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FLOW IN GEOTHERMAL WELLS:
PART II. WORKING EQUATIONS FOR
ONE-DIMENSIONAL FLOW THROUGH A
CIRCULAR VERTICAL CHANNEL

Z. Bilicki and J. Kestin

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1. Introduction

Part I [16] of the series of three reports, of which this is the second[#], contained a derivation of the field equations for two-phase flow. The derivation was based on the classical methods of thermodynamics of irreversible processes [9,10,11] and resulted in a *homogeneous diffusion model*. The present Part II specializes these equations to a point where the equations can be applied to the calculation of well flows in the form of an "engineering approximation" in that all fields are simplified to become functions of a single variable z - the distance along the well axis.

The equations of Part I were local and instantaneous and ignored turbulent fluctuations as well as the observable fluctuations of the phase boundaries. As the equations are simplified, we assume, heuristically, that all quantities represent time-averages and restrict ourselves to *steady flows*.

Nevertheless, we do not explicitly introduce the Reynolds stresses of turbulent flow but rely on the fact that the system of working equations must be supplemented with the aid of experimentally justified empirical closures. We suppose that these adequately account for the stresses and entropy production associated with turbulence.

2. Nature of the one-dimensional (engineering) approximation

The complexity of the problem at hand has forced all investigators interested in practical applications to have recourse to the simplest possible form of the theory in which the field variables are lumped over a cross-section. In our version we assume that the flow is steady and that it occurs in a very long vertical pipe of given (usually stepped) cross-

[#]Equations, figures and literature references are numbered consecutively throughout all parts.

section $A(z)$ which is a function of a single space variable. Consequently, we shall integrate all quantities over the cross-section and leave the single field variable, the vertical distance z , in the equations.

To proceed systematically, we choose a volume element

$$dV = A(z) dz \quad (52)$$

and integrate all conservation laws with respect to it, replacing the integration with respect to

$$dA = r dr d\theta \quad (53)$$

by mean values (denoted by $\langle \dots \rangle$). This procedure applies only to extensive quantities and can be carried out formally in the equations of conservation of mass and energy. The momentum equation and the expression for entropy production will require the addition of a heuristic input. In general, we simplify the integral of any extensive quantity (per unit mass) as

$$\begin{aligned} \Psi &= \int_V \rho(r, \theta, z) \psi(r, \theta, z) dV \\ &= \int_{z_1}^{z_2} dz \int_A \rho(r, \theta, z) \psi(r, \theta, z) r dr d\theta . \end{aligned} \quad (54)$$

We now replace the integrand $\rho\psi$ by the mean value

$$\langle \rho\psi \rangle = \frac{1}{A} \int_A \rho(r, \theta, z) \psi(r, \theta, z) r dr d\theta , \quad (55)$$

and note that the integrand

$$A(z) [\langle \rho\psi \rangle (z)]$$

which appears in the integral

$$\Psi = \int_{z_1}^{z_2} A(z) [\langle \rho\psi \rangle (z)] dz \quad (56)$$

is a function of the single variable, z .

Whenever an integrand appears in the form of a divergence of a vectorial quantity $\rho\psi v_i$, where v_i is an arbitrary vector, it is convenient to apply Gauss's theorem

$$\int_V \frac{\partial}{\partial x_i} (\rho\psi v_i) dV = \int_A \rho\psi v_i n_i dA . \quad (57)$$

In this equation we replace the symbol for velocity, v_i , by w_i and confine the further derivation to the case when the only nonvanishing component of w_i is the axial component w , Fig. 6. Thus

$$\rho\psi v_i \text{ becomes } \rho\psi w , \quad (58a)$$

and

$$\frac{\partial}{\partial x_i} (\rho\psi v_i) \text{ becomes } \frac{\partial}{\partial z} (\rho\psi w) . \quad (58b)$$

As our volume V in eqn. (57), we choose the volume confined between cross-sectional areas $A_1=A(z_1)$ and $A_2=A(z_2)$ and the circumferential area A_0 ,

