Distribution Category:
Magnetic Fusion Energy
(UC-20)

ANL/FPP/TM-211

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DE87 007716

**3-D TRANSIENT EDDY CURRENT CALCULATIONS FOR
THE FELIX CYLINDER EXPERIMENTS**

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December 1986

Work supported by

Office of Fusion Energy
U.S. Department of Energy

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TABLE OF CONTENTS

	<u>Page</u>
List of Figures.....	iii
List of Tables.....	v
Abstract.....	1
1. Introduction.....	2
2. The Second Order Vector Potential Formulation - The U-V Method..	3
3. Basic Functions and the Null Field Integral Technique.....	5
4. Pseudo-Analytical Formulation.....	13
5. Total Transient Response.....	15
6. Results.....	16
7. Conclusions.....	22
8. References.....	37

LIST OF FIGURES

<u>No.</u>	<u>Page</u>
1. FELIX Cylinder Stressed by a Vertical B Field; Inner Radius a, Outer Radius b.....	7
2. FELIX Cylinder Experiment. A Slit Cylinder is Immersed in a Homogeneous Field $H = H_0$ at $t = 0$; $a = 57.1$ mm, $b = 69.8$ mm, length = 300 mm, $\sigma = 2.538 \times 10^7$ mho/m; field collapses with time constant $T = 39.68$ m sec.....	8
3. Small Cylinder, Induced Field as a Function of Time, 39.68 ms. Solid Line: Predicted Field (one eigenvalue). Dashed Line: Measured Field Fit to Two Exponentials.....	17
4. Medium Cylinder (60 cm) Induced Field as a Function of Time, 39.68 ms.....	18
5. Large Cylinder Induced Field as a Function of Time, 39.68 ms.....	19
6. Large Cylinder, 39.68 ms. (a) Fit with One Eigenvalue, (b) Fit with Two Eigenvalues, (c) Fit with Four Eigenvalues.....	21
7. Medium Cylinder (60 cm) 39.68 ms. (a) Fit with One Eigenvalue, (b) Fit with Two Eigenvalues.....	23
8. Small Cylinder, 39.68 ms. (a) Maximum Induced Field at Different Axial Positions, (b) Time of Maximum Induced Field at Different Axial Positions. Solid Curve: Predicted Values (one eigenvalue). Dots: Experimental Results.....	24
9. Medium Cylinder (60 cm), 39.68 ms. (a) Maximum Induced Field, (b) Time.....	25
10. Large Cylinder, 39.68 ms. (a) Maximum Induced Field, (b) Time.....	26
11. Small Cylinder, Induced Field as a Function of Time, 6.87 ms.....	27
12. Medium Cylinder (20 cm), Induced Field as a Function of Time, 6.87 ms.....	28
13. Medium Cylinder (40 cm), Induced Field as a Function of Time, 6.87 ms.....	29
14. Large Cylinder, 6.87 ms. (a) Fit with One Eigenvalue, (b) Fit with Two Eigenvalues, (c) Fit with Three Eigenvalues.....	30
15. Large Cylinder, 6.87 ms. (a) Fit with One Eigenvalue, (b) Fit with Two Eigenvalues, (c) Fit with Three Eigenvalues.....	31
16. Small Cylinder, 6.87 ms. (a) Maximum Induced Field at Different Axial Positions, (b) Time of Maximum Induced Field at Different Axial Positions.....	33

LIST OF FIGURES (cont'd)

<u>No.</u>	<u>Page</u>
17. Medium Cylinder (20 cm), 6.87 ms. (a) Maximum Induced Field, (b) Time.....	34
18. Medium Cylinder, (40 cm), 6.87 ms. (a) Maximum Induced Field, (b) Time.....	35
19. Large Cylinder, 6.87 ms. (a) Maximum Induced Field, (b) Time.....	36

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LIST OF TABLES

	<u>Page</u>
Table I. Relation of A, B, and J to u and v.....	6
Table II. Representation of ψ , u, and v and Boundary Condition Requirements.....	10
Table III. Calculation of Eigenvalues.....	12
Table IV. Analytic Boundary Conditions and Constraints.....	14

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ABSTRACT

The three-dimensional eddy current transient field problem is formulated first using the U-V method. This method breaks the vector Helmholtz equation into two scalar Helmholtz equations. Null field integral equations and the appropriate boundary conditions are used to set up an identification matrix which is independent of null field point locations. Embedded in the identification matrix are the unknown eigenvalues of the problem representing its impulse response in time. These eigenvalues are found by equating the determinant of the identification matrix to zero. When this initial forcing function is Fourier decomposed into its spatial harmonics, each Fourier component can be associated with a unique eigenvalue by this technique. The true transient solution comes through a convolution of the impulse response so obtained with the particular external field decay governing the problem at hand. The technique is applied to the FELIX cylinder experiments; computed results are compared to data. A pseudoanalytic confirmation of the eigenvalues so obtained is formulated to validate the procedure.

