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Maximum Entropy Signal Restoration With Linear Programming

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MAXIMUM ENTROPY SIGNAL RESTORATION WITH LINEAR PROGRAMMING

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Abstract

Dantzig's bounded-variable method is used to express the maximum entropy restoration problem as a linear programming problem. This is done by approximating the nonlinear objective function with piecewise linear segments, then bounding the variables as a function of the number of segments used. The use of a linear programming approach allows equality constraints found in the traditional Lagrange multiplier method to be relaxed. A robust revised simplex algorithm is used to implement the restoration. Experimental results from 128- and 512-point signal restorations are presented.

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MAXIMUM ENTROPY SIGNAL RESTORATION WITH LINEAR PROGRAMMING

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Abstract: Dantzig's bounded-variable method is used to express the maximum entropy restoration problem as a linear programming problem. This is done by approximating the nonlinear objective function with piecewise linear segments, then bounding the variables as a function of the number of segments used. The use of a linear programming approach allows equality constraints found in the traditional Lagrange multiplier method to be relaxed. A robust revised simplex algorithm is used to implement the restoration. Experimental results from 128- and 512-point signal restorations are presented.

1 Introduction

"...when we make inferences based on incomplete information, we should draw them from that probability distribution that has the maximum entropy allowed by the information we do have."[1]

The fundamental task in signal restoration is to replace information which is lost through a signal degradation process. This must be done within constraints imposed by the signal and the degradation process. Only in rare cases is there exact knowledge of the missing data. We are usually resigned to making the "best possible" estimate of what should be used to restore the signal. "Best" is a qualitative judgement and there are many quantitative criteria for defining "best." One definition is based on the well known fact that natural systems tend toward energy states which maximize entropy [2]. Maximum entropy signal restoration techniques are an attempt to mimic nature by maximizing the entropy of the restored information subject to constraints imposed by the problem.

An important point must be made which can affect the layman's belief about the appropriateness of the maximum entropy technique. A maximum entropy restoration is not necessarily a maximum probability restoration. In other words, the signal restored by a maximum entropy technique is usually not the most probable restored signal. It is the restoration which can be combinatorily formed in the most number of ways [3]. If maximum probability is not the required criteria, then maximum entropy restoration is appropriate.

There is considerable controversy over the form of the entropy expression. The issue is whether $-f \ln f$ or $-\ln f$ is the proper form, f representing signal amplitude. Kikuchi

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and Soffer [4] examine this argument in great depth and provide detailed mechanisms for determining which form is appropriate for a given problem. This analysis is well beyond the scope of this report. The convention in the image restoration field is to use the form $-f \ln f$. The majority of the image restoration problems posed as maximum entropy problems have occurred in optical astronomy and the $-f \ln f$ form is supported by Kikuchi and Soffer for this application. Experience has shown that even a crude approximation to $-f \ln f$ works well, as long as the function remains convex. Results presented later in this report show that reducing the number of piecewise linear segments used to approximate the entropy function to just two segments has little effect on the restoration.

Some readers may not come from an image or signal processing background, so some discussion is necessary of signal degradations and parameters which must be supplied to the restoration processes. Consider, for example, the case of imaging dim stars through an optical telescope. The image captured by the telescope is degraded by the turbulent atmosphere and by unique properties of the optical telescope. We don't really care which degradations occur in the telescope and which occur in the atmosphere. We are interested in modeling the entire degradation. If we were to image a point source, then the resulting image is a quantitative measure of how the telescopic/atmospheric system degrades the perfect input. This is known as the point-spread function. The image we see through the telescope results from a convolution of the point-spread function with the perfect image.

An additional degradation may come from the apparatus we use to boost the intensity of the image captured by the telescope. The apparatus may add noise to the already convolution-degraded image. Experience with the image intensifier or even an educated guess will be used to determine a signal-to-noise ratio. If we guess at the signal-to-noise ratio and restore the image only to find the resulting image to be noisier or smoother than experience tells us it should be, then we will select a different signal-to-noise ratio and reprocesses the image. Likewise, we may have some error in the point-spread function we used in a restoration. We may also revise it and reprocesses the image. The mathematical model for restoration which will be presented in this paper assumes the point-spread function and signal-to-noise ratio are known. These parameters play a central role in LP constraints upon the maximum entropy objective function.