Fig. 6. Thus

$$\begin{aligned} \int_V \frac{\partial}{\partial x_i} (\rho\psi v_i) dV & \text{ becomes} \\ \int_V \frac{\partial}{\partial z} (\rho\psi w) dV & = - \int_{A_1} \rho\psi w dA + \int_{A_2} \rho\psi w dA , \end{aligned} \quad (59a)$$

because the flux across A_0 vanishes. Introducing the mean value

$$\langle \rho\psi w \rangle = \frac{1}{A} \int_A \rho\psi w dA , \quad (59b)$$

we get

$$\int_V \frac{\partial}{\partial z} (\rho\psi w) dV = A_2 [\langle \rho\psi w \rangle (z_2)] - A_1 [\langle \rho\psi w \rangle (z_1)] . \quad (59c)$$

Fig. 6. Control surface and notation for mass conservation.

3. Conservation of mass

In order to write the one-dimensional form of the equation of mass conservation, we refer again to Fig. 6 and confine our system between cross-sections A_1 and A_2 a distance $z_2 - z_1$ apart, the only variable coordinate, z , being chosen vertical for well-flow calculations. The circumferential area A_0 is chosen to coincide with the wetted wall. We now employ eqn. (22) with $\partial/\partial t \equiv 0$ and write

$$\int_V \frac{\partial}{\partial x_i} (\rho w_i) dV = 0 \quad (60a)$$

in the form

$$\int_{A_1} \rho w dA - \int_{A_2} \rho w dA = 0 \quad (60b)$$

that is

$$A \langle \rho w \rangle = \text{const} \quad , \quad \text{or} \quad \frac{d}{dz} (A \langle \rho w \rangle) = 0 \quad , \quad (60c, d)$$

where w is the barycentric velocity of the mass contained between cross-sections A_1 and A_2 . Here the mass flux is

$$\psi(z) = \langle \rho w \rangle = \frac{1}{A} \int_A \rho w dA \quad \text{and} \quad \psi = \text{const.} \quad (61)$$

The mass flux of the liquid and vapor can be represented in an analogous fashion, when we obtain

$$\psi' = \langle w' \rho' \phi' \rangle = \langle w' \rho' (1 - \phi) \rangle = \dot{m}' / A \quad , \quad (62a)$$

$$\psi'' = \langle w'' \rho'' \phi'' \rangle = \langle w'' \rho'' \phi \rangle = \dot{m}'' / A \quad , \quad (62b)$$

where $\dot{m}'$ and $\dot{m}''$ denote the respective flow rates.

In the analysis of two-phase flow it is often useful to introduce the *dynamic dryness fractions*

$$y' = \frac{\dot{m}'}{\dot{m}} \quad \text{and} \quad y'' = \frac{\dot{m}''}{\dot{m}} \quad \text{with} \quad y \equiv y'' \quad . \quad (63)$$

These definitions are analogous to the ordinary dryness fractions γ , γ' and γ'' introduced earlier in eqn. (3). However, being based on mass-flow rates rather than masses, they are conceptually different.

Making use of equations (63), (62a,b) and approximating[#]

$$\left. \begin{aligned} \langle w' \rho' \phi' \rangle &\text{ by } \langle w' \rangle \langle \rho' \rangle \langle \phi' \rangle \\ \text{and } \langle w'' \rho'' \phi'' \rangle &\text{ by } \langle w'' \rangle \langle \rho'' \rangle \langle \phi'' \rangle, \end{aligned} \right\} \quad (64)$$

we can relate the void fraction to the dynamic dryness fraction and the slip ratio

$$\beta_s(z) = \frac{w''(z)}{w'(z)}. \quad (65)$$

Thus, we obtain

$$\langle \phi \rangle = \frac{1}{1 + \frac{1-y}{y} \beta_s \frac{\rho'}{\rho''}}. \quad (66)$$

The relation between the mean void fraction and dynamic dryness fraction is the same as that between the local void fraction and the local static dryness fraction, eqn. (6), if $\beta_s = 1$.

