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TITLE: VORTEX LATTICES IN THEORY AND PRACTICE

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Vortex Lattices in Theory and Practice

Laurence J. Campbell*

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Abstract. The formal simplicity of ideal point vortex systems in two dimensions has long attracted interest in both their exact solutions and in their capacity to simulate physical processes. Attention here is focused on infinite, two-fold periodic vortex arrays, including an expression for the energy density of an arbitrary vortex lattice (i.e., an arbitrary number of vortices with arbitrary strengths in a unit cell parallelogram of arbitrary shape). For the case of two vortices per unit cell, the morphology of stable lattices can be described completely. A non-trivial physical realization of such lattices is a rotating mixture of ^{3}He and ^{4}He at temperatures so low that *both* isotopic components are superfluid. The structure of the expected lattices is quite different from the usual triangular structure. Magnetic flux lines in high-temperature superconductors show a one-parameter family of degenerate ground states of the lattice due to the anisotropy of the vortex-vortex interaction. A final topic, closely related to Josephson-junction arrays, is the case of vortices confined to a grid. That is, the vortices interact pair-wise in the usual manner but are constrained to occupy only locations on an independent periodic grid. By using vortex relaxation methods in the continuum and then imposing the grid it is possible to find low-lying states extremely rapidly compared to previous Monte Carlo calculations.

1. Point Vortex Lattices, *per se*. Consider N vortices of circulation strengths Γ_j at positions $\mathbf{r}_j$, $j = 1,..,N$ in the unbounded plane. To study their stationary configurations it is necessary to find the extrema of the function,

$$F = -\frac{d}{4\pi}\sum_{i<j}^{N-1}\sum^{N}\Gamma_i\Gamma_j \log|\mathbf{r}_i-\mathbf{r}_j| - \Omega\sum_{i=1}^{N}\Gamma_i(1-r_i^2) - \sum_{i=1}^{N} c_i, \qquad (1)$$

where d is the fluid density, Ω is a Lagrange multiplier (proportional to the angular rotation) that constrains the configuration to constant angular momentum, and the c_i are constant self-energies for each vortex. Obviously, the c_i do not affect the extrema and are

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included only to show where assumptions about the core structure would be buried. For the cases considered here the cores do not overlap, so it is correct to use the logarithmic potential - it is sufficient that F be defined within an additive constant. Global and local minima of F are sought; this requires vortices of the same sign, except on the grid in Section 3(c).

The effect of Ω in Eq. (1) is to fix the vortex density n of the minimal F configurations, $n \propto \Omega/\Gamma$, where Γ is an average strength. On the lattice, however, the density of vortices will be kept constant explicitly, so the Lagrange multiplier Ω can be eliminated, leaving only the double sum of $\log|\mathbf{r}_i - \mathbf{r}_j|$ to be evaluated.

Before proceeding with the evaluation of the energy for a lattice it will give some perspective on the current state of ignorance to state two open questions:

I. What is the smallest number of vortices per unit cell in a non-triangular, stable lattice of identical vortices?

II. Are all stable, space-filling configurations lattices?

It has been known for some years that for equal Γ_j the triangular vortex lattice has the lowest energy.[10] What is not known, or at least not proven, is the existence of *any* other lattice that is a local minimum of the energy, although Tkachenko[11] suggests that all lattices "sufficiently close to triangular" are stable. The square lattice is disqualified - it is a saddle point of the energy. Numerical searches by the present author using eleven or fewer vortices per unit cell have failed to find non-triangular stable lattices.

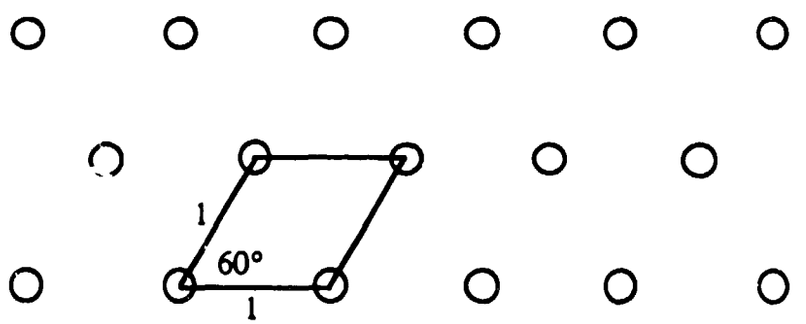

Fig. 1 The triangular lattice; the only stable vortex lattice known for equal Γ_j.