With the question, "Why maximum entropy as a restoration criteria?", addressed and main controversies mentioned for completeness, we will now formulate the maximum entropy signal restoration problem. Once posed, the applicability of Dantzig's bounded-variable linear programming (LP) approach will be addressed. Posing a problem is only half the battle: implementing it is the next issue. We will discuss the intricacies of using a large revised-simplex LP package to execute the bounded-variable method. The report will then be concluded with test results and analysis.

2 Problem Formulation

Our problem is formulated in the spirit of Frieden [5] who was the first to pose the signal restoration problem as a maximum entropy problem. A number of other formulations exist and the reader should refer to [6-10] for details. Let us begin by defining the signal values

$$f_j ; 0 < f_j \leq 1 ; j = 1, \dots, J \quad (1)$$

where each f_j is an unperturbed, noiseless input to the imaging system. The actual signal amplitude is simply rescaled between 0 and 1 here for use in the entropy relation. The input signal undergoes perturbation by a linear detection system. The point-spread function, which is a matrix representation for the detection system degradation characteristics, is known or has been estimated. The degraded signal may then be represented by the relation

$$\sum_{j=1}^J f_j H_{jk} + n_j = g_j ; j, k = 1, \dots, J \quad (2)$$

where H_{jk} is a space-invariant point-spread function (Toeplitz form), n_j is a positive additive noise, and g_j (the output of the convolution equation) is the detected, degraded signal. By adding an amplitude bias, the requirement $n_j > 0$ does not impose any restrictions on real-world data.

Our goal is to maximize the entropy of the restored signal given constraints imposed by the convolution equation (2). We may identify two components in the objective function to be maximized: one due to the input signal and one due to noise. We assume that the signal and noise are independent. With this in mind, we may state our restoration objective as

$$MAX \{z = \sum_{j=1}^J -f_j \ln f_j + \sum_{j=1}^J -n_j \ln n_j\} \quad (3)$$

or

$$MIN \{z = \sum_{j=1}^J f_j \ln f_j + \sum_{j=1}^J n_j \ln n_j\} \quad (4)$$

where the MAX and MIN are taken over the variables $f_j, n_j, j = 1, 2, \dots, J$.

Two more constraints are needed. The first is an energy constraint to assure that there is no gain in the restored signal. The total energy, P_0 , has the form

$$\sum_{j=1}^J f_j = P_0. \quad (5)$$

In reality, we don't know the exact value of P_0 . We estimate P_0 by summing the degraded signal intensities, g_j . An exact value of P_0 is not critical. The second relates to the signal-to-noise ratio of the degraded signal relative to the input signal. Let

$$\rho = \sum_{j=1}^J f_j / \sum_{j=1}^J n_j \quad (6)$$

be our signal-to-noise ratio estimate. Because of the positivity constraint imposed for linear programming, we know that any noise corrupted signal value, g_j , will have an amplitude greater than or equal to that of the signal value, f_j . We may set an upper bound on f_j based on our signal-to-noise estimate, which is

$$f_j \leq g_j - g_j/\rho ; j = 1, 2, \dots, J. \quad (7)$$

The final form of the problem is, therefore,

$$MIN \{z = \sum_{j=1}^J f_j \ln f_j + \sum_{j=1}^J n_j \ln n_j\}$$

subject to

$$\begin{aligned} \sum_{j=1}^J f_j H_{jk} + n_j &= g_j ; j, k = 1, \dots, J \\ \sum_{j=1}^J f_j &= P_0 \\ g_j - g_j/\rho &\geq f_j \\ f_j > 0; n_j > 0; j &= 1, \dots, J. \end{aligned} \tag{8}$$

Most researchers pose this problem with equality constraints and solve it using Lagrange multipliers. If, however, one can approximate the objective function with piecewise linear segments, the resulting system can be solved by an LP technique. Posing the problem in this way allows one to introduce inequality constraints which can be a distinct advantage for very large problems.

