GYROKINETIC ENERGY CONSERVATION AND POISSON-BRACKET FORMULATION

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ABSTRACT

An integral expression for the gyrokinetic total energy of a magnetized plasma with general magnetic field configuration perturbed by fully electromagnetic fields was recently derived through the use of a gyro-center Lie transformation. We show that the gyrokinetic energy is conserved by the gyrokinetic Hamiltonian flow to all orders in perturbed fields. This paper is concerned with the explicit demonstration that a gyrokinetic Hamiltonian containing quadratic nonlinearities preserves the gyrokinetic energy up to third order. The Poisson-bracket formulation greatly facilitates this demonstration with the help of the Jacobi identity and other properties of the Poisson brackets.

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I. Introduction

Conservation laws provide us with powerful tools in the study of nonlinear dynamical systems, as the symmetry characteristics of the former allow reduction (in the sense of dynamical variables) of the latter. In the case of a Hamiltonian system, the reduced dynamics are generated by a reduced Hamiltonian vector field.^{1,2} It is important, however, that the original conservation laws be also preserved in their reduced form, i.e., that the reduced Hamiltonian flow satisfies the reduced version of the original conservation law.

Approximate conservation laws are also of interest. Indeed, because of their associated asymptotic scale separation, they allow the asymptotic expression of reduced dynamics. The pioneering work of Littlejohn³⁻⁵ on the application of Lie transformation methods to the development of a Hamiltonian guiding-center theory of charged particle motion has provided much of the impetus for the establishment of Hamiltonian gyrokinetics. Within Hamiltonian guiding-center theory, Littlejohn demonstrated both energy and (angular) momentum conservation to all orders in single particle motion,⁵ and elegantly rederived expressions for the first and second adiabatic invariants.³⁻⁶

In Hamiltonian gyrokinetics, the unperturbed guiding-center Hamiltonian system is perturbed by electromagnetic fields with spatial and time scales that permit asymptotic analysis.⁷ The asymptotic reduction is carried through by a transformation to gyro-center coordinates, characterized by the adiabatic invariance of the gyrokinetic magnetic moment, canonically conjugate to the gyro-angle. The reduced dynamics are, therefore, expressed in terms of a gyrokinetic Hamiltonian with no gyro-angle dependence.⁸⁻¹⁵

The issue of the preservation of the gyrokinetic total energy by the gyrokinetic Hamiltonian flow was discussed by various authors, but due to the tediousness of the calculations involved, mixed results have been obtained.^{9,13,14,16} By using the algebraic simplicity associated with the Poisson-bracket formulation, we express our previous gyrokinetic treatment¹⁵ in terms of Poisson brackets, and show, conclusively, that the total gyrokinetic energy is conserved by the gyrokinetic Hamiltonian flow, to all orders in our asymptotic analysis.

This result is, in fact, not surprising at all, due to the Hamiltonian nature of the method used.³⁻⁵ The necessity, however, to keep all relevant nonlinear terms in our gyrokinetic Maxwell-Vlasov system,¹⁵ makes the explicit demonstration of this result valuable and represents an important test on the correctness of our nonlinear gyrokinetic system.

II. Poisson-bracket Formulation of Gyrokinetics

1. Gyro-center Transformation. The gyro-center Lie transformation introduced in Ref. 15, using the action-variational Lie perturbation method of Littlejohn¹⁶ and Cary and Littlejohn,¹⁷ was devised so that the gyrokinetic Hamiltonian (and, consequently, the equations of motion) would not depend on the gyro-angle. The generating vector fields

carrying the transformation to gyro-center coordinates $Z^a = (\mathbf{X}, \rho_0, \mu, \zeta, w, t)$, using the compact Poisson-bracket formulation, are given as

$$G_{10}^* = \delta \mathbf{A}_0 \cdot \{\mathbf{X} - \rho_0, Z^a\}_0 - \{S_{10}, Z^a\}_0, \quad (1)$$

for the first-order generating vector field, and

$$G_{20}^* = \{S_{20}, Z^a\}_0 - \frac{1}{2} \{S_{10}, \delta \mathbf{A}_0\}_0 \cdot \{\mathbf{X} + \rho_0, Z^a\}_0 + \frac{1}{2} \{S_{10}, \mathbf{X} + \rho_0\}_0 \cdot \{\delta \mathbf{A}_0, Z^a\}_0, \quad (2)$$

for the second-order generating vector field. The operator $(T_{GY0}^*)^{-1}$, involved in the evaluation of the pull-back of an arbitrary scalar function F ,¹⁶ is given by the expression

$$(T_{GY0}^*)^{-1} F = F - G_{10}(F) - (G_{20} - \frac{1}{2} G_{10}^2)(F) + \dots \quad (3)$$

