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SEMICLASSICAL APPROACH TO LARGE N EXPANSION

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ABSTRACT

We review some recent work done in collaboration with B. Sakita, H. Levine and N. Papanicolaou on developing new techniques for studying the planar limit of QCD.

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EFFECTIVE HAMILTONIANS FOR $N = \infty$

Quantum chromodynamics at large N provides phenomenologically an appealing picture of strong interactions.¹ In contrast, however, the actual problem of performing the large N expansion represents a rather formidable task. One would certainly like to avoid a direct summation of large N Feynman diagrams (which is obviously hopeless); as a possible procedure we have been formulating a nonperturbative semiclassical approach to this problem.

The first stage of this approach deals with summarizing the large N graphs in terms of an effective Hamiltonian.² This effective Hamiltonian is based on the collective field formalism, which represents a quantum change of variables to appropriate collective degrees of freedom. The usefulness of this procedure lies in the fact that in this new representation the large N limit becomes particularly simple: it is determined by classical stationary points of the effective Hamiltonian.

Specifically, in an $SU(N)$ Yang-Mills theory one starts with the standard $A_0 = 0$ gauge canonical formalism and the change is made to the gauge invariant loop-space variables

$$\phi(c) = \text{Tr} (e^{i\oint \vec{A} \cdot d\vec{x}}) \quad ; \quad \pi(c) = \frac{\hbar}{i} \frac{\partial}{\partial \phi(c)} \quad (1)$$

Here $\phi(c)$ are arbitrary space-like loops. To exemplify how one goes about deriving the new effective Hamiltonian let us describe for notational simplicity the analog example of a single $U(N)$ matrix degree of freedom:

$$H = - \frac{\hbar^2}{2} \frac{\partial^2}{\partial M^2} + V(M) \quad (2)$$

Corresponding to the Gauss law here we have the restriction to $SU(N)$ singlet states:

$$\hat{J}|> = 0 \quad ; \quad \hat{J} = [M, \frac{\partial}{\partial M}] \quad (3)$$

The $U(N)$ invariant collective variables are $\phi_k = \text{Tr exp}(ikM)$ and the conjugate π_k . The change of variables involves the chain rule of differentiation and the evaluation of the Jacobian which is found in a self-consistent way. This leads to the effective Hamiltonian which essentially has the form

$$H_{\text{eff}} = \sum \left\{ -\frac{kk'}{2} \pi_k \phi_{k+k'} \pi_{k'} + \frac{\hbar^2}{2} \omega \Omega^{-1}(k, k') \omega' \right\} + V[\phi] \quad (4)$$

Here $\omega(k)$ and $\Omega(k, k')$ are functionals of the field ϕ_k and read:

$$\Omega(k, k') = kk' \phi_{k+k'} \quad ; \quad \omega = k \sum_{k'} \phi_k \phi_{k-k'} \quad (5)$$

Of special importance in the above form is the appearance of the quantum $\hbar^2$ term which is an analog of the centrifugal barrier in quantum mechanics. The $N \rightarrow \infty$ limit is determined by stationary points of this effective Hamiltonian. The $\hbar^2$ -term is of the same order as the potential term and, therefore, it plays a crucial role in this limit. For example, the vacuum is given by the static equation which is obtained by varying V_{eff} :

$$\hbar^2 \left\{ \frac{1}{2} \left(\int \frac{\phi(y)}{x-y} \right)^2 - \int \frac{\phi}{x-y} \int \frac{\phi}{y-y'} \right\} + V(x) - c = 0 \quad (6)$$

One can explicitly solve this nonlinear, singular integral equation and the result agrees with the original work of Ref. 4; on the other hand, this example also exhibits the complexity of the full Yang-Mills problem. Namely, in that case one has a nonlinear equation for a field defined on loops since essentially $\phi(k)$ gets replaced with $\phi(c)$. For the explicit form see the paper of Sakita.