This is quite unlike the situation with finite N for which the number of stable states grows rapidly as N increases.[2,8] Yet, only *one* stable configuration is known for a vortex lattice, which contains an infinite number of vortices. In Open Question II the possibility is raised that infinite configurations other than lattices (by definition periodic) might be stable. These could be as exotic as an aperiodic tiling or as prosaic as a single dislocation in an otherwise perfect lattice. It is intuitively obvious to crystallographers that stable dislocations will exist in the vortex lattice. Nevertheless, there is no proof and it is not so obvious, because the atomic interactions in crystals are short-ranged. (However, dislocations in crystals, like vortices, interact according to $\log|\mathbf{r}|$ which raises the amusing prospect of studying dislocation lattices of dislocation lattices of atomic lattices.)

Returning to the energy, it is necessary to evaluate the energy density defined as

$$E = -\lim_{M\to\infty} \frac{1}{JM} \sum_{i<j}^{N-1} \sum^{N} \Gamma_i \Gamma_j \log|\mathbf{r}_i - \mathbf{r}_j| \,, \tag{2}$$

where J is the number of vortices per unit cell, M is the number of cells, and $N = JM$. Strictly speaking, E is the energy per vortex (apart from a dimensional constant) but it is

also the energy per unit area when the vortex density is constant. To eliminate M, divide the sum over N into sums over each of the J vortices,

$$\sum_{j=1}^{N} = \sum_{j_1=1}^{M} + \sum_{j_2=1}^{M} + \cdots + \sum_{j_J=1}^{M} . \tag{3}$$

The double sum in Eq. (2) consists of pairs,

$$\sum_{j_\alpha}^{M} \sum_{j_\beta}^{M\prime} \Gamma_\alpha \Gamma_\beta \log|\mathbf{r}_{j_\alpha} - \mathbf{r}_{j_\beta}| = -M \Gamma_\alpha \Gamma_\beta \sum_{\mathbf{n}}{}' \log|\mathbf{r}^o_\alpha - \mathbf{r}^o_\beta + \mathbf{L}_\mathbf{n}| , \tag{4}$$

where $\mathbf{r}^o_\alpha$ and $\mathbf{r}^o_\beta$ are in the same unit cell and the vector $\mathbf{L}_\mathbf{n}$ ranges over all M cells,

$$\mathbf{L}_\mathbf{n} = n_1 L_1 \mathbf{e}_1 + n_2 L_2 \mathbf{e}_2 \tag{5}$$

Here $L_1\mathbf{e}_1$ and $L_2\mathbf{e}_2$ are the generators of the unit cell and $n_1, n_2 = 0, \pm1, \pm2, \ldots$, except that $n_1 = n_2 = 0$ must be omitted if $\alpha = \beta$.

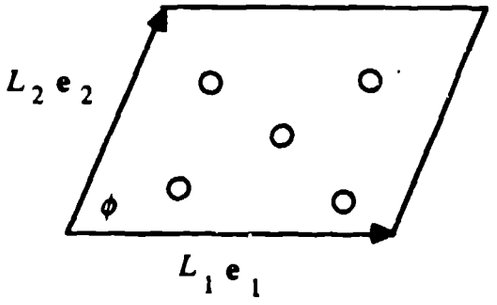

Fig. 2 The unit cell containing $J = 5$ vortices. $\mathbf{e}_1 \cdot \mathbf{e}_2 = \cos\phi$

In this notation, Eq. (2) becomes

$$E = \overline{\Gamma^2} \sum_{\mathbf{n}} \log|\mathbf{L}_\mathbf{n}| - \frac{2}{J} \sum_{\alpha<\beta}^{J} \Gamma_\alpha \Gamma_\beta \sum_{\mathbf{n}} \log|\mathbf{r}^o_\alpha - \mathbf{r}^o_\beta + \mathbf{L}_\mathbf{n}|, \qquad \overline{\Gamma^2} \equiv \frac{1}{J} \sum_{\alpha=1}^{J} \Gamma_\alpha^2 . \tag{6}$$

The principal ingredients to evaluating Eq. (6) include the following. (This derivation has appeared elsewhere[3,6] and an elegant, independent version by O'Neil[7] is included in these proceedings.)

i. Expansion of $\log|\mathbf{x}|$ in "box" normalization.

$$-\log|\mathbf{x}| = \lim_{\mu\to0} \lim_{s_1,s_2\to\infty} \frac{2\pi}{s_1 s_2} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2+\mu^2}, \qquad \mathbf{k} = 2\pi\left(\frac{n_1}{s_1}, \frac{n_2}{s_2}\right) . \tag{7}$$

ii. Transformation from $\mathbf{k}$ to "reciprocal" lattice $\mathbf{g}$.

