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APPLICATION OF THE GREEN'S FUNCTION METHOD TO SOME NONLINEAR PROBLEMS
OF AN ELECTRON STORAGE RING*

PART II - CHECKING THE METHOD BY A QUADRUPOLE PERTURBATION

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1. INTRODUCTION

In the first part¹ of this work I described a general approach to some storage ring problems. The basic concept and the main equations were developed there. The next natural step in this work should be an application of the developed technique to some particular nonlinear problem. Instead I found it useful to apply the method to the case of a field gradient perturbation first. This happened to be plausible not only as a check of the method, but, more importantly, as a way to resolve several troublesome difficulties encountered in subsequent calculations. The present note does not contain any new results (or only few, perhaps). Still, the work is felt to be necessary as a support for all the future applications of the method.

I consider here the effect of a global gradient error in a weak-focusing machine (Section 2) and the effect of a local gradient error in a strong focusing machine (Section 3). The distribution function of a particle bunch for a perturbed lattice in these cases can be written explicitly. The expansion of this distribution function as a series of perturbation terms produces the first and the second order corrections to the unperturbed function. From them we calculate then corresponding second moments.

A similar expansion of the distribution function is found by the Green's function method described in Part I of this work. The second moments are found independently with the help of this distribution function. The comparison of these two results is one way to check the method and the correctness of the calculations. This comparison is done for the first and second order perturbations i.e. up to the order which is sufficient for subsequent applications.

One may ask the question, "why not to compare the two expansions of the distribution function directly instead of dealing with their moments".

The answer to this is given in Part I: The calculation of any moment of the distribution function expressed in terms of the Green's function is much simpler than finding the distribution function.

For the same reason, in subsequent parts of this work we deal with the moments of the distribution function.

The horizontal (vertical) component of the field representing the gradient perturbation depends on the horizontal (vertical) coordinate only. Hence we may consider a one-dimensional case representing the motion in either of these planes.

In the subsequent calculations x and $\dot{x}$ represent a displacement and a velocity of the particle (in any lateral plane), ω (or ν) and α the frequency (or tune) and the damping constant for the unperturbed machine.

2. FIELD GRADIENT PERTURBATION IN A WEAK FOCUSING MACHINE

Let us start with the case of a weak-focusing machine. One reason to consider this case is its simplicity. Another, perhaps more important reason is that this case can be solved exactly, i.e. without the use of the perturbation theory. The exact solution is given in Appendix A since it is outside the scope of the present work.

The equation describing the motion of a particle in the presence of a global gradient perturbation to the magnetic field may be written as follows

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = -\Delta\omega^2 x, \quad (2.1)$$

where $\Delta\omega^2$ is proportional to the gradient error.

The corresponding Fokker-Planck equation for the distribution function $\psi(x, \dot{x}, t)$ is

$$\frac{\partial \psi}{\partial t} + \dot{x} \frac{\partial \psi}{\partial x} - 2\alpha\dot{x} - 2\alpha x \frac{\partial \psi}{\partial \dot{x}} - \omega^2 x \frac{\partial \psi}{\partial \dot{x}} - q \frac{\partial^2 \psi}{\partial \dot{x}^2} = \Delta\omega^2 x \frac{\partial \psi}{\partial \dot{x}} \quad (2.2)$$

We assume here that neither the damping constant α , nor the radiation noise constant q in this equation changes due to the gradient perturbation.

Equation (2.2) can also be written in the equivalent form

$$\frac{\partial \psi}{\partial t} + \dot{x} \frac{\partial \psi}{\partial x} - 2\alpha \psi - 2\alpha x \frac{\partial \psi}{\partial \dot{x}} - (\omega^2 + \Delta\omega^2) x \frac{\partial \psi}{\partial \dot{x}} - q \frac{\partial^2 \psi}{\partial \dot{x}^2} = 0 \quad (2.2')$$

which allows to express the distribution function in terms of a perturbed oscillation frequency.

2.1 Expansion of the Explicit Distribution Function

The steady state solution of equation (2.2') has the following form

(cf. equations (4.3 - 4.5) Part I):

$$\psi(x, \dot{x}) = \frac{2\alpha \sqrt{\omega^2 + \Delta\omega^2}}{\pi q} \exp \left\{ -\frac{\alpha(\omega^2 + \Delta\omega^2)x^2}{q} - \frac{\alpha}{q} \dot{x}^2 \right\} \quad (2.3)$$

This function is the explicit distribution function for a perturbed machine.

