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HADRON STRUCTURE IN THE LADDER MODEL *

MASTER

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Abstract

The (flavor non-singlet) Green's function to find a far off shell quark in a hadron is obtained in the renormalization group improved ladder model for QCD in the space-like axial gauge. Particular attention is paid to the role of the singularity in the gluon propagator.

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I. Introduction

There has been great progress in recent years in understanding the basis for the parton model in quantum chromodynamics. Among important early work one may cite the 1974 work of Applequist and Poggio,¹ Mueller,² and Gribov and Lipatov.³ In 1977, Altarelli and Parisi⁴ argued that the pattern of scaling violation in deep inelastic lepton scattering in QCD might be understood in terms of parton decays. The most recent group of papers,⁵⁻¹² taken collectively, has two themes. The first theme is the argument that all mass singularities can be factored from hard inclusive quark cross sections into the quark distribution functions, thereby establishing a parton model interpretation for hard processes. Most of these arguments use, and demonstrate the relevance of, the axial gauge ladder model. The second theme is the use of this model to calculate hard inclusive cross sections.

The point of view I adopt in the present paper is that if the papers cited above are right, then the function $\phi(k)_{\alpha\beta} = \langle P | \bar{\psi}\psi | P \rangle$ is of direct physical significance in the axial gauge, being essentially the probability to find an off shell quark in a hadron. Furthermore, to leading order in large k^2 , ϕ can be computed in the (normalization group improved) axial gauge ladder model. This computation is the main subject discussed here.

This paper shares much in spirit with the papers cited above. The chief difference in approach compared to the earlier literature is that I am concerned with the full behavior of $\phi_{\alpha\beta}(k)$ before it is attached to a hard cross section, the Dirac indices are traced over, and the variable $k_{\perp}$ is integrated over. I thus add rungs to the large momentum end rather than the small momentum end of the ladder. This method would lead, in a naive calculation, to an infinity of the form $\int_0^1 dx/(1-x)$, associated with the singularities in the gauge propagator. However the large terms

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turn out to be quite harmless and sum to a well behaved function of k^2 and the hadro energy that multiplies $\phi(k)$. The method of adding rungs to the large momentum end of the Bethe-Salpeter equation was used by Dokshitzer, D'Yakonov and Troyan,¹⁰ although their approach differs from that of this paper in several substantial details. The result obtained here appears to be consistent with that of these authors.

Let us now consider deep inelastic electron scattering as a simple example to illustrate in more detail the motivation for this paper. One can write $\nu W_2(x, Q^2)$ as a convolution of the two factors ϕ and H shown in Fig. 1. By definition ϕ contains its quark leg propagators,

$$\Phi_{\alpha\beta}(k, P) = \sum_s \int d^4x e^{ik \cdot x} \langle P, s | \psi_\beta(0) \bar{\psi}_\alpha(x) | P, s \rangle \quad (1.1)$$

and $H_{\alpha\beta}^{\mu\nu}(Q, k)$ is amputated on its quark legs and is two particle irreducible. There should also be color and flavor indices and functions describing gluons in place of quarks, but I will ignore these indices and will restrict my attention to the color non-singlet contribution to νW_2 so as to eliminate the contributions from the two gluon channel.

If we choose a reference frame such that $P^\mu \equiv (P^+, P^1, P^2, P^-) = (P^+, 0, M^2/2P^+, 0)$ and $Q^\mu = (0, Q, Q \cdot P/P^+)$, then the convolution of ϕ and H that gives νW_2 is

$$\nu W_2(x, Q^2) = \frac{2Q \cdot P}{2\pi} \int \frac{d^4k}{(2\pi)^4} \frac{1}{4(P^+)^2} \text{Tr} \{ H^{++}(Q, k) \Phi(k, P) \} \quad (1.2)$$

(for the argument that follows, we should take $P^+ = \text{const. } \sqrt{-Q^2}$).

We are interested in the scaling limit of νW_2 : $Q^\mu = e^t \tilde{Q}^\mu$; $P^\mu = e^t \tilde{P}^\mu + e^{-t} \tilde{a}^\mu$ with $\tilde{P}^2 = 0$, $2\tilde{a} \cdot \tilde{P} = M^2$; $t \rightarrow \infty$ with $\tilde{Q}^\mu, \tilde{P}^\mu, \tilde{a}^\mu$ fixed. To discuss this limit, let us change integration variables from k^μ to $\tilde{k}^\mu$ with $k^\mu = e^t \tilde{k}^\mu$. Now we can use the renormalization groups to replace $H(Q, k; g)$ by $e^{-t} [C_F(2t)^{\gamma_F/b}]^{-1} H(\tilde{Q}, \tilde{k}; g(t))$ where my conventions are: $g^2(t)/(4\pi)^2 \sim 1/2bt$ $b = 11 - \frac{2}{3}N_F$; $\gamma_F(g) \sim \gamma_F g^2/(4\pi)^2$ is the quark anomalous dimension; and C_F is defined by

$$\text{quark propagator} \propto e^{2 \int_0^t d\tau \gamma_F(g(\tau))} \sim C_F (2t)^{\gamma_F/b} \text{ as } t \rightarrow \infty \quad (1.3)$$

I take the quark mass to be zero to simplify the argument; otherwise there is another variable $m(t)$. Since the running coupling constant approaches zero as $t \rightarrow \infty$, we can replace the hard scattering function $H(\tilde{Q}, \tilde{k}; g(t))$ by its expression in lowest order perturbation theory (in this case, zeroth order), thus simplifying the problem enormously.

There is a well known snag hidden just beneath the surface of this quickly flowing argument. One soon discovers that $\tilde{k}$ integral is controlled by the region $\tilde{k}^2 \approx 0$, so that the argument that the lowest order term in H dominates breaks down unless the higher order terms in H are finite as $\tilde{k}^2 \rightarrow 0$. In general, H is singular as $\tilde{k}^2 \rightarrow 0$. However, R.K. Ellis, Georgi, Machacek, Politzer, and Ross⁵ have shown (among other things) that the 2PI amplitude H is free of $\tilde{k}^2 \rightarrow 0$ singularities order by order in perturbation theory in the axial gauge: $\eta_\mu A^\mu = 0$, $\eta^2 = -1$.