3 Implementing Dantzig's Bounded-Variable Method

The problem posed by equations (8) with a nonlinear objective function, but linear constraints, is not new to LP applications. Kuhn and Tucker [11] and Tucker [12] showed that if the concave objective function were nonnegative and differentiable, then the method of Lagrange multipliers could be extended to inequality constraints on concave functions. Dantzig [13] considered convex functions having nonpositive values (negative or concave functions) and relaxed the requirement of differentiability provided that several conditions on the convex set are satisfied. Dantzig [13] further developed the convex objective function approach to include separable convex objectives. The bounded variable method is one such approach. It has the advantage of keeping the number of equations fixed, while increasing the number of variables with the number of separable objectives. When dealing with LP techniques, increasing tableau columns is always a computational advantage over increasing tableau rows.

We may state the LP restoration problem of equations (8) using Dantzig's bounded-variable method fairly simply. The properties of convex functions which form the theoretical basis of this method are not discussed here. The reader is referred to reference [13] for a complete discussion. We shall begin with a mathematical description of the piecewise linear approximation to a general convex function, extend the description to our maximum entropy function, then present our LP restoration problem in the form used in the experiments which are discussed in the following section.

Let $\bar{\varphi}(x)$ be a piecewise linear approximation to a general continuous convex function $\varphi(x)$. We will momentarily restrict our discussion to x having only one component, x_j . This is in contrast to the real restoration problem in which x is a vector having J components. Let $\bar{\varphi}(x_j)$ have break-point coordinates $(f_{rj}, \bar{\varphi}(f_{rj}))$, $r = 0, 1, \dots, k+1$, where k is the number of piecewise linear segments (Figure 1). Any x_j bounded by f_{0j} and f_{rj} may be

Figure 1. Arbitrary objective function having break-point coordinates $(f_{rj}, \varphi(f_{rj}))$. x_j is bounded by f_{1j} and f_{2j} . The superscript 0 denotes simplex iteration number 0.

represented as

$$x_j = \sum_{r=0}^k \lambda_{rj} f_{rj} \quad (9)$$

where

$$\sum_{r=0}^k \lambda_{rj} = 1 \quad (10)$$

and

$$\lambda_{rj} \geq 0. \quad (11)$$

Note that for the case shown in Figure 1, λ_{3j} through λ_{kj} are zero and $\lambda_{2j} = (1 - \lambda_{1j})$.

The general LP problem, where x is now a vector having J components, may then be stated as

$$\begin{aligned} \text{MIN } \{z = \sum_{j=1}^J \overline{\varphi}_j(x_j)\} \\ = \text{MIN } \{z = \sum_{j=1}^J \sum_{r=0}^k \lambda_{rj} \overline{\varphi}(f_{rj})\} \end{aligned}$$

where

$$x_j = \sum_{r=0}^k \lambda_{rj} f_{rj}$$

subject to

$$\begin{aligned} \sum_{j=1}^J a_{ij} [\lambda_{0j} f_{0j} + \lambda_{1j} f_{1j} + \dots + \lambda_{kj} f_{kj}] &= g_j \\ \sum_{r=0}^k \lambda_{rj} &= 1 ; j = 1, \dots, J \\ \lambda_{rj} &\geq 0. \end{aligned} \quad (12)$$

where a_{ij} is the cost matrix. We solve the LP problem, then compute x_j from equation (9) to complete the solution.

The primary drawback to posing the signal restoration problem in the form of equations (12) is that we increase the number of equations by one for each $\overline{\varphi}_j(x_j)$ as well as increasing the number of variables by the number of segments. We may take advantage of the linearization resulting from a segmented convex function while keeping the original number of equations by exploiting Dantzig's upper bounding method. Each variable in the constraint equations is replaced by a set of bounded variables, the number of bounded variables being a function of the number of convex segments. Figure 2 graphically shows a convex segmented objective function and it's associated bounded variables. Dantzig's formulation of the linear programming problem follows.