Expanding this function in a power series of $\Delta\omega^2/\omega^2$ terms to the order of $(\Delta\omega^2/\omega^2)^2$, one finds:

$$\psi_0 = \frac{1}{2\pi \sigma^2 \omega} \exp \left\{ -\frac{x^2}{2\sigma^2} - \frac{\dot{x}^2}{2\sigma^2 \omega^2} \right\}, \quad (2.4)$$

$$\psi_1 = \psi_0 \frac{\Delta\omega^2}{2\omega^2} \left(1 - \frac{x^2}{\sigma^2} \right), \quad (2.5)$$

$$\psi_2 = \psi_0 \left(\frac{\Delta\omega^2}{2\omega^2} \right)^2 \left(\frac{x^4}{2\sigma^2} - \frac{x^2}{\sigma^2} - \frac{1}{2} \right), \quad (2.6)$$

where $\psi = \psi_0 + \psi_1 + \psi_2$ and

$$\sigma^2 = q/2 \omega^2 \alpha = \iint_{-\infty}^{\infty} x^2 \psi_0 dx d\dot{x} \quad (2.7)$$

is the square of the unperturbed rms size of the bunch. Now it is easy to find the second moments of the first and the second order corrections of the explicit distribution function:

$$E_1 = \iint \dot{x}^2 \psi_1 \, dx \, d\dot{x} = -\sigma^2 \Delta \omega^2 / \omega^2 \quad (2.8)$$

$$E_1' = \iint \ddot{x}^2 \psi_1 \, dx \, d\dot{x} = 0 \quad (2.9)$$

$$E_2 = \iint \dot{x}^2 \psi_2 \, dx \, d\dot{x} = \sigma^2 (\Delta \omega^2 / \omega^2)^2 \quad (2.10)$$

$$E_2' = \iint \ddot{x}^2 \psi_2 \, dx \, d\dot{x} = 0 \quad (2.11)$$

Expressions (2.8 - 2.11) may be compared with the results of further calculations by means of the Green's function as well as with the non-perturbation solution given in Appendix A.

2.2 The first order Green's function calculation

As it is shown in Part I, the first order solution of equation (2.2) can be written in the form (cf. equation (3.8)*, Part I):

$$\psi_1(x, \dot{x}) = \int_0^\infty dt \iint_{-\infty}^\infty dx_0 d\dot{x}_0 G(x, \dot{x}, x_0, \dot{x}_0, t) \Delta \omega^2 x_0 \frac{\partial \psi_0}{\partial \dot{x}_0} \quad (2.12)$$

Now we can find the second moments of ψ_1 . The second moment of $\dot{x}^2$ can be expressed in terms of the second moment $P_2(x_0, \dot{x}_0, t)$ of the Green's function, which is found in Appendix B, Part I. Only coefficient $p_3(t)$ comes into the final result:

$$E_1 \equiv \iint \dot{x}^2 \psi_1 \, dx \, d\dot{x} = -2 \Delta \omega^2 \sigma^2 \int_0^\infty p_3(t) \, dt \quad (2.13)$$

Substitute now the value of the integral (see Appendix C) one finds:

$$E_1 = -\sigma^2 \Delta \omega^2 / \omega^2 \quad (2.14)$$

* The summation in equation (3.8), Part I, which the reader will find, is due to the δ -function time dependence of the right hand side of the Fokker-Planck equation (2.7), Part I. In the case considered here, the right hand side of equation (2.2) does not contain δ -functions in time. Hence instead of the summation the solution contains an integration over time, cf. formula (3.5), Part I.

To find the second moment of $\dot{x}^2$ over ψ_1 one needs to know the second moment of $\dot{x}^2$ over the Green's function. This moment Q_2 and its coefficients are found in Appendix B of this note. Again only coefficient q_3 plays a role:

$$E_1' \equiv \iint \dot{x}^2 \psi_1 dx d\dot{x} = -2 \Delta \omega^2 \sigma^2 \int_0^\infty q_3(\tau) d\tau \quad (2.15)$$

Since the integral here is zero (Appendix C) we finally obtain:

$$E_1' = 0 \quad (2.16)$$

Both results (2.14) and (2.16) comply with the results of the direct calculations (2.8) and (2.9), respectively.

2.3 The second order Green's function calculation

The second order solution of equation (2.2) for our case is:

$$\psi_2(x, \dot{x}) = \int_0^\infty dt \int_0^\infty d\dot{x}_1 \iint_{-\infty}^\infty dx_1 d\dot{x} f(x, \dot{x}, x_1, \dot{x}_1, \tau) \Delta \omega^2 x \frac{\partial}{\partial \dot{x}_1} \iint_{-\infty}^\infty dx_0 d\dot{x}_0 G(x_1, \dot{x}_1, x_0, \dot{x}_0, \tau) \frac{\partial \psi_0}{\partial \dot{x}_0} \quad (2.17)$$

(cf. (3.9), Part I).

The second moment of $\dot{x}^2$ over ψ_2 is:

$$E_2 \equiv \iiint \dot{x}^2 \psi_2 dx d\dot{x} = 4 \frac{(\Delta \omega^2)^2}{\omega^2} \sigma^2 \left[\int_0^\infty p_2^2(t) dt \int_0^\infty r_3(t) dt + \left(\int_0^\infty p_3(t) dt \right)^2 \right] \quad (2.18)$$

Similarly, the second moment of $\dot{x}^2/\omega^2$ over ψ_2 is:

$$E_2' \equiv \iiint (\dot{x}^2/\omega^2) \psi_2 dx d\dot{x} = 4 \frac{(\Delta \omega^2)^2}{\omega^2} \sigma^2 \left[\int_0^\infty q_2^2 dt \int_0^\infty r_3 dt + \int_0^\infty q_3 dt \int_0^\infty p_3 dt \right] \quad (2.19)$$

In expressions (2.18) and (2.19) coefficients $p_2(t), p_3(t), q_2(t), q_3(t)$ and $r_3(t)$ are those of corresponding second moments of the Green's function $\dot{p}_2, Q_2$ and P_2 . They are given in Appendix B.

