**Mode-locking and the Transition to Chaos
in Dissipative Systems**

CONF-8406186--1

Per Bak
Physics Department
Brookhaven National Laboratory
Upton, New York 11973

BNL--35071

DE84 015723

Tomas Bohr
Laboratory of Atomic and Solid State Physics
Cornell University
Ithaca, New York 14853

Hogens Høgh Jensen
H. C. Ørsted Institute
Universitetsparken 5
Copenhagen, Denmark

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To be published in proceedings of the 59th Nobel symposium "The physics of chaos and related phenomena", Gräfteavallen, Sweden, 1984, Physica Scripta.

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Per Bak

Physics Department
Brookhaven National Laboratory
Upton, New York 11973

Tomas Bohr

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Mogens Høgh Jensen

H. C. Ørsted Institute
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Abstract

Dissipative systems with two competing frequencies exhibit transitions to chaos. We have investigated the transition through a study of discrete maps of the circle onto itself, and by constructing and analyzing return maps of differential equations representing some physical systems. The transition is caused by interaction and overlap of mode-locked resonances and takes place at a critical line where the map loses invertibility. At this line the mode-locked intervals trace up a complete Devil's Staircase whose complementary set is a Cantor set with universal fractal dimension $D \sim 0.87$. Below criticality there is room for quasiperiodic orbits, whose measure is given by an exponent $\beta \sim 0.34$ which can be related to D through a scaling relation, just as for second order phase transitions. The Lebesgue measure serves as an order parameter for the transition to chaos.

The resistively shunted Josephson junction, and charge density waves (CDWs) in rf electric fields are usually described by the differential equation of the damped driven pendulum. The 2d return map for this equation collapses to 1d circle map at and below the transition

to chaos. The theoretical results on universal behavior, derived here and elsewhere, can thus readily be checked experimentally by studying real physical systems. Recent experiments on Josephson junctions and CDWs indicating the predicted fractal scaling of mode-locking at criticality are reviewed.

1. Introduction

In the 17th century the dutch physicist Christiaan Huyghens noted that two clocks hanging back-to-back on the wall tend to synchronize their motion [1]. This phenomenon is known as phase-locking and is generally present in dissipative systems with competing frequencies. The two frequencies may arise dynamically within the system (as for Huyghens coupled clocks, figure 1) or through the coupling of an oscillating or rotating motion to an external periodic force—as in a swing, figure 2. In many-dimensional systems the effective loss of degrees of freedom through dissipation may reduce the phase-locking phenomena to basically two coupled oscillators.

If some parameter is varied (as for instance the eigenfrequency ω_1 of the swing) the system may pass through resonant regimes which are phase locked and regimes which are not. For weakly nonlinear coupling between the oscillators the phase locked intervals will have small measure. The motion is either (with small probability) periodic or, more likely, quasiperiodic, i.e. the ratio between the two frequencies ω_1/ω_2 is irrational. As the nonlinearity increases, the phase locked portions increase, and eventually chaotic motion may occur in addition to the periodic and quasiperiodic (incommensurate) motion. The mechanism, in these systems, leading eventually to chaotic behavior is interaction between the different resonances caused by non-linear couplings, and overlap between the resonant regions when the coupling exceeds a certain critical value. In some sense the mechanism is the

analog for dissipative systems of Chirikov's [2] instability of quasiperiodic orbits in Hamiltonian systems. In this article our recent work on the transition to chaos caused by overlap of resonances [3-6] will be reviewed.

The layout of the article is as follows. In section 2 we shall discuss briefly the Josephson junction in a microwave field, and sliding CDW's (as found for instance in niobium triselenide, NbSe_3) in applied dc plus ac electric fields, and the concept of a return map is introduced. It is shown that the return map exhibits "dimensional reduction" from $d=2$ to $d=1$, and acquires the form of a mapping of the circle onto itself. In section 3 the mode-locking structure of the simple "sine" circle map is investigated, and critical exponents characterizing scaling of mode-locking at criticality and the measure of quasiperiodic orbits below criticality are defined and estimated. The measure of quasiperiodic orbits may serve as an order parameter for the transition to chaos. It is argued that the exponents are universal so the theory is predictive for real physical systems where the return map is a circle map, although not necessarily the sine map.