Let me mention here two caveats, which are associated with the fact that higher order corrections to H in the axial gauge contain factors like $\ln[(\eta \cdot Q)^2/\eta^2 Q^2]$. First, one may be tempted to choose η_μ to be lightlike. But then the factors of $\ln \eta^2$ appear as (infrared) divergences that cannot be absorbed into the usual renormalization counter terms. Thus QCD is not (conventionally) renormalizable in the lightlike axial gauge. Happily, QCD is renormalizable in the space-like axial gauge.¹³ We therefore choose $\eta^2 = -1$. The second caveat is that our scaling argument remains valid after the introduction of η_μ only if η_μ is fixed (in the reference frame in which $\tilde{Q}^\mu$ and $\tilde{P}^\mu$ are fixed) at $t \rightarrow \infty$. Thus scale invariant combinations like $(\eta \cdot Q)^2/\eta^2 Q^2$ remain fixed as $t \rightarrow \infty$, so that $\ln^N[(\eta \cdot Q)^2/\eta^2 Q^2]/\ln(-Q^2/\mu^2) \rightarrow 0$.

The result of the not very novel argument outlined above is that the hard scattering factor in the cross section can be replaced by its form in lowest

order (renormalization group improved) perturbation theory, provided one works in axial gauge. In the case of vW_2 , this replacement gives

$$vW_2(x, Q^2) \sim 2Q \cdot P \, e_g^2 \int \frac{d^4 k}{(2\pi)^4} \delta((k+Q)^2) [C_F (2t)^{4/b}]^{-1} \\ \times \frac{1}{4(P \cdot k)^2} \text{Tr} \{ \gamma^+ (\not{k} + \not{Q}) \gamma^+ \not{P} \Phi(k, P) \} \quad (1.4)$$

Now all we need is $\Phi(k, P)$, which is the subject of this paper. Of course, one does not need to know the full structure of Φ to obtain vW_2 , since most of the structure gets averaged out in doing the integration in (1.4). In fact, there are elegant methods for dealing with parton hard scattering problems that finesse the need to know Φ at all.^{5,6} Nevertheless, one can give a direct physical interpretation to expressions like (1.4): they are the average over k_μ of the probability Φ to find an off shell quark with momentum k_μ , times a simple cross section for this quark to scatter. The existence of this interpretation for Φ provides, I think, sufficient motivation for studying Φ in as much detail as possible.

In this paper I study Φ in a model, the renormalization group improved ladder model. I now turn to the motivation for the use of this model. The function Φ (or, more properly, its flavor non-singlet part) obeys the Bethe-Salpeter equation shown in Fig. 2. Here ϕ_B is the two particle irreducible version of Φ , which I assume can be neglected in comparison to Φ when k^2 is large. The kernel $K(k, \ell)$ is the full two particle irreducible kernel. We are interested in $\Phi(k)$ for large k , so we set $k^\mu = e^{t\ell^\mu}$ and use the renormalization group to replace $K(e^{t\ell^\mu}, \ell^\mu; g)$ by $K(\tilde{k}^\mu, e^{-t\ell^\mu}, g(t))$. As in the case of vW_2 , we argue that the lowest order term in K (in this case, second order) will dominate at large t because $g(t) \rightarrow 0$ as $t \rightarrow \infty$. Again there is a potential problem arising from the fact that the dominant integration region is $\ell^2 \equiv (e^{-t}\ell)^2 \approx 0$; but we proceed with the assurance that the higher order terms in K are finite as $\ell^2 \rightarrow 0$.

Again there is another potential problem arising from terms proportional to $[g^2(t)]^{N+1} [\ln((\eta \cdot k)^2/\eta^2 k^2)]^L$ in K evaluated at order $2N + 2 > 2$ in perturbation theory. The most interesting region of k^μ is the region that dominates the integration (1.4) to form $vW_2(x, Q^2)$. In this region $(\eta \cdot k)^2 = x^2 (\eta \cdot P)^2 = x^2 (-Q^2)$, but $k^2 \ll Q^2$. Thus $\ln((\eta \cdot k)^2/\eta^2 k^2)$ is large in this region. On the reasonable assumption that $L \leq N$ always, we can still neglect the higher order terms in K as long as $\ln[(\eta \cdot k)^2/\eta^2 k^2] \ll \ln(-k^2/\mu^2)$. (This assumption implies that the Bethe-Salpeter kernel is better behaved for large $(\eta \cdot k)^2/\eta^2 k^2$ than is the hard scattering function which, I suspect, contains corrections of order $\ln^2((\eta \cdot k)^2/\eta^2 k^2)/\ln(-k^2/\mu^2)$.)

In this paper I simply regard the replacement of the full $K(k, \ell, g(t))$ by its lowest order term as a model and derive the consequences for Φ in that model. The range of k_μ for which corrections to the approximate solution Φ to be obtained are negligible will turn out to be the range $\ln((\eta \cdot k)^2/\eta^2 k^2) \ll \ln(-k^2/\mu^2)$ mentioned above. The range of k_μ for which corrections to the model itself are negligible may turn out to be either larger or smaller than this range. I hope to return to this question in a future publication.

To complete the specification of the model, we must specify what the scaling variable e^{2t} is to be. In the usual treatment of this kind of problem, in which one adds rungs to the small momentum (k_μ) end of a ladder with a big momentum Q_μ at the fixed end, it is useful to let $e^{2t} = -Q^2/\mu^2$ rather than, say, $-2Q \cdot k/\mu^2$, because Q^2 is invariant under a scaling of the active momentum k_μ ($k_\mu \rightarrow \lambda k_\mu$). In the present case we are adding rungs to the large momentum (k_μ) end of the ladder with a fixed small momentum P_μ at the other end. Here again it will prove convenient to choose a scaling variable that is invariant under a scaling of the active momentum, namely $e^{2t} \equiv (2k \cdot P)/(-k^2)_\mu^2$.