Now that one has succeeded in formulating the summation of planar diagrams in terms of an effective equation the basic difficulty is that this still represents a rather complex nonlinear problem. Due to a nonstandard nature of this equation a direct approach to solving it does not seem to offer much hope. However, we have also introduced an alternative line of development which leads to the possibility that the problem may be reduced to ordinary space, in fact to the original classical equation. This will be described in the following section.

As a final comment we would now like to show that the collective field procedure can also be used to derive an effective action for Yang-Mills theory which then provides nonequal time correlation functions at large N . Namely, in the Euclidean formalism one would like to change integration variables to the general Wilson loop variables $W(c)$ so that

$$Z = \int d\Lambda_\mu e^{-S(A)} = \int dW(c) J e^{-S} \quad (7)$$

The essential problem now is how to determine the Jacobian J , however, in the Hamiltonian approach one had to do the same and consequently, we just generalize the Jacobian used there to four dimensions. We therefore obtain an effective action for large N : $S_{\text{eff}} = S - \log J$ with the corresponding equations of motion:

$$\frac{\partial S}{\partial W(c)} = - \sum \Omega^{-1}(c, c') \omega(c') \quad (8)$$

where

$$\Omega(c, c') = \frac{\partial W(c)}{\partial A_\mu} \frac{\partial W(c')}{\partial A_\mu} ; \quad \omega = \frac{\partial^2 W(c)}{\partial A_\mu^2} \quad (9)$$

These are certain functionals of Wilson loop variable and for example explicitly:

$$\omega(c) = \oint dx_\mu \oint dx'_\mu \delta(x-x') W(c_{xx'}) W(c_{x'x}) \quad (10)$$

where the loops on the right hand side represent splitting of the loop c at point of intersection. We mention that Eq. 8 agrees with an integrated form of the Migdal-Makkeenko equation.⁵ Also instead of (8) one could differentiate the effective action with respect to the field $A_\mu(x)$ and this then produces an equation exactly equivalent to M-M equation.

CLASSICAL DYNAMICS AT LARGE N

The semiclassical quantization method is an extremely useful technique for studying nonlinear Hamiltonian systems. We have established based on several different models that there is a direct relevance of this method to the $1/N$ expansion.⁶ Essentially what we show is that the large N effective Hamiltonians are nothing but the original classical Hamiltonians with certain boundary conditions. Consequently, some special real-time classical solutions directly determine the sum of large N diagrams.

For example, in the matrix model problem the effective equations obtained from the effective Hamiltonian of Eq. (4) read:

$$\frac{d}{dt} (\Omega^{-1} \phi) = \frac{\partial V_{\text{eff}}}{\partial \phi} \quad (11)$$

where V_{eff} contains the $\hbar^2$ term. On the other hand, the classical equations of motion are

$$\ddot{M} = \frac{\partial V}{\partial M} \quad (12)$$

Obviously if $M(t)$ is a classical solution with vanishing angular momentum the associated field $\phi_k = \text{Tr} \exp(ikM)$ obeys Eq. (11) but without the $\hbar^2$ term. However, one can show that a classical solution with

$$J \equiv i [M, \dot{M}] = \hbar (1 - \delta_{ij}) \quad (13)$$

precisely reproduces the effective Eq. (11). A simple way to demonstrate this is to consider the constraint (Eq. (13)) writing it in the adjoint representation:

$$\tilde{M}_{\alpha\beta} P_{\beta} = \kappa C_{\alpha} \quad (14)$$

where $\alpha \equiv (ij)$; $P_{\alpha} = \dot{M}_{ij}$. Imposing a gauge condition $M = \text{diag}(\lambda)$ we see that the constraint can be solved by

$$P_{\alpha} = \delta_{ij} \dot{\lambda}_j + \kappa \tilde{M}_{\alpha\beta}^{-1} C_{\beta} \quad (15)$$

This is a canonical transformation and now the kinetic term of the Hamiltonian becomes

$$\text{Tr}(\dot{M}^2) = \sum_i \dot{\lambda}_i^2 + \kappa^2 C_{\alpha} (\tilde{M}^{-2})_{\alpha\beta} C_{\beta} \quad (16)$$

Comparing this with the first two terms in the effective Hamiltonian one finds that with the special choice for C (given in Eq. (13)) the two coincide. What happens is that the nonzero angular momentum contribution exactly simulates the κ^2 centrifugal barrier term. Therefore, we have reached the following simple conclusion: classical solutions with special quantum boundary conditions (Eq. (13)) provide a sum of planar diagrams.