In section 4 we shall reconsider the return map for the damped driven pendulum in view of the theoretical results from the study of circle maps. It is shown that the return map remains one-dimensional up to and including the transition point. At criticality the curve "crinkles up" instead of just forming a third order inflection point (as the circle maps do). It will be shown that this is a common feature of 2d maps collapsing to 1d, and

that criticality can be defined generally by the point where the return map loses its analyticity. In section 5, as a concrete example, the "Chirikov map with dissipation" will be studied. Numerical evidence is presented that the critical line is smooth and that there is scaling behavior of the mode-locked intervals as for the circle map.

Finally, in section 6, specific experiments to investigate the transition to chaos are discussed and reviewed.

2. Phase locking in Josephson junctions and CDW systems

Fig. 3 shows the equivalent electric diagram for the Josephson junction. Here, θ is the phase difference across the junction. The time dependence of the phase is governed by the differential equation.

$$\alpha \ddot{\theta} + \beta \dot{\theta} + \gamma \sin \theta = A + B \cos \omega t, \quad (1)$$

where $\alpha = \hbar c / 2e$, $\beta = \hbar / 2eR$ and γ is the critical current I_c . Finally, A and B are the amplitudes of the dc and ac microwave component of the current through the junction. The model is usually referred to as the resistively shunted Josephson junction (RSJ) model, and a vast literature exists about it [7]. The differential equation is that of the forced, damped pendulum with mass α , damping coefficient β , and gravitational field γ . For certain values of the parameters the junction can be driven to a "noisy" state [8,9] and indeed numerical simulations have indicated that the noise arises as chaotic solutions to the differential equation [10-12].

In the CDW systems, θ is the position of a sliding CDW relative to an "impurity" pinning potential (Fig. 4). The parameters α , β and γ are phenomenological constants representing the effective mass, damping, and pinning potential [13,14]. A is a dc electric field which depins the CDW, and B is the amplitude of an oscillating rf electric field.

In those systems the phase-locking phenomenon shows up as tendency of the average (angular) velocity $\langle \dot{\theta} \rangle$ to lock into rational multiples of the frequency of the external field,

$$\langle \dot{\theta} \rangle = \frac{N}{M} \omega \quad (2)$$

For small torque A on the pendulum (or dc current in the Josephson junction, or dc voltage in the CDW system) the pendulum stays near its downward position. When A exceeds a critical value the pendulum enters a running "rotating" mode with average velocity $\sim A/B$ (for γ and β not too large).

For the Josephson junction the voltage V is given by the Josephson relation

$$V = \frac{\hbar}{2e} \dot{\theta} \quad (3)$$

so a locking of $\langle \dot{\theta} \rangle$ implies a locking of $\langle V \rangle$ and steps will be seen in the IV characteristics. For $M = 1$ these are the Shapiro steps [15], but in between them subharmonic steps (with $M > 1$) can often be seen. Fig. 5 shows the striking experimental observation by Belykh et al. [9] of a multitude of such sub-steps in the IV characteristics of an Nb-Nb Josephson junction.

In the CDW systems the current carried by the sliding charge density wave is proportional to its velocity $\dot{\theta}$, so the average current is

$$I_{CDW} = \langle \dot{\theta} \rangle. \quad (4)$$

Hence, a locking of $\langle \dot{\theta} \rangle$ implies a locking of the current carried by the CDW. (The current carried by the normal electrons behaves in a smooth way, being proportional to the dc electric field.) The roles of currents and voltages are the reverse for Josephson junctions and CDW systems. Very recently, a multitude of subharmonic steps have been observed by Brown, Grüner, and Mozurkewich [16] in the sliding CDW system niobium-triselenide, NbSe₃.

The most effective way of studying phase locking of differential equations such as (1) is through their return maps. Let us view the system with "stroboscopic light" at the discrete times $T_n = 2\pi n/\omega$, i.e. at the completion of the cycles of the external field. Since the differential equation is of second order the values of the phase θ_n and its derivative $\dot{\theta}_n$ at T_n contain all information about the system, and the values θ_{n+1} and $\dot{\theta}_{n+1}$ at T_{n+1} must be unique functions of θ_n and $\dot{\theta}_n$:

$$\begin{pmatrix} \theta_{n+1} \\ \dot{\theta}_{n+1} \end{pmatrix} = \begin{pmatrix} G_1(\theta_n, \dot{\theta}_n) \\ G_2(\theta_n, \dot{\theta}_n) \end{pmatrix} = R \begin{pmatrix} \theta_n \\ \dot{\theta}_n \end{pmatrix}. \quad (5)$$

Equation (5) defines the two dimensional return map for the differential equation. It is essential that the stroboscopic period be chosen as that of the external force. In general the functions G_1 and G_2 are not known analytically but must be determined numerically. The functions G_1

explicit in eq. (1). Thus the map R can be thought of either as mapping the plane into itself or identifying θ -values differing by multiples of 2π —as a mapping of a cylinder (or an annulus) to itself as shown in Fig. 6.