II. Solution of the Model

In this section we solve (approximately) for $\Phi(k)$ in the ladder model discussed in the introduction and illustrated in Fig. 3:

$$\Phi(k) = \frac{4}{3} \frac{g^2((-2k \cdot P)^2/(-k^2)\mu^2)}{(4\pi)^2} \frac{2}{\pi} \int d^4\ell \delta_+(l-k^2) \times \frac{1}{k} \gamma_\alpha \Phi(\ell) \gamma_\beta \frac{1}{k} D^{\alpha\beta}(\ell-k^2) \quad (2.1)$$

where $\delta_+(q^2)$ is a delta function on the forward light cone and

$$\frac{g^2(\Lambda^2/\mu^2)}{(4\pi)^2} \sim \frac{1}{b \ln(\Lambda^2/\mu^2)} \quad (2.2)$$

Here $b = 11 - \frac{2}{3}N_F = 25/3$ for QCD with four flavors and $4/3 = C_2$ is the appropriate color weighting factor. The axial gauge polarization sum $D^{\alpha\beta}$ is

$$D^{\alpha\beta}(q) = -g^{\alpha\beta} + \frac{q^\alpha q^\beta \cdot \eta^\alpha \eta^\beta}{[\eta \cdot q]_P} - \frac{\eta^2}{[(\eta \cdot q)^2]_P} q^\alpha q^\beta \quad (2.3)$$

The "P's" here indicate that the inverse powers of $\eta \cdot q$ are to be treated as distributions with a principle value prescription at the singularity.¹³

It will prove convenient to study this equation in a reference frame in which $P_T = \eta_T = 0$, η^+ and η^- are fixed and $P^+ e^t$ is very large. In this frame $P^- = M^2/2P^+$ is very small, and we will simplify the kinematics by setting $P^- = 0$. (That is, we neglect M^2 compared to every momentum in the problem).

A scalar amplitude in the expansion of $\Phi(k)$ can depend on k_μ through three Lorentz invariants, for instance, k^2 , $k \cdot P$, and $k \cdot \eta$. It will prove convenient to use the following three variables: $-2k \cdot P$; the Bjorken scaling variable

$$y = \frac{k^2}{2k \cdot P} \quad (2.4)$$

and another scaling variable $(k \cdot \eta/P \cdot \eta) = \eta^2 k \cdot P / (2P \cdot \eta)$, which equals k^+/P^+ in the reference frame we are using. Some important restrictions on the range of these variables follow from the fact that $\Phi(k)$ must vanish unless the momentum $P^\mu - k^\mu$ of the intermediate state lies within the forward light cone: $0 < P^- - k^- = -k^-$ implies $0 < -2k \cdot P$; and $0 < (P-k)^2 = (-2k \cdot P)(1-y)$ implies $y < 1$. Also, the relation $0 < k_\perp^2 = (-2k \cdot P)(y - k^+/P^+)$ implies $k^+/P^+ < y$.

We will be interested in the behavior of Φ in the region of large $-2k \cdot P$ with y and k^+/P^+ fixed. There is another invariant, $-(2P \cdot \eta)^2/\eta^2 = -\eta^2(P^+/ \eta^+)^2$, which is approximately the large momentum squared of a problem into which Φ is to be inserted. The range of $-2k \cdot P$ of interest is $1 < \ln(-2k \cdot P/\mu^2)$ and $-2k \cdot P \leq -(2P \cdot \eta)^2/\eta^2$.

Before we proceed to solve Eq. (2.1), it is useful to rewrite the integration over ℓ^μ in terms of the variables $-2\ell \cdot P$,

$$z = \frac{\ell^2}{2\ell \cdot P} \quad (2.5)$$

and an angle ϕ defined by $S_T \cdot k_T = |S_T| |k_T| \cos \phi$ where $S_T = \ell_T - (\ell \cdot P/k \cdot P)k_T$.

One finds in a straightforward fashion that for any $F(\ell_\mu)$ with the support properties of Φ mentioned above,

$$\begin{aligned} \int d^4\ell \delta_+((\ell-k)^2) F(\ell) &= \int_0^{-2k \cdot P} d(-2\ell \cdot P) \frac{(-2\ell \cdot P)}{(-2k \cdot P)} \int_y^1 dz \int_0^{2\pi} d\phi F(\ell) \\ &= \frac{1}{4} \int_0^{-2k \cdot P} d(-2\ell \cdot P) \frac{(-2\ell \cdot P)}{(-2k \cdot P)} \int_y^1 dz \int_0^{2\pi} d\phi F(\ell) \end{aligned} \quad (2.6)$$

(The lower endpoint on the $-2\ell \cdot P$ integration and the upper endpoint of the z integration are due to the support properties of F . The other endpoints follow from the requirement that $\ell_\mu - k_\mu$ lies on the forward light cone.)

To solve the equation in the large $(-2k \cdot P)$ limit we begin with an assumption: that in the limit of interest $(-2\ell \cdot P)\Phi(\ell)$ varies very slowly over (increasingly)

many units of $\ln(-2\ell \cdot P/\mu^2)$. Then the integration in Eq. (2.1), which is schematically $\int d\ell \ln(-2\ell \cdot P/\mu^2)(-2\ell \cdot P)\Phi(\ell)$, is dominated by values of $-2\ell \cdot P$ that are much less than $-2k \cdot P$ and we may approximate under the integral sign

$$(-2\ell \cdot P)/(-2k \cdot P) \ll 1. \quad (2.7)$$

This assumption is self consistent: the solution we will find does have this property as long as $-2k \cdot P$ is not too small. I return to this question in Sec. III. I have not investigated whether other possible solutions emerge if this assumption is relaxed, but I do not expect any.