Let s_r represent the slopes of the piecewise linear segments approximating the convex function and let α_r represent the widths (upper bounds) of the trapazoids defined by the segments in Figure 2. If the segmented objective function is convex, then

Figure 2. Convex segmented objective function and its associated bounded variables [13].

The signal-to-noise constraint is not explicitly expressed in equations (17) and (18). It is incorporated in bounds set on the b_i in the revised simplex LP implementation. Just as we set bounds on the variables comprising x_j , we may set bounds on b_i . In this case, we will set a lower bound

$$LB(b_i) = 0 \quad (21)$$

and an upper bound

$$UB(b_i) = b_i - b_i/\rho. \quad (22)$$

In summary, the bounded variable, separable convex function formulation of the restoration problem is

$$MIN \{z = \sum_{i=1}^{2J} [s_{1j}\Delta_{1j} + s_{2j}\Delta_{2j} + \dots + s_{kj}\Delta_{kj}]\} \quad (17)$$

such that

$$\sum_{j=1}^{2J} a_{ij} x_j = b_i \quad ; i = 1, \dots, J+1 \quad (18)$$

where

$$\vec{x} = [f_1, f_2, \dots, f_J, n_1, n_2, \dots, n_J]^T \quad (16)$$

$$\mathbf{b} = [q_1, q_2, \dots, q_J, P_0]^T \quad (19)$$

$$\{a_{ij}\} = \left[\begin{array}{c|cccccc} & & & & & \\ & H & & & I & \\ & & & & & \\ \hline 1 & \dots & \dots & \dots & 1 & 0 & \dots & \dots & \dots & 0 \end{array} \right] \quad (20)$$

$$\mathbf{x}_j \geq \mathbf{0} \quad (23)$$

$$0 \leq \Delta_{ri} \leq \alpha_{ri} \quad (15)$$

s_r = slopes of the piecewise-linear segments defined by the break-points in the convex objective function.

Because x_j in equation (18) is formed by the sum of k segmented variables, the constraint relations have the form

$$\left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \\ H & & & I & & \\ & & & & & \\ & & & & & \\ \hline 1 & \dots & 1 & 0 & \dots & 0 \end{array} \right] \begin{bmatrix} \Delta_{1,1} + \Delta_{2,1} + \dots + \Delta_{k,1} \\ \Delta_{1,2} + \Delta_{2,2} + \dots + \Delta_{k,2} \\ \vdots \\ \Delta_{1,2J} + \Delta_{2,2J} + \dots + \Delta_{k,2J} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{J+1} \end{bmatrix}. \quad (24)$$

4 Experimental Results

Restorations were attempted on one-dimensional signals having 128 and 512 points. All restorations in this report were performed on a VAX 8550 super-minicomputer. The largest LP problem which could reasonably be performed on the VAX 8550 was 513 equations and 3072 unknowns. This corresponds to a 512-point signal with three segments approximating the objective function. From a computer memory size standpoint, a similar problem is 129 equations and 3072 unknowns which corresponds to a 128-point signal and twelve segments. This problem was also attempted.

It is important at the outset to keep in mind that the memory requirements for any 128- or 512-point restoration are a function of the signal content. Experimentation has shown that the larger the blurring window in degraded signals, the more work space memory required by the revised simplex algorithm. Also, the more complex the signal, i.e. number of delta functions or size of extended signal components, the more memory required for deblurring. One can only estimate work array sizes, attempt a restoration, then adjust array sizes in response to error diagnostics. There is also a trade-off between array dimensions, which define page sizes in the revised simplex algorithm, and the amount of paging done by the algorithm. This affects processing time and impacts certain computer hardware limitations.