$$\mathbf{g} = \frac{2\pi}{\sin\phi}\left(\frac{m_1}{L_1}\mathbf{v}_1 + \frac{m_2}{L_2}\mathbf{v}_2\right), \qquad \mathbf{v}_i \cdot \mathbf{e}_j = \delta_{ij} \sin\phi . \tag{8}$$

iii. Elimination of $\mathbf{g} = 0$ term in the lattice sum. Eq. (6) is formally divergent, but when

rewritten as a sum over **g** this divergence is concentrated in the **g** = 0 term, which can be "thrown away". More correctly, it can be explained away as an artifact of neglecting the Ω term of Eq. (1), which also diverges as $N \to \infty$ and would cancel the **g** = 0 term. Vortices of the same sign require a rigid-body counter-rotation of the background fluid to contain them, i.e., a rotating reference frame is necessary for them to appear stationary. In the language of charges (the system here is equivalent to a collection of line charges of the same sign) a neutralizing background charge of opposite sign is necessary to prevent divergence of the electrostatic energy density.

iv. Application of Glasser's method for phase modulated sums.[4] Using Jacobi's transformation from the theory of theta functions, Glasser shows how to reduce sums of the form $\sum_{\mathbf{k}}{}' \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^{2n}}$.

v. Constraint of constant vortex density. The above ingredients give an expression for E that contains $\log L_1$ (or $\log L_2$) as a parameter. To fix the density, L_1 must scale properly with the number of vortices per unit cell J. This ensures that the same energy is obtained for equivalent lattices, i.e., a lattice whose unit cell is doubled, with no other changes, must have the same energy density as before. Multiply L_1 and L_2 by a scaling parameter q and set the density to unity to obtain invariance for equivalent lattices,

$$\frac{J}{qL_1 qL_2 \sin\phi} = 1 \Rightarrow qL_1 = \sqrt{\frac{L_1 J}{L_2 \sin\phi}} \,. \tag{9}$$

The final result for the energy density is

$$E = \overline{\Gamma^2}\left\{\frac{1}{\rho}\frac{\pi}{6}\sin\phi - \log\left(2\pi\sqrt{\sin\phi/J\rho}\right) - \log\prod_{m=1}^{\infty} H(0,0,m)\right\}$$

$$+ \frac{2}{J}\sum_{i<j}^{J}\Gamma_i\Gamma_j\left\{\frac{1}{\rho}\left[\frac{|z_{2,ij}|}{4\pi}\left(\frac{|z_{2,ij}|}{\sin\phi} - 2\pi\right) + \frac{\pi}{6}\sin\phi\right] - \frac{1}{2}\log\prod_{m=-\infty}^{\infty} H(z_{1,ij}, z_{2,ij}, m)\right\}, \tag{10}$$

where $z_{1,ij} = \dfrac{2\pi(\mathbf{r}^o_i - \mathbf{r}^o_j)\cdot\hat{\mathbf{x}}}{L_1}$, $z_{2,ij} = \dfrac{2\pi(\mathbf{r}^o_i - \mathbf{r}^o_j)\cdot\hat{\mathbf{y}}}{L_2}$, $\rho = \dfrac{L_1}{L_2}$, $\overline{\Gamma^2} = \dfrac{1}{J}\sum_{i=1}^{J}\Gamma_i^2$ and

$$H(z_1, z_2, m) = 1 - 2e^{-\frac{1}{\rho}|z_2 + 2\pi m \sin\phi|}\cos\left(z_1 + \frac{2\pi m}{\rho}\cos\phi\right) + e^{-\frac{2}{\rho}|z_2 + 2\pi m \sin\phi|} \,. \tag{11}$$

E is the relative energy density of lattices containing fixed ratios of vortex species with fixed strengths. To compare the energies of lattices with different vortex species or species ratios requires assumptions or physical information about the self-energies of the vortices. What makes Eqs. (10) and (11) useful for numerical evaluation is the fast convergence of the function $H(z_1, z_2, m)$; $m \leq 5$ is usually sufficient. Explicit periodicity in the y-direction has been removed for simplification, but is retained in the x-direction. Also, it is convenient for calculating the partial derivatives of E (used in the conjugate gradient method) to change the unit cell variables ρ and ϕ to $\sigma = 2\pi \sin\phi/\rho$ and $\chi = 2\pi \cos\phi/\rho$.

2. The Two-vortex Lattice and Two More Questions. With the help of Eq. (10) we may exhibit the complete morphology of the minima of E for the case of two vortices per unit cell; it is only necessary to consider $0 \le \gamma \le 1$ where $\gamma = \Gamma_2/\Gamma_1$. The numerical results, using the conjugate gradient method, are shown in Fig. 3. (The lattice parameters ϕ and ρ must be varied along with $N - 1$ vortex positions.)