1. INTRODUCTION

Approaches to transient eddy current problems to date tend to fall into one of two categories:

1. Time domain developed by a forward difference procedure, explicit or implicit. Spatial discretization is pursued in much the same way as in time harmonic problems [1,2].
2. Time domain developed via the characteristic eigenvalues and eigenvectors of the system. The spatial domain is characteristically pursued via either finite difference or finite element techniques [3,4].

The first approach is computationally intensive, requiring the solution of the entire spatial domain recursively throughout the time period of interest. The second approach is theoretically more efficient, but is fraught with many other problems. To approach the problem using large scale spatial discretizations, as is characteristic of most finite element approaches, necessitates large identification matrices (the matrices used to determine the eigenvalues and the weightings of eigenvectors in the problem). Such a large matrix has a host of eigenvalues; most of them are spurious and subdominant. Since most real problems characteristically have a small set of dominant eigenvalues, it is natural to capitalize on a technique which extracts those dominant eigenvalues. To capitalize on the positive features of the eigenvalue approach, while minimizing the size of matrices and thus the number of eigenvalues, the present research was undertaken. A general theory involving the use of null field integral equations determining 3-D eigenvalues is developed. The theory is applied to the cylinder experiments performed with the FELIX (Fusion ELectromagnetic Induction eXperiments) facility at Argonne National Laboratory (ANL) [5-7].

The general formulation preceding the null field integral technique involves the use of a second order vector potential, a method discussed by [8-10] but as far as the authors are aware, never implemented to date. After the eigenvalues are found through the null field integral technique, the total transient field solution is obtained through a convolution of the impulse response with a particular external field decay generating the transient.

A general overview of the process is as follows:

1. Start with the eigenvalue formulation of the eddy current problem.
2. Break the vector Helmholtz equation into scalar Helmholtz equations using the U-V method, i.e., the second order vector potential developed in [8-10].
3. Use the null field equations and the boundary conditions to set up the identification matrix, which is independent of null field point locations.
4. Find the eigenvalues corresponding to the shape of the initial field. Then find the transient response to an impulse from those eigenvalues as a Fourier sum.
5. Derive the transient solution as a convolution of the impulse response derived in step 4 with the actual external field decay.

Steps 1 through 5 were applied to the FELIX cylinder experiments, and the results are compared with experimental data. In addition, the eigenvalues obtained with the integral technique are compared to those predicted with a pseudoanalytic analysis of the cylinder.

2. THE SECOND ORDER VECTOR POTENTIAL FORMULATION - THE U-V METHOD

The vector Helmholtz equation for the magnetic vector potential is

$$\nabla^2 \bar{A} - \mu \sigma \frac{\partial \bar{A}}{\partial t} = 0. \quad (1)$$

Assume that $\bar{A} = \vec{A} e^{-\lambda t}$. The parameter λ is an eigenvalue of the problem representing the characteristic temporal decay time of the field. It depends on the spatial distribution of the forcing function giving rise to the transient. It is this eigenvalue λ (actually a set of them) that we seek. For $k^2 = \mu \sigma \lambda$, (1) is written as

$$\nabla^2 \vec{A} + k^2 \vec{A} = 0 \quad (2)$$

where $\vec{B} = \nabla \times \vec{A}$. We begin by defining a second vector potential

$$\vec{A} = \nabla \times \vec{W} \quad (3)$$

where

$$\vec{W} = \hat{e}u + \hat{e} \times \nabla v \quad (4)$$

and $\hat{e}$ is a fixed unit vector (e.g. $\hat{a}_r, \hat{a}_z$, or $\hat{a}_\theta$). Inserting (3) into (2) gives

$$-\nabla \times [\nabla \times \nabla \times \vec{W} - k^2 \vec{W}] = 0 \quad (5)$$

or

$$\nabla \times [\nabla^2 \vec{W} + k^2 \vec{W} - \nabla(\nabla \cdot \vec{W})] = 0 \quad (6)$$

or finally (to within the gradient of a scalar)

$$\nabla^2 \vec{W} + k^2 \vec{W} = 0. \quad (7)$$

No gauge has been applied on $\vec{W}$. In fact, as P. Hammond [8] points out, it is inconsistent to impose one since it forces u and v to have certain spatial dependencies assumed a priori.