Performing integrations (they are done in Appendix C) in expressions

(2.18), (2.19) one finds^{*}

$$E_2 = \left(\frac{\Delta\omega}{\omega} \right)^2 \sigma^2 \quad (2.20)$$

$$E_2' = 0 \quad (2.21)$$

Again these results agree with the results of the direct calculation (2.10) and (2.11), respectively.

Both the first and the second order calculations are in agreement also with the exact nonperturbation solution given in Appendix A.

3. FIELD GRADIENT PERTURBATION IN A STRONG FOCUSING MACHINE

We consider a local δ -functionlike gradient perturbation at the azimuth s of a strong focusing machine. Such a perturbation is fundamentally different from other nonlinear perturbations. Namely, a linear perturbation not only changes all the machine functions (beta, eta-and alpha-functions and tunes), but also changes the damping constants and the rate of the radiation noise. Consequently, the beam emittance changes also. Hence, the Fokker-Planck equation itself in this case (cf. equation (5.11), Part I) changes due to the dependence of its coefficients (such as $\bar{\alpha}$, ν and $\bar{q}$) on the machine functions. As we shall see further, this dependence influences the second order effects. From these considerations it is clear that the calculation which does not take into account the longitudinal motion (or particle energy oscillations), can not give a correct answer in the second order. In particular, the expression of the beam emittance in terms of the machine functions can not

* Note that in r_2 , terms to the second order in α/ω should be kept since both $\int p_2^2 dt$ and $\int q_2^2 dt$ are proportional to $\frac{1}{\alpha}$.

be obtained in the frame of two dimensional model. Indeed, as we know the emittance depends not only on beta- and alpha-functions, but also on the machine dispersion function².

3.1 Changes of the tune, the beta and the alpha-functions due to a local gradient error

We need to find expressions for the change of the tune and the beta function up to the second order in the perturbation. For the alpha function the first order expression is sufficient. This can be done most easily by a matrix method.

The transformation matrix for one turn around the unperturbed machine at an azimuth s is:

$$M = \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -(1+\alpha^2)(\sin \mu)/\beta & \cos \mu - \alpha \sin \mu \end{pmatrix} \quad (3.1)$$

where $\mu (=2\pi\nu)$, β and $\alpha = (-\beta'/2)$ are values of the phase advance, the beta- and the alpha functions at the point s . The same expression holds for the perturbed matrix M' if it is expressed in terms of the perturbed values $\mu' = \mu + \Delta\mu$, $\beta' = \beta + \Delta\beta$ and $\alpha' = \alpha + \Delta\alpha$. On the other hand M' can be obtained as a product of M and a perturbation matrix

$$\begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix} \quad (3.2)$$

where Q is the integrated strength of the quadrupole perturbation.

Comparing the elements of the matrix derived in both ways one obtains:

$$\cos \mu' = \cos \mu - \frac{Q\beta}{2} \sin \mu \quad (3.3)$$

$$\alpha' \sin \mu' = \alpha \sin \mu + \frac{Q\beta}{2} \sin \mu \quad (3.4)$$

$$\beta' \sin \mu' = \beta \sin \mu \quad (3.5)$$

Expanding the left hand sides of these equations up to the second order in the small perturbation parameter $Q\beta$ we find:

$$\frac{\Delta\mu}{\mu} = \frac{Q\beta}{2} - \frac{(Q\beta)^2}{8 \tan \mu} \quad (3.6)$$

$$\frac{\Delta\beta}{\beta} = -\frac{Q\beta}{2 \tan \mu} + \frac{(Q\beta)^2}{4 \tan^2 \mu} \left(1 + \frac{1}{2 \cos^2 \mu} \right) \quad (3.7)$$

$$\Delta\alpha = \frac{Q\beta}{2} - \frac{Q\beta\alpha}{2 \tan \mu} \quad (3.8)$$

The first order terms in (3.6), (3.7) and (3.8) can be obtained also with the help of the corresponding formulae from the works by M. Sands² or E. Courant and H. Snyder³.

3.2 Expansion of the explicit distribution function

For a strong focusing machine the steady state distribution function is a Gaussian function if it is expressed in terms of the Courant-Snyder variables (cf. (5.16), Part I):

$$\psi(u, u') \, du \, du' = \frac{\exp \left\{ -\frac{u^2}{2\epsilon} - \frac{u'^2}{2\epsilon\nu^2} \right\}}{2\pi\epsilon\nu} \, du \, du' \quad (3.9)$$

where ϵ and ν are the perturbed beam emittance and the machine tune,

$$u = x/\sqrt{\beta} \quad (3.10)$$

$$u' = \nu (\sqrt{\beta} x' + \alpha x/\sqrt{\beta}) \quad (3.11)$$

(prime denotes a derivative over betatron phase ϕ for u and over s for x).³

Both u and u' depend on perturbed ν , β and α . Hence, to obtain a valid expansion of ψ in a perturbation series one should take into account this implicit dependence on perturbed machine functions. It can be done first by changing variables to x , x' , expanding the machine functions in a series and then expressing x , x' in terms of u , u' with the unperturbed machine functions. Let us denote:

$$\epsilon' = \epsilon \left(1 + \frac{\Delta\epsilon_1}{\epsilon} + \frac{\Delta\epsilon_2}{\epsilon} \right) \quad (3.12)$$

$$\beta' = \beta \left(1 + \frac{\Delta\beta_1}{\beta} + \frac{\Delta\beta_2}{\beta} \right) \quad (3.13)$$

$$v' = v \left(1 + \frac{\Delta v_1}{v} + \frac{\Delta v_2}{v} \right) \quad (3.14)$$

$$\alpha' = \alpha + \Delta\alpha \quad (3.15)$$

where subscript 1 and 2 denote the first and the second order terms, respectively. Then, the first and the second order terms in the expansion of ψ are:

$$\begin{aligned} \psi_1 &= \psi_0 \left[\frac{u^2}{2\epsilon} \left(\frac{\Delta\epsilon_1}{\epsilon} + \frac{\Delta\beta_1}{\beta} \right) + \frac{u^2}{2\epsilon v^2} \left(\frac{\Delta\epsilon_1}{\epsilon} - \frac{\Delta\beta_1}{\beta} \right) - \Delta\alpha \frac{u u'}{\epsilon v} + \frac{\Delta\epsilon_1}{\epsilon} \right] \quad (3.16) \\ \psi_2 &= \psi_0 \left\{ \frac{u^2}{2\epsilon} \left[\frac{\Delta\epsilon_2}{\epsilon} - 2 \left(\frac{\Delta\epsilon_1}{\epsilon} \right)^2 + \frac{\Delta\beta_2}{\beta} - \left(\frac{\Delta\beta_1}{\beta} \right)^2 - 2 \frac{\Delta\epsilon_1}{\epsilon} \frac{\Delta\beta_1}{\beta} - (\Delta\alpha)^2 \right] \right. \\ &\quad + \frac{u^2}{2\epsilon v^2} \left[\frac{\Delta\epsilon_2}{\epsilon} - 2 \left(\frac{\Delta\epsilon_1}{\epsilon} \right)^2 - \frac{\Delta\beta_2}{\beta} + 2 \frac{\Delta\epsilon_1}{\epsilon} \frac{\Delta\beta_1}{\beta} \right] \\ &\quad + \frac{u^4}{8\epsilon^2} \left[\left(\frac{\Delta\epsilon_1}{\epsilon} \right)^2 + \left(\frac{\Delta\beta_1}{\beta} \right)^2 + 2 \frac{\Delta\epsilon_1}{\epsilon} \frac{\Delta\beta_1}{\beta} \right] \\ &\quad + \frac{u^4}{8\epsilon^2 v^4} \left[\left(\frac{\Delta\epsilon_1}{\epsilon} \right)^2 + \left(\frac{\Delta\beta_1}{\beta} \right)^2 - 2 \frac{\Delta\epsilon_1}{\epsilon} \left(\frac{\Delta\beta_1}{\beta} \right) \right] \\ &\quad + \frac{u^2 u'}{4\epsilon^2 v^2} \left[\left(\frac{\Delta\epsilon_1}{\epsilon} \right)^2 - \left(\frac{\Delta\beta_1}{\beta} \right)^2 \right] - \frac{\Delta\epsilon_1}{\epsilon} 2 + \left(\frac{\Delta\epsilon_1}{\epsilon} \right)^2 \\ &\quad + \Delta\alpha \frac{u u'}{\epsilon v} \frac{\Delta\epsilon_1}{v} - \Delta\alpha \frac{u^3 u'}{2\epsilon^2 v} \left(\frac{\Delta\epsilon_1}{\epsilon} + \frac{\Delta\beta_1}{\beta} \right) - \Delta\alpha \frac{u u'}{2\epsilon^2 v^3} \left(\frac{\Delta\epsilon_1}{\epsilon} - \frac{\Delta\beta_1}{\beta} \right) \\ &\quad \left. + (\Delta\alpha)^2 \frac{u^2 u'^2}{2\epsilon^2 v^2} \right\} \quad (3.17) \end{aligned}$$

It is easy to prove that $\int \psi_1 du du'$ and $\int \psi_2 du du'$ both are equal to zero. The second moments of ψ_1 and ψ_2 is also easy to find:

$$L_+ \equiv \int \psi_1 u^2 du du' = \epsilon \left(\frac{\Delta\epsilon_1}{\epsilon} + \frac{\Delta\beta_1}{\beta} \right) \quad (3.18)$$

$$E_1' \equiv \int \psi_1 \frac{u'^2}{v^2} du du' = \epsilon \left(\frac{\Delta \epsilon_1}{\epsilon_1} - \frac{\Delta \beta_1}{\beta_1} \right) \quad (3.19)$$

$$E_2 \equiv \int \psi_2 u^2 du du' = \epsilon \left[\frac{\Delta \epsilon_2}{\epsilon} + \frac{\Delta \beta_2}{\beta} + \frac{\Delta \epsilon_1}{\epsilon} \frac{\Delta \beta_1}{\beta} \right] \quad (3.20)$$

$$E_2' \equiv \int \psi_2 \frac{u'^2}{v^2} du du' = \epsilon \left[\frac{\Delta \epsilon_2}{\epsilon} - \frac{\Delta \beta_2}{\beta} + \left(\frac{\Delta \beta_1}{\beta} \right)^2 - \frac{\Delta \epsilon_1}{\epsilon} \frac{\Delta \beta_1}{\beta} + (\Delta \alpha)^2 \right] \quad (3.21)$$

Staying in the limits of the present approach (i.e. considering the transverse motion and neglecting all the effects arising from the energy oscillations), it is impossible to find the dependence of ϵ on the machine functions. Hence, we are unable to compare expressions (3.18) - (3.21) with the corresponding formulae found from the Green's function calculations. The only thing we can do is to express $\Delta \epsilon_1/\epsilon$ and $\Delta \epsilon_2/\epsilon$ in terms of the machine functions. This will be done in the next two sections.