The above phenomena appears to be of quite general nature, for instance we have established that it holds for arbitrary $U(N)$ invariant vector type models. Here one has a complex scalar field multiplet $(\phi_a(x), a = 1, 2, \dots, N)$ and the $U(N)$ charge is given by

$$Q_{ab} = \frac{i}{2} \int (\dot{\phi}_a^* \phi_b - \dot{\phi}_a \phi_b^*) \quad (17)$$

We find that the classical equations again reproduce the large N quantum theory provided that the classical solutions obey the constraints

$$\begin{aligned} \phi^*(x) \cdot Q \cdot \phi(y) &= -2iN\hbar \phi^*(x) \cdot \phi(y) \\ \dot{\phi}(x) \cdot Q \cdot \dot{\phi}(y) &= -2iN\hbar \dot{\phi}^*(x) \cdot \dot{\phi}(y) \end{aligned} \quad (18)$$

plus a third one with mixed time derivatives. Now one still has to show that the above relations do not contradict the equations of motion: this works out basically because Q_{ab} is a conserved quantity. In fact, for the case of infinitely many degrees of freedom the constraints are simply solved by $Q_{ab} = \hbar N \delta_{ab}$. This simple constraint then provides the quantum boundary conditions, and the corresponding classical solutions determine the large N limit. For example, for $(\phi^* \cdot \phi)^2$ theory the lowest energy solution reads:

$$\phi_n^{cl}(x, t) = A_n \exp\{i(k_n x - \omega_n t)\} \quad (19)$$

with $\omega_n^2 = k_n^2 + m^2 + g^2/2N |A|^2$. Imposing the constraint we obtain the well known gap equation, and the above solution obviously represents the large N vacuum. The quantity $\phi^*(x,t) \cdot \phi(y,t)$ is nothing but the equal time correlation function.

For $O(N)$ invariant vector models one has a similar situation. For example, in the case of Helium-like Hamiltonians with two degrees of freedom $\vec{x}, \vec{y}$ the constraints are found to be:⁷

$$\begin{aligned} L^2 &= (\pi N)^2 \quad ; \quad L_{ij} x_i \dot{x}_j = L_{ij} y_i \dot{y}_j \\ L_{ij} x_i y_j &= L_{ij} \dot{x}_i \dot{y}_j = 0 \end{aligned} \quad (20)$$

where L_{ij} is the conserved $O(N)$ generator.

For Yang-Mills theory we also see a possibility for this phenomena. Namely, in analogy with the discussion following Eq. (14) we may have:

$$J_o^\alpha \left(\frac{1}{D^2} \right)_{\alpha\beta} J_o^\beta = \pi^2 \sum_{c,c'} \omega(c) \Omega^{-1}(c,c') \omega(c') \quad (21)$$

where J_o^α is an external density and $D^2 = D_i D_i$. However, due to the complexity of the right hand side we have not yet managed to work out the constraints which need to be imposed on the classical field $A_i(x,t)$ in order to make the above identification possible. Furthermore, as we have done in the above examples it is important to show the consistency of constraints with the equations of motion, this would then assure that the solution exists.

The equations presented in this section remind one of the Bohr-Sommerfeld method where certain quantization conditions have to be imposed on the classical dynamics. In effect, what we have obtained are some type of Bohr-Sommerfeld conditions for these more complex systems.⁹ The correspondence with the effective large N equations essentially means that this semiclassical quantization method becomes exact in the $N \rightarrow \infty$ limit.

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