An important simplification arises because of (2.7). The variable ℓ^+/P^+ is given in terms of the variables we have adopted by

$$\frac{\ell^+}{P^+} = y - \left[\frac{-2\ell \cdot P}{-2k \cdot P} \right]^{\frac{1}{2}} \left[\left(y - \frac{k^+}{P^+} \right) \left(1 - \frac{-2\ell \cdot P}{-2k \cdot P} \right) (z - y) \right]^{\frac{1}{2}} 2 \cos \phi + \left[\frac{-2\ell \cdot P}{-2k \cdot P} \right] \left[z + \frac{k^+}{P^+} - 2y \right] \quad (2.8)$$

Using (2.7), we can replace

$$\frac{\ell^+}{P^+} \rightarrow y \quad (2.9)$$

under the integration. Similarly, the transverse part of ℓ^μ is given by

$$\underline{\ell}_T = [-2\ell \cdot P]^{\frac{1}{2}} \left\{ \left[1 - \frac{-2\ell \cdot P}{-2k \cdot P} \right] (z - y) \frac{\underline{S}_T}{|\underline{S}_T|} + \left[\frac{-2\ell \cdot P}{-2k \cdot P} \right]^{\frac{1}{2}} \left[y - \frac{k^+}{P^+} \right]^{\frac{1}{2}} \frac{\underline{k}_T}{|\underline{k}_T|} \right\} \quad (2.10)$$

where $\underline{S}_T$ is the vector used above in defining the angle ϕ . Thus $\underline{\ell}_T$ can be neglected compared to terms of size $\sqrt{-2k \cdot P}$. To summarize, inside the important region of the ℓ -integration we have $\ell_\mu \approx \text{const. } P_\mu$.

Let us consider now the Dirac structure of $\Phi(k)$. It suffices to assume that $\Phi(k)$ has a one gamma matrix structure, $\Phi(k) = A_\mu(k) \gamma^\mu$, since when this assumption

is made for $\Phi(\ell)$, then $\Phi(k)$ has exactly this structure. (I have not investigated the possibility of solutions with other Dirac structures). Thus we may decompose Φ in terms of three invariant amplitudes ϕ , ϕ_T , and ϕ_L :

$$\Phi(k) = \not{P} \phi(k) + \left(\not{k} - \frac{k^+}{P^+} \not{P} \right) \phi_T(k) + \frac{-2k \cdot P}{2P \cdot \eta} \not{\eta} \phi_L(k) \quad (2.11)$$

When this structure is inserted for $\Phi(\ell)$ in Eq. (2.1) and the replacements described above for ℓ_μ are made, one finds for the invariant amplitude $\phi(k)$ (after a certain amount of algebra):

$$\phi(k) \sim A \left\{ \left[1 + \left(\frac{k^+}{y} \right)^2 + \left(\frac{k^+}{y} \right) \Delta \right] \left[\frac{1}{d} \right]_P + 2\Delta \left[\frac{1}{d^2} \right]_P \right\} \psi(k) \quad (2.12)$$

where

$$A = \frac{4}{3} \cdot 32 \pi^3 \frac{g^2(-2k \cdot P/\mu^2)}{(4\pi)^2} \frac{1}{y(-2k \cdot P)^2}$$

$$\Delta = \frac{\eta^2 2k \cdot P}{y(2P \cdot \eta)^2}$$

$$d = 1 - \frac{k^+}{y} - \Delta$$

and

$$\psi(k) = (-2k \cdot P) \int \frac{d^4 \ell}{(2\pi)^4} 2 \delta_+(1 - k^2) \phi(\ell) \quad (2.13)$$

For the other two invariant amplitudes one finds

$$\phi_T(k) \sim A \left[\frac{k^+}{y} + \Delta \right] \left[\frac{1}{d} \right]_P \frac{1}{y} \psi(k)$$

$$\phi_L(k) \sim A \left[\frac{1}{d} \right]_P \frac{1}{y} \psi(k) \quad (2.14)$$

Notice that the three output functions $\phi(k)$, $\phi_T(k)$, $\phi_L(k)$ depend in the limit we consider on only one of the three input functions $\phi(\ell)$, $\phi_T(\ell)$, $\phi_L(\ell)$ -- namely $\phi(\ell)$. This result is a consequence of the fact that, although $\phi_T(\ell)$ and $\phi_L(\ell)$ are just as large as $\phi(\ell)$, the coefficient functions multiplying $\phi_T(\ell)$ and $\phi_L(\ell)$ vanish as ℓ_μ becomes parallel to P_μ .

Now that the Dirac structure has been dealt with, we are faced with the task of solving Eq. (2.12) for the scalar function $\phi(k)$. To do so, introduce another variable, q_μ , which varies over the same range as k_μ , and define

$$\psi(q) = (-2q \cdot P) \int \frac{d^4 k}{(2\pi)^4} 2\delta_+((k-q)^2) \phi(k) \quad (2.15)$$

Now insert the expression (2.12) for $\phi(k)$ in terms of $\psi(k)$ into (2.15). This gives us an integral equation for ψ . Using Eq. (2.2) for the running coupling constant and Eq. (2.6) for the change of integration variables, one obtains

$$\psi(q) \sim \frac{4}{3} \frac{1}{b} \int_x^1 \frac{dy}{y} \int_{\mu^2}^{-2q \cdot P} \frac{d(-2k \cdot P)}{(-2k \cdot P) \ln(-2k \cdot P / y \mu^2)} \frac{1}{2\pi} \int_0^{2\pi} d\phi$$

$$\times 2 \left\{ \frac{1 + \left(\frac{k^+ / P^+}{y}\right)^2 + \left(\frac{k^+ / P^+}{y}\right) \Delta}{\left[1 + \frac{k^+ / P^+}{y} - \Delta\right]_P} + \frac{\Delta}{\left[1 - \frac{k^+ / P^+}{y} - \Delta\right]_P} \right\} \psi(k) + E(x) \quad (2.16)$$