The cylindrical conductor of interest in this paper suggests the assignment of $\hat{a}_z$ for $\hat{e}$. It follows then that

$$\vec{W} = \hat{a}_z u + \hat{a}_z \times \nabla v = \hat{a}_z u + \hat{a}_\theta \frac{\partial v}{\partial r} + \hat{a}_r \left(-\frac{1}{r} \frac{\partial v}{\partial \theta}\right). \quad (8)$$

The expansion of $\nabla^2 \vec{W}$ in cylindrical coordinates gives

$$\begin{aligned} \nabla^2 \vec{W} = & \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(-\frac{\partial v}{\partial \theta} \right) \right) - \frac{1}{r^3} \frac{\partial^3 v}{\partial \theta^3} - \frac{2}{r^2} \frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial^3 v}{\partial z^2 \partial \theta} \right] \hat{a}_r \\ & + \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \frac{\partial v}{\partial r} + \frac{\partial^2}{\partial z^2} \frac{\partial v}{\partial r} - \frac{2}{r^3} \frac{\partial^2 v}{\partial \theta^2} \right] \hat{a}_\theta \\ & + \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \right] \hat{a}_z. \end{aligned} \quad (9)$$

The r component of (9) is just $\left(-\frac{1}{r} \frac{\partial}{\partial \theta} \nabla^2 v\right)$. The θ component of (9) is $\frac{\partial}{\partial r} (\nabla^2 v)$. The z component of (9) is $\nabla^2 u$. Thus, from (7) it is clear that the total problem is solved by insuring that both u and v satisfy the Helmholtz equation

$$\nabla^2 v + k^2 v = 0 \quad (10)$$

$$\nabla^2 u + k^2 u = 0 \quad (11)$$

From the fact that $\vec{B} = \nabla \times \nabla \times \vec{W}$ and $\vec{J} = \frac{1}{\mu} \nabla \times \vec{B}$, it is possible to relate these quantities $\vec{A}$, $\vec{B}$, and $\vec{J}$ to u and v . Table I summarizes these relationships.

3. BASIS FUNCTIONS AND THE NULL FIELD INTEGRAL TECHNIQUE

We apply the method to the FELIX cylinder discussed in [6,7] and adopted as a 3-D test problem for the international eddy current workshops [12]. Figure 1 depicts the geometry of the problem. A y directed field is applied to the cylindrical shell of inner radius a , outer radius b , length l . Beginning at time $t = 0$, this external field collapses to zero with a time constant τ (5 to 40 msec). As shown by Figure 2 this can be treated as a three region problem. (Note that regions 1 and 3 come in contact at $z = \pm l/2$). In regions 1 and 3, we let $\vec{H} = -\nabla\psi$ and solve

$$\nabla^2 \psi = 0. \quad (12)$$

In region 2, we solve

$$\nabla^2 u + k^2 u = 0 \quad (13)$$

$$\nabla^2 v + k^2 v = 0. \quad (14)$$

The integral solutions of (12)-(14) are respectively

$$\left. \begin{array}{l} \psi(r) ; r \in V_{\text{Air}} \\ \frac{\psi(r)}{2} ; r \text{ surface} \\ 0 ; r \notin V_{\text{Air}} \end{array} \right\} = \oint_S \left[\frac{\partial \psi(r')}{\partial n} G(r, r') - \psi(r') \frac{\partial G(r, r')}{\partial n} \right] ds \quad (15)$$

$$\text{where } G(r, r') = \frac{1}{4\pi |\vec{r} - \vec{r}'|} ;$$

TABLE I

A, B, AND J EXPRESSED IN TERMS OF u AND v

$$A_r = \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{\partial^2 v}{\partial r \partial z}$$

$$A_\theta = -\left[\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 v}{\partial \theta \partial z}\right]$$

$$A_z = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2}$$

$$B_r = \frac{\partial^2 u}{\partial r \partial z} - \frac{k^2}{r} \frac{\partial v}{\partial \theta}$$

$$B_\theta = \frac{1}{r} \frac{\partial^2 u}{\partial \theta \partial z} + k^2 \frac{\partial v}{\partial r}$$

$$B_z = \frac{\partial^2 u}{\partial z^2} + k^2 u$$

$$J_r = k^2 \left[\frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{\partial^2 v}{\partial r \partial z} \right]$$

$$J_\theta = -k^2 \left[\frac{1}{r} \frac{\partial^2 v}{\partial \theta \partial z} + \frac{\partial u}{\partial r} \right]$$

$$J_z = k^2 \left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} \right]$$

Figure 1. FELIX Cylinder Stressed by a Vertical B Field; Inner Radius a , Outer Radius b .