The Jacobian of the mapping

$$J = \det \begin{pmatrix} \frac{\partial G_1}{\partial \theta_n} & \frac{\partial \dot{G}_1}{\partial \dot{\theta}_n} \\ \frac{\partial G_2}{\partial \theta_n} & \frac{\partial \dot{G}_2}{\partial \dot{\theta}_n} \end{pmatrix} \quad (6)$$

is simply related to the parameters of the differential equation, namely

$$J = e^{-\frac{2\pi\beta}{\omega\alpha}} \equiv b \quad (7)$$

so that because of dissipation ($\beta > 0$) the map is area-contracting, $|J| < 1$. The initial condition will soon be forgotten due to the damping of the motion, and one might hope that the motion will asymptotically be confined to a smooth invariant curve $\theta(t)$ on a torus rather than a cylinder. This means that asymptotically $\dot{\theta}$ is just a given function of θ so the map R has a smooth invariant curve

$$\dot{\theta}_n = g(\theta_n) \quad (8)$$

on which the asymptotic behavior takes place. Going back to eq. (5) and inserting eq. (8) we find a unique relation

$$\theta_{n+1} = f(\theta_n) = G_1(\theta_n, g(\theta_n)), \quad (9)$$

where f maps the circle $0 < \theta_n \leq 2\pi$ to itself, i.e. $f(\theta_n)$ is a circle map.

Whether this dimensional reduction actually takes place depends crucially on the assumption (8) and for given values of the parameters we do not know whether or not it is satisfied, except in the limit $\alpha \ll 1$ where the connection with the circle map has been established analytically [17]. The best we can do is to generate the return map (9) by solving the differential equation (1) numerically. Figure 7 shows the reduced 1d map in a situation where this does indeed happen. The curve seems to be "filled up" ergodically as the integration proceeds, so the asymptotic form is quasiperiodic and the map one-dimensional (see inset). Changing some parameters (for instance increasing A) we can generate plots that asymptotically display only a discrete set of points, namely the Q points $\theta_1^*, \theta_2^*, \dots, \theta_Q^*$ which are stable periodic points of the map R (or f in (9)). Increasing A still further leads again to quasiperiodic motion.

In general, $f(\theta)$ could be any periodic function of θ . However, it will be argued later that the specific form of f is not important for the critical behavior at the transition to chaos, so in the following chapter we shall consider a specific class of return maps, namely the sine circle maps.

3. Mode-locking and universality in circle maps

The "sine" circle maps are defined by the equation

$$\theta_{n+1} = f_{\Omega}(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi} \sin 2\pi\theta_n \quad (10)$$

where we have chosen the periodicity to be 1 instead of 2π . The function f is periodic since we identify the points θ and $\theta + 1$. The advantage of studying simple maps on this form is obvious. It is much easier to identify periodic, quasiperiodic and chaotic solutions by iterating the map than by a cumbersome numerical integration of the underlying differential equation. The map has a linear term θ_n and a bias term Ω representing the frequency of the system in the absence of nonlinear coupling K . To study the mode-locking in the circle map we consider iterations of the map, $\theta, f(\theta), f^2(\theta) \dots$, or $\theta_1, \theta_2, \theta_3$. The frequency of the dynamical system is given by the winding number of the mapping,

$$W = \lim_{n \rightarrow \infty} \frac{f^n_\Omega - \theta}{n} . \quad (11)$$

Clearly, $W = \Omega$ in the absence of nonlinear coupling. Under iteration the variable θ_n may converge to a series which is either periodic, $\theta_{n+Q} = \theta_n + P$, with rational winding number P/Q , quasiperiodic, with irrational winding number $W = q$, or chaotic where the series behaves irregularly.