3.3 The first order Green's function calculation

The equation of motion in any of the lateral planes, which takes into account a local quadrupole perturbation Q (cf. Section 5, Part I) is:

$$u'' + 2\alpha u' + v^2 u = \sum_k \delta(\phi - \phi_k) F, \quad (3.22)$$

where

$$F = -v\beta Qu \quad (3.23)$$

v , ϕ , β and α denote the tune, the phase, the beta-function and the damping of the betatron oscillations respectively. Note, that the motion is described thus in the unperturbed Courant-Snyder variables u and $u' = du/d\phi$ i.e. the variables, defined by expressions (3.10), (3.11) with unperturbed machine functions. The phase value ϕ_k , at which a particle experiences a "kick" from the local perturbation is

$$\phi_k = \phi_0 + 2\pi k, \quad (3.24)$$

where k assumes all integer values such that ϕ_k does not exceed the current phase ϕ .

As it is shown in Part I, the first order perturbation distribution function in this case has the following form:

$$\psi_1(u, u', \phi_k) = - \sum_{m=-\infty}^{k-1} \int du_0 du'_0 G(u, u', u_0, u'_0, \phi_k - \phi_m) F \frac{\partial \psi_0}{\partial u}, \quad (3.25)$$

where $G(u, u', u_0, u'_0, \phi_k - \phi_m)$ is the Green's function of the Fokker-Planck equation and $\psi_0(u, u')$ is the steady state unperturbed distribution function.

Each of the second moments of ψ_1 , E_1 and E_1' , can be expressed through the second moments P_2 and Q_2 of the Green's function G . Performing integration by parts one then finds:

$$E_1 = \int u^2 \psi_1(u, u') du du' = - 2\beta Q E \sum_{m=-\infty}^{\infty} (v p_3) \quad (3.26)$$

$$E_1' = \int \frac{u'^2}{v^2} \psi_1(u, u') du du' = - 2\beta Q E \sum_{m=-\infty}^{\infty} \left(\frac{q_3}{v} \right) \quad (3.27)$$

where p_3 and q_3 are the corresponding coefficients of P_2 and Q_2 . One can find expressions for them in Appendix B.

The sums in the expressions (3.26), (3.27) along with several others, which are needed for the second order calculations, are evaluated in Appendix C. We get:

$$E_1 = - \frac{\beta Q E}{4\pi v} - \frac{\beta Q E}{2 \tan 2\pi v} = - \epsilon \frac{\Delta v_1}{v} + \epsilon \frac{\Delta \beta_1}{\beta_1} \quad (3.28)$$

$$E_1' = - \frac{\beta Q E}{4\pi v} + \frac{\beta Q E}{2 \tan 2\pi v} = - \epsilon \frac{\Delta v_1}{v} - \epsilon \frac{\Delta \beta_1}{\beta} \quad (3.29)$$

where $\Delta v_1/v$ and $\Delta \beta_1/\beta$ are the corresponding first order corrections to the tune and the beta-function from (3.6) and (3.7).

Expressions (3.28) and (3.29) agree with the formulae (3.18) and (3.19), respectively, if

$$\frac{\Delta \epsilon_1}{\epsilon} = - \frac{\Delta \nu}{\nu} \quad , \quad (3.30)$$

or what is the same

$$\epsilon = \text{const}/\nu \quad (3.31)$$

Such a relation is consistent with the one dimensional model we are considering. Indeed, expression (3.31) can be obtained from the invariant of the motion (i.e. the ratio of the oscillation energy to its frequency).

In an actual strong focusing machine the dependence of the emittance on the tune might deviate from that of expression (3.31) due to a contribution from the eta-function. Such contributions can not be taken into account without including the longitudinal motion into consideration.

3.4 The second order Green's function calculation

The second order perturbation distribution function according to the formula (3.9), Part I is:

$$\begin{aligned} \psi_2(u, u', \phi_k) &= \sum_{k=-\infty}^{k-1} \int du_1 du_1' G(u, u', u_1, u_1', \phi_k - \phi_k) \\ \sum_{m=-\infty}^{k-1} F(u_1) \frac{\partial}{\partial u_1} &\int du_0 du_0' G(u_1, u_1', u_0, u_0', \phi_k - \phi_m) F(u_0) \frac{\partial \psi_0}{\partial u_0} \end{aligned} \quad (3.32)$$