$$\vec{H} = H_0 \hat{a}_y \text{ at } t=0.$$

Figure 2. FELIX Cylinder Experiment. Slit Cylinder is Immersed in a Homogeneous Field $\vec{H} = H_0 \hat{a}_y$ at $t = 0$; $a = 57.1$ mm, $b = 69.8$ mm, length = 300 mm, $\sigma = 2.538 \times 10^7$ mho/m; field collapses with time constant $T = 39.68$ ms.

$$\left[\begin{array}{l} u(r) ; r \in V_{\text{conductor}} \\ \frac{u(r)}{2} ; r \text{ surface} \\ 0 ; r \notin V_{\text{conductor}} \end{array} \right] = \oint_S \left[\frac{\partial u(r')}{\partial n} G_c(r, r') - u(r') \frac{\partial G_c}{\partial n}(r, r') \right] ds \quad (16)$$

$$\left[\begin{array}{l} v(r) ; r \in V_{\text{conductor}} \\ \frac{v(r)}{2} ; r \text{ surface} \\ 0 ; r \notin V_{\text{conductor}} \end{array} \right] = \oint_S \left[\frac{\partial v(r')}{\partial n} G_c(r, r') - v(r') \frac{\partial G_c}{\partial n}(r, r') \right] ds \quad (17)$$

where

$$G_c(r, r') = \frac{e^{jk|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|}.$$

We choose to solve (15)-(17) using (15c), (16c), and (17c). The approach is as follows:

1. Assign appropriate basis functions to the interfacial fields ψ , u , v , and their normal derivatives.
2. Pick arbitrary null field points outside the volumes of interest.
3. Impose the boundary conditions that tangential H and normal B be continuous across each interface.
4. Set up a matrix, the identification matrix, using the equations from (15c)-(17c).
5. Determine the eigenvalues k for which the determinant of the identification matrix is zero.
6. Reconstruct the transient response in terms of these eigenvalues.

For the problem at hand, we know a priori that there is no θ directed current at $\theta = \pm \pi/2$. By symmetry, it is also obvious that J_z is zero at $\theta = 0, \pi$. Furthermore, $J_r = 0$ at $r = a$ and $r = b$. Both J_θ and J_z can be expressed as a combination of sinusoidal functions as follows:

$$J_\theta = \sum C_{\theta n} \sin \frac{n\pi}{L} z \cos \theta \quad (18)$$

$$J_z = \sum C_{zn} \cos \frac{n\pi}{L} z \sin \theta \quad (19)$$

Table II summarizes the appropriate basis functions for ψ , u , and v consistent with (18), (19), and Table I, as well as the boundary conditions

TABLE II
REPRESENTATION OF ψ , u , and v AND BOUNDARY
CONDITION REQUIREMENTS

Scalar Variable	Functional Representation	
	$r = a$	$r = b$
ψ	$A \cos \frac{n\pi}{l} z \cos \theta$	$B \cos \frac{n\pi}{l} z \cos \theta$
$\frac{\partial \psi}{\partial r}$	$C \cos \frac{n\pi}{l} z \cos \theta$	$D \cos \frac{n\pi}{l} z \cos \theta$
u	$E \sin \frac{n\pi}{l} z \cos \theta$	$F \sin \frac{n\pi}{l} z \cos \theta$
$\frac{\partial u}{\partial r}$	$G \sin \frac{n\pi}{l} z \cos \theta$	$H \sin \frac{n\pi}{l} z \cos \theta$
v	$I \cos \frac{n\pi}{l} z \sin \theta$	$J \cos \frac{n\pi}{l} z \sin \theta$
$\frac{\partial v}{\partial r}$	$K \cos \frac{n\pi}{l} z \sin \theta$	$L \cos \frac{n\pi}{l} z \sin \theta$

Boundary conditions $\hat{n} \cdot \vec{B} = 0$, $\hat{n} \times \vec{H} = 0$ require

$$A = \frac{El}{n\pi} (k^2 - (\frac{n\pi}{l})^2)$$

$$K = \frac{El}{an\pi}$$

$$C = \frac{k^2}{a} I - G(\frac{n\pi}{l})$$

$$B = \frac{Fl}{n\pi} (k^2 - (\frac{n\pi}{l})^2)$$

$$L = \frac{Fl}{bn\pi}$$

$$D = \frac{k^2}{b} J - H \frac{n\pi}{l}$$

$\hat{n} \cdot ||\vec{E}|| = 0$ and $\hat{n} \times ||\vec{E}|| = 0$. We use the notation $||\vec{E}|| = (\vec{E}_1 - \vec{E}_2)$ to designate the jump in the vector across the interface.