Several test signals were generated which, after blurring and/or random noise addition, were processed by the algorithm. These test signals were created to test both isolated delta functions and extended Gaussian or sine functions. This is because maximum entropy practitioners have often commented that maximum entropy techniques work well on isolated points on a flat background (as one would find in optical astronomical images) and poorly on extended scenes (such as city photographs). The 512-point test signal is shown in Figure 3. Moving left-to-right, we see a progression beginning with different amplitude point functions, changing to narrow Gaussians and adjacent narrow Gaussians, and ending with a sine lobe. Two 128-point test signals were used. The first is shown in Figure 4 and it is similar in nature to Figure 3. Figure 5 shows the second signal, a biased sinusoid.

The first restoration test run on each signal was a restoration of the test signal without perturbation. The point-spread-function was set to a delta function and the signal-to-noise ratio was set to 1.0×10^{12} . Our purpose was to detect any anomalies associated with the algorithm.

One important discovery from these tests was the effect of inequality constraints on the restoration. An advantage of the LP implementation over the traditional Lagrange multiplier implementation is the opportunity for inequality constraints. In signals where the vast majority of signal values are zero, degeneracy in the revised simplex implementation will occur. By relaxing equality constraints, it is easier for the algorithm to converge to a feasible solution. One must, however, be careful when relaxing these constraints. Figure 6 shows the restoration of a 128-point signal where the convolution constraint

$$\sum_{j=1}^J f_j H_{jk} + n_j = g_j ; j, k = 1, \dots, J \quad (8)$$

was relaxed to

**MAXIMUM ENTROPY RECONSTRUCTION OF
THE ORIGINAL INPUT SIGNAL,
NO BLUR, NO NOISE**

Figure 3. 512-point maximum entropy test signal.

Figure 4. 128-point maximum entropy test signal.

MAXIMUM ENTROPY TEST SIGNAL

Figure 5. 128-point sinusoidal maximum entropy test signal.

$$0.9 g_j \leq \sum_{j=1}^J f_j H_{jk} + n_j \leq 1.1 g_j. \quad (26)$$

Amplitudes are attenuated and the sine lobe is perturbed. Figure 7 shows the effect on the sine-wave signal. When equality constraints are imposed, the problem disappears. This phenomenon appears to be related to the way the objective function is minimized for any given x_j by allowing Δ_1 through Δ_{k-1} to assume the maximum bounded value α , then, for the final segment, setting $\Delta_k < \alpha_k$ to satisfy the objective function (See section on Dantzig's bounded-variable method). Note that the lobes in Figures 6 and 7 are attenuated only near their tops. For all subsequent tests, equality constraints were used.

A number of restorations were performed on signals degraded by a sliding window average blur, additive uniformly distributed random noise, and by a blur followed by additive noise. The number of segments approximating the objective function was changed, too, in order to assess the impact on restorations. Figures 8 through 9 demonstrate restorations of blurred signals only. Regardless of the number of objective function segments, the restorations are correct. Figures 10 through 11 demonstrate restorations of blurred and noise corrupted signals. The noise components do change with the number of segments, but qualitatively a twelve segment restoration is as "good" as a two segment restoration.

It is difficult to reliably quantify the performance of the algorithm in the presence of noise. The criteria for evaluation depends heavily upon exactly what one wishes to recover from the signal and upon the nature of the signal. Is the position of high amplitude signal components enough? Are accurate amplitudes for the signal components needed? Is the resolution of adjacent signal components the key, and if so, are other factors aside from noise involved? Is the signal comprised of point sources or extended sources? The signal in Figure 3 was blurred and corrupted with various amplitudes of additive noise as a test. Generally speaking, restorations of these signals having signal-to-noise ratios of 10 or larger were restored very well. Point source and narrow lobe locations were resolved, as were the adjacent narrow lobes. Restored amplitudes, however, were often attenuated. Below ratios of 10, lower amplitude components were often relegated to noise or were buried in any residual noise in the restored signal.

Professor B. R. Frieden, upon reviewing the mathematics of our algorithm, has suggested the use of a multiplicative constant in the signal-to-noise constraint (equation 7) for fine tuning. The larger the tuning factor amplitude, more weight is given to maximizing noise entropy and less to maximizing object entropy. This allows higher fluctuation, and hence resolution, in the object estimate. Future versions of the algorithm will incorporate this concept.