Let us find the second moments of u^2 and u'^2/ν^2 over ψ_2 . Integrating first over $du du'$ we will express E_2 and E_2' in terms of the second moments of the Green's function $P_2(u_1, u_1')$, $\dot{P}_2(u_1, u_1')$ and $Q_2(u_1, u_1')$. Use now the formula for integration by parts over u_1 and then integrate over $du_1 du_1'$. It is seen that now appear the moments $P_2(u_0, u_0)$, $\dot{P}_2(u_0, u_0')$ and $Q_2(u_0, u_0')$. We get finally:

$$E_2 \equiv \int u^2 \psi_2 du du' = 4\epsilon \beta^2 Q^2 \left\{ \sum_k^{\infty} v_{p_2}^2(\phi_k) \cdot \sum_k^{\infty} r_3(\phi_k) + \left[\sum_k^{\infty} v_{p_3}(\zeta_k) \right]^2 \right\} \quad (3.33)$$

$$E_2' \equiv \int \frac{u^2}{v^2} \psi_2 du du' = 4\epsilon \beta^2 Q^2 \left\{ \sum_k^{\infty} q_2^2(\phi_k) \cdot \sum_k^{\infty} r_3(\phi_k) + \sum_k^{\infty} v_{p_3}(\phi_k) \cdot \sum_k^{\infty} \frac{q_3(\zeta_k)}{v} \right\} \quad (3.34)$$

The sums entering these expression can be found in Appendix C. We get:

$$E_2 = \epsilon \beta^2 Q^2 \left[\frac{1 + \cos^2 \mu}{\sin^2 \mu} + \frac{2 \cos \mu}{\mu \sin \mu} \right] / 4 \quad (3.35)$$

$$E_2' = \epsilon \beta^2 Q^2 / 4 \quad (3.36)$$

Comparing (3.35) with (3.20) or (3.36) with (3.21) and substituting corresponding expressions for $\Delta\beta_1/\beta$, $\Delta\beta_2/\beta$ and $\Delta\alpha$ from (3.7), (3.8) one finds consistently:

$$\Delta\epsilon_2/\epsilon = \beta^2 \frac{Q^2}{4} \left(\frac{\cos \mu}{\mu \sin \mu} + \frac{1}{2 \sin^2 \mu} \right) \quad (3.37)$$

4. Concluding remarks

We considered here the application of the Green's function method to the case of a linear perturbation. Using this case as check for the method, we also learned two important things.

One is the rule of summation over subsequent kicks experienced by a particle going through a perturbation. The necessity to subtract 1/2 from the sum of cosines is not immediately clear although may be understood both from physical and mathematical considerations.

From the physical point of view only half of the kick at the time 0 should be taken into account. Perhaps, it becomes more clear if we consider instead of the δ -function kick, a less abrupt smooth δ -functionlike kick, center of which is positional at the time 0. Only that part of such a kick

which acts at $t < 0$ can influence the particle motion at $t = 0$. More rigorous proof is needed here. In any case, to obtain correct results up to the second order in perturbation the sum of r_j should be a small number in the limit $\alpha/v \rightarrow 0$ as it is shown in Appendix A. This is true only if we use the above mentioned rule of summation. From the mathematical point of view such a subtraction is needed to provide for a discontinuous function the value equal to $1/2$ at the point where it jumps from 0 to 1 . Similar agreement is used in the theory of the Fourier series.

The second important lesson is related to the linear part of the nonlinear perturbation. Since the linear force changes the machine functions on which implicitly depends the distribution function, more accurate results may be obtained by including the linear part of the perturbation into unperturbed solution, or in the other words, it might be better to use for the unperturbed distribution function that obtained by including the linear part of the perturbing force into the machine lattice. The perturbation should then contain only the nonlinear part of the force.

The subtraction of the linear part resembles the procedure used in the previous work⁴ but has no ambiguity and no fitting parameter used there.

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References

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Appendix A

Nonperturbation Treatment of the Green's Function Integral Equation

An integral equation which is fully equivalent to the Fokker-Planck equation has its Green's function as the kernel (see equation (3.5), Part I). In our case (i.e. for equation (2.2) with a linear perturbing term) the integral equation has the following form:

$$\psi(x, \dot{x}, t) = \psi_0(x, \dot{x}) + \int_{-\infty}^t d\tau \int_{-\infty}^{\infty} dx_0 d\dot{x}_0 G(x, \dot{x}, x_0, \dot{x}_0, t-\tau) \Delta\omega^2 x_0 \frac{\partial \psi}{\partial \dot{x}_0}, \quad (A1)$$

where ψ_0 is the initial distribution function. Let us find the second moment of x^2 over ψ . Multiplying both sides of equation (A1) by x^2 and integration over $dx d\dot{x}$ we get:

$$E(t) = \sigma^2 + \int_{-\infty}^t d\tau \int dx_0 d\dot{x}_0 P_2(x_0, \dot{x}_0, t-\tau) \Delta\omega^2 x_0 \frac{\partial \psi}{\partial \dot{x}_0}, \quad (A2)$$

where P_2 is the second moment of the Green's function which is evaluated in Appendix B, Part I (see also Appendix B of this note). Take integral in (A2) by parts:

$$E(t) = \sigma^2 - 2\Delta\omega^2 \int_{-\infty}^t d\tau \int dx_0 d\dot{x}_0 \left[P_2^2(t-\tau) \dot{x}_0 x_0 + P_3(t-\tau) x_0^2 \right] \psi(x_0, \dot{x}_0, \tau) \quad (A3)$$