Authors working in the field of two-phase flow often make use of so-called *superficial velocities*

$$w'_0 = \psi''/\rho'' \quad \text{and} \quad w''_0 = \psi'/\rho'. \quad (67)$$

It is worth noticing here that ρ' and ρ'' are assumed constant in a cross-section. This is certainly the case when the pressure and temperature are uniform in a cross-section and when surface tension effects are negligible. On the other hand the density, ρ , is *not* uniform because, in principle, the void fraction varies throughout the cross-section, and not only with z , so that

$$[\langle \rho \rangle(z)] = \frac{1}{A} \int_A \rho(r, \theta, z) dA. \quad (68)$$

[#] We give a fuller discussion of the process of averaging in Section 7.

4. Conservation of momentum

In order to obtain the one-dimensional form of the equation of motion, we integrate eqn. (24) over the same boundary as was used in Section 3. This is repeated in Fig. 7 which also illustrates the notation employed for the normal and shearing stresses present in cross-sections A_1 and A_2 as well as in the surface A_0 . Stipulating steady flow in eqn. (24), and applying Gauss's theorem to the gradient of the pressure tensor, we obtain

$$\int_A (\rho w_i^2) n_i dA + \int_V \rho g_i dV + \int_A P^* dA + \int_A \pi_{ji} n_j dA = 0 . \quad (69)$$

Here $P^* = \frac{1}{3} P_{ii}$ is the hydrostatic pressure and π_{ji} denotes the deviator of the pressure tensor P_{ji} . We now specialize our equations to assume axial symmetry ($\partial/\partial\theta \equiv 0$) and note that the resultant of all normal components of the pressure tensor acting on the surface A_0 vanish. However, the axial components contribute a nonvanishing term ([17]p.256). We assume that

$$\int_{A_1}^{A_2} (\pi_{zz} - P^*) dA \quad (69a)$$

is small and can be neglected, and stipulate that the normal stress acting on A_0 is also equal to P^* . Thus the normal component of the pressure acting on A_0 is

$$\int_{z_1}^{z_2} P^* \frac{dA}{dz} dz . \quad (69b)$$

Following our established procedure, we introduce the mean values

$$\left[\langle \rho w^2 \rangle (z) \right] = \frac{1}{A} \int_A \rho w^2 dA , \quad (70)$$

together with

$$\left[\langle P^* \rangle (z) \right] = \frac{1}{A} \int_A P^*(r, \theta, z) dA . \quad (71)$$

Fig. 7. Control surface and notation for momentum conservation.

Projecting eqn. (69) in the z-direction, we have

$$\begin{aligned}
 & A_2 \left[\langle \rho w^2 \rangle (z_2) \right] - A_1 \left[\langle \rho w^2 \rangle (z_1) \right] + g \int_{z_1}^{z_2} A \langle \rho \rangle dz + \\
 & + A_2 \left[\langle P^* \rangle (z_2) \right] - A_1 \left[\langle P^* \rangle (z_1) \right] - \int_{z_1}^{z_2} P^*(z) \frac{dA}{dz} dz + \\
 & + \int_{A_0} \tau_w \cos \alpha dA = 0 .
 \end{aligned} \tag{72}$$

Here $\tau_w = \tau_{rz} \cos \alpha$ is the component of shear stress acting on surface A_0 along its generator that is the wall shear stress. The angle α is formed between the generator and axis z . We note for future reference that

$$\int_{A_0} \tau_w \cos \alpha dA = \int_{z_1}^{z_2} \tau_w(z) C(z) dz , \tag{72a}$$

where $C(z)$ is the circumference of area $A(z)$. The differential form of this equation is

$$\frac{d(A \langle P^* \rangle)}{dz} - \langle P^* \rangle \frac{dA}{dz} + \frac{d}{dz} (A \langle \rho w^2 \rangle) + Ag \langle \rho \rangle + C \tau_w = 0$$

or

$$A \frac{d \langle P^* \rangle}{dz} + \frac{d}{dz} (A \langle \rho w^2 \rangle) + Ag \langle \rho \rangle + C \tau_w = 0 . \tag{72b}$$

In some treatises on the subject the equation of motion is written in the form of a sum of local pressure gradients, namely as

$$\left(\frac{dP}{dz} \right)_{\text{tot}} = \left(\frac{dP}{dz} \right)_{\text{accel}} + \left(\frac{dP}{dz} \right)_{\text{grav}} + \left(\frac{dP}{dz} \right)_{\text{fric}} . \tag{73}$$