The unknown constant $E(x)$ here reflects the contribution to the integral from the small $-2k \cdot P$ integration region, where the approximations that were made in the integrand are not valid. In addition, the Born term ϕ_B that has been neglected in the integral equation contributes to $E(x)$. Notice that, for the reasons discussed earlier, the dominant region of the k_μ integration when $-2q \cdot P$ is large is $(-2k \cdot P) \ll (-2q \cdot P)$. Therefore we may set

$$\frac{k^+}{P^+} \rightarrow x \equiv \frac{q^2}{2q \cdot P} \quad (2.17)$$

underneath the integral sign. The resulting equation for ψ is much simpler than the equation for ϕ : The variables q^+ / P^+ , k^+ / P^+ do not appear in the ψ -equation, so that $\psi(q)$ depends only on $-2q \cdot P$ and x but not on q^+ / P^+ .

To be more precise about this, refer to Eq. (2.8), which, after appropriate substitutions, gives k^+ / P^+ in terms of $-2k \cdot P$, y , ϕ and $-2q \cdot P$, x , q^+ / P^+ . When $-2k \cdot P \ll -2q \cdot P$ the dependence of k^+ / P^+ on q^+ / P^+ disappears, leaving only $k^+ / P^+ = y$. Therefore, if we were to suppose that $\phi(k)$ depended on k^+ / P^+ , we would find that $\phi(q)$ was independent of q^+ / P^+ and thus that this assumption was inconsistent.

For the same reasons, nothing in the integrand depends on the variable ϕ , so we can replace $(2\pi)^{-1} \int d\phi$ by 1.

One more simplification occurs in the integrand of (2.16) in the limit being considered. Since $-2k \cdot P \ll -2q \cdot P \leq -(2P \cdot \eta)^2 / \eta^2$ we have $\Delta \equiv \eta^2 2k \cdot P / y (2P \cdot \eta)^2 \ll 1$. In the numerator of the first term in (2.16) we can simply set Δ to zero compared to $1 + x^2 / y^2$. However, it would evidently not be correct to set Δ to zero compared to $1 - x/y$ in the denominators, since $1 - x/y$ can be arbitrarily small; nor is it correct to ignore the second term, even though it contains a factor of Δ . To take the $\Delta \rightarrow 0$ limit, we use the following easily proved distribution theoretic (i.e. true inside an integral) theorems:

$$\frac{\Theta(\xi)}{[\xi - \Delta]_P} \rightarrow \left[\frac{\Theta(\xi)}{\xi} \right]_R - \delta(\xi) \ln\left(\frac{1}{\Delta}\right) \quad \text{as } \Delta \rightarrow 0, \quad (2.17)$$

$$\frac{\Theta(\xi) \Delta}{[\xi - \Delta]_P^2} \rightarrow -\delta(\xi) \quad \text{as } \Delta \rightarrow 0. \quad (2.18)$$

Here the distribution $[\Theta(\xi)/\xi]_R$ is defined to be the derivative of the distribution $\Theta(\xi) \ln \xi$:

$$\int_{-\infty}^{\infty} d\xi \left[\frac{\Theta(\xi)}{\xi} \right]_R h(\xi) \equiv - \int_{-\infty}^{\infty} d\xi \ln \xi \frac{d}{d\xi} h(\xi) \quad (2.19)$$

Before going further it is convenient to make a minor change in the endpoint of the $-2k \cdot P$ integration from $(-2q \cdot P)$ to $(-2q \cdot P)(y/x)$. The justification is the same as for the previous approximations: The predominant contribution to the integral comes from the many units of $\ln(-2k \cdot P / \mu^2)$ below the endpoint.

With these replacements made, the integral equation is

$$\psi(q) \sim \frac{4}{3} \frac{1}{b} \int_x^1 \frac{dy}{y} \int_{\mu^2}^{\frac{y}{x} (-2q \cdot P)} \frac{d(-2k \cdot P)}{(-2k \cdot P) \ln(-2k \cdot P / y \mu^2)}$$

$$\times \left\{ \frac{2(1+x^2/y^2)}{\left[1 - \frac{x}{y}\right]_R} + 4\delta\left(1 - \frac{x}{y}\right) \ln\left(\frac{y(2P \cdot \eta)^2}{e \eta^2 2k \cdot P}\right) \right\} \psi(k) + E(x). \quad (2.20)$$

It will prove convenient to define a parameter $\sigma \equiv -(2P \cdot \eta)^2 / e\eta^2 \mu^2$. Now to proceed we regard $\psi(q)$ as a function of x and the variable

$$\omega \equiv \frac{-2q \cdot P}{x \mu^2} \quad (2.22)$$

We differentiate with respect to ω and take moments with respect to x ,

$$\psi_N(\omega) \equiv \int_0^1 \frac{dx}{x} x^N \psi(x, \omega) \quad (2.23)$$

This yields a differential equation that is diagonal in N and can be solved exactly:

$$\omega \ln \omega \frac{d}{d\omega} \psi_N(\omega) = \left\{ \frac{4}{3} \frac{2A_N}{b} + \frac{4}{3} \frac{4}{b} \ln \left(\frac{\sigma}{\omega} \right) \right\} \psi_N(\omega) \quad (2.24)$$

where

$$A_N = \int_0^1 \frac{dz}{z} z^N \frac{1+z^2}{[1-z]_R} = \frac{1}{N(N+1)} - 2 \sum_{l=1}^N \frac{1}{l} \quad (2.25)$$

The solution of this differential equation is

$$\psi_N(\omega; \sigma) = e^{-F(\omega, \sigma)} C_N(\sigma) [\ln \omega]^{\frac{4}{3} \frac{2A_N}{b}} \quad (2.26)$$

where

$$\begin{aligned} F(\omega, \sigma) &= \frac{4}{3} \frac{4}{b} \int_{\omega}^{\sigma} \frac{d\omega'}{\omega'} \frac{\ln(\frac{\sigma}{\omega'})}{\ln \omega'} \\ &= \frac{4}{3} \frac{4}{b} \left\{ \ln \sigma \ln \left(\frac{\ln \sigma}{\ln \omega} \right) - \ln \left(\frac{\sigma}{\omega} \right) \right\} \end{aligned} \quad (2.27)$$

(A graph of $F(\omega; \sigma)$ appears in the following section.)