The subscript (0) in the previous expressions is used to indicate that the unperturbed magnetic field is taken as a uniform field.

The gyro-angle-dependent phase-space gauge functions S_{10} and S_{20} , in the previous expressions, are chosen so that the extended phase-space canonical pair (w, t) is not affected by the gyro-center transformation.¹⁵ The uniform-field Poisson bracket of two arbitrary phase-space functions is given as¹⁵

$$\begin{aligned} \{F, G\}_0 = & \left(\frac{\partial G}{\partial \mu} - B \frac{\partial G}{\partial w} \right) \frac{\partial F}{\partial \zeta} - \frac{\partial G}{\partial \zeta} \frac{\partial F}{\partial \mu} \\ & + \left(\frac{\dot{b}}{B} \times \nabla_0 G + \frac{\dot{b}}{B} \frac{\partial G}{\partial \rho_{||}} - \rho_{||} \mathbf{B} \frac{\partial G}{\partial w} \right) \cdot \nabla_0 F - \left(\frac{\dot{b}}{B} \cdot \nabla_0 G \right) \frac{\partial F}{\partial \rho_{||}} \\ & + \left(\frac{\partial G}{\partial t} + \rho_{||} \mathbf{B} \cdot \nabla_0 G + B \frac{\partial G}{\partial \zeta} \right) \frac{\partial F}{\partial w} - \frac{\partial G}{\partial w} \frac{\partial F}{\partial t}. \end{aligned}$$

Finally, the perturbation fields $(\delta \chi_0, \delta \phi_0 \text{ or } \delta \mathbf{A}_0)$ have the following functional dependence: $\delta \chi_0 = \delta \chi_0(\mathbf{X} + \rho_0, t)$, where the gyroradius is denoted ρ_0 . We note that because of this particular dependence, we have $\{\delta \chi_0, \mathbf{X} + \rho_0\}_0 = 0$, and $\{\delta \phi_0, \delta \mathbf{A}_0\}_0 = 0$.

2. Gyrokinetic Hamiltonian. The gyrokinetic Hamiltonian is given by the expression (containing quadratic nonlinearities)¹⁵

$$H = H_0 - w + \langle \delta \psi_{10} \rangle + \langle \delta \psi_{20} \rangle + \mathcal{O}(\epsilon^3), \quad (4)$$

where

$$\delta \psi_{10} = \delta \phi_0 - \mathbf{V}_{gc0} \cdot \delta \mathbf{A}_0, \quad (5)$$

$$\delta \psi_{20} = \frac{1}{2} \langle |\delta \mathbf{A}_0|^2 \rangle - \frac{1}{2B} \langle \{ \delta \tilde{\Psi}_{10}^{(1)}, \delta \tilde{\psi}_{10} \}_0 \rangle + \delta \tilde{\psi}_{20}, \quad (6)$$

with

$$\begin{aligned}\delta\tilde{\Psi}_{10}^{(m-1)} &= \int d\zeta \delta\tilde{\Psi}_{10}^{(m)}, \\ \delta\tilde{\Psi}_{10}^{(0)} &= \delta\tilde{\psi}_{10} \equiv \delta\psi_{10} - \langle \delta\psi_{10} \rangle,\end{aligned}$$

and $\mathbf{V}_{gc0}$ is the unperturbed particle velocity. The angular brackets $\langle \dots \rangle$ in Eq. (4) represent gyro-angle averaging. Finally, the expressions for S_{10} and S_{20} are given as

$$\begin{aligned}S_{10} &= \frac{1}{B} \delta\tilde{\Psi}_{10}^{(1)} - \frac{1}{B^2} \left(\frac{\partial}{\partial t} - \rho_l \mathbf{B} \cdot \nabla_0 \right) \delta\tilde{\Psi}_{10}^{(2)} + \mathcal{O}(\epsilon^3), \\ S_{20} &= \frac{1}{B} \delta\tilde{\Psi}_{20}^{(1)} + \mathcal{O}(\epsilon^3).\end{aligned}\tag{7}$$

where ϵ is the small parameter involved in the definition of the (nonlinear) gyrokinetic ordering.⁷