The following guidelines are offered for choosing global basis function representations of the interfacial variables for generalized problems:

1. Use the physical constraints of the problem (i.e., imposed boundaries and generic symmetries) to characterize the spatial dependence of the current density as much as possible (e.g., $\cos \theta$ or $\sin \theta$ in this problem).
2. Express the remaining dimensional dependence of $\vec{J}$ as a superposition of modes into which the excitation source can be decomposed (in this case, the Fourier decomposition $\sin \frac{n\pi z}{L}$).
3. Assign spatial dependence to the field variables consistent with the current density characterized in (1) and (2).
4. If guidelines (1) and (2) fail to provide sufficient insight, examine pseudo-analytical or experimental solutions.

We have 12 unknowns (A,B,C,D,E,F,G,H,I,J,K,L), 6 boundary conditions (Table II), and 6 integral equations ((15c), (16c), (17c)) written both at $r = a$ and $r = b$. Note, the integral equations involve integration around the ends. The fields at the ends must be a combination of Bessel functions (J_1 and Y_1); the weighting of the J and Y Bessel functions is chosen so as to yield the assumed fields of Table II at $r = b$ and $r = a$. Unless the cylinder is very short or very fat, these end effects are small. Thus we have a 6×6 identification matrix. The zeros of its determinant were calculated for modal surfaces $n = 1, 3, 5, 7, 9$, and 11. The results are shown in Table III.

The experience of the scattering research community [11] suggests that the use of equations (15b), (16b), and (17b) to yield six boundary equations as well as (16c) to give an additional null field equation may give improved results. Thus, we formulate seven equations for six unknowns. The matrix (now seven by six) must be multiplied by its transpose before searching for the eigenvalues of this modified matrix. This technique can stabilize the problem greatly. In fact, with this change, the problem can be solved as a two region problem without artificial interfaces.

TABLE III
CALCULATION OF EIGENVALUES

Modal Value n	Integral Value	Pseudoanalytical Value
1	52.0	51.99
3	61.0	61.42
5	74.0	74.48
7	87.0	87.13
9	99.0	99.02
11	110.0	110.32

4. PSEUDO-ANALYTICAL FORMULATION

As an alternative analytical result, we sought a solution assuming the fields have periodicity 2ℓ in z space. Solutions to (12)-(14) are

$$\psi(r, \theta, z) = AI_1\left(\frac{n\pi}{\ell} r\right) \cos\theta \cos\left(\frac{n\pi}{\ell} z\right) ; \text{ region 1} \quad (20)$$

$$BK_1\left(\frac{n\pi}{\ell} r\right) \cos\theta \cos\left(\frac{n\pi}{\ell} z\right) ; \text{ region 3} \quad (21)$$

$$u(r, \theta, z) = [C_1 J_1(\beta r) + C_2 Y_1(\beta r)] \cos\theta \sin\left(\frac{n\pi}{\ell} z\right) ; \text{ region 2} \quad (22)$$

$$v(r, \theta, z) = [D_1 J_1(\beta r) + D_2 Y_1(\beta r)] \sin\theta \cos\left(\frac{n\pi}{\ell} z\right) ; \text{ region 2} \quad (23)$$

where

$$\beta^2 = k^2 - \left(\frac{n\pi}{\ell}\right)^2 .$$

Boundary conditions require that $\nabla \times \vec{A}|_{\text{conductor}} = -\nabla\psi|_{\text{Air}}$ on the interface. These conditions and their commensurate requirements on A , B , C_1 , C_2 , D_1 , and D_2 are summarized in Table IV. The six equations in six unknowns can be solved exactly as with the integral technique for the unknown eigenvalues. The results are listed in Table III for comparison with the integral technique formalism; a discrepancy of less than one percent was observed.

One important case develops when $k^2 = (n\pi/\ell)^2$. The solutions corresponding to (20)-(23) collapse to

$$\psi(r, \theta, z) = AI_1\left(\frac{n\pi}{\ell} r\right) \cos\theta \cos \frac{n\pi}{\ell} z ; \text{ region 1} \quad (24)$$

$$BK_1\left(\frac{n\pi}{\ell} r\right) \cos\theta \cos \frac{n\pi}{\ell} z ; \text{ region 3} \quad (25)$$

$$u(r, \theta, z) = \left(\frac{C_1}{r} + C_2 r\right) \cos\theta \sin\left(\frac{n\pi}{\ell} z\right) ; \text{ region 2} \quad (26)$$

$$v(r, \theta, z) = \left(\frac{D_1}{r} + D_2 r\right) \sin\theta \cos\left(\frac{n\pi}{\ell} z\right) ; \text{ region 2} \quad (27)$$

