, X_n are associated and we can conclude the following:

Theorem 1: Define an autoregressive process by $X_0 = \varepsilon_0$. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots$ be a sequence of independent random variables (not necessarily identically distributed so X_n needn't be stationary), $X_n = \rho X_{n-1} + \varepsilon_n$ for $n \geq 1$. We need $0 \leq \rho$. Let $M_n = \max(X_0, X_1, \dots, X_n)$ then $P(M_n \leq t) \geq P(M_n^* \leq t)$ where $M_n^* = \max(X_0^*, X_1^*, \dots, X_n^*)$ and X_i^* has the same distribution as X_i . $X_0^*, X_1^*, \dots, X_n^*$ are independent.

Of course the above theorem holds as long as $(X_0, X_1, \dots, X_n)$ are associated and hence it can be applied to any process with a moving average representation $X_n = \sum_{r=0}^{\infty} \alpha_r \varepsilon_{n-r}$ with $\alpha_r \geq 0 \forall r$ and ε_i 's independent. It is also true that $P(X_0 \leq u_0, X_1 \leq u_1, \dots, X_n \leq u_n) \geq \prod_{i=0}^n P(X_i^* \leq u_i)$. This follows by restating Lemma 1 with $s=t$ and applying induction.

Returning to the first order exponential autoregressive process we see that $P[M_n \leq x + \ln n] \geq P[M_n^* \leq x + \ln n] \forall n$ and hence $\lim_{n \rightarrow \infty} P[M_n \leq x + \ln n] \geq e^{-e^{-x}}$ or at least $\liminf_{n \rightarrow \infty} P[M_n \leq x + \ln n] \geq e^{-e^{-x}}$.

4. Loynes' Theorems

R. M. Loynes (1965, Annals of Mathematical Statistics) showed that for stationary stochastic processes which satisfy a uniform mixing condition, the maximum M_n appropriately normalized can only converge to one of the three extreme value distributions. These are the three types given in the extreme value theory for independent random variables and are as follows:

$$\text{I. } G(x) = e^{-e^{-x}}, \quad -\infty < x < \infty$$

$$\begin{aligned} \text{II. } G(x) &= 0 & x &\leq 0 \\ &= e^{-x^{-\alpha}} & x &> 0, \quad \alpha > 0 \end{aligned}$$

$$\begin{aligned} \text{III. } G(x) &= e^{-(-x)^\alpha} & x &< 0, \quad \alpha < 0 \\ &= 1 & x &\geq 0. \end{aligned}$$

In addition he proves the following theorem:

Theorem 2: Let $\{X_n : n \geq 1\}$ be a uniformly mixing strictly stationary stochastic process, and let $C_n(\xi)$ satisfy

$$P[X_0 > C_n(\xi)] \leq \xi/n \leq P[X_0 \geq C_n(\xi)]$$

then $P[M_n \leq C_n(\xi)] \rightarrow e^{-k\xi}$ as $n \rightarrow \infty$ where k is a positive constant less than or equal to one.

In Section 5 we shall verify that the exponential first order autoregressive process is uniformly mixing. Let $C_n(\xi) = \ln n - \ln \xi$, then $P[X_0 > C_n(\xi)] = \xi/n$. So by virtue of this theorem of Loynes', $P[M_n \leq \ln n - \ln \xi] \rightarrow e^{-k\xi}$ as $n \rightarrow \infty$.

Setting $x = -\ln \xi$ or $\xi = e^{-x}$ we have that $\lim_{n \rightarrow \infty} P[M_n \leq x + \ln n] = e^{-ke^{-x}}$ for some $0 < k \leq 1$. For this process we know from Section 3 that $e^{-e^{-x}} \leq \lim_{n \rightarrow \infty} P[M_n \leq x + \ln n] \leq e^{-(1-\rho)e^{-x}}$. So $1-\rho \leq k \leq 1$.

Our limit theorem will be complete when we can specify k . Loynes gives a sufficient condition for $k=1$. In Section 5 we will see that the process satisfies the sufficient condition, as well as the uniform mixing condition, and hence $\lim_{n \rightarrow \infty} P[M_n \leq x + \ln n] = e^{-e^{-x}}$.

Before stating the sufficient condition we shall define the uniform mixing condition.