The last term can be expressed through $E(\tau)$. In the same way we can find $\dot{E}(t)$ (the second moment of $x\dot{x}$ over ψ). Repeat now the previous calculations almost exactly. The only difference will be that instead of P_2 one will get $\dot{P}_2$ which also can be found in Appendix B, Part I. The result will be:

$$\dot{E}(t) = -2\Delta\omega^2 \int_{-\infty}^t d\tau \left[r_2(t-\tau) \dot{E}(\tau) + r_3(t-\tau) E(\tau) \right] \quad (A4)$$

Equation (A3) can be rewritten in the similar form:

$$E(t) = \sigma^2 - 2\Delta\omega^2 \int_{-\infty}^t d\tau \left[p_2^2(t-\tau) \dot{E}(\tau) + p_3(t-\tau) E(\tau) \right] \quad (A5)$$

Let us introduce the Fourier transformations of E and $\dot{E}$:

$$e_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{ikt} dt \quad (A6)$$

$$e'_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \dot{E}(t) e^{ikt} dt \quad (A7)$$

Using the convolution theorem one get:

$$e_k (1 + 2\Delta\omega^2 p_{3k}) + 2\Delta\omega^2 e'_k (p_2^2)_k = \sigma^2 \delta(k), \quad (A8)$$

$$2\Delta\omega^2 e_k r_{3k} + e'_k (1 + 2\Delta\omega^2 r_{2k}) = 0, \quad (A9)$$

where p_{3k} , $(p_2^2)_k$, r_{3k} and r_{2k} are the Fourier components of the functions $p_3(t)$, $p_2^2(t)$, $r_3(t)$ and $r_2(t)$, respectively.

System of algebraic equations (A8), (A9) with e_k and e'_k as unknowns is easy to solve. Since both of them are proportional to $\delta(k)$, expressions for $E(\infty)$ and $\dot{E}(\infty)$ contain values p_{3k} , $(p_2^2)_k$, r_{3k} and r_{2k} at $k = 0$ (i.e. $\int_0^\infty p_3(t)dt$ etc.) only. These integrals are easy to find. In particular,

$$\int_0^\infty r_3(t)dt = 0, \quad (A10)$$

$$\int_0^\infty p_3(t)dt = 1/2\omega^2 \quad (A11)$$

(See Appendix C)

Hence,

$$E = \frac{\sigma^2}{1 + \Delta\omega^2/\omega^2} \quad (A12)$$

$$\dot{E} = 0 \quad (A13)$$

Expansions of (A12) into series gives

$$E \approx \sigma^2 \left[1 - \Delta\omega^2/\omega^2 + (\Delta\omega^2/\omega^2)^2 \right] \quad (A14)$$

In the similar way one can evaluate $E'(t)$ (the second moment of $(\dot{x}/\omega)^2$ over ψ).

$$E'(t) = \sigma^2 - 2\Delta\omega^2 \int_{-\infty}^t d\tau \left[\frac{q_2^2(t-\tau)}{\omega} \dot{E}(\tau) + \frac{q_3(t-\tau)}{\omega^2} E(\tau) \right] \quad (A15)$$

Again using the Fourier transformation it is easy to show that

$$E'(\infty) = \sigma^2 \quad (A16)$$

$$\left(\text{since } \int_0^\infty q_3(t) dt = 0 \right)$$

Both results (A14) and (A16) agree with the results (2.8 - 2.11) and (2.14), (2.16), (2.20) and (2.21) obtained by perturbation method.

Appendix B The Second Moment $\dot{x}^2$ of the Green's Function

In Appendix B, Part I two second moments of the Green's function are found

$$P_2(x_0, \dot{x}_0, t) = \int dx d\dot{x} \ x^2 G(x, \dot{x}, x_0, \dot{x}_0, t) \quad (B1)$$

and

$$\dot{P}_2(x_0, \dot{x}_0, t) = \int dx d\dot{x} \ x \dot{x} G(x, \dot{x}, x_0, \dot{x}_0, t) \quad (B2)$$

It is shown there that

$$P_2 = p_0 + p_1^2 x_0^2 + p_2^2 \dot{x}_0^2 + 2p_3 x_0 \dot{x}_0, \quad (B3)$$

$$\dot{P}_2 = r_0 + r_1 x_0^2 + r_2 \dot{x}_0^2 + 2r_3 x_0 \dot{x}_0, \quad (B4)$$

Coefficients $p_i(t)$ and $r_i(t)$ ($i = 0, 1, 2, 3$) can be found there. I repeat here expressions for $p_2(t)$, $p_3(t)$ and $r_3(t)$ for the readers convenience:

$$p_2 = -\frac{1}{\omega} e^{-\alpha t} \sin \omega t \quad (B5)$$

$$p_3 = \frac{1}{\omega} e^{-2\alpha t} (\sin \omega t \cos \omega t + \frac{\alpha}{\omega} \sin^2 \omega t) \quad (B6)$$

$$r_3 = \frac{1}{2} e^{-2\alpha t} (\cos 2\omega t - \frac{\alpha^2}{\omega^2} + \frac{\alpha^2}{\omega^2} \cos 2\omega t) \quad (B7)$$