The integration constants $C_N(\sigma)$ depend, in principle, on the parameter $\sigma = -(2P \cdot \eta)^2 / e\eta^2 \mu^2$. In order to fix their σ dependence and to relate their values to experiment, let us compute the deep inelastic structure function νW_2 from ψ , using Eq. (1.4):

$$\nu W_2(Q, P) \sim e_g^2 [C_F(\ln \omega)^{4_F/b}]^{-1} x \psi(x, \omega; \sigma)$$

where $\omega \equiv 2Q \cdot P / x \mu^2$. Recall that the simple relation Eq. (1.4) used here is valid in the scaling limit, $Q_\mu \sim e^{\tilde{t}} \tilde{Q}$, $P_\mu \sim e^{\tilde{t}} \tilde{P}$, with η_μ fixed. Thus σ/ω is fixed in this limit. For our purposes it is convenient to take $\sigma = \omega$, since $F(\omega, \omega)$ equals zero exactly. (More generally, $F(\sigma, \omega) = O[\ln^2(\sigma/\omega)/\ln \sigma]$.) Then we have for the N th moment of $x^{-1} \nu W_2$

$$\left[\frac{1}{x} \nu W_2 \right]_N \sim e_g^2 [C_F(\ln \omega)^{4_F/b}]^{-1} C_N(\omega) [\ln \omega]^{\frac{4}{3} \frac{2A_N}{b}}$$

On the other hand, the known behavior of νW_2 in QCD is¹⁴

$$\left[\frac{1}{x} \nu W_2 \right] \sim e_g^2 a_N [\ln \omega]^{\frac{4}{3} 2(A_N - A_1)/b} \quad (2.28)$$

Comparing these expressions gives

$$C_N(\omega) = C_F a_N [\ln \omega]^{(4_F - \frac{4}{3} 2A_1)/b}$$

Explicit calculation of the quark wave function renormalization constant in the axial gauge gives

$$Z_F - 1 = \frac{2}{\epsilon} \frac{g^2}{(4\pi)^2} \frac{4}{3} 3 + O(g^4) \quad (2.29)$$

in 4-ε dimensions. Thus $\gamma_F(g^2) = \gamma_F g^2 / (4\pi)^2 + \dots$ where

$$\gamma_F = -\frac{4}{3} \quad (2.30)$$

Remarkably enough, $2A_1 = -3$, so that the integration constants $C_N(\sigma)$ turn out to be independent of the parameter σ : $C_N = C_F a_N$. Thus (for $\sigma \neq \omega$ now).

$$\psi_N(\omega, \sigma) \sim C_F [\ln \omega]^{\gamma_F/b} e^{-F(\omega, \sigma)} \left[\frac{1}{e^{\frac{1}{x}}} \frac{1}{x} \nu w_2 \right]_N \quad (2.31)$$

III. Conclusion

Let me summarize the result of the previous section. The full function $\phi_{\alpha\beta}(k)$ that approximately solves the ladder equation has the form $\bar{\psi}\phi(k)$ plus terms with other Dirac structures that are less important in the limit considered and are given in Eqs. (2.11) and (2.14). The scalar function $\phi(k)$ equals the product of three factors: the running coupling constant; a certain rational function (Eq. (2.12)) of the three invariants $-2k \cdot P$, k^2 , and $k^+ / P^+ \equiv k \cdot \eta / P \cdot \eta - \eta^2 k \cdot P / 2P \cdot \eta$; and a function $\Psi(k)$ that depends on only two of these invariants, $-2k \cdot P$ and k^2 . Finally, $\Psi(k)$ has the form

$$\psi_{a/A}^{N.S.}(k) \sim [C_F (\ln \omega)^{\gamma_F/b}] e^{-F(\omega, \sigma)} \bar{\psi}_{a/A}^{N.S.}(y, k^2) \quad (3.1)$$

Here $\bar{\psi}_{a/A}^{N.S.}(y, k^2)$ is the non-singlet part of the probability to find a quark of flavor a in hadron A as measured in deep inelastic lepton scattering. (Recall

*Some authors have obtained a Q^2 dependent or infinite quark anomalous dimension in axial gauge,^{4,10,11} but this is not the standard γ_F related to the removal of ultraviolet divergences. It should be understood as an "effective" anomalous dimension.

that this paper has considered only the flavor non-singlet channel). The variables used are $\gamma = k^2 / 2k \cdot P$, $\omega = -2k \cdot P / y \mu^2 = -k^2 / y^2 \mu^2$, and $\sigma = -(2P \cdot \eta)^2 / e \eta^2 \mu^2$. The first factor, $C_F (\ln \omega)^{\gamma_F/b}$, simply cancels the similar wave function renormalization factor that appears in whatever (amputated) hard scattering function one might wish to attach to ϕ . The function $F(\omega, \sigma)$ in the second factor is

$$F(\omega, \sigma) = \frac{4}{3} \frac{4}{b} \left\{ \ln \sigma \ln \left(\frac{\ln \sigma}{\ln \omega} \right) - \ln \left(\frac{\sigma}{\omega} \right) \right\} \quad (3.2)$$

where $b = 11 - \frac{2}{3} N_F$.