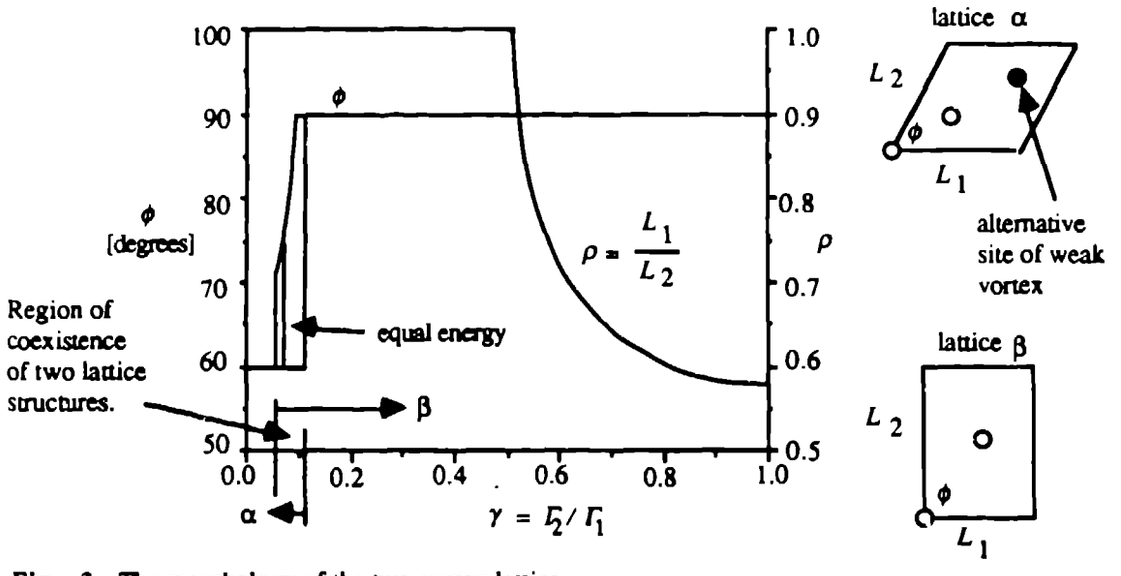

Fig. 3 The morphology of the two-vortex lattice.

At small γ the strong vortex dominates, resulting in a triangular lattice for itself, to which the weak vortex accommodates by choosing either of two equivalent positions 1/3 or 2/3 along the long diagonal of the unit cell (lattice α). These positions are also the centers of the triangles formed by the strong vortices and, thus, the positions of maximum distance from the strong vortices. At larger γ the weak vortex discontinuously switches to the center of the unit cell (lattice β) and remains there, while the shape of the unit cell continues to change, until a triangular lattice is again attained at $\gamma = 1$. It is natural to ask if this behavior is generic. In the space of the $\gamma_j = \Gamma_j / \Gamma_1$, are the minima of E never connected by a curve on which the lattice parameters and vortex positions change continuously?

Open Question III. Is the lattice space $\mathcal{L}_N(\gamma_2, \ldots, \gamma_N)$ always disjoint?

For a given set of Γ_j, it is natural to ask how the minima change, if at all, as the unit cell is doubled, tripled, etc. Is there a tendency for identical vortices to clump together and gain the benefit of forming their own triangular lattice or, on the contrary, do they diffuse as widely as possible? A disadvantage of clumping is the additional boundary energy between regions of different vortices, but if these regions are not discommensurate perhaps that energy penalty is small. Suppose there are m_j vortices of strength Γ_j, $j = 1,..M$ per unit cell. Do larger cells have smaller energy?

Open Question IV. For what m_j, Γ_j is $E[nm_j] < E[m_j]$, $n = 2,3,..$?

It seems almost nothing is known about vortex lattices.

3. Physical Applications.

a. Lattices of unequal vortices in quantum liquids. Superfluid ^{4}He in a rotating container mimics the solid-body rotation of an ordinary liquid by nucleating, parallel to Ω, rectilinear vortices of quantized strength $\Gamma_4 = h/m_4$ and density $n = 2\Omega/\Gamma$, where h is Planck's constant, m_4 the mass of a ^{4}He atom, and Ω is the angular velocity.[2] The other isotope, ^{3}He, also becomes superfluid at low temperatures but responds to a rotating container by nucleating vortices of a different strength, $\Gamma_3 = h/2m_3$. (The factor of 1/2 in Γ_3 arises from the fermi statistics of ^{3}He atoms; ^{4}He atoms are bosons.) In a fluid mixture of ^{4}He and ^{3}He both components should become superfluid at sufficiently low temperature (not yet attained) and each component will support its own type of quantized vortex line. What will be the equilibrium vortex lattice structure in a rotating, superfluid ^{3}He-^{4}He mixture? The relative densities of the two species of vortices is $n_3/n_4 = \Gamma_4/\Gamma_3 = 2m_3/m_4 = 1.50700 \approx 3/2$. Thus, for small unit cells there are 3 ^{3}He vortices for every 2 ^{4}He vortices. Examples of energy minima for the smallest unit cell and $\Gamma_3/\Gamma_4 = 2/3$ are shown in Fig. 4.