This form is identical with the more rigorously derived eqn. (72b) if we identify terms as follows:

$$\left(\frac{dP}{dz}\right)_{\text{accel}} = - \frac{d}{dz} (A \langle \rho w^2 \rangle) , \quad (74a)$$

$$\left(\frac{dP}{dz}\right)_{\text{grav}} = - Ag \langle \rho \rangle , \quad (74b)$$

$$\left(\frac{dP}{dz}\right)_{\text{fric}} = - C \tau_w , \quad (74c)$$

where $P \equiv P^*$. Thus

$$A \frac{d \langle P \rangle}{dz} + \frac{d}{dz} (A \langle \rho w^2 \rangle) + Ag \langle \rho \rangle + C \tau_w = 0 . \quad (74d)$$

5. Conservation of energy

In order to derive a workable equation of energy conservation, we take as our point of departure eqn. (33) of Part I and apply it to an identical element of volume as that shown in Figs. 6 and 7. We begin by stipulating steady flow and by applying Gauss's theorem to all terms containing divergences. We further introduce the *stress* deviator, τ_{ij} , and put

$$P_{ij} - P^0 \delta_{ij} = - \tau_{ij} + (P^* - P^0) \delta_{ij} . \quad (75)$$

Thus we obtain

$$\begin{aligned} & \int_A \rho w_i \left(h + \frac{1}{2} w_i^2 \right) n_i dA + \int_V \rho w_i g_i dV - \int_A (\tau_{ji} w_i) n_j dA + \\ & + \frac{1}{2} \int_A w_i \{ J_j (w_j'' - w_j') \} n_i dA + \int_A (P^* - P^0) w_i n_i dA = \\ & = \int_A q_i n_i dA . \end{aligned} \quad (76)$$

We now introduce the coordinates r, θ, z and assume axial symmetry which leads us to consider the stress components acting on an element of volume $r dr d\theta dz$ shown sketched in Fig. 8. These are the normal stresses

Fig. 8. Element of volume showing stresses.

$$\tau_{rr} = -P'_{rr} + P^* \text{ and } \tau_{zz} = -P'_{zz} + P^* ,$$

and the shearing stresses

$$\tau_{zr}, \tau_{\theta r}, \tau_{z\theta}, \tau_{r\theta} .$$

The work rate of the stress deviator represented by the third term in eqn.

(76) leads to the following partial work terms:

on the cylindrical face

$$\int_{A_0} (\tau_{rz} w_z + \tau_{rr} w_r) dA \quad (77a)$$

(because $w_\theta=0$ by assumption), and on the two flat faces

$$\int_{A_1}^{A_2} (\tau_{zr} w_r + \tau_{zz} w_z) dA . \quad (77b)$$

We introduce equations (77a,b) into eqn. (76) and change notation to that applicable to the cylindrical system of coordinates of Fig. 9. We further simplify the problem by neglecting the axial heat flux, and denote the heat flux normal to the wall of the channel by q_w . With these changes we get

$$\begin{aligned} & \int_{A_1}^{A_2} \rho w (h + \frac{1}{2} w^2) dA + g \int_V \rho w dV - \int_{z_1}^{z_2} (\tau_{rz} w + \tau_{rr} w_r) C / \cos \alpha dz + \\ & \quad (1) \quad (2) \quad (3) \\ & - \int_{A_1}^{A_2} (\tau_{zr} w_r + \tau_{zz} w) dA = - \int_{z_1}^{z_2} q_w C / \cos \alpha dz - \frac{1}{2} \int_A \{ w_j J_i (w_i'' - w_i') \} n_j dA + \\ & \quad (4) \quad (5) \quad (6) \\ & - \int_{A_1}^{A_2} (P^* - P^0) w dA . \quad (7) \end{aligned} \quad (78)$$