Def: A strictly stationary stochastic process is said to be uniformly mixing if $|P(A \cap B) - P(A)P(B)| < \phi(k)$ for any $A \in \sigma(X_0, X_1, \dots, X_n)$, $B \in \sigma(X_{n+k+1}, X_{n+k+2}, \dots)$ for some $n \geq 0$ and $\phi(k)$ is a function depending only on k with $\phi(k) \rightarrow 0$ as $k \rightarrow \infty$. $\sigma(X_0, X_1, \dots, X_n)$ denotes the sigma algebra generated by the random variables $X_0, X_1, \dots, X_n$.

Lemma 2: (Loynes, 1965)

Let $\{X_n\}$ be a uniformly mixing process. If there exist sequences of integers $\{p_m\}$ and $\{q_m\}$ satisfying $m\phi(q_m) \rightarrow 0$, $q_m/p_m \rightarrow 0$, and $p_{m+1}/p_m \rightarrow 1$ as $m \rightarrow \infty$, then letting $t_m = m(p_m + q_m)$, if

$$\sum_{i=1}^{p_m-1} \left(\frac{p_m - i}{p_m} \right) \frac{P(X_1 > Ct_m(\xi), X_{i+1} > Ct_m(\xi))}{P(X_1 > C_{t_m}(\xi))} \rightarrow 0$$

as $m \rightarrow \infty$ where ξ is chosen. $P[X_0 > C_n(\xi)] \leq \xi/n \leq P[X_0 \geq C_n(\xi)]$, $P[M_n \leq C_n(\xi)] \rightarrow e^{-\xi}$ as $n \rightarrow \infty$, (i.e. $k=1$).

5. Verifying Loynes' Conditions.

a) The Uniform Mixing Condition

Recently, M. R. Leadbetter has shown that Loynes' theorem about convergence to one of the extreme value types holds under conditions weaker

than uniform mixing. This result can be found in SIMS Technical Report No. 10, August 1, 1977. We shall not be concerned with this result in this paper since it is possible to verify the uniform mixing condition for this process. It also appears that the proof would not be made simpler by verifying Leadbetter's conditions.

Let $A \in \sigma(X_0, X_1, \dots, X_n)$, $B \in \sigma(X_{n+j+1}, \dots)$. We seek to find a function $\phi(j)$ independent of A , B and n such that $|P(A \cap B) - P(A)P(B)| \leq \phi(j)$ and $\phi(j) \rightarrow 0$ as $j \rightarrow \infty$. We shall rely heavily on the fact that this process is a Markov process, and we will need to compute the joint distribution of X_n and X_{n+j+1} . The following lemma is useful toward this goal:

Lemma 3: Let $W_j = \sum_{r=1}^j \rho^{j-r} \epsilon_r$ where the ϵ_r 's are those defined for the exponential autoregressive process. W_j has the following distribution

$W_j = 0$ with probability ρ^j ,

$W_j \sim \exp(1)$ with probability $1-\rho^j$, and

$W_j \xrightarrow{d} \exp(1)$ as $j \rightarrow \infty$.

Proof: $X_j = \rho^j X_0 + W_j$ and W_j is independent of X_0 . $\phi_{X_j}(t) = \frac{1}{1-it}$ and

$$\begin{aligned} \phi_{W_j}(t) &= \frac{\phi_{X_j}(t)}{\phi_{\rho^j X_0}(t)} = \frac{\phi_{X_j}(t)}{\phi_{X_0}(\rho^j t)} = \frac{\frac{1}{1-it}}{\frac{1}{1-i\rho^j t}} \\ &= \frac{1-i\rho^j t}{1-it} = \frac{1-\rho^j + \rho^j - i\rho^j t}{1-it} = (1-\rho^j) \frac{1}{(1-it)} + \rho^j. \end{aligned}$$

This we recognize is the characteristic function of a random variable which puts mass ρ^j at zero and is $\exp(1)$ with probability $1-\rho^j$.

Now $\lim_{j \rightarrow \infty} \phi_{W_j}(t) = \lim_{j \rightarrow \infty} (1-\rho^j) \frac{1}{1-it} + \rho^j = \frac{1}{1-it}$. Since $\frac{1}{1-it}$ is continuous at $t=0$, we have $P[W_j \leq x] \rightarrow 1-e^{-x}$.

At first it might not seem clear that a function ϕ with the required properties for uniform mixing exists. The following lemma shows that $\phi(j) = \rho^j$ works for special sets, and we might hope that a function ϕ proportional to ρ^j will work for general sets A and B .