Although the question of the existence of smooth behavior in circle maps has very much the flavor of the general problem of the existence of smooth invariant tori in dynamical systems (the KAM problem), much stronger theorems due to Arnold [18] and Herman [19] exist for the one-dimensional circle maps. As long as $f(\theta)$ is monotonic these theorems guarantee that no chaotic motion can occur. The transition to chaos, and the associated nontrivial scaling behavior that we shall discuss occurs precisely at the point in parameter space where $f(\theta)$

looses its invertibility. In that case the theorems above break down and not much is known in general. The first indication of interesting scaling behavior was given in a numerical investigation by Shenker [20] followed by renormalization group treatments by Feigenbaum, Kadanoff and Shenker [21] and Rand et al. [22]. These studies concentrate on quasiperiodic behavior with specific well-behaved winding numbers.

In our work we focus on the mode-locking structure at and below the transition to chaos. The mapping (10) is sketched in Fig. 8a for $\Omega = 0.2$ and $K = 0.9$. Again, we have reduced it to the square $0 \leq \theta_n < 1$. We see two branches in the unit square. When $K < 1$ the map is strictly monotonic, just as for the return map calculated for pendulum equation, figure 7. At $K = 1$ the map develops a cubic inflection point at $\theta = 0$ so the map is still invertible but the inverse has a singularity. For $K > 1$ (Fig. 8c) the map develops a local maximum and is no longer invertible. It will be seen later that a very similar scenario occurs for the return map of the differential equation (1) as the parameters are varied.

We shall here be concentrating on the situation for K equal to or slightly below 1. For $0 < K < 1$ it has been shown [19] that the winding number locks-in at every single rational number P/Q in a non-zero interval of Ω , $\Delta\Omega(P/Q)$. (See Fig. 9). For K close to zero all intervals are quite small so the probability that the winding number is rational is almost zero (as for the Huyghens clocks), and the probability of hitting an irrational winding number is almost 1. However, with increasing K the width of all intervals increase. The

regimes in (Ω, K) space where W assumes rational values are called "Arnold tongues". Clearly, the widths of the resonances can not grow indefinitely; at some point they will interact and overlap.

One might speculate that at $K = 1$ the resonances will fill-up the critical line, and confine the quasiperiodic orbits to a Cantor set of zero measure. To test this conjecture we have calculated the widths $\Delta\Omega(P/Q)$ for all intervals with $Q \leq 95$. Figure 10 shows the widths of the "steps" vs. P/Q . Note the self-similarity of the function under rescaling.

Figure 11 shows the "staircase" function formed by plotting the winding number vs. Ω . We conjecture that eventually $\Delta(P/Q) > 0$ for all P and all Q . By including more and more steps the Ω axis becomes more and more "filled up." To investigate whether or not the mode-locked intervals will eventually cover the entire Ω -axis we have calculated the total width of all steps which are larger than a given scale r . We are interested in the space between the steps, $1 - S(r)$ and have measured it on the scale r to find the number of holes, $N(r) = [1 - S(r)]/r$. On figure 12, $\log N(r)$ has been plotted versus $1/r$ for 40 values of r in the interval $[0.0009, 0.000017]$. The points fall excellently on a straight line indicating a power law

$$N(r) \sim (1/r)^D \quad (12)$$

From the slope of the straight line we find $D = 0.8700 \pm 4 \times 10^{-4}$. The result (12) means that the space between the steps vanishes as

$$1 - S(r) \sim r^{1-D} \quad (13)$$

as $r \rightarrow 0$. Thus, there is no room for quasiperiodic motion at criticality, and the staircase figure 11 is called complete. The

exponent D is the fractal dimension of the staircase, or rather the fractal dimension of the Cantor set which is the complementary set to the mode locked intervals on the Ω axis.

The fractal dimension can be determined by an alternative method simply by counting the number of steps $N_1(r)$ which are wider than a given scale r . This number is given by the equation

$$\frac{\partial N_1}{\partial r} = \frac{1}{r} \frac{\partial S(r)}{\partial r} = \frac{1}{r} \frac{\partial r^{1-D}}{\partial r} = r^{-1-D} \quad (13a)$$

so

$$N_1(r) = r^{-D} \quad (13b)$$

The latter method of determining D seems easier since uncertainties in the determination of the stepwidth are not accumulated as when $S(r)$ is calculated. On the other hand, even if $N_1(r)$ obeys the simple power law (13b) there is no guarantee that the staircase is complete. Integration of (13b) lead to

$$S(r) = r^{1-D} + C, \quad (14)$$

where C is an integration constant. The two methods are equivalent only when the staircase is known to be complete, i.e. when $C = 0$.