III. Gyrokinetic Energy Conservation

1. Energy Conservation Equation. The expression for the gyrokinetic total energy was derived in Ref. 15 and was given as

$$E_{G10} = \int B^2 \langle (T_{GY0}^*)^{-1} H_0 \rangle F d^3\mathbf{z} + \frac{1}{8\pi} \int (|\delta\mathbf{E}|^2 + |\mathbf{B} + \delta\mathbf{B}|^2) d^3\mathbf{x},$$

where $\langle (T_{GY0}^*)^{-1} H_0 \rangle = H - \langle (T_{GY0}^*)^{-1} \delta\phi_0 \rangle$. The gyro-particle distribution function is denoted by F , and the perturbation electromagnetic fields satisfy the gyrokinetic version of Maxwell's equations. From this expression, the energy conservation equation was given as

$$\frac{\delta E}{\delta t} \equiv \frac{\partial H}{\partial t} - \frac{d}{dt} \langle (T_{GY0}^*)^{-1} \delta\phi_0 \rangle - \langle (T_{GY0}^*)^{-1} (\mathbf{V}_{gc0} \cdot \delta\mathbf{E}_0) \rangle.\tag{8}$$

Note that this equation depends entirely on the single gyro-particle dynamics and, therefore, does not involve the gyro-particle distribution function (compare with Ref. 18).

Using the expression for the gyrokinetic Hamiltonian, Eq. (4), it is easy to show that

$$\frac{\partial H}{\partial t} = \langle (T_{GY0}^*)^{-1} \frac{\partial}{\partial t} \delta\psi_{10} \rangle,$$

where

$$\frac{\partial}{\partial t} \delta\psi_{10} = \left(\frac{\partial}{\partial t} \delta\phi_0 + \{ \delta\phi_0, H_0 \}_0 \right) + \mathbf{V}_{gc0} \cdot \delta\mathbf{E}_0,$$

so that the energy conservation equation is, therefore, given as

$$\frac{\delta E}{\delta t} = \langle (T_{GY0}^*)^{-1} \frac{d}{dt} \delta\phi_0 \rangle - \frac{d}{dt} \langle (T_{GY0}^*)^{-1} \delta\phi_0 \rangle.\tag{9}$$

The total time derivative in the first term of the right-hand side of Eq. (9) is the unperturbed time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} - \rho_0 \mathbf{B} \cdot \nabla - B \frac{\partial}{\partial \zeta},$$

whereas the total time derivative in the second term of the right-hand side of Eq. (9) is the perturbed (gyrokinetic) time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} - \rho_0 \mathbf{B} \cdot \nabla + \{, \langle \delta \psi_{10} - \delta \psi_{20} \rangle_0 \}.$$

The explicit expression for the energy conservation equation is given as

$$\begin{aligned} \frac{\delta E}{\delta t} = & \{ \langle \delta \psi_{10} \rangle, \langle \delta \phi_0 \rangle \}_0 - \langle \{ \delta \psi_{10}, \delta \phi_0 \}_0 \rangle \\ & + \langle \{ \delta \bar{\psi}_{10} - \{ S_{10}, \langle \delta \psi_{10} \rangle_0, \delta \phi_0 \}_0 \}, \delta \phi_0 \rangle_0 + \langle \{ S_{10}, \{ \delta \phi_0, \langle \delta \psi_{10} \rangle_0 \}_0 \}_0 \rangle \\ & + \langle \{ \delta \psi_{20} \rangle, \langle \delta \phi_0 \rangle \}_0 - \langle (G_{20} - \frac{1}{2} G_{10}^2) (B \frac{\partial \delta \phi_0}{\partial \zeta}) \rangle + \mathcal{O}(\epsilon^4). \end{aligned} \quad (10)$$