Lemma 4: $|P[X_{0-} \leq x, X_j \leq y] - P[X_{0-} \leq x] P[X_j \leq y]| = |P[X_{n-} \leq x, X_{n+j+1} \leq y] - P[X_{n-} \leq x] P[X_{n+j+1} \leq y]| < \rho^j$.

Proof: $P(X_{0-} \leq x, X_j \leq y) \geq P(X_{0-} \leq x) P(X_j \leq y)$ by associativity. Also

$$P(X_{0-} \leq x, X_j \leq y) = P(X_{0-} \leq x, W_j + \rho^j X_{0-} \leq y) < P(X_{0-} \leq x, W_j \leq y) = P(X_{0-} \leq x) P(W_j \leq y).$$

Hence

$$P[X_{0-} \leq x] [P(W_j \leq y) - P(X_j \leq y)] > P(X_{0-} \leq x, X_j \leq y) - P(X_{0-} \leq x) P(X_j \leq y) \geq 0.$$

Now

$$\begin{aligned} P(X_{0-} \leq x) [P(W_j \leq y) - P(X_j \leq y)] &= (1-e^{-x}) (1-e^{-y} + \rho^j e^{-y} - (1-e^{-y})) \\ &= \rho^j e^{-y} (1-e^{-x}) < \rho^j \quad \forall x, y > 0. \end{aligned}$$

$$\therefore |P(X_{0-} \leq x, X_j \leq y) - P(X_{0-} \leq x) P(X_j \leq y)| < \rho^j.$$

Now let us compute

$$\begin{aligned} P(X_{0-} \leq x, X_j \leq y) &= P(X_{n-} \leq x, X_{n+j+1} \leq y) \\ &= \rho^j \int_0^x e^{-u} du + (1-\rho^j) \int_0^x \int_0^{y-\rho^j u} e^{-w} e^{-u} dw du \\ &= \rho^j (1-e^{-x}) + (1-\rho^j) (1-e^{-x}) - (1-\rho^j) e^{-y} \left(\frac{1-e^{-(1-\rho^j)x}}{1-\rho^j} \right) \\ &= (e^{-x} - e^{-y} (1-e^{-(1-\rho^j)x})) \quad \text{for } y > \rho^j x. \end{aligned}$$

Now let $h(x,y)$ denote the joint density of X_n, X_{n+j+1} at $X_n = x$, $X_{n+j+1} = y$ for $y > \rho^j x$. We note $h(x,y) = 0$ for $y < \rho^j x$ and since $W_j = 0$ with probability ρ^j there is mass ρ^j concentrated on the line $y = \rho^j x$.

For $y > \rho^j x$,

$$\begin{aligned} h(x,y) &= \frac{\partial}{\partial x} \frac{\partial}{\partial y} P[X_n \leq x, X_{n+j+1} \leq y] = \frac{\partial}{\partial x} \frac{\partial}{\partial y} (1 - e^{-x} - e^{-y}(1 - e^{-(1-\rho^j)x})) \\ &= \frac{\partial}{\partial x} (e^{-y}(1 - e^{-(1-\rho^j)x})) = (1 - \rho^j) e^{-y} e^{-(1-\rho^j)x}. \end{aligned}$$

Let $f(x) = e^{-x}$ the density of X_n and $f(y) = e^{-y}$ the density for X_{n+j+1} . Let $A \in \sigma(X_0, X_1, \dots, X_n)$ and $B \in \sigma(X_{n+j+1}, X_{n+j+2}, \dots)$

$$\begin{aligned} |P(A \cap B) - P(A)P(B)| &= |E\{P(A \cap B | X_n, X_{n+j+1})\} - E\{P(A | X_n)\} E\{P(B | X_{n+j+1})\}| \\ &= \left| \int_0^\infty \int_{\rho^j u}^\infty P(A \cap B | X_n = u, X_{n+j+1} = v) h(u,v) dv du + P(A \cap B \cap \{X_{n+j+1} = \rho^j X_n\}) \right. \\ &\quad \left. - \int_0^\infty P(A | X_n = u) f(u) du \int_0^\infty P(B | X_{n+j+1} = v) f(v) dv \right| \\ &\leq \left| \int_0^\infty \int_{\rho^j u}^\infty P(A \cap B | X_n = u, X_{n+j+1} = v) h(u,v) dv du \right. \\ &\quad \left. - \int_0^\infty \int_0^\infty P(A | X_n = u) P(B | X_{n+j+1} = v) f(u) f(v) dv du \right| + \rho^j P(A \cap B | X_{n+j+1} = \rho^j X_n) \\ &\leq \rho^j + \int_0^\infty \int_{\rho^j u}^\infty P(A \cap B | X_n = u, X_{n+j+1} = v) h(u,v) \\ &\quad - P(A | X_n = u) P(B | X_{n+j+1} = v) f(u) f(v) dv du. \end{aligned}$$