When passing beyond the $K = 1$ line the steps continue to increase. Since they fill-up the whole Ω axis for $K = 1$ they must necessarily overlap for $K > 1$ (see fig. 9). In an experimental situation the transition to chaos is most easily identified by considering locating the onset of hysteresis at the edges of the larger steps. The transition to chaos is basically caused by overlap of resonances and one can visualize the chaotic motion as an erratic jumping between the resonances. Most dissipative non-linear periodic systems perturbed by an

external periodic field will probably exhibit a transition to chaos caused by overlap of resonances as described here.

It is important to know whether or not the critical behavior at the transition is universal, i.e. whether or not it depends on the specific function $f(\theta)$ in (10). To check universality we have studied a class of mappings

$$f_{\Omega,a}(\theta) = \theta + \Omega - \frac{K}{2\pi} (\sin 2\pi\theta + a \sin^3 2\pi\theta). \quad (15)$$

Generally, the details are different from the staircase shown in figure 11. Some steps become wider, some become narrower. The scaling, however, remains the same, with $D \approx 0.87$. For $a = 1/6$ where the lowest order term in an expansion of $f(\theta)$ vs. θ is of 5th order there is also a complete staircase, but the fractal dimension $D \approx 0.81$. Thus, the fractal dimension depends on the nature of the inflection point, but the generic critical exponent to be expected in an experiment is $D \approx 0.87$.

We would like to stress that although the dimension D was calculated by considering steps in a large Ω interval it is a well-defined and universal number at any infinitesimal interval. The self-similarity is a local property of the staircase.

The steps do not fill-up the entire Ω axis for $K < 1$ and the slope D in the $\log N(r)$ vs. $\log (1/r)$ plot must necessarily converge towards $D \approx 1$ for small r . When K is slightly smaller than 1 it seems that the scaling follows $D \approx 0.87$ down to a certain scale (depending on $1-K$) and then makes a smooth cross-over to the trivial scaling characterized by $D = 1$. We shall now define and estimate an exponent ν which

characterizes this crossover, and another exponent β which governs the measure of the quasiperiodic orbits for $K < 1$. Since the measure, M , is zero at and above the transition, and nonzero below the transition, it may serve as an order-parameter for the transition to chaos, or rather for the transition away from chaos, with the regular phase playing the role of the ordered phase and the chaotic phase corresponding to the disordered phase at a thermodynamic second order transition. Hence, β is the order-parameter exponent just as the exponent characterizing the magnetization in a magnetic system. The quantity $(1-K)$ plays the role of the reduced temperature.

In order to quantify the cross-over we have calculated the scale $r(N_1, 1-K)$ such that the number of resonances which are wider than r is precisely N_1 . Figure 13a shows $\log r(N_1, 1-K)$ versus $(1-K)$ for several values of N_1 . The straight line indicates exponential behavior:

$$r(N_1, 1-K) = r(N_1, 0) \exp[-b(N_1)(1-K)]. \quad (16)$$

Figure 13b shows $\log b(N_1)$ vs. $\log N_1$. The linear behavior allows us to define an exponent ν :

$$b(N_1) \sim N_1^{\frac{1}{D\nu}}, \quad \frac{1}{D\nu} = 0.44 \pm 0.02 \quad (17)$$

Eqs. (16) and (17) yield a cut-off, N_0 , in the number of resonances which give a contribution to the integrated measure of periodic orbits below criticality:

$$N_0 \sim (1-K)^{-D\nu}. \quad (18)$$

This cutoff in N_0 can be related to a cutoff scale r_0 through (13b):

$$r_0 \sim (1-K)^\nu, \quad \nu \sim 2.63. \quad (19)$$

For a given value of K the resonances, which at $K = 1$ are narrower than the critical scale (19), are effectively cut-off. The measure of quasiperiodic orbits is precisely the measure of the cut-off periodic orbits:

$$M(1-K) = \int_{r=0}^{(1-K)^v} \frac{\partial N_1}{\partial r} r \, dr$$

$$\sim (1-K)^{v(1-D)} \equiv (1-K)^\beta. \quad (20)$$

where $\beta = 0.34 \pm 0.02$.

Equation (20) defines a scaling relation,

$$D = 1 - \beta/v \quad (21)$$

which is very similar to the relation

$$D = d - \beta/v$$

which has been derived for second order phase transitions. Here, d is the Euclidean dimension. It would be interesting to determine the measure of quasiperiodic orbits independently in order to check the scaling relation.