In deriving this result we used repeatedly the assumption that Ψ is slowly varying: $\Delta \ln \omega \equiv \psi(d\psi/d \ln \omega)^{-1} \gg 1$. The functions $[\ln \omega]^{\gamma_F/b}$ and $\bar{\psi}(y, k^2)$ are indeed slowly varying in this sense when ω is large. Thus the solution obtained is valid in the range of ω such that $-dF(\omega, \sigma)/d \ln \omega \ll 1$. That is

$$\frac{4}{3} \frac{4}{b} \frac{\ln(\sigma/\omega)}{\ln \omega} \ll 1 \quad (3.3)$$

If we take " $\ll 1$ " to mean " $< 1/2$ " and choose four flavors, the allowed range of ω is $\sigma^{.75} < \omega < \sigma$. In principle, this allowed range of ω includes the entire range over which e^{-F} is appreciably larger than zero, as long as σ is large enough: at $\omega = \sigma^{.75}$, $e^{-F} \approx \sigma^{-0.024}$. With protons produced by real accelerators, σ will never be large enough in this sense, but one can hope that even with modestly large σ the approximations made here do not too seriously deteriorate until e^{-F} is rather small. It would certainly be of interest to investigate this question numerically, as well as to improve the rigor and precision of the derivation presented here.

The factor $e^{-F(\omega, \sigma)}$ obtained in this paper appears to play the same role as the "exact" quark propagator $d_F(k^2, (2\eta \cdot k)^2 / \eta^2)$ which appears (but is not derived) in the paper of Dokshitzer, D'Yakonov, and Troyan.¹⁰ To facilitate comparison, let us replace ω by $\hat{\omega} \equiv -k^2 / \mu^2$ and σ by $\hat{\sigma} \equiv -(2k \cdot \eta)^2 / \eta^2 \mu^2 (= \sigma_{DT} / \mu^2)$. Since $\hat{\omega} = y^2 \omega$

and $\hat{\sigma} = (k \cdot \eta / P \cdot \eta)^2 \sigma$, one finds that $F(\hat{\omega}, \hat{\sigma}) \approx F(\omega, \sigma)$ for large ω and σ with y and $(k \cdot \eta / P \cdot \eta)$ fixed, as long as $\ln(\omega/\sigma)/\ln \omega$ can be neglected. With this approximation, $\exp(-F(\hat{\omega}, \hat{\sigma}))$ is a function only of k_μ and η_μ and could be considered to be a correction due to the propagators attached to $\Phi(k)$. Thus one expects $e^{-F(\hat{\omega}, \hat{\sigma})} \sim d_F(\hat{\omega}, \hat{\sigma})$. Indeed, the function d_F given by DDT is (DDT Eq. 111 and the first of Eq. 110 with neglect of finite terms in " γ_F "):

$$\ln d_F(\hat{\omega}, \hat{\sigma}) \sim -\frac{4}{3} \frac{4}{b} \int_{\hat{\omega}}^{\hat{\sigma}} \frac{d\hat{\omega}'}{\hat{\omega}' \ln \hat{\omega}'} \ln \left(\frac{\hat{\omega}'}{\hat{\sigma}} + \frac{1}{\hat{\omega}'} \right) \quad (3.4)$$

This differs from e^{-F} only by the addition of the term $1/\hat{\omega}'$ in the argument of the logarithm, which DDT have apparently introduced by means of an ad hoc cutoff in an integration over gluon transverse momentum. The functions $e^{-F(\omega, \sigma)}$ and $d_F(\omega, \sigma)$ are shown in Fig. 4 for $y = 10^4$. Also shown is the approximation introduced by DDT for d_F (DDT Eq. 114):

$$\begin{aligned} \ln d_F(\hat{\omega}, \hat{\sigma}) &= -\frac{4}{3} \frac{2}{b \ln \hat{\omega}} \ln^2 \frac{\hat{\sigma}}{\hat{\omega}} \quad \hat{\sigma} < \hat{\omega} \\ &= -\frac{4}{3} \frac{2}{b \ln \hat{\omega}} \left\{ \frac{1}{2} \ln^2 \hat{\sigma} - \ln^2 \hat{\omega} \right\} \quad \hat{\omega} < \hat{\sigma} \end{aligned} \quad (3.5)$$

Evidently the agreement between the three curves is good in the region of small $\ln(\omega/\sigma)/\ln \omega$, which is the only region in which any of the curves should be considered highly reliable.

I have benefitted in writing this paper from having performed several calculations of one loop graphs in the light-like axial gauge in collaboration with J.F. Gunion. I also thank him and R.C. Hwa for helpful conversations about the ladder model and the leading logarithm approximation, and S.J. Brodsky for a helpful conversation about the Born term Φ_B .

APPENDIX

In the body of this paper I have made two crucial approximations: first $\ell^+/p^+ + y = k^2/2k \cdot P$, then $k^+/P^+ + q^2/2q \cdot P$. These approximations are easily justified in the numerators. On the other hand the propriety of these approximations in the singular gauge denominator is less self evident, and is worth a more detailed discussion here.

Let us first review the overall sequence of steps. The singular factor in question is $\mathcal{J}^{-1}$ where $P \cdot \eta \mathcal{J} = (\ell \cdot k) \cdot \eta$ and a principle value instruction is understood when integrating over $\mathcal{J}^{-1}$. The exact expression for $\mathcal{J}$ is

$$\mathcal{J} = \frac{\ell^+}{P^+} - \frac{k^+}{P^+} - \frac{(-2k \cdot P) - (-2\ell \cdot P)}{\zeta} \quad (A.1)$$

where

$$\zeta = (2P \cdot \eta)^2 / (-\eta^2) \quad (A.2)$$

In the first step we integrate over ℓ_μ , the important integration region is $-2\ell \cdot P \ll -2k \cdot P$, and $\mathcal{J}$ becomes

$$\mathcal{J}_1 = y - \frac{k^+}{P^+} - \frac{(-2k \cdot P)}{\zeta} \quad (A.3)$$

In the second step we integrate over k_μ , the important integration region is $-2k \cdot P \ll -2q \cdot P$, and $\mathcal{J}_1$ becomes

$$\mathcal{J}_2 = y - x - \frac{(-2k \cdot P)}{\zeta} \quad (A.4)$$

Let us now examine the first step, which leads to Eq. (2.12). We write $\phi(k)$ as an integral over ℓ of $\phi(\ell)$ and make the relevant $-2\ell \cdot P \ll -2k \cdot P$ approximations everywhere except in the singular factor $\mathcal{A}^{-1}$. Also, we exhibit the form as given by Eq. (2.12) for the proposed solution $\phi(\ell)$. Finally, we omit some terms that are not relevant to the present analysis, as indicated by the "+ ...".