Now

$$\begin{aligned} P(A \cap B | X_n = u, X_{n+j+1} = v) &= P(A | X_n = u, X_{n+j+1} = v) P(B | A, X_n = u, X_{n+j+1} = v) \\ &= P(B | X_n = u, X_{n+j+1} = v) P(A | X_n = u, X_{n+j+1} = v) \end{aligned}$$

since A and B are conditionally independent given X_n and X_{n+j+1} .

Now $\forall v > \rho^j u$, $P(A | X_n = u, X_{n+j+1} = v) = P(A | X_n = u)$ and $\forall u < v/\rho^j$

$$P(B | X_n = u, X_{n+j+1} = v) = P(B | X_{n+j+1} = v).$$

$$\therefore P(A \cap B | X_n = u, X_{n+j+1} = v) = P(A | X_n = u) P(B | X_{n+j+1} = v).$$

So the last expression simplifies to give

$$|P(A \cap B) - P(A)P(B)| \leq \rho^j + \int_0^\infty \int_{\rho^j u}^\infty P(A \cap B | X_n = u, X_{n+j+1} = v)$$

$$|h(u, v) - f(u)f(v)| dv du \leq \rho^j + \int_0^\infty \int_{\rho^j u}^\infty |h(u, v) - f(u)f(v)| dv du.$$

$$\text{Now } |h(u, v) - f(u)f(v)| = |(1-\rho^j)e^{-v} e^{-(1-\rho^j)u} - e^{-u}e^{-v}| = e^{-v} |(1-\rho^j)e^{-(1-\rho^j)u} - e^{-u}| \text{ and}$$

$$\begin{aligned} |(1-\rho^j)e^{-(1-\rho^j)u} - e^{-u}| &= e^{-u} - (1-\rho^j)e^{-(1-\rho^j)u} \text{ for } u < \frac{-\ln(1-\rho^j)}{\rho^j} \\ &= (1-\rho^j)e^{-(1-\rho^j)u} - e^{-u} \text{ for } u \geq \frac{-\ln(1-\rho^j)}{\rho^j}. \end{aligned}$$

So

$$\begin{aligned} \int_0^\infty \int_{\rho^j u}^\infty |h(u, v) - f(u)f(v)| dv du &= \int_0^\infty \int_{\rho^j u}^\infty e^{-v} |(1-\rho^j)e^{-(1-\rho^j)u} - e^{-u}| dv du \\ &= \int_0^\infty e^{-\rho^j u} |(1-\rho^j)e^{-(1-\rho^j)u} - e^{-u}| du \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{-\ln(1-\rho^j)}{\rho^j}} [e^{-(1+\rho^j)u} - e^{-u(1-\rho^j)}] du \\
&\quad + \int_{\frac{-\ln(1-\rho^j)}{\rho^j}}^{\infty} ((1-\rho^j)e^{-u} - e^{-(1+\rho^j)u}) du \\
&= \frac{1}{1+\rho^j} (1 - (1-\rho^j)^{(1+\rho^j)/\rho^j}) + (1-\rho^j) [(1-\rho^j)^{1/\rho^j} - 1] + (1-\rho^j)(1-\rho^j)^{1/\rho^j} \\
&\quad - \frac{(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} = \frac{1}{1+\rho^j} - (1-\rho^j) + 2(1-\rho^j)(1-\rho^j)^{1/\rho^j} \\
&\quad - \frac{2(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} = \frac{1-(1+\rho^j)}{1+\rho^j} + \rho^j + 2 \left[(1-\rho^j)(1+\rho^j)/\rho^j - \frac{(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} \right] \\
&= \rho^j \left(1 - \frac{1}{1+\rho^j} \right) + 2(1-\rho^j)(1+\rho^j)/\rho^j \left(1 - \frac{1}{1+\rho^j} \right) \\
&= \rho^j \left(\frac{\rho^j}{1+\rho^j} \right) + 2(1-\rho^j)(1+\rho^j)/\rho^j \left(\frac{\rho^j}{1+\rho^j} \right).
\end{aligned}$$