4. Transition to chaos in Josephson junctions, charge-density waves and damped driven pendula

Having conjectured the scaling behavior at the critical curve for the circle map, it is of interest to go back to the real physical systems considered in section 2 (which are described by differential equations with 1d return circle maps) and investigate to what extent the results from the circle map can be taken over. First of all, it is important to establish the critical curve for the transition to chaos.

For the circle map this line is just $K = 1$ but for higher dimensional systems no such simple line exists.

The return map depicted in figure 7 is monotonically increasing and thus corresponds to regular behavior only. To approach the regime with chaotic behavior we reduce the damping and vary A to keep the winding number roughly constant, $W = 0.38$. Indeed it appears (figure 14) that the map acquires a zero slope at the transition to chaos so the transition seems closely related to the one occurring in the circle map. If the damping is reduced further (fig. 15) the curve develops a local maximum allowing chaotic behavior.

However, as seen from the insets, the behavior of $f(\theta)$ is more complicated around the transition point: instead of just turning over to form isolated local maxima and minima, the curve "crinkles up" and seems to be filled-up in an uneven nonergodic way. This is not accidental but represents the typical critical behavior for return maps generated through the collapse of higher dimensional systems. Assuming a 2d return map of the form (5) it can be shown for a wide class of functions G_j that if $f'(\theta_n) \rightarrow 0^+$ then $f'(\theta_{n+1}) \rightarrow -\infty$, as indicated numerically for the return map in figure 14. For a proof of this statement, see ref. [6] or the article by T. Bohr in these proceedings [23].

Consider now the graph of f as a function of some parameter and assume to start with that f is an increasing circle map. Varying the parameter decreases $f'(\theta)$ towards zero and therefore $f'(f(\theta))$ must approach minus infinity. But then $f'(\theta)$ must be zero somewhere in the interval $[0, f(\theta)]$ giving another infinite slope, etc. The whole curve crinkles up precisely as found in figures 14 and 15. Thus, the loss of

monotonicity is tied to the loss of smoothness for the invariant circle: if there exists a value of θ for which $f'(\theta) = 0$ then the invariant circle has already broken up and the initial assumption of a smooth orbit $\dot{\theta} = g(\theta)$ is contradictory.

Then, when precisely does the map lose its smooth monotonic behavior? It can be shown that for limit-cycle fixed points this happens precisely when the eigenvalues of the product of the Jacobians (6) at the Q limit cycle points become identical, $\lambda_1 = \lambda_2 = b^{Q/2}$, where b is defined by eq. 7. In the following section we shall see that this allows us to define a smooth transition curve by considering a specific two dimensional map.

5. Scaling behavior of the "Dissipative Standard Map"

Since we are looking for universal behavior we might as well choose a simple analytic map instead of doing the integration of the differential equation. For that particular map we shall try to locate the "critical line", i.e. the line in parameter space where the smoothness breaks down.

The fundamental question is whether the "critical line" in any sense defines a smooth curve. One might fear that since the definition of the criticality condition depends on the denominator Q of the winding number that the curve is fundamentally fractal. Our finding for the particular map which will be studied is that it is not fractal: The dissipation present in the map assures that the critical line defined through the very high order rational steps is smooth. This finding is explored further in ref. [23].

The map is defined by recursion relations in two variables θ and r :

$$\begin{aligned}\theta_{n+1} &= \theta_n + \Omega - \frac{K}{2\pi} \sin 2\pi\theta_n + br_n \\ r_{n+1} &= br_n - \frac{K}{2\pi} \sin 2\pi\theta_n\end{aligned}\tag{22}$$

where b is between 0 and 1. The equation has the required symmetry in θ and the variable r_n plays a similar role as $\dot{\theta}_n$ in (6). The Jacobian matrix of the mapping is

$$\bar{D} = \begin{pmatrix} 1 - K \cos 2\pi\theta & b \\ -K \cos 2\pi\theta & b \end{pmatrix}\tag{23}$$

with determinant b , showing indeed that (22) is area contracting when $b < 1$. When $b \rightarrow 0$ we recover the "sine" circle map (10) and when $b \rightarrow 1$ we obtain the so-called standard map. The conclusions derived in section 4 are certainly valid for the effective reduced 1d map of (23), if such a map exists.