This gives

$$\phi(k) \sim \frac{4}{3} \frac{g^2}{(4\pi)^2} (-2k \cdot P)^{-2} \left[1 + \left(\frac{k^+ / P^+}{y} \right)^2 \right] \int_0^{-2k \cdot P} \frac{d(-2\ell \cdot P)}{(-2\ell \cdot P)} \int d\ell' \cdot \mathcal{A}^{-1} \left[\ell'^2 - \frac{(-2\ell \cdot P)}{5} \right]^{-1} \left[1 + \left(\frac{y}{\frac{4}{3}} \right)^2 \right] \frac{4}{3} 2\pi g^2 \psi(z, -2\ell \cdot P) + \dots \quad (A.5)$$

Here ℓ' is a dimensionless transverse momentum vector

$$\ell' = [-2\ell \cdot P]^{-1/2} \ell$$

(Vectors $\ell' = [-2\ell \cdot P]^{-1/2} \ell$ and $q' = [-2q \cdot P]^{-1/2} q$ will also be defined.) The Bjorken scaling variable $z \equiv \ell^2 / 2\ell \cdot P$ is a rather complicated function of the integration variables used here:

$$z = y + \left[1 + \frac{(-2\ell \cdot P)}{(-2k \cdot P)} \right]^{-1} \left(\ell' - \left[\frac{-2\ell \cdot P}{-2k \cdot P} \right]^{1/2} k' \right)^2$$

The outer limit of the ℓ' integration, which is reached when $z = 1$, is contained inside a circle of radius 1. (This is most easily seen from the relation $\ell'^2 = z - \ell^+ / P^+ < 1$)

The singular denominator on which we wish to concentrate our attention is, when written without approximation in terms of $-2\ell \cdot P$ and ℓ' ,

$$\mathcal{A} = \left(1 - \frac{(-2\ell \cdot P)}{(-2k \cdot P)} \right) \left(k' - \left[\frac{-2\ell \cdot P}{-2k \cdot P} \right]^{1/2} \ell' \right)^2 - \frac{(-2k \cdot P) - (-2\ell \cdot P)}{5}$$

In the important integration region $-2\ell \cdot P \ll -2k \cdot P$ this immediately becomes

$$\mathcal{A} \sim \left(k' - \left[\frac{-2\ell \cdot P}{-2k \cdot P} \right]^{1/2} \ell' \right)^2 - \frac{(-2k \cdot P)}{5} \quad (A.7)$$

Let us now think of $\mathcal{A}^{-1}$ as a function of k' . It is a singular function that would not be integrable over k' if it were not for the effective cutoff $(-2k \cdot P)/5$. When $-2\ell \cdot P \ll -2k \cdot P$, integration over ℓ' smears this singular function out over a tiny region in k' space. However, we are not interested in $\mathcal{A}^{-1}(k')$ point by point for very tiny k' , but only in its effective behavior when multiplied by a slowly varying function of k' ($\partial/\partial k' \sim O(1)$) and integrated over k' . Therefore the slight smearing is unimportant and we approximate $\mathcal{A}$ by

$$\mathcal{A}_1 = k'^2 - \frac{(-2k \cdot P)}{5} \quad (A.8)$$

This is the same as Eq. (A.3). This completes the first step.

In the second step we integrate over k to form $\psi(q)$:

$$\psi(z) \sim \frac{4}{3} \frac{2}{\pi} \int_0^{-2q \cdot P} \frac{d(-2k \cdot P)}{(-2k \cdot P)} \int d\ell' \mathcal{A}_1^{-1} \frac{g^2}{(4\pi)^2} \left(1 + \frac{x^2}{y^2} \right) \psi(k). \quad (A.9)$$

(Here some approximations not involving the singular denominator $\mathcal{A}^{-1}$ have already been made and some terms that are not relevant to the present discussion have been left out). The approximation under discussion consists of substituting

$$\mathcal{A}_2 = \left(k' - \left[\frac{-2k \cdot P}{-2q \cdot P} \right]^{1/2} q' \right)^2 - \frac{(-2k \cdot P)}{5} \quad (A.10)$$

(This is the same as Eq. (A.4))

For $\mathcal{A}_1$. Here $q' \equiv [-2q \cdot P]^{-1/2} q$ and we recall that $\ell'^2 = x \cdot q^+ / P^+$ is always smaller than 1. Since the function that multiplies $\mathcal{A}^{-1}$ is a slowly varying function of ℓ' , this is a good approximation under the ℓ' integral in the important region of the $-2k \cdot P$ integration, $-2k \cdot P \ll -2q \cdot P$. This completes the second step.

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Figure Captions

1. Deep inelastic lepton scattering as a convolution of H and Φ .
2. The Bethe-Salpeter equation for Φ .
3. The ladder model Bethe-Salpeter equation for Φ .
4. The function $e^{-F(\omega, \sigma)}$ (solid line) for $\sigma = 10^4$. Also shown for comparison is the function $d_F(\omega, \sigma)$ introduced by Dokshitzer, D'Yakanov, and Troyan (dashed line) and their approximation to $d_F(\omega, \sigma)$ (dotted line).

Fig 1

Fig 2

Fig 3

Fig 4