If we attempt to match boundary conditions (Table IV, $\frac{\partial^2 u}{\partial z^2} + k^2 u = -\frac{\partial \psi}{\partial z}$), there results

$$AI_1\left(\frac{n\pi}{\ell} a\right) = 0 \quad (28)$$

TABLE IV
ANALYTIC BOUNDARY CONDITIONS AND CONSTRAINTS

Boundary Conditions

$$r \text{ component ; } \frac{\partial^2 u}{\partial r \partial z} - \frac{k^2}{r} \frac{\partial v}{\partial \theta} = - \frac{\partial \psi}{\partial r} \left| \begin{array}{l} r = a \\ r = b \end{array} \right.$$

$$\theta \text{ component ; } \frac{1}{r} \frac{\partial^2 u}{\partial \theta \partial z} + k^2 \frac{\partial v}{\partial r} = - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \left| \begin{array}{l} r = a \\ r = b \end{array} \right.$$

$$z \text{ component ; } \frac{\partial^2 u}{\partial z^2} + k^2 u = - \frac{\partial \psi}{\partial z} \left| \begin{array}{l} r = a \\ r = b \end{array} \right.$$

Commensurate Constraints
 $r = a$

$$\frac{n\pi}{\ell} [C_1 J_1'(\beta a) + C_2 Y_1'(\beta a)] - \frac{k^2}{a} [D_1 J_1(\beta a) + D_2 Y_1(\beta a)] = -A I_1'(\frac{n\pi}{\ell} a)$$

$$- \frac{1}{a} [C_1 J_1(\beta a) + C_2 Y_1(\beta a)] (\frac{n\pi}{\ell}) + k^2 [D_1 J_1'(\beta a) + D_2 Y_1'(\beta a)] = \frac{A}{a} I_1(\frac{n\pi}{\ell} a)$$

$$-(\frac{n\pi}{\ell})^2 [C_1 J_1(\beta a) + C_2 Y_1(\beta a)] + k^2 [C_1 J_1'(\beta a) + C_2 Y_1'(\beta a)] = \frac{n\pi}{\ell} A I_1(\frac{n\pi}{\ell} a)$$

$$r = b$$

$$\frac{n\pi}{\ell} [C_1 J_1'(\beta b) + C_2 Y_1'(\beta b)] - \frac{k^2}{b} [D_1 J_1(\beta b) + D_2 Y_1(\beta b)] = -B K_1'(\frac{n\pi}{\ell} b)$$

$$- \frac{1}{b} [C_1 J_1(\beta b) + C_2 Y_1(\beta b)] (\frac{n\pi}{\ell}) + k^2 [D_1 J_1'(\beta b) + D_2 Y_1'(\beta b)] = \frac{B}{b} K_1(\frac{n\pi}{\ell} b)$$

$$-(\frac{n\pi}{\ell})^2 [C_1 J_1(\beta b) + C_2 Y_1(\beta b)] + k^2 [C_1 J_1'(\beta b) + C_2 Y_1'(\beta b)] = \frac{n\pi}{\ell} B K_1(\frac{n\pi}{\ell} b)$$

$$BK_1\left(\frac{n\pi}{L}b\right) = 0. \quad (29)$$

This yields the trivial result that $A = B = 0$.

These roots appeared for both the integral and analytical formulations. They must be discarded. For $k = \frac{n\pi}{L}$, even the determinant of the non-null field identification matrix is zero; no information about the constants A-L can ever be inferred using these degenerate values for k.

5. TOTAL TRANSIENT RESPONSE

The choice of eigenvalue is dictated entirely by the initial field causing the transient. If the field at $t = 0$ were $\vec{H} = \hat{a}_y H_0 \cos(\frac{3\pi}{L}z)$, we would use only the $n = 3$ eigenvalue and its associated temporal decay $(\frac{k^2}{\mu\sigma})$. (Actually there are higher order eigenvalues ($k > 150$) with each individual mode n ; we did not bother to calculate them since their temporal decay is so rapid.) In general, one must decompose the initial field into its spatial Fourier components. The field $H = H_0$ for all z ; can be written

$$\vec{H} = \sum_{n=1, \text{odd}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{4H_0}{n\pi} \cos\left(\frac{n\pi}{L}z\right). \quad (30)$$

Thus, the response to each component of the initial field is calculated separately (each with its own decay time) and weighted by the value $\frac{4}{n\pi}$. The higher order modal surfaces decay rapidly, but since the higher order Fourier components contribute most significantly to the regions of the field near $z = \pm L/2$, one expects these regions to fall off more quickly due to the higher k values associated with the higher order modes.