Using the Poisson-bracket property known as the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \quad (11)$$

this expression reduces to (ignoring terms of order ϵ^4)

$$\frac{\delta E}{\delta t} = \{ \langle \delta \psi_{20} \rangle, \langle \delta \phi_0 \rangle \}_0 - \langle (G_{20} - \frac{1}{2} G_{10}^2) (B \frac{\partial \delta \phi_0}{\partial \zeta}) \rangle + \langle \{ \{ S_{10}, \delta \phi_0 \}_0, \langle \delta \psi_{10} \rangle_0 \}_0 \rangle. \quad (12)$$

2. Energy Conservation up to third order. We wish to show that Eq. (12) vanishes at order ϵ^3 . The second term on the right-hand side of Eq. (12) is given, explicitly, by the expression

$$\begin{aligned} - \langle (G_{20} - \frac{1}{2} G_{10}^2) (B \frac{\partial \delta \phi_0}{\partial \zeta}) \rangle = & \langle \{ \delta \bar{\psi}_{20}, \delta \phi_0 \}_0 \rangle + \langle \{ \delta \bar{\Psi}_{10}^{(1)}, \delta \mathbf{A}_0 \cdot \{ \mathbf{X} + \rho_0, \frac{\partial \delta \phi_0}{\partial \zeta} \}_0 \}_0 \rangle \\ & + \frac{1}{2B} \langle \{ \delta \bar{\Psi}_{10}^{(1)}, \{ \delta \bar{\Psi}_{10}^{(1)}, \frac{\partial \delta \phi_0}{\partial \zeta} \}_0 \}_0 \rangle \\ & - \frac{1}{2B} \langle \{ \{ \delta \mathbf{A}_0 \cdot \{ \mathbf{X} + \rho_0, \delta \bar{\Psi}_{10}^{(1)} \}_0, H_0 \}_0, \delta \phi_0 \}_0 \rangle \\ & + \frac{1}{2} \langle \delta \mathbf{A}_0 \cdot \{ \mathbf{X} + \rho_0, \{ \mathbf{X} + \rho_0, B \frac{\partial \delta \phi_0}{\partial \zeta} \}_0 \}_0 \cdot \delta \mathbf{A}_0 \rangle. \end{aligned}$$

It can be further simplified by using the following identities. In the second and fifth terms on the right-hand side of this expression, we use the Jacobi identity to show that

$$\{ \mathbf{X} + \rho_0, B \frac{\partial \delta \phi_0}{\partial \zeta} \}_0 = \{ \mathbf{X} + \rho_0, \{ \delta \phi_0, H_0 \}_0 \} = \{ \delta \phi_0, \mathbf{V}_{\text{gc}0} \}_0,$$

so that

$$\langle \{\delta\tilde{\Psi}_{10}^{(1)}, \delta\mathbf{A}_0 \cdot \{\delta\phi_0, \mathbf{V}_{gc0}\}_0\}_0 \rangle = \langle \{\delta\tilde{\Psi}_{10}^{(1)}, \{\delta\psi_{10}, \delta\phi_0\}_0\}_0 \rangle,$$

and

$$\{\mathbf{X} - \rho_0, \{\mathbf{X} - \rho_0, B \frac{\partial \delta\phi_0}{\partial \zeta}\}_0\}_0 = \{\delta\phi_0, \{\mathbf{X} - \rho_0, \mathbf{V}_{gc0}\}_0\}_0 = \{\delta\phi_0, \mathbf{I}\}_0 = 0,$$

Finally, using again the Jacobi identity and integrating the gyro-angle by parts, the third term on the right-hand side becomes

$$\frac{1}{2} \langle \{\delta\tilde{\Psi}_{10}^{(1)}, \{\delta\tilde{\Psi}_{10}^{(1)}, \frac{\partial \delta\phi_0}{\partial \zeta}\}_0\}_0 \rangle = - \langle \{\delta\tilde{\Psi}_{10}^{(1)}, \{\delta\tilde{\psi}_{10}, \delta\phi_0\}_0\}_0 \rangle + \frac{1}{2} \langle \{\{\delta\tilde{\Psi}_{10}^{(1)}, \delta\tilde{\psi}_{10}\}_0, \delta\phi_0\}_0 \rangle.$$