It should be noted that $w_r = w = 0$ at the wall and this cancels term (3) in eqn. (78). In the engineering approximation, the difference between the hydrostatic pressure, P^* , and the thermodynamic pressure, P° , is disregarded, which cancels term (7). Finally, it is argued that the work terms (4) and (6) are small compared with (1) which justifies their omission. With all these simplifications, the energy equation normally encountered in our subject assumes the form

$$A_2 \left[\langle \rho w (h + \frac{1}{2} w^2) \rangle (z_2) \right] - A_1 \left[\langle \rho w (h + \frac{1}{2} w^2) \rangle (z_1) \right] + \\ + g \int_{z_1}^{z_2} A \langle \rho w \rangle dz = - \int_{z_1}^{z_2} (C q_w / \cos \alpha) dz \quad (79)$$

or, in differential form

$$\frac{d}{dz} \left[A \langle \rho w (h + \frac{1}{2} w^2 + gz) \rangle \right] = - C \langle q_w \rangle / \cos \alpha . \quad (80)$$

6. Entropy production

In order to understand the nature of the irreversible processes which contribute to the production of entropy, we refer to eqn. (36) of Part I and re-write it for steady flow in the form

$$\int_A \rho s w_i n_i dA = \int_V \sigma dV - \int_A J_i^S n_i dA . \quad (81)$$

Limiting ourselves to adiabatic flow, we further specialize this form to apply to the system of Figs. 6 and 7 used earlier, and obtain

$$A_2 \left[\langle \rho w s \rangle (z_2) \right] - A_1 \left[\langle \rho w s \rangle (z_1) \right] = \\ = \int_{z_1}^{z_2} [\langle \sigma \rangle (z)] A(z) dz - \int_A J_i^S n_i dA . \quad (82)$$

The next task is to re-write eqns. (35) and (42a-c) in one-dimensional form. This will entail the additional simplification that the chemical potentials of the two phases are equal so that the entropy fluxes across the flat faces are

$$J^S(z) = q(z) / T(z), \text{ (with } T \equiv T^0) , \quad (83)$$

those crossing the cylindrical surface having vanished owing to the restriction to adiabatic flow. In terms of a Fourier-type linear assumption

$$q(z) = - \lambda \, dT(z) / dz , \quad (84)$$

and

$$\int_A J_i^S n_i dA = - \int_{A_1}^{A_2} \left\langle \frac{\lambda}{T} \frac{dT}{dz} \right\rangle dA. \quad (85)$$

It should be recalled at this stage that the *apparent* thermal conductivity λ is not equal either to that of the fluid or to that of the vapor, but must be determined by a separately arranged program of measurement.

Reverting to the equations of entropy production (42a-c), we note that our simplifications lead to

$$\begin{aligned} \sigma_o &= 0 , \\ \langle \sigma_1 \rangle &= \left\langle \frac{\lambda}{T^2} \left(\frac{dT}{dz} \right)^2 \right\rangle + \left\langle \frac{J_z}{T} \frac{d}{dz} (w'' - w') \right\rangle , \\ &= \left\langle \frac{\lambda}{T^2} \left(\frac{dT}{dz} \right)^2 \right\rangle + \left\langle \frac{1}{T} \frac{d}{dz} J_z (w'' - w') \right\rangle , \end{aligned} \quad (86)$$

because J_z is independent of z . The last term, eqn. (42c), assumes the explicit local form

$$\begin{aligned} T\sigma_2 &= \tau_{rz} \left(\frac{\partial w_z}{\partial r} + \frac{\partial w_r}{\partial z} \right) + \tau_{\theta z} \left(\frac{\partial w_\theta}{\partial z} + \frac{\partial w_z}{\partial \theta} \right) + \\ &+ \tau_{\theta r} \left[r \frac{\partial}{\partial r} \left(\frac{w_\theta}{r} \right) + \frac{1}{r} \left(\frac{\partial w_r}{\partial \theta} \right) \right] . \end{aligned} \quad (87)$$

Since the local barycentric components of velocity are related to the slip by the general equation

$$\rho w_i = \rho'' \phi'' w_i'' + \rho' \phi' w_i' , \quad (87a)$$

it is apparent that the entropy production term σ_2 takes into account the dissipation due to slip as well as to changes in the void fraction ϕ .