The procedure for finding the transient field is as follows:

1. Set up five null field equations: (16c) ($r < a$ and $r > b$); (17c) ($r < a$ and $r > b$); (15c) ($a < r < b$ only).
2. Set up one non-null field equation using (15a) ($r < a$). For $r < a$ we know that H at $t = 0^+$ is identical to H_{external} at $t = 0^-$.
3. Solve the six equations (and boundary conditions) for the unknowns (A-L) for the modal component n of interest.

4. The field found using these values of (A-L) for the mode n and eigenvalue k defines the cylinder response to a step change in the n^{th} component of external field

$$H_{y_n} = H_{o_n} (1 - u_{-1}(t)) . \quad (31)$$

The actual source field is

$$H_{y_n} = H_{o_n} \{1 - u_{-1}(t) + e^{-\alpha t} u(t)\} \quad (32)$$

where α is 5 to 40 ms for the FELIX experiments at ANL.

The total field response is realized through a convolution of $(\frac{-dH_{yn}}{dt})$ with $e^{-\lambda t}$, i.e.,

$$\begin{aligned} H_{\text{response}} &= \int_0^t \left(-\frac{d}{dt} (H_y)_{t=\tau} \right) e^{-\lambda(t-\tau)} d\tau \\ &= \frac{\frac{k^2}{\mu\sigma} t}{\frac{k^2}{\mu\sigma} - \alpha} \left\{ e^{\left(\frac{k^2}{\mu\sigma} - \alpha \right) t} - 1 \right\} . \end{aligned} \quad (33)$$

Thus (33) is the response (i.e., induced) field; the total field is (33) plus the external field $H_{o_n} e^{-\alpha t} \hat{a}_y$.

6. RESULTS

Figures 3, 4, and 5 show the predicted induced field (solid) and measured data (dashed) from the ANL experiments, fitted with a two exponential fit [7], for the small, medium, and large cylinders ($\tau = 39.68$ ms).

As evident in the above plots, the results agree reasonably well. Improvement was observed, especially for the large cylinder field predictions, by employing seven null field equations to derive the identification matrix rather than six.

A second improvement resulted after formulating the problem as a two region one and making the matrix nonsquare. In addition to the base resonances found for each mode as before, additional resonances (eigenvalues) appeared. For the 2-D problem we found 2 eigenvalues ($k = 51$ and $k = 150$).

Figure 3. Small Cylinder, Induced Field as a Function of Time, 39.68 ms.
Solid Line: Predicted Field (one eigenvalue). Dashed Line:
Measured Field Fit to Two Exponentials.

Figure 4. Medium Cylinder (60 cm) Induced Field as a Function of Time, 39.68 ms.

Figure 5. Large Cylinder (60 cm) Induced Field as a Function of Time, 39.68 ms.

The second value is quite far removed and it so happens that the weighting constant multiplying the second eigenvalue (based on the initial conditions) is nearly zero.

The 3-D eigenvalues for mode 1 ($\cos \frac{\pi}{L} z$) are $k = 57, 107, 128, 151, 178$. We seek the eigenvectors for each eigenvalue mode by mode exactly as explained above.

$$\underline{x} = \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} = c_{11} x_{11} e^{-\lambda_{11} t} + c_{21} x_{21} e^{-\lambda_{21} t} + \dots c_{n1} e^{-\lambda_{n1} t}$$

(for mode 1: $\cos(\frac{\pi}{L} z)$ or $\sin(\frac{\pi}{L} z)$)

$$+ c_{13} x_{13} e^{-\lambda_{13} t} + c_{23} x_{23} e^{-\lambda_{23} t} + \dots c_{n3} e^{-\lambda_{n3} t}$$

(for mode 2: $\cos(\frac{3\pi}{L} z)$ or $\sin(\frac{3\pi}{L} z)$)

+ higher order modes in z space

(34)

where the constants A-F are defined in Table II. Each eigenvector is weighted with a constant c chosen to satisfy the initial conditions. Constant c_n in Table II at $\theta = 0, z = 0$, must be equal to that part of the initial field which has a $\cos(\frac{n\pi}{L} z)$ dependence, i.e. $H_0 \frac{4}{n\pi} (-1)^{n+1/2}$ for a spatial Fourier decomposition. Likewise, the constant A_n at $\theta = 0, z = 0$, must equal the same field index $H_0 \frac{4}{n\pi} (-1)^{n+1/2}$ times the radius. With 2 eigenvectors/mode, we can use these conditions to specify the weighting constants in (34).