The energy conservation equation, Eq. (12), is thus given by the expression

$$\begin{aligned} \frac{\delta E}{\delta t} &= \langle \{\delta\psi_{20} + \{S_{10}, \langle \delta\psi_{10} \rangle\}_0 + \frac{1}{2} \{S_{10}, \delta\tilde{\psi}_{10}\}_0 \\ &\quad - \frac{1}{2} \{\delta\mathbf{A}_0 \cdot \{\mathbf{X} + \rho_0, S_{10}\}_0, H_0\}_0, \delta\phi_0\}_0 \rangle. \end{aligned}$$

Finally, using the Poisson-bracket expression for ψ_{20} ,

$$\begin{aligned} \delta\psi_{20} &= \frac{1}{2} \cdot \delta\mathbf{A}_0 \cdot \mathbf{I}^2 - \frac{1}{2} \{S_{10}, \delta\tilde{\psi}_{10}\}_0 - \{S_{10}, \langle \delta\psi_{10} \rangle\}_0 \\ &\quad - \frac{1}{2} \{\delta\mathbf{A}_0 \cdot \{\mathbf{X} + \rho_0, S_{10} H_0\}_0, \end{aligned}$$

where $S_{10} = \frac{1}{B} \delta\tilde{\Psi}_{10}^{(1)}$ is used [see Eq. (7)], we obtain

$$\frac{\delta E}{\delta t} = \langle \delta\mathbf{A}_0 \cdot \{\delta\mathbf{A}_0, \delta\phi_0\}_0 \rangle = 0.$$

IV. Discussion

Looking at the gyrokinetic ordering of the two terms on the right hand side of Eq. (9), we see that the first term contains all terms of order one, two and three, and the second term contains all terms of order two, three, and four. Showing that the gyrokinetic total energy is preserved at orders one, two, and three is, therefore, the best that can be achieved with a gyrokinetic Hamiltonian containing quadratic nonlinearities.

The claim that the gyrokinetic total energy is preserved to all orders of perturbation by the gyrokinetic Hamiltonian flow is, consequently, made in the following sense: the gyrokinetic dynamics generated by the Hamiltonian

$$H = H_0 + H_1 + \dots + H_n,$$

where $n = 1, 2, \dots$, is such that

$$\dot{\mu} = \mathcal{O}(\epsilon^{n+1}),$$

and

$$\frac{\delta E}{\delta t} = \mathcal{O}(\epsilon^{n-2}).$$

The energy conservation equation, for the case of a general magnetic field configuration, has again the same form as Eq. (9). It is easy to show that the gyrokinetic total energy is preserved at orders $\epsilon\epsilon_B$ and $\epsilon^2\epsilon_B$ (provided the first-order generating vector field contains all terms of order $\epsilon\epsilon_B$), where ϵ_B is a small parameter used to describe the magnetic nonuniformity, introduced in Ref. 15.

In the work by Dubin et al.,⁹ gyrokinetic total energy is claimed not to be conserved at order three. Indeed, if the expression for S_0 , Eq. (7), is truncated so that only the first-order term is kept, and is substituted into Eq. (1), then, the energy conservation equation, Eq. (9), does not vanish at order three.

In the work of Hagan,¹⁸ the proof of the energy conservation property is made difficult through the introduction of the gyro-particle distribution function (more precisely, its unperturbed and perturbed parts). It is claimed, however, that the proof, after tedious algebra, is carried through to fourth order by using a gyrokinetic Hamiltonian containing cubic nonlinearities.

In our previous work,¹⁵ we have shown that the energy conservation equation can be entirely written in terms of the single gyro-particle dynamics. In addition, in this present work, we have shown that the proof that the gyrokinetic total energy is preserved by the gyrokinetic Hamiltonian flow, is simplified by the use of the Poisson-bracket formulation.

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