The times needed to process a number of signals were compiled for the sake of comparison. Variables in the comparisons are signal length, signal-to-noise ratio, blur extent, signal content, and the number of segments approximating the objective function. All tests were performed on a VAX 8550 super-minicomputer running the VAX VMS 4.5 virtual operating system. The version of the revised simplex routine used here comes from release 4 of the SLATEC common mathematical subroutine library [14]. This version contains the enhancements mentioned in Section 3. Other factors which can affect execution times are 1) computer memory size, 2) VMS working set sizes and page quotas, 3) VMS oper-

**MAXIMUM ENTROPY RESTORATION OF
ORIGINAL TEST SIGNAL, NO BLUR, NO NOISE
 $f * h + n = \text{RHS}(J) \pm 10\%$**

Figure 6. Maximum entropy restoration of the test signal in Fig. 4 assuming no blur and a delta function point-spread-function. The convolution constraints were relaxed significantly. A six segment approximation to the objective function was used.

Figure 7. Maximum entropy restoration of the test signal in Fig. 5 assuming no blur and a delta function point-spread-function. Same relaxation of convolution constraints as in Fig. 6. Note the amplitude attenuation. A six segment approximation to the objective function was used.

Figure 8. Maximum entropy restoration of a blurred signal. a) After convolution of the test signal in Fig. 3 with a 5-element rectangular point-spread-function.

b) The restored signal with a three segment approximation to the objective function.

**INPUT SIGNAL
7 ELEMENT BLUR, NO NOISE**

Figure 9. Maximum entropy restoration of a blurred signal. a) After convolution of the test signal in Figure 4 with a 7-element rectangular point-spread-function.

**MAXIMUM ENTROPY RESTORATION OF
A BLURRED SIGNAL, NO NOISE
7 ELEMENT BLUR**

b) The restored signal with a two segment approximation to the objective function.

**MAXIMUM ENTROPY RESTORATION OF
A BLURRED SIGNAL, NO NOISE
7 ELEMENT BLUR**

c) The restored signal with a twelve segment approximation to the objective function.

**INPUT SIGNAL
5 ELEMENT BLUR, SNR = 7.36**

Figure 10. Maximum entropy restoration of a blurred and noisy signal. a) After convolution of the test signal in Fig. 3 with a 5-element rectangular point-spread-function. Uniformly distributed random noise was added to give a signal-to-noise ratio of 7.36.

**MAXIMUM ENTROPY RECONSTRUCTION OF
A NOISY, BLURRED SIGNAL
5 ELEMENT BLUR, SNR = 7.36
SIGNAL COMPONENT**

b) The restored signal with a three segment approximation to the objective function.

**MAXIMUM ENTROPY RECONSTRUCTION OF
A NOISY, BLURRED SIGNAL
5 ELEMENT BLUR, SNR = 7.36
NOISE COMPONENT**

c) The recovered noise component.

**INPUT SIGNAL
7 ELEMENT BLUR, SNR = 11.2**

Figure 11. Maximum entropy restoration of a blurred and noisy signal. a) After convolution of the test signal in Fig. 4 with a 7-element rectangular point-spread-function. Uniformly distributed random noise was added to give a signal-to-noise ratio of 11.2.

**MAXIMUM ENTROPY RECONSTRUCTION OF
A NOISY, BLURRED SIGNAL
7 ELEMENT BLUR, SNR = 11.2
SIGNAL COMPONENT**

b) The restored signal with a twelve segment approximation to the objective function.

MAXIMUM ENTROPY RESTORATION OF
A NOISY, BLURRED SIGNAL
7 ELEMENT BLUR, SNR = 11.2
SIGNAL COMPONENT

c) The restored signal with a two segment approximation to the objective function.

ating system version number, and 4) working space sizes which translate into page sizes within the revised simplex routine. CPU times listed in Table 1 were acquired from batch executions.