To test how important these additional eigenvalues are, the additional eigenvalues for mode $n = 1$ were included. A dramatic improvement is evident from Figure 6a to 6b for the large cylinder ($\tau = 39.6$ ms) as an additional eigenvector is included. One would hope the trend might continue as four eigenvalues (Figure 6c) per mode are included; clearly it does not. It should be remembered that the third eigenvalue for mode 1 is of the same order of magnitude as the second eigenvalue for mode $n = 3; \dots n=3$. Thus the inclusion of the third eigenvalue for mode one must be accompanied with the inclusion of the second eigenvalue for mode two; the fourth for mode one must be coupled with the third for mode three and the second for mode four. The good agreement in Figure 6b did not warrant the additional work.

Figure 6. Large Cylinder, 39.68 ms. (a) Fit with One Eigenvalue, (b) Fit with Two Eigenvalues, (c) Fit with Four Eigenvalues.

As a check, we see that when reasonable agreement exists using one eigenvalue as is the case for the medium cylinder (Figure 7a, $\tau = 39.68$ ms), adding an additional eigenvalue to the problem does not disturb the agreement (Figure 7b).

To illustrate the effect of higher order nodes, Figures 8-10 are included showing a plot of the peak field (normalized to reference value 0.02118 Tesla) versus axial position. As mentioned above, in the cylinder end regions, higher order n modes have a large influence, and the field decays more rapidly.

Figures 11-14 show the predicted (solid curve) and measured (dashed curve) induced field for a field decay time τ of 6.87 ms. Before these measurements were made, the 60 cm long medium cylinder was cut into two cylinders of length 20 cm and 40 cm, to emphasize 3-D end effects. The figures show results for the small, medium (20 cm), medium (40 cm), and large cylinders. The agreement between computed and measured fields was improved in going from one eigenvalue (Figure 15a) to two (Figure 15b), though not so much as in the 39.68 ms case (Figure 6). Again, adding a third eigenvalue (Figure 15c) gave no further appreciable improvement.

Figures 16-19 show the peak fields and the times those peak fields occur for the four cylinders at a field decay time τ of 6.87 ms.

7. CONCLUSIONS

The contributions of this work are as follows:

1. Utilization of the $u-v$ method to formulate a truly 3-D cylindrical problem in terms of scalar potentials without Dyadic Green's function.
2. Using the null field integral technique to calculate the characteristic eigenvalues of the cylinder.
3. Associating different eigenvalues with excitation field modal shape via Fourier decomposition.
4. Evaluation of the total 3-D field response using only 5×6 matrices via the use of smart basis functions.
5. Improvement of the results by adding additional null field point(s), making the identification matrix nonsquare.

Figure 7. Medium Cylinder (60 cm) 39.68 ms. (a) Fit with One Eigenvalue, (b) Fit with Two Eigenvalues.

Figure 8. Small Cylinder, 39.68 ms. (a) Maximum Induced Field at Different Axial Positions, (b) Time of Maximum Induced Field at Different Axial Positions. Solid Curve: Predicted Values (one eigenvalue). Dots: Experiment Results.

Figure 9. Medium Cylinder (60 cm), 39.68 ms. (a) Maximum Induced Field, (b) Time.

Figure 10. Large Cylinder, 39.68 ms. (a) Maximum Induced Field, (b) Time.

Figure 11. Small Cylinder, Induced Field as a Function of Time, 6.87 ms.

Figure 12. Medium Cylinder (20 cm), Induced Field as a Function of Time, 6.87 ms.

Figure 13. Medium Cylinder (40 cm), Induced Field as a Function of Time, 6.87 ms.

Figure 14. Large Cylinder, 6.87 ms. (a) Fit with One Eigenvalue, (b) Fit with Two Eigenvalues, (c) Fit with Three Eigenvalues.

Figure 15. Large Cylinder, 6.87 ms. (a) Fit with One Eigenvalue, (b) Fit with Two Eigenvalues.

Figure 15. Large Cylinder, 6.87 ms. (c) Fit with Three Eigenvalues.

Figure 16. Small Cylinder, 6.87 ms. (a) Maximum Induced Field at Different Axial Positions, (b) Time of Maximum Induced Field at Different Axial Positions.

Figure 17. Medium Cylinder (20 cm), 6.87 ms. (a) Maximum Induced Field, (b) Time.

Figure 18. Medium Cylinder, (40 cm), 6.87 ms. (a) Maximum Induced Field, (b) Time.

Figure 19. Large Cylinder, 6.87 ms. (a) Maximum Induced Field, (b) Time.

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