We need now to evaluate one more moment of the Green's function:

$$Q_2(x_0, \dot{x}_0, t) = \int dx d\dot{x} \ \dot{x}^2 G(x, \dot{x}, x_0, \dot{x}_0, t) \quad (B8)$$

Performing calculation similar to those in Part I, we find:

$$Q_2 = q_0 + q_1^2 x_0^2 + q_2^2 \dot{x}_0^2 + 2q_3 x_0 \dot{x}_0, \quad (B9)$$

Where

$$q_0 = \frac{1}{2A_2} \quad (B10)$$

$$q_1 = \frac{\bar{A}_9}{2A_2} = -\omega e^{-\alpha t} \sin \omega t \quad (B11)$$

$$q_2 = \frac{\bar{A}_{10}}{2A_2} = e^{-\alpha t} \left(\cos \omega t - \frac{\alpha}{\omega} \sin \omega t \right) \quad (B12)$$

$$q_3 = q_1 q_2 = -\omega e^{-2\alpha t} \left(\sin \omega t \cos \omega t - \frac{\alpha}{\omega} \sin^2 \omega t \right) \quad (B13)$$

Here

$$\bar{A}_2 = A_2 - A_3^2/4A_1, \quad (B14)$$

$$\bar{A}_9 = A_9 - A_3 A_7/2A_1, \quad (B15)$$

$$\bar{A}_{10} = A_{10} - A_3 A_8/2A_1, \quad (B16)$$

where coefficients A_i are defined in appendix A, Part I.

Appendix C

Integrals and Sums of the Coefficients of the Second Moments of the Green's Function

I tabulate here for the readers convenience several integrals and sums of the coefficients which enter expressions discussed in the present note. All calculations are done in the lowest order of the small parameter α/ω .

Table I

Integrals

$f(t)$	$\int_0^\infty f(t)dt$
$p_2^2 = \frac{1}{2\omega^2} e^{-2\alpha t} (1 - \cos 2\omega t)$	$1/4 \alpha$
$p_3 = \frac{1}{2\omega} e^{-2\alpha t} (\sin 2\omega t + \frac{\alpha}{\omega} - \frac{\alpha}{\omega} \cos 2\omega t)$	$1/2 \omega^2$
$r_3 = \frac{1}{2} e^{-2\alpha t} (\cos 2\omega t - \frac{\alpha^2}{\omega^2} + \frac{\alpha^2}{\omega^2} \cos 2\omega t)$	0
$q_2^2 = \frac{1}{2} e^{-2\alpha t} (1 + \cos 2\omega t - \frac{2\alpha}{\omega} \sin 2\omega t)$	$1/4 \alpha$
$q_3 = -\frac{\omega}{2} e^{-2\alpha t} (\sin 2\omega t - \frac{\alpha}{\omega} + \frac{\alpha}{\omega} \cos 2\omega t)$	0

Table 2

Sums

$f(2\pi k)$	$\sum_k f(2\pi k)$
$e^{-4\pi\alpha k}$	$\frac{1}{2} + \sum_{k=1}^{\infty} e^{-4\pi\alpha k} = \frac{1}{4\pi\alpha}$
$e^{-4\pi\alpha k} \cos 4\pi\nu k$	$\frac{1}{2} + \sum_{k=1}^{\infty} e^{-4\pi\alpha k} \cos 4\pi\nu k = 2\pi\alpha / (1 - \cos 4\pi\nu)$
$e^{-4\pi\alpha k} \sin 4\pi\nu k$	$\sum_{k=1}^{\infty} e^{-4\pi\alpha k} \sin 4\pi\nu k = 1/2 \tan 2\pi\nu$
$p_2^2 = \frac{1}{2\nu^2} e^{-4\pi\alpha k} (1 - \cos 4\pi\nu k)$	$1/8\pi\alpha\nu^2$
$p_3 = \frac{1}{2\nu} e^{-4\pi\alpha k} (\sin 4\pi\nu + \frac{\alpha}{\nu} - \frac{\alpha}{\nu} \cos 4\pi\nu k)$	$1/4 \nu \tan 2\pi\nu + 1/8\pi\nu^2$
$r_3 = \frac{1}{2} e^{-4\pi\alpha k} (\cos 4\pi\nu k - \frac{\alpha^2}{\nu^2} + \frac{\alpha^2}{\nu^2} \cos 4\pi\nu k)$	$\pi\alpha/2 \sin^2 2\pi\nu - \alpha/8\pi\nu^2$
$q_2^2 = \frac{1}{2} e^{-4\pi\alpha k} (1 + \cos 4\pi\nu k - 2\frac{\alpha}{\nu} \sin 4\pi\nu k)$	$1/8\pi\alpha$
$q_3 = -\frac{\nu}{2} e^{-4\pi\alpha k} (\sin 4\pi\nu - \frac{\alpha}{\nu} + \frac{\alpha}{\nu} \cos 4\pi\nu k)$	$-\nu/4 \tan 2\pi\nu + 1/8\pi$

All the sums are calculated to the lowest order of the small parameter α/ν . We assume also that the unperturbed tune is far from the resonance values (all the integers and halfintegers).