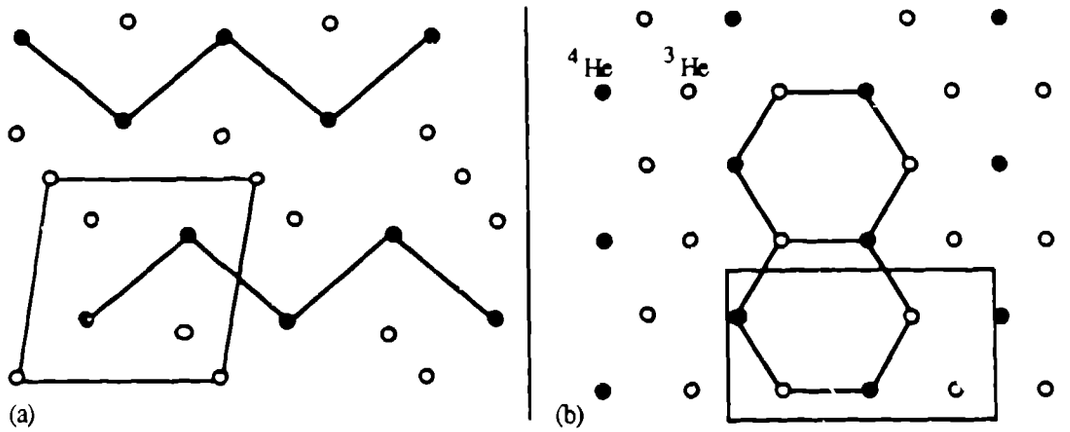

Fig. 4 Vortex lattices for ^{3}He-^{4}He mixtures of equal fluid densities. Lattice (a) has lower energy.

This, however, is wrong for the physical helium mixture because it overlooks the relative miscible densities of the isotopic components. As seen from Eq. (1) the energy of vortices is proportional to the fluid density d. The energy of a fluid mixture is a sum of their separate energies plus their interaction. If the fluids were entirely noninteracting then each would form its own, independent triangular vortex lattice. However, the helium isotopic components interact and there is a maximum amount of ^{3}He that can be mixed with ^{4}He. The energy of the system can be expressed according to Eq. (1),

$$F = -\frac{d_4}{4\pi}\sum_{(4)}\Gamma_4^{\,2}\log|\mathbf{r}_i-\mathbf{r}_j| - \frac{d_3}{4\pi}\sum_{(3)}\Gamma_3^{\,2}\log|\mathbf{r}_k-\mathbf{r}_l| - \frac{\sqrt{d_3 d_4}}{4\pi}\sum_{(3)}\sum_{(4)}\Gamma_3\Gamma_4\log|\mathbf{r}_i-\mathbf{r}_k| - \Omega(\textit{angular momentum}) \ , \qquad (12)$$

where a hydrodynamic model has been used in the third term on the right to express the ^{3}He-^{4}He interaction. This is equivalent to a single fluid containing vortices of strengths $\Gamma_4' = d_4^{1/2}\Gamma_4$ and $\Gamma_3' = d_3^{1/2}\Gamma_3$. For the maximum ^{3}He solubility this gives the ratio

$\Gamma_3'/\Gamma_4' = 0.18$. Note that the relative fluid densities do not affect the relative vortex densities. Quite different lattices are obtained using the effective strengths corrected for the fluid densities, as shown in Fig. 5.