We adopt analogous simplifications to those accepted in Section 5, namely that the entropy created by the shear stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ is negligible and introduce the linear, Newton-like assumption

$$S_{ji} = \tau_{ji} / \eta \quad (88a)$$

with η denoting an *apparent* viscosity. The expression for the rate of deformation can be simplified in the spirit of boundary-layer theory and approximated by

$$S_{ji} \approx \partial w / \partial r \quad (88b)$$

on the assumption that

$$\partial w / \partial r \gg \partial w_r / \partial z .$$

This yields

$$\sigma_2 = \tau_{rz}^2 / \eta T = \frac{\eta}{T} \left(\frac{\partial w}{\partial r} \right)^2 . \quad (89)$$

Before we proceed to integrate the expression in eqn. (89), we must realize that the velocity gradients which multiply τ_{rz} in eqn. (87), namely $\partial w / \partial r$ and $\partial w_r / \partial z$, are very small in the bulk of the flow, but assume large values in the boundary layer near the wall where $\tau_{rz} = \tau_w \cos \alpha$. Hence

$$\langle \sigma_2 \rangle = \frac{1}{A} \int_A \frac{\tau_w^2 \cos^2 \alpha dA}{\eta T} . \quad (90)$$

Collecting terms, and noting that $\dot{m} = A \rho w$, we can write the entropy balance

equation

$$\begin{aligned} \frac{d}{dz} (A \langle \rho w s \rangle) &= A \left\langle \frac{\lambda}{T^2} \left(\frac{dT}{dz} \right)^2 \right\rangle + A \left\langle \frac{1}{T} \frac{d}{dz} J_z (w'' - w') \right\rangle + \\ &+ A \left\langle \frac{\eta}{T} \tau_w^2 \cos^2 \alpha \right\rangle + \frac{d}{dz} \left(A \left\langle \frac{\lambda}{T} \frac{dT}{dz} \right\rangle \right). \end{aligned} \quad (91)$$

In the further development of the subject it is customary to neglect the first and last term of entropy production, both caused by the existence of an axial temperature gradient dT/dz . The remaining two terms require the introduction of two empirical statements. In the third term, we have formally postulated a linear relation between the wall shearing stress and the rate of deformation

$$S = \frac{1}{2} \left[\frac{\partial w}{\partial r} + \frac{\partial w_r}{\partial z} \right]. \quad (92)$$

In actual calculations this phenomenological statement is replaced by a heuristic one for example that due to Lottes and Flinn [18], to Petrick [19] or to Dukler [20]. All of them correlate the wall shearing stress with the dynamic slip ratio to be introduced later. The second term relates the flux

$$J = \phi' \rho' (w' - w) + \phi'' \rho'' (w'' - w') \quad (92a)$$

with the slip $w''-w'$ or its ratio

$$\beta_s = w'' / w'. \quad (92b)$$

7. Summary of equations

The preceding considerations demonstrate that a rigorous one-dimensional theory operates with averages of a number of products of local quantities. Each product constitutes an extensive thermodynamic quantity as a

whole, but contains intensive terms. It does not appear possible to make further progress without at this stage admitting that the averages of products may be replaced by products of average quantities taken over a cross-sectional area and conceived of as field variables which are functions of a single coordinate z . The intensive quantities are deemed to have been defined by the conservation laws as already stated earlier. This means that care must be exercised in selecting proper size-scales for the instruments to measure them. Each such product varies over the cross-section (or just with r in the case of axial symmetry) as well as with the axial coordinate z . If we could solve the field equations completely, we would form the averages, and thus make explicit the coupling between the simplified equations. In this case we could write:

$$\begin{aligned} \langle \rho w \rangle &\equiv \frac{1}{A} \int [\rho' w' (1 - \phi) + \rho'' w'' \phi] dA \\ &= K_1(z) \langle \rho' \rangle \langle w' \rangle \langle (1 - \phi) \rangle + K_2(z) \langle \rho'' \rangle \langle w'' \rangle \langle \phi \rangle \end{aligned} \quad (93a)$$

$$\begin{aligned} \langle \rho w^2 \rangle &\equiv \frac{1}{A} \int [\rho' w'^2 (1 - \phi) + \rho'' w''^2 \phi] dA \\ &= K_3(z) \langle \rho' \rangle \langle w' \rangle^2 \langle (1 - \phi) \rangle + K_4(z) \langle \rho'' \rangle \langle w'' \rangle^2 \langle \phi \rangle \end{aligned} \quad (93b)$$