We now have for $A \in \sigma(X_0, X_1, \dots, X_n)$, $B \in \sigma(X_{n+j+1}, X_{n+j+2}, \dots)$

$$\begin{aligned}
|P(A \cap B) - P(A)P(B)| &< \rho^j + \rho^j \left[\frac{\rho^j}{1+\rho^j} + \frac{2(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} \right] \\
&= \rho^j \left[1 + \frac{\rho^j}{1+\rho^j} + \frac{2(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} \right] = \frac{\rho^j}{1+\rho^j} \left[1 + 2\rho^j + 2(1-\rho^j)(1+\rho^j)/\rho^j \right] \\
&\quad \frac{1+2\rho^j + 2(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} \rightarrow 1 + 2/e
\end{aligned}$$

since $(1-\rho^j)^{1/\rho^j} \rightarrow 1/e$ as $j \rightarrow \infty$. Now

$$\frac{1+2\rho^j + 2(1-\rho^j)(1+\rho^j)/\rho^j}{1+\rho^j} < \frac{5}{1+\rho^j} < 5 \quad \forall j.$$

So $|P(A \cap B) - P(A)P(B)| < 5\rho^j \quad \forall j$. We can take $\phi(j) = 5\rho^j$ and $\phi(j) \rightarrow 0$ as $j \rightarrow \infty$. Since we are only interested in finding a ϕ with the required properties, we did not take care to get the best upper bound on

$$|P(A \cap B) - P(A)P(B)| \quad \text{for } A \in \sigma(X_0, X_1, \dots, X_n), \quad B \in \sigma(X_{n+j+1}, \dots).$$

For example, since $1 + 2/e < 2$, $2\rho^j$ would be a sufficient bound for large j .

b) The Sufficient Condition for M_n to Have the Same Limit as M_n^* .

We have $\phi(j) = 5\rho^j \quad \forall j \geq 0$,

$$\frac{P(X_1 > C_{t_m}(\xi); X_{i+1} > C_{t_m}(\xi))}{P(X_1 > C_{t_m}(\xi))} \leq e^{-(1-\rho^i)C_{t_m}(\xi)}$$

since

$$P[X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi)] = P[X_1 > C_{t_m}(\xi), W_i + \rho^i X_1 > C_{t_m}(\xi)]$$

and $W_i + \rho^i X_1 > W_i + \rho^i C_{t_m}(\xi)$ if $X_1 > C_{t_m}(\xi)$. So $W_i > (1-\rho^i)C_{t_m}(\xi) \Rightarrow W_i + \rho^i X_1 > C_{t_m}(\xi)$ if $X_1 > C_{t_m}(\xi)$. Hence the event

$$\{X_1 > C_{t_m}(\xi), W_i + \rho^i X_1 > C_{t_m}(\xi)\} \subseteq \{X_1 > C_{t_m}(\xi), W_i > (1-\rho^i)C_{t_m}(\xi)\}$$

$$\therefore P\{X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi)\} \leq P\{X_1 > C_{t_m}(\xi), W_i > (1-\rho^i)C_{t_m}(\xi)\}$$

$$= P\{X_1 > C_{t_m}(\xi)\} P\{W_i > (1-\rho^i)C_{t_m}(\xi)\}$$

$$= e^{-C_{t_m}(\xi)} P\{W_i > (1-\rho^i)C_{t_m}(\xi)\}.$$

Recalling that $W_i = 0$ with probability ρ^i and $W_i \sim \exp(1)$ with probability $1-\rho^i$ we have $P\{W_i > (1-\rho^i)C_{t_m}(\xi)\} = (1-\rho^i)e^{-(1-\rho^i)C_{t_m}(\xi)}$.