The behavior of a limit cycle with period Q is characterized by the eigenvalues λ_1, λ_2 of the matrix

$$\bar{M} = \bar{D}(\theta_Q^*) \bar{D}(\theta_{Q-1}^*) \dots \bar{D}(\theta_1^*)\tag{24}$$

and the condition for criticality is simply $\lambda_1 = \lambda_2 = b^{Q/2}$. This condition determines for large Q a curve in (K, Ω) space (equivalent to the (β, A) space for eq. (1)) as shown schematically in figure 16. Generally we wish to assign only one critical point to each cycle (and not a curve) and we chose this point to be the lowest point on the hyperbola. Having these points for several P/Q cycles we define the critical curve as the curve determined by the accumulation points of these critical points. Figure 17 shows critical curves calculated for $b = 0.25$ and $b = 0.5$. We first of all notice that the critical points

converge towards a smooth critical curve. A few points fall outside the curve but since they are associated with low Q-cycles they do not affect the limiting behavior.

Having identified a smooth critical curve the next problem is the scaling properties of P/Q steps. A particular limit cycle point or "step" is stable as long as the largest eigenvalue, say λ_1 , is smaller than 1; the condition $\lambda_1 = 1$ is obtained when

$$\text{Tr}M = 1 + b^Q \quad (25)$$

For high Q-cycles the quantity b^Q becomes very small so the condition for criticality is essentially $\lambda_1 = \lambda_2 = 0$, and the condition for stability, $\text{Tr}M = 1$, approach the corresponding conditions for the one-dimensional circle map. This gives a hint that the critical behavior is that of the circle map.

Figure 18 is a plot of the winding number versus Ω at criticality which again forms a staircase. Again, we measure the accumulated width of all steps larger than r in an Ω interval of length Ω_0 . The "number of holes" plotted versus $1/r$ follow a power law

$$\frac{\Omega_0 - S(r)}{r} \sim \left(\frac{1}{r}\right)^D \quad (26)$$

with $D = 0.86 \pm 0.01$. The staircase is therefore complete and the dimension of the Cantor set is within the numerical accuracy identical to the universal exponent found for circle maps. We therefore believe that the critical behavior at the transition to chaos is that of the circle maps.

6. Conclusion and discussion of experiments

We have investigated the transition to chaos by overlap of resonances by studying circle maps, differential equations representing real physical systems, and 2d discrete dissipative maps. Our conjecture is that the critical behavior of all these systems is the same, so theoretical results on circle maps, derived by us and others, can be taken over to explain and predict critical behavior at the transition to chaos for specific experimental systems. We urge that experiments on Josephson junctions, charge-density-wave systems and other systems with two competing periodicities be performed to check our predictions.

We are aware of three experiments which have been performed since we first announced our results: Kao et al. [24] and Alstrom et al. [25] have studied the differential equation (1) with a "Josephson junction simulator"; they find scaling behavior as predicted here, with $D \sim 0.91 \pm 0.04$ and $D \sim 0.87 \pm 0.02$, respectively. More importantly, Brown, Mozurkewich and Grüner [16] have measured subharmonic steps in the CDW system NbSe_3 in ac plus dc electric fields; they also find scaling behavior in agreement with our conjecture, with $D \sim 0.91 \pm 0.03$. We feel that the accuracy of the experiments could be improved; in particular more work should be done to locate the critical line, following, for instance, the ideas presented here. It would be of interest to measure the return map directly. The most precise measurements can probably be performed on Josephson junctions in microwave fields, but no experiment on the scaling behavior near the critical point has been reported so far.

At this symposium we have seen a number of other exciting experimental systems with competing frequencies for which the theory presented here can readily be applied. Jeffries [26] has found frequency locking and transitions to chaos in a resonantly driven p-n junction in silicon. Arecchi [27] has observed deterministic chaos and resonances for a homogeneously broadened laser with injected signal. Gollub has studied mode-locking and quasi periodic behavior in Rayleigh-Benard experiment [28]. The transition to chaos in this system is precisely of the type discussed here. Swinney [29] has found phase locking and a "devil's staircase" in the Belusov-Zhabotinsky reaction. The staircase seems to have a different structure than the one found here, so it is not yet clear to what extent the theory applies in this case.

Acknowledgement

Work supported by the Division of Materials Sciences U.S. Department of Energy under contract DE-AC02-76CH00016, by National Science Foundation grant DMR-8314625 and by the Danish Natural Science Research Council.