5 Conclusion

The most obvious conclusion is that the LP approach using Dantzig's bounded-variable method is a viable alternative to the traditional Lagrange multiplier approach using Newton-Raphson. For larger problems, allowing even an infinitesimal amount of deviation from equality can aid in achieving a basic feasible solution. Too much deviation, however, can introduce anomalies in the restoration as mentioned in the previous section.

A second conclusion is that there is nothing sacred about the maximum entropy form used in the objective function. Figure 12 shows a two segment approximation to $x \ln x$ which is a crude representative of the true continuous function. Nevertheless, the two segment restoration is qualitatively as "good" as the twelve segment restoration. This leads us to assert that the principal rule for this type of restoration is that the objective function be convex. Purists, however, will undoubtedly qualify this statement.

Our experiments have verified a common understanding of maximum entropy practitioners that pure maximum entropy restorations work best for point sources on a flat dark background. Extended scenes, such as our broad lobe, are not well restored. A few experiments with broad lobes on amplitude biases showed that biases were often detrimental to restorations. These problems are to be expected, however, since each signal point has an independent imaging constraint, and the only global constraints are total energy and the maximum entropy objective function. No global smoothing constraints exist.

The LP approach has not addressed a critical problem for practical image restoration, namely size limitations. Restoration of the simple one-dimensional signals presented here requires approximately 7 megabytes of data and program memory. (Smaller array sizes may be tolerated at the expense of additional computation time due to paging.) More complex one-dimensional signals require even more memory. The source of the size problem is the existence of one convolution constraint per signal point. In order to apply the LP approach to reasonably sized images using today's technology, more global constraint relationships will need to be created. The challenge, of course, is to reduce the number of constraint equations without ignoring too much of the necessary information residing in the degraded signal.

Any analysis of an LP problem these days would be incomplete without some speculation about improvements which could be realized by the Karmarkar algorithm [17,18]. Unfortunately, experience with the Karmarkar algorithm is small when compared to experience with the revised simplex routine. The Karmarkar algorithm is touted to execute LP problems more efficiently than revised simplex. If this is true, the LP approach to maximum entropy restoration could rival Frieden's efficient Lagrange multiplier/Newton-Raphson algorithm which requires time proportional to the problem size squared [19]. Optimism, however, must be tempered by unknown aspects of the Karmarkar algorithm such as computational complexity per iteration and memory requirements.

Table 1

Execution Times

<i>Signal</i>	<i>Length (points)</i>	<i>Blur (elements)</i>	<i>SNR</i>	<i>No. Convex Segments</i>	<i>No. Simplex Iterations</i>	<i>CPU Time (Min:Sec)</i>
Fig. 3	128	0	∞	12	1058	1:51.15
Fig. 3	128	0	∞	12	2077	4:33.39
Fig. 3	128	7	4.96	12	1790	4:46.21
Fig. 3	128	7	11.2	12	1793	4:27.39
Fig. 3	128	7	∞	2	366	0:22.88
Fig. 3	128	7	4.96	2	301	0:17.14
Fig. 3	128	7	11.2	2	298	0:17.31
Delta	128	7	∞	12	413	1:43.77
Fig. 2	512	0	∞	3	986	4:03.70
Fig. 2	512	5	∞	3	2234	8:14.87
Fig. 2	512	5	7.36	3	1257	5:48.49
Fig. 2	512	5	2.95	3	1016	4:05.04
Fig. 2	512	0	2.56	3	1205	4:29.74
Delta	512	7	∞	3	545	4:01.19

All blurs resulted from convolution with a rectangular point-spread-function.

Computer: VAX 8550, Oper. Syst.: VMS 4.5, Oper. Syst. Parm.: WSdef=150 WSquo=500, WSextent=1000, Software: SPLP from SLATEC release 4, revised 3/22/87, PRGOPT(*) key 53 min. size = 400,000 and max. size = 450,000.

TWO SEGMENT APPROXIMATION

Figure 12. Two-segment approximation of the objective function superimposed on the continuous objective function. This is a very crude approximation to the objective function, but restorations based on two segments are qualitatively "good".

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