Hence

$$\frac{P(X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi))}{P(X_1 > C_{t_m}(\xi))} \leq (1-\rho^i)e^{-(1-\rho^i)C_{t_m}(\xi)} < e^{-(1-\rho^i)C_{t_m}(\xi)}.$$

Now

$$\sum_{i=1}^{P_m-1} \left(\frac{P_m-i}{P_m} \right) \frac{P(X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi))}{P(X_1 > C_{t_m}(\xi))} < \sum_{i=1}^{P_m-1} \left(\frac{P_m-i}{P_m} \right) e^{-(1-\rho^i)C_{t_m}(\xi)}$$

$$< e^{-(1-\rho)C_{t_m}(\xi)} \left[\sum_{i=1}^{P_m-1} \left(\frac{P_m-i}{P_m} \right) \right],$$

since $e^{-(1-\rho)C_{t_m}(\xi)} \geq e^{-(1-\rho^i)C_{t_m}(\xi)}$ $\forall i \geq 1$ and $C_{t_m}(\xi) = \ln t_m - \ln \xi \geq 0$ since $t_m > \xi$.

$$e^{-(1-\rho)C_{t_m}(\xi)} \left[\sum_{i=1}^{P_m-1} \left(\frac{P_m-i}{P_m} \right) \right] = e^{-(1-\rho)(\ln t_m + \ln(P_m+q_m) - \ln \xi)}$$

$$\begin{aligned} \left(\sum_{i=1}^{P_m-1} 1 - \sum_{i=1}^{P_m-1} \frac{1}{P_m} \right) &= \left(\frac{P_m-1}{2} \right) e^{-(1-\rho)(\ln t_m + \ln(P_m+q_m) - \ln \xi)} \\ &= \frac{P_m-1}{2} \xi^{(1-\rho)} e^{-(1-\rho)(\ln t_m + \ln(P_m+q_m))} \end{aligned}$$

since

$$\sum_{i=1}^{P_m-1} 1 - \frac{1}{P_m} \sum_{i=1}^{P_m-1} i = P_m - 1 - \frac{1}{P_m} \frac{(P_m-1)P_m}{2} = P_m - 1 - \left(\frac{P_m-1}{2} \right) = \frac{P_m-1}{2}.$$

We have for each fixed ξ

$$\begin{aligned} \sum_{i=1}^{P_m-1} \left(\frac{P_m-i}{P_m} \right) \frac{P(X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi))}{P(X_1 > C_{t_m}(\xi))} &< \frac{P_m-1}{2} \xi^{(1-\rho)} \\ e^{- (1-\rho)(\ell_n m + \ell_n(P_m + q_m))} &< P_m \xi^{(1-\rho)} e^{- (1-\rho)\ell_n m} \\ &= \frac{P_m \xi^{(1-\rho)}}{m^{(1-\rho)}} \end{aligned}$$

Let $P_m = [m^{(1-\rho)/2}]$ where $[X]$ is the greatest integer less than or equal to X . Now

$$\frac{P_m \xi^{(1-\rho)}}{m^{(1-\rho)}} < \frac{m^{(1-\rho)/2} \xi^{(1-\rho)}}{m^{(1-\rho)}} = \frac{\xi^{(1-\rho)}}{m^{(1-\rho)/2}}$$

and

$$\lim_{m \rightarrow \infty} \frac{\xi^{(1-\rho)}}{m^{(1-\rho)/2}} = 0 \quad \forall \xi > 0$$

and $0 < \rho < 1$. So

$$\sum_{i=1}^{P_m-1} \left(\frac{P_m-i}{P_m} \right) \frac{P(X_1 > C_{t_m}(\xi), X_{i+1} > C_{t_m}(\xi))}{P(X_1 > C_{t_m}(\xi))} \rightarrow 0$$

as $m \rightarrow \infty$ for this choice of $\{P_m\}$. Let $q_m = [m^{(1-\rho)/4}]$, then $0 < \frac{q_m}{P_m}$

$$< \frac{m^{(1-\rho)/4}}{m^{(1-\rho)/2} - 1} \quad \text{and} \quad \frac{m^{(1-\rho)/4}}{m^{(1-\rho)/2} - 1} \rightarrow 0 \quad \text{as } m \rightarrow \infty, \text{ so } \frac{q_m}{P_m} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

$$\frac{(m+1)^{(1-\rho)/2} - 1}{m^{(1-\rho)/2}} < \left(\frac{P_{m+1}}{P_m} \right) < \frac{(m+1)^{(1-\rho)/2}}{(m^{(1-\rho)/2}) - 1} \quad \text{and} \quad \frac{(m+1)^{(1-\rho)/2}}{m^{(1-\rho)/2} - 1} \rightarrow 1,$$