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Figure Captions

- Figure 1. The Huyghens clocks. Two weakly interacting clocks tend to synchronize each other.
- Figure 2. The swing. A system with two coupled frequencies. If a torque A is applied to the pendulum the situation is very similar to the ac-driven Josephson junction.
- Figure 3. The resistively shunted Josephson junction, driven by a constant current A and a microwave current with amplitude B.
- Figure 4. Sliding charge density wave in ac plus dc electric field, and a pinning potential which could be a surface potential. The motion is that of a particle rolling down an oscillating washboard.
- Figure 5. IV characteristics of an Nb-Nb Josephson point junction in 295 GHz microwave field at $T = 4.2$ K. (Belykh et al., Ref.[9]).
- Figure 6. Annular map (schematic) showing the invariant circle and the invariant manifolds. The simplest locked state, $P/Q = 1/2$ is shown with a stable and an unstable fixed point.
- Figure 7. Return map calculated numerically for $B = \alpha = \gamma = 1$, $\omega = 1.76$, $\beta = 1.576$, $A = 1.4$. The function $f(\theta_n)$ is monotonically increasing. The inset is a magnification, emphasizing the one-dimensionality of the map. Because of periodicity, only the square $0 < \theta_n, \theta_{n+1} \leq 1$ is shown.
- Figure 8. Evolution of iterations of the circle map for $\Omega = 0.2$ and a) $K = 0.9$ (quasiperiodic orbit), b) $K = 1.0$ (limit cycle) and c) $K = 1.1$ (chaotic orbit). For $K > 1$ the map develops local maxima and minima and chaotic behavior may occur.

Figure 9. Phase diagram for the "sine" circle map. Note the Arnold tongues where the winding number assumes locked rational values. The winding number assumes irrational values along one-dimensional curves ending at $K = 1$. The dotted lines indicate overlap of resonances.

Figure 10. Widths of intervals $\Delta\Omega(P/Q)$. The diagram is self-similar under magnification.

Figure 11. Winding number W vs. Ω for $K = 1$. Note the self-similar nature of the staircase.

Figure 12. Plot of $\log N(r)$ vs. $\log (1/r)$ for the critical circle map. The slope of the straight line yields $D = 0.8700 \pm 3.7 \times 10^{-4}$.

Figure 13. a) Scale r for which there are N intervals wider than r , plotted vs. $(1-K)$. b) Plot of $\log b(N_1)$ defined by eq. (16) vs. $\log N_1$.

Figure 14. Return map for eq. (7) calculated for $\beta = 1.253$, $A = 1.2$. The function develops a cubic inflection point indicating the transition to chaos. The inset shows an enlargement of the curve around $f^2(\theta_I)$ where θ_I is the inflection point ($f'(\theta_I) \rightarrow 0$, $f^{2'}(\theta_I) \rightarrow \infty$).

Figure 15. Return map calculated for $\beta = 1.081$, $A = 1.094$. The map develops a local minimum and "wiggles" (insets) indicating chaotic behavior.

Figure 16. Schematic diagram showing the hyperbolae where the invariant circle becomes critical at the stable periodic points. The critical curve is approximated by the minimum points.

Figure 17. Critical curves for 2d standard map. The curve a) was calculated for $b = 0.25$; the curve b) is a magnification of a). The curve c) was calculated for $b = 0.5$.

Figure 18. The widths of phase locked steps at criticality for the map (22) with $b = 0.25$. Note the similarity with the circle map, figure 11.

Fig. 1

Fig. 2

$$I = I_c \sin \theta$$

$$V = \frac{\hbar}{2e} \dot{\theta}$$

$$A + B \cos \omega t = C \frac{dV}{dt} + \frac{V}{R} + I_c \sin \theta$$

$$= \frac{\hbar C}{2e} \ddot{\theta} + \frac{\hbar}{2eR} \dot{\theta} + I_c \sin \theta$$

Fig. 3

$$E = A + B \sin \omega t$$

$$\alpha \ddot{\theta} + \beta \dot{\theta} + \gamma \sin \theta = E$$

Fig. 4

Fig. 5

Fig. 6

Fig. 8a

Fig. 8b

Fig. 8c

Fig. 9

Fig. 10

Fig. 11

Fig. 12

Fig. 13a

Fig. 13b

Fig. 14

Fig. 15

Fig. 16

Fig. 17a

Fig. 17b

Fig. 17c

Fig. 18