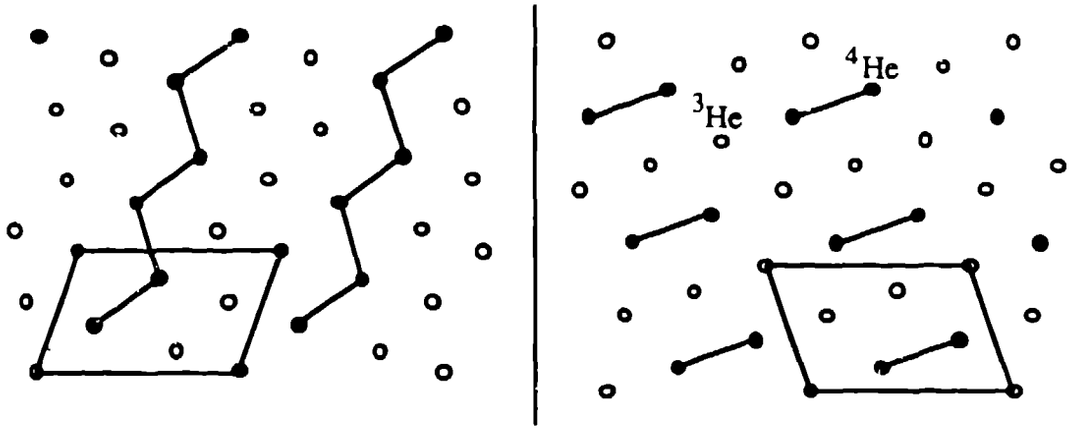

Fig. 5 Vortex lattices for ^{3}He - ^{4}He superfluid mixtures. The lattice on the left has lower energy.

These results could change further with improvements to the ^{3}He-^{4}He interaction model.

b. Lattices of anisotropic vorticity in superconductors. Type II superconductors are another example of a quantum fluid that supports quantized vorticity. In this case the circulating current in the vortex is charged, which creates a magnetic field resulting in an integrated flux over a plane perpendicular to the vortex line of $\Phi_0 = 2.07\ 10^{-7}$ gauss-cm^2. The current associated with a superconducting vortex or flux line does not extend to infinity, so the interaction between flux lines is not strictly logarithmic but rather that of the modified Bessel function $K_0(|\mathbf{r}_i - \mathbf{r}_j|/\lambda)$ where the so-called penetration depth λ sets the range of the current and the vortex-vortex interaction. This interaction is obtained in Eq. (7) by setting $\mu = \mu_0 > 0$. However, if the inter-vortex spacing l is much smaller than the range of interaction λ then Eq. (10) remains a good physical approximation.

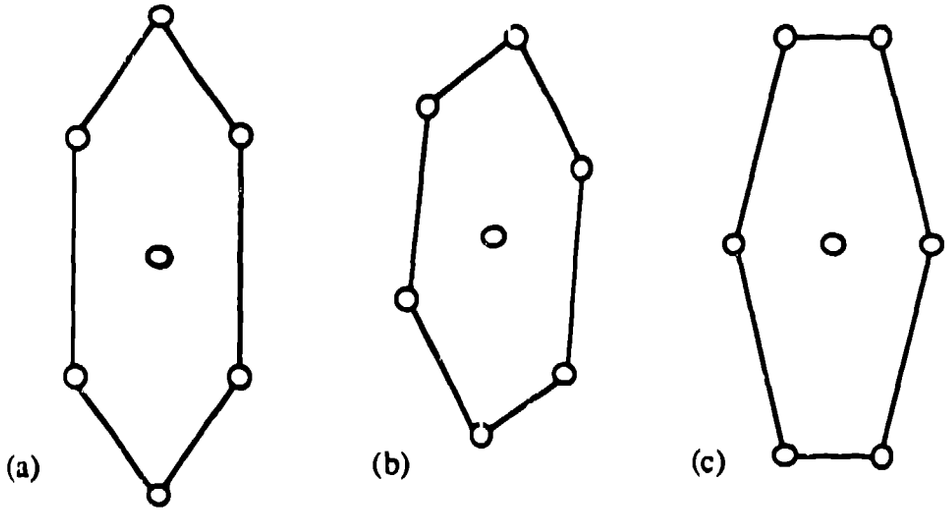

Fig. 6 Flux lattices (vortices of supercurrent) in an anisotropic superconductor arising from different orientations of a triangular lattice. All orientations between (a) and (b) are possible.

Because the new high-temperature superconductors are strongly type II (which means λ is much larger than the size of the vortex core), the condition $l << \lambda$ holds for a wide range of applied magnetic field (which creates the flux lines). Unlike the traditional superconductors, the new ones are highly anisotropic in the sense that λ is direction dependent. What are the consequences of this for the energy and the structure of the vortex lattices? A detailed answer would involve technical issues not appropriate here,[1] but a major part can be deduced immediately from the earlier discussion. The anisotropy essentially amounts to enhancing the interaction in one spatial direction, i.e., the interaction between vortices becomes $\log|\varepsilon(x_i-x_j)^2 + (y_i-y_j)^2|$, where ε is the anisotropy parameter in the x-direction. Obviously, this can be removed by a simple scale transformation, $x' = x/\varepsilon$, and everything follows as before. In particular, the minimum is the triangular lattice in the scaled space (x',y), or a "stretched" triangular lattice in physical space (x,y). The rotational continuum of triangular lattices in the scaled space are trivially equivalent, but in the physical space they are inequivanent, as shown in Fig. 6, although they have equal energy. This degeneracy in first order allows higher order physical processes to be studied by experimentally observing which lattice is preferred under different conditions.