$\frac{(m+1)^{(1-\rho)/2}}{m^{(1-\rho)/2}} - 1 \rightarrow 0$ as $m \rightarrow \infty$. So $1 \leq \lim_{m \rightarrow \infty} \frac{P_{m+1}}{P_m} \leq \lim_{m \rightarrow \infty} \frac{P_{m+1}}{P_m} \leq 1$ implying $\lim_{m \rightarrow \infty} \frac{P_{m+1}}{P_m} = 1$. It also remains to check that $m\phi(q_m) \rightarrow 0$ as $m \rightarrow \infty$. $m\phi(q_m) = 5m\rho^{m(1-\rho)/4} < 5m\rho^{m((1-\rho)/4)}$ and $0 < m\rho^{m((1-\rho)/4)} < m\rho^m$ since $m^{(1-\rho)/4} < m$ for $0 \leq \rho < 1$, $\lim_{m \rightarrow \infty} m\rho^m = \lim_{m \rightarrow \infty} \frac{m}{\rho^{-m}} = \lim_{m \rightarrow \infty} \frac{m}{e^{-m \ln \rho}}$ and by L'Hospital's rule, $= \lim_{m \rightarrow \infty} \frac{1}{-\ln \rho e^{-m \ln \rho}} = \frac{-1}{\ln \rho} \lim_{m \rightarrow \infty} e^{m \ln \rho} = 0$ since $\ln \rho < 0$ for $0 \leq \rho < 1$.

We now have the following limit theorem as a special case of Loynes' theorems.

Theorem 3: For the stationary first order autoregressive exponential process defined by $X_n = \rho X_{n-1} + \varepsilon_n$, $0 \leq \rho < 1$, $M_n = \max(X_0, X_1, \dots, X_n)$, ε_n i.i.d. with $\varepsilon_n = 0$ with probability ρ , $\varepsilon_n \sim \exp(1)$ with probability $1-\rho$, $P\{M_n \leq x + \ln n\} \rightarrow e^{-e^{-x}}$ as $n \rightarrow \infty$ for each x .

In 1964 Berman showed that M_n and M_n^* with the same normalization converge to $e^{-e^{-x}}$ for stationary Gaussian sequences when $r_n \ln n \rightarrow 0$ as $n \rightarrow \infty$ where $r_n = E\{X_0 X_n\}$ with $EX_0 = 0$, $EX_0^2 = 1$. In 1975 Mittal and Ylvisaker showed that if $r_n \ln n \rightarrow \infty$ a different normalization is appropriate for M_n , and M_n converges to the normal distribution. The exponential autoregressive process studied in this paper has $r_n = \rho^n$ and hence $r_n \ln n \rightarrow 0$. It would be nice to classify stationary processes and determine the limit laws for the maxima of these processes as they have done so nicely for Gaussian processes. All stationary finite autoregressive processes have $r_n \ln n \rightarrow 0$. It is still an open question whether or not there exists a stationary process with $r_n \ln n \rightarrow 0$ and $P[M_n^* \leq a_n X + b_n] \rightarrow G(X)$, but $P[M_n \leq a_n X + b_n] \not\rightarrow G(X)$.

We have seen a particular case where the limit law for the maximum parallels the result for Gaussian processes. It seems possible that under conditions less stringent than uniform mixing or even Leadbetter's condition a limit theorem can be obtained.

REFERENCES

1. Barlow, R. E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing, Probability Models. Holt, Rinehart and Winston, pp. 29-31.
2. Berman, S. M. (1964). Limit theorems for the maximum term in stationary sequences. AMS, Vol. 35.
3. Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of random variables, with applications. AMS, Vol. 38, pp. 1466-1474.
4. Gaver, D. P. and Lewis, P. A. W. (1975). First order autoregressive gamma sequences and point processes. Unpublished.
5. Leadbetter, M. R. (1977). On extreme values of sampled and continuous stochastic data. SIMS Technical Report No. 10, Stanford University.
6. Loynes, R. M. (1965). Extreme values in uniformly mixing stationary stochastic processes. AMS, Vol. 36, pp. 993-999.
7. Mittal, Y., and Ylvisaker, D. (1975). Limit distributions for the maxima of stationary Gaussian processes. Stochastic Processes and Their Applications, Vol. 3, pp. 1-18.
8. Slepian, D. (1962). The one-sided barrier problem for Gaussian noise. Bell Systems Tech. J., Vol. 41, pp. 463-501.