$$\begin{aligned} \langle \rho w^3 \rangle &\equiv \frac{1}{A} \int [\rho' w'^3 (1 - \phi) + \rho'' w''^3 \phi] dA \\ &= K_5(z) \langle \rho' \rangle \langle w' \rangle^3 \langle (1 - \phi) \rangle + K_6(z) \langle \rho'' \rangle \langle w'' \rangle^3 \langle \phi \rangle \end{aligned} \quad (93c)$$

$$\begin{aligned} \langle \rho h w \rangle &\equiv \frac{1}{A} \int [\rho' h' w' (1 - \phi) + \rho'' h'' w'' \phi] dA \\ &= K_7(z) \langle \rho' \rangle \langle h' \rangle \langle w' \rangle \langle (1 - \phi) \rangle + K_8(z) \langle \rho'' \rangle \langle h'' \rangle \langle w'' \rangle \langle \phi \rangle . \end{aligned} \quad (93d)$$

This procedure would introduce a set of coefficients

$$K_i = \frac{\langle \alpha_1 \alpha_2 \dots \alpha_n \rangle}{\langle \alpha_1 \rangle \langle \alpha_2 \rangle \dots \langle \alpha_n \rangle} \quad (93e)$$

about which it is necessary to make simplifying assumptions.

The *homogeneous diffusion model*[#] is obtained when we postulate that each

$$K_i \equiv 1 .$$

Performing the obvious substitutions, we calculate the averages:

$$\langle \rho w \rangle = \dot{m}/A \quad (94a)$$

$$\langle \rho w^2 \rangle = \frac{\dot{m}^2}{A^2 \rho'} \left[\frac{(1-y)^2}{1-\phi} + \frac{y^2}{\phi} \cdot \frac{\rho'}{\rho''} \right] \quad (94b)$$

$$\frac{1}{2} \langle \rho w^3 \rangle = \frac{\dot{m}^3}{A^3 \rho'^2} \left[\frac{(1-y)^3}{(1-\phi)^2} + \frac{y^3}{\phi^2} \cdot \left(\frac{\rho'}{\rho''} \right)^2 \right] \quad (94c)$$

$$\langle \rho h w \rangle = \frac{\dot{m}}{A} [h'(1-y) + h''y] . \quad (94d)$$

In turn, we are led to the following set of simplified balance equations (in which we drop the averaging operator $\langle \dots \rangle$ for simplicity)

$$\frac{d\dot{m}}{dz} = 0 \quad (95a)$$

$$A \frac{dP}{dz} + \dot{m}^2 \frac{d}{dz} \left\{ \frac{1}{A \rho'} \left[\frac{(1-y)^2}{1-\phi} + \frac{y^2}{\phi} \left(\frac{\rho'}{\rho''} \right) \right] \right\} = - C \tau_w - \rho g A \quad (95b)$$

$$\frac{d}{dz} \left\{ h'(1-y) + h''y + \frac{\dot{m}^2}{A^2 \rho'^2} \left[\frac{(1-y)^3}{(1-\phi)^2} + \frac{y^3}{\phi^2} \left(\frac{\rho'}{\rho''} \right)^2 \right] \right\} = - \frac{q_w C}{\dot{m} \cos \alpha} - g \quad (95c)$$

For adiabatic channels with constant cross-sections, we write

$$\frac{d\psi}{dz} = 0 \quad (\psi = \rho w = \dot{m}/A) \quad (96a)$$

$$\frac{dP}{dz} + \psi^2 \frac{d}{dz} \left\{ \frac{1}{\rho'} \left[\frac{(1-y)^2}{1-\phi} + \frac{y^2}{\phi} \left(\frac{\rho'}{\rho''} \right) \right] \right\} = - \frac{C}{A} \tau_w - \rho g \quad (96b)$$

[#]Some authors, e.g. [2-8], refer to this as the two-fluid model.

$$\frac{d}{dz} \left\{ h'(1-y) + h''y + \frac{\psi^2}{\rho'^2} \left[\frac{(1-y)^3}{(1-\phi)^2} + \frac{y^3}{\phi^2} \left(\frac{\rho'}{\rho''} \right)^2 \right] \right\} = -g . \quad (96c)$$