c. Lattices constrained on grids: Josephson junction arrays. A Josephson junction array is a collection of superconductors that are interconnected by bonds that permit imperfect supercurrent flow (tunneling supercurrent). A generic characteristic of superconductors and superfluids is the existence of a new thermodynamic parameter, the phase θ. The energy of two superconductors connected with such a bond (Josephson junction) is proportional to $-\cos(\theta_1-\theta_2)$. The energy of an array of Josephson junctions is[5,9]

$$E_a = -K_0 \sum_{\langle ij \rangle} \cos(\theta_i-\theta_j-\psi_{ij}) , \tag{13}$$

where $\langle i\,j \rangle$ are nearest neighbors,

$$\psi_{ij} = \frac{2\pi}{\Phi_0} \int_i^j \mathbf{A}\cdot d\mathbf{s} , \tag{14}$$

and the vector potential A is given by the applied magnetic field, $\mathbf{H} = \nabla\times\mathbf{A}$. The integral $\int \mathbf{A}\cdot d\mathbf{s}$ is over the bond between superconductors i and j. This is illustrated schematically in Fig. 7(a), where periodic boundary conditions are assumed. The energy function E_a can be mapped to a new function E_b on the dual lattice illustrated in Fig. 7(b),

$$E_b = -K_1 \sum_{i,j} q_i q_j \log|\mathbf{r}_i-\mathbf{r}_j| , \tag{15}$$

where the "charges" or vortices have mixed signs, $q_i = 1-f$ or $-f$. The ratio of the number of $1-f$ vortices to $-f$ vortices is simply $f/(1-f)$, i.e., the net charge or circulation is zero. The parameter f, the so-called filling factor, is the ratio $f = \mathbf{H}_z a^2/\Phi_0$, where a is the length of the grid spacing; in Fig. 7(b), $f = 1/6$. At this stage one could apply Eq. (10) to find the minimum of E_b by constraining the shape of the unit cell (to be square) and the positions of the vortices to be on the grid. [Note that Eq.(10) is perfectly valid for vortices of mixed sign, and can be used to find their minimum energy configurations in this case because the grid constraint prevents singularities.] Some permutations of the 36 vortices would give minimum E_b. However, one can do better on two counts. First, I have found a further mapping from the two-species to a single-species system, thereby reducing the length of

the calculation. Second, with a system of vortices having the same sign the minimum energy can first be found quickly in a unit cell without grid points and the resulting pattern then attached to the grid. (Incidentally, there is no fundamental reason to restrict either the unit cell of grid points or their symmetry to be square.)

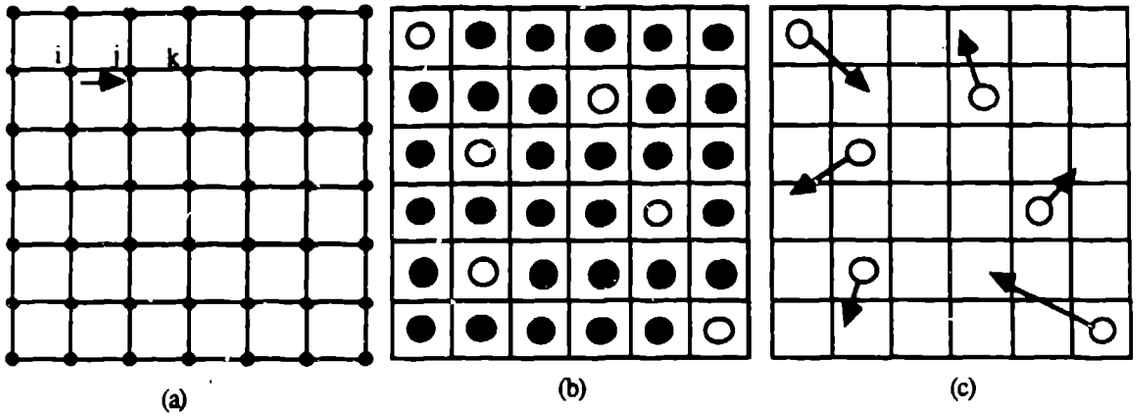

Fig. 7 Grids representing the mapping of a Josephson junction array to a single-species vortex system.

To specify the mapping from (b) to (c) in Fig. 7 (ignore the arrows for the moment) let $E(\Gamma_1,\Gamma_2)$ be the energy from Eq.(10) of two vortex species that fill all the grid points of an $I \times I$ square grid, $N_1 + N_2 = I^2$. The filling factor (sometimes called the frustration) is $f \equiv N_1/I^2 \le 1/2$. The principal mapping is

$$E(1-f,-f) = E(1,0) - f^2 E_{sq} \ , \tag{16}$$

and, for completeness, I list a similar one, $E(1,-1) = 4E(1,0) - (4f-1)E_{sq}$. Here, $E_{sq} = -1.3105329259$ is the energy of the square lattice. (For comparison, the energy of the triangular lattice is $E_{tr} = -1.3211174284$.) The two-species energy $E(1,0)$ is simple to evaluate compared to $E(1-f,-f)$ and can be further reduced to an explicitly one-species form $E(\Gamma_1)$ by $E(1,0) = f[E(1) - (1/2)\log f]$. That is, for $E(1,0)$ the number of vortices per unit cell is $J = I^2$, while for $E(1)$, $J = N_1$.

The task of finding low energy states of the Josephson junction array has been reduced to finding low energy states of N_1 unit vortices on a grid. These states cannot be local minima of the energy because, as far as we know, only the triangular lattice is a minimum for the single-species case. Therefore, the vortex-vortex interactions will not usually cancel at the vortex positions, but result in a "force" on each vortex [indicated by the arrows in Fig. 7(c)] unless the configuration is a saddle point of E, which can occur if the configuration is particularly symmetric or "commensurate". (These remnant forces become important for the dynamical properties of the system.) As mentioned above, a particularly efficient way to find low energy states is to first find minimal energy configurations of N_1 vortices in a square unit cell (obviously, such minima can never be triangular) and then overlay a grid on the pattern and move each vortex to the nearest grid point. This does not guarantee the lowest energy grid state at a given f, but typically comes close. (Another method is to overlay a grid on the triangular lattice - in practice this actually gives relatively poor results, i.e., high energies.) Fig. 8 shows values of low energy vortex grid states for two hundred values of the filling factor f. The energy is normalized to the square and triangular lattices, $E_C = [E(1,0) - E_{tr}]/[E_{sq} - E_{tr}]$. As $f \to 0$ the vortices become dilute on the grid and can always approach the favorable triangular and square lattices. Special

values of f allow an exact square lattice. The relative sparseness in the number of values plotted at higher f merely reflects the fewer number of low energy states found there. Obviously, an energy can be given to every f, but the problem is to find the lowest energy states. The connectivity drawn between e (f) values is only a guide to the eye.

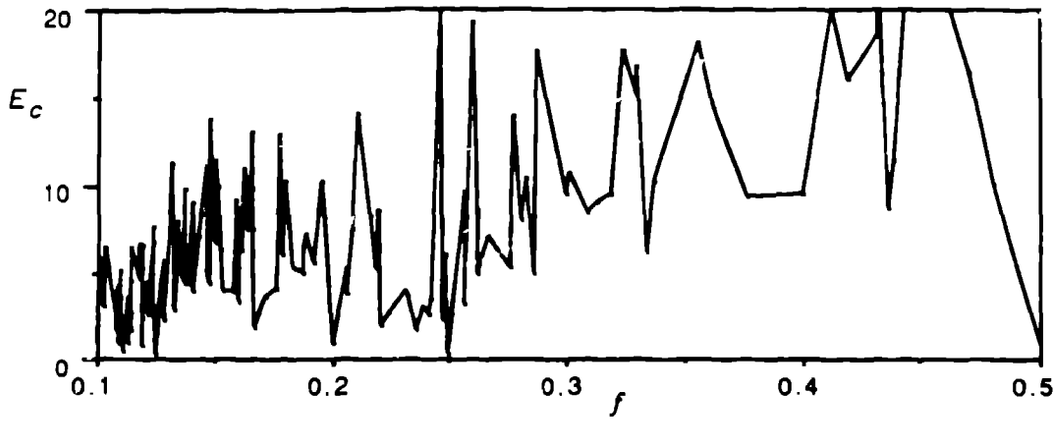

Fig. 8 Vortex grid energy vs. filling fraction.

4. Summary. An expression is now available that is well suited for numerical evaluation of the energy density of arbitrary vortex lattices. The basic properties of stable vortex lattices are currently unknown and, judging from the two-vortex lattice, their morphology promises to be rich. Quantum fluids, in the general sense, offer a direct application of vortex lattice theory. Further applications can be found in systems of logarithmically interacting objects (dislocations, line currents, line charges, etc.) or by mapping to a vortex lattice, as illustrated by the Josephson junction array.

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