The term "homogeneous diffusion model" employed for the sets of equations (95) and (96) is justified, because the state of an element is referred to its barycentric velocity, w , each subelement (liquid or gas) being endowed with its own velocity (w' or w''). The matter of averaging can be approached differently. We notice that all state quantities can be referred to the barycentric velocity, w , without the need to introduce slip (in the form of the two velocities, w' and w'') *explicitly* as long as the shearing stress at the wall, τ_w , is not assumed to depend on them *separately*. In the very special case when $w'=w''=w$ we would retrieve the standard theory of one-dimensional gas dynamics. In either case we would write

$$\langle \rho w \rangle = \dot{m}/A , \quad (97a)$$

$$\langle \rho w^2 \rangle = \dot{m}^2 / \rho A , \quad (97b)$$

$$\frac{1}{2} \langle \rho w^3 \rangle = \dot{m}^3 / 2 \rho^2 A^3 , \quad (97c)$$

$$\langle \rho h w \rangle = \dot{m} h / A . \quad (97d)$$

The system of differential equations for the balance equations is thus

$$\frac{d\dot{m}}{dz} = 0 , \quad (98a)$$

$$\frac{dP}{dz} + \frac{\dot{m}^2}{A} \frac{d}{dz} \left(\frac{1}{A\rho} \right) = - \frac{C}{A} \tau_w - \rho g , \quad (98b)$$

$$\frac{dh}{dz} + \frac{\dot{m}^2}{2} \frac{d}{dz} \left(\frac{1}{A^2 \rho^2} \right) = -g - \frac{C q_w}{\dot{m} \cos \alpha} . \quad (98c)$$

In the literature, this set of equations is sometimes described as the *homogeneous equilibrium model*. We emphasize that there is no need to stipulate that there is no slip as long as τ_w is made to depend on it. The designation "equilibrium" refers to thermodynamic equilibrium between the phases.

As treated here, this is a *simplified version* of the *homogeneous diffusion model*.

In all preceding models, the lowest level of description on the scale V discussed in Part II envisaged elements large enough to contain subelements of both phases whose mechanical interactions need not be taken into account *explicitly*. An alternative, represented by a number of authors [2,3], consists in choosing a sufficiently small scale of initial description. This requires that each phase must be treated separately and that the discontinuities at the boundary between them ("jump conditions") must be included *explicitly*. This results in a true *two-fluid model*.

In concrete problems it is necessary to decide on a choice of independent functions of the single space variable, z , which are to be determined. There is a certain latitude to this choice, but the basis is always provided by the *three* conservation laws. By themselves, these are not sufficient to give a complete solution. The system can be closed only by the addition of an appropriate number of equations of state and one empirical closure condition. This set of additional statements provides us with the "constitutive equations".

For example, in the case of the *homogeneous diffusion model* represented by eqns. (95a-c) we may choose as our mathematically dependent variables the following functions:

$$\dot{m}(z) = \psi(z) A(z) ; \quad y(z) \quad \text{and} \quad P(z),$$

where $A(z)$ is given. The additional equations of state can take the form of:

$$\rho', \rho'', h', h'', \text{ all functions of } P$$

and an empirical closure equation in the form of the relation

$$F(\phi, y) = 0 .$$

In the case of the *simplified homogeneous diffusion model* we may choose the mathematically dependent functions:

$$\dot{m}(z) = \psi(z) A(z) ; \quad h(z) \quad \text{and} \quad P(z)$$

together with the equation of state

$$\rho = \rho(h, P)$$

and a closure condition for the shearing stress, τ_w .

The numerous calculational models encountered in the literature differ with respect to the choice of the dependent functions of z , the appropriate equation of state and the preferred closure condition. Part III of the report will contain a description of several such schemes [21] together with a statement of our current preference.

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Figure Captions

- Fig. 6. Control surface and notation for mass conservation.
- Fig. 7. Control surface and notation for momentum conservation.
- Fig. 8. Element of volume showing stresses.
- Fig. 9. Control surface and